Blossoming trees and planar maps

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Séminaire Philippe Flajolet, 6th February 2014
Planar Maps – Definition.

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Plane maps are rooted: by orienting an edge.
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Plane maps are **rooted**: by orienting an edge.

Distance between two vertices = number of edges between them.

Planar map = **Metric space**
Which maps?

- **Quadrangulations**
- **4-regular maps**
- **Simple triangulations** (no loops nor multiple edges)
Why maps?

What the motivation for studying maps instead of graphs?

Because maps have more structure than graphs, they are actually simpler to study.
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Euler Formula: \( \# \text{vertices} + \# \text{faces} = 2 + \# \text{edges} \)

A quadrangulation with \( n \) faces has \( 2n \) edges and \( n + 2 \) vertices.
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A quadrangulation with \( n \) faces has \( 2n \) edges and \( n + 2 \) vertices.

Structure allows recursive decomposition \( \Rightarrow \) enumeration [Tutte, ’60s].

Two possibilities:
- The root edge is an isthmus
- The root edge is not an isthmus

\[ q_n = \text{number of quadrangulations with } n \text{ faces} = \frac{2}{n+2} \cdot \frac{3^n}{n+1} \binom{2n}{n} \]
Random maps

$\mathcal{Q}_n = \{\text{Quadrangulations of size } n\}$

$= n + 2$ vertices, $n$ faces, $2n$ edges

$\mathcal{Q}_n = \text{Random element of } \mathcal{Q}_n$

$(V(\mathcal{Q}_n), d_{gr})$ is a random compact metric space
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Simulations by N.Curien
Random maps

What is the behavior of $Q_n$ when $n$ goes to infinity? typical distances? convergence towards a continuous object?
Random maps

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typical distances?

convergence towards a continuous object?

well understood:

• Schaeffer’s bijection : quadrangulations $\leftrightarrow$ labeled trees.
  Labels in the trees = distances in the map.

• distance between two random points $\sim n^{1/4} +$ law of the distance
  [Chassaing-Schaeffer ’04]

• convergence of normalized quadrangulations $+$ limiting object: Brownian map.
  [Marckert-Mokkadem ’06], [Le Gall ’07], [Miermont ’08],
  [Miermont 13], [Le Gall 13]
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+ what if quadrangulations are replaced by triangulations, simple triangulations, 4-regular maps?
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Idea: The Brownian map is a universal limiting object. All "reasonable models" of maps (properly rescaled) are expected to converge towards it.

Problem: These results rely on nice bijections between maps and labeled trees [Schaeffer '98], [Bouttier-Di Francesco-Guitter '04].
Which maps?

Quadrangulations
Number of quadrangulations with \( n \) faces:

\[
q_n = \frac{2 \cdot 3^n}{(n + 2)(n + 1)} \binom{2n}{n}
\]

[Cori-Vauquelin '81], [Schaeffer '98]

4-regular maps

Simple triangulations (no loops nor multiple edges)
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[Tutte, 60], [Cori-Vauquelin '81], [Schaeffer '98]

**4-regular maps**
Number of rooted 4-regular maps with $n$ vertices:
\[
R_n = \frac{2 \cdot 3^n}{n + 1} \binom{2n}{n}
\]
[Tutte, 62], [Schaeffer '97]

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Which maps?

**Quadrangulations**

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**Simple triangulations** (no loops nor multiple edges)

Number of simple triangulations with $n + 2$ vertices:

$$\Delta_n = \frac{2 \cdot (4n - 3)!}{n!(3n - 1)!}$$

[Tutte, 62], [Poulalhon-Schaeffer '05]
History: what questions about maps?

- **Enumerate them**: a lot of different techniques

  - Recursive decomposition: [Tutte, '60]
  - Matrix integrals: [t’Hooft, '74], [Brézin, Itzykson, Parisi and Zuber ’78]
  - Representation of the symmetric group: [Goulden and Jackson ’87].

  - Bijective approach with labeled trees: [Cori-Vauquelin ’81], [Schaeffer ’98], [Bouttier, Di Francesco and Guitter ’04], [Bernardi and Fusy], ...
  - Bijective approach with blossoming trees: [Schaeffer ’98], [Schaeffer and Bousquet-Mélou ’00], [Poulalhon and Schaeffer ’05], [Fusy, Poulalhon and Schaeffer ’06], [Bernardi and Fusy]
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- **Sample them** (efficiently)
  - Take a bijection between maps and trees, sample a tree (easy), you’re DONE.
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- **Understand random ones**
  Take a bijection between maps and trees, study the trees (complicated but doable), relate the distances in the maps and in the trees (sometimes OK, sometimes not), work a lot, you’re DONE (maybe).
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  **Bijective approach with blossoming trees.**

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Today: what’s the plan?

What is a blossoming tree?

Can we unify the constructions involving blossoming trees?

Can we prove some convergence results to the Brownian map using blossoming trees? i.e. can we put ”distances” on trees?
**Today: what’s the plan?**

What is a blossoming tree? Wait a second

Can we unify the constructions involving blossoming trees?

Yes, cf also [Bernardi, Fusy]

Can we prove some convergence results to the Brownian map using blossoming trees?

i.e. can we put ”distances” on trees? Yes ... for some models
What is a blossoming tree?

A blossoming tree is a plane tree where vertices can carry opening stems or closing stems, such that:

\[ \# \text{ closing stems} = \# \text{ opening stems} \]
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![Diagram of blossoming tree with opening and closing stems]
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A plane map can be canonically associated to any blossoming tree by making all closures clockwise.
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If the tree is rooted and its edges oriented towards the root + closure edges oriented naturally

⇒ Accessible orientation of the map without ccw cycles.
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⇒ Accessible orientation of the map without ccw cycles.
Can we transform a plane map into a blossoming tree?

**Theorem**: [Bernardi '07], [A., Poulalhon 14+]

If a plane map $M$ has a marked vertex $v$ is endowed with an orientation such that:

- there exists a directed path from any vertex to $v$,
- there is no counterclockwise cycle,

then there exists a **unique** blossoming tree rooted at $v$ whose closure is $M$ endowed with the same orientation.
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Proof by induction on the number of faces + identification of closure edges ....
Orientations

**Orientation** = orientation of the edges of the map.

To apply the construction: need to find **canonical orientations**
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4-regular maps

2 outgoing edges/vertex
2 ingoing edges/vertex
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To apply the construction: need to find **canonical orientations**

- **4-regular maps**
  - 2 outgoing edges/vertex
  - 2 ingoing edges/vertex

A map is 4-regular iff it admits an orientation with indegree 2 and outdegree 2 for each vertex.
Orientations

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To apply the construction: need to find **canonical orientations**

**4-regular maps**
- 2 outgoing edges/vertex
- 2 ingoing edges/vertex

**Simple triangulations**
- 3 outgoing edges / non-root vertex
- 1 outgoing edge / root vertex
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### 4-regular maps

- 2 outgoing edges/vertex
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### Simple triangulations

- 3 outgoing edges / non-root vertex
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A triangulation is simple iff it admits an orientation with:
- Outdegree 3 for each non-root vertex
- Outdegree 1 for each vertex on the root face.
Orientations

4-regular maps

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Simple triangulations

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Many families admit a characterization via orientations
(description of the orientation = outdegree for each vertex is prescribed)
Orientations

4-regular maps

2 outgoing edges/vertex
2 ingoing edges/vertex

Simple triangulations

3 outgoing edges / non-root vertex
1 outgoing edge / root vertex

Theorem requires accessible orientation without ccw cycles:
Too much too ask?
Orientations

4-regular maps
2 outgoing edges/vertex
2 ingoing edges/vertex

Simple triangulations
3 outgoing edges / non-root vertex
1 outgoing edge / root vertex

Theorem requires accessible orientation without ccw cycles: NO!
Too much too ask?

Proposition: [Felsner ’04]
For a given map and orientation, there exists a unique orientation with the same outdegrees and without ccw cycles.
If there exists one accessible such orientation, all of them are accessible.
Summary

- Take a family of maps,

Maps with even degrees.
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• Take a family of maps,
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• Apply the bijection,

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Summary

- Take a family of maps,
- Try to find a characterization of the family by an orientation,
- Consider the unique orientation without counterclockwise cycles,
- Apply the bijection,
- Study the family of blossoming trees.
Distances in blossoming trees: simple triangulations

Simple Triangulation:
- no multiple edges
- no loops

Euler Formula: \( v + f = 2 + e \)

Triangulation: \( 2e = 3f \)

\( \mathcal{M}_n = \{ \text{Simple triangulations of size } n \} \)
= \( n + 2 \) vertices, \( 2n \) faces, \( 3n \) edges

\( M_n = \text{Random element of } \mathcal{M}_n \)

What is the behavior of \( M_n \) when \( n \) goes to infinity?
Typical distances? Scaling limit of \( M_n \)?
From simple triangulations to blossoming trees

Simple triangulation endowed with its unique orientation such that:
- no counterclockwise cycle
- $\text{out}(v) = 3$ for $v$ an inner vertex
- $\text{out}(v) = 1$ for $v$ an outer vertex
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The closure operation is a bijection between balanced 2-blossoming trees and simple triangulations.

**Theorem:** [Poulalhon, Schaeffer ’05]

The closure operation is a bijection between balanced 2-blossoming trees and simple triangulations.
Same bijection with corner labels

- Start with a planted 2-blossoming tree.
- Give the root corner label 2.
Same bijection with corner labels

- Start with a planted 2-blossoming tree.
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In contour order, apply the following rules:

- Non-leaf to leaf, label does not change.
- Leaf to non-leaf, label increases by 1.
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Aside: Tree is balanced $\iff$

\[ \text{all labels } \geq 2 \]

+ root corner incident to two stems

Closure: Merge each leaf with the first subsequent corner with a smaller label.
Same bijection with corner labels

Aside: Tree is balanced ⇔
all labels ≥ 2
+root corner incident to two stems
Closure: Merge each leaf with the first subsequent corner with a smaller label.
Same bijection with corner labels

Aside: Tree is balanced ⇔
    all labels \( \geq 2 \)
    + root corner incident to two stems

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From blossoming trees to labeled trees

In the following:
Labels gives approximate distances to the root **in the map**

**label of a vertex =**
minimum label of its corners
From blossoming trees to labeled trees

Generic vertex:

\[ \begin{align*}
&i - 1 \quad i \\
&i - 1 \quad i + 1 \\
&i - 1 \quad i + 1
\end{align*} \]
From blossoming trees to labeled trees

Generic vertex:

- Can retrieve the blossoming tree from the labeled tree.
- Labeled tree = GW trees + random displacements on edges uniform on \((-1, -1, \ldots, -1, 0, 0, \ldots, 0, 1, 1 \ldots, 1)\).

almost the setting of [Janson-Marckert] and [Marckert-Miermont] but r.v are not ”locally centered” ⇒ symmetrization required
Distances in simple triangulations

**Claim 1:** \(3d_M(root, u) \geq \text{Label of } u\)

First observation: In the tree, the labels of two adjacent vertices differ by at most 1. **What can go wrong with closures?**
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- Consider the Left Most Path from $(u, v)$ to the root face.
- For each inner vertex: 3 LMP
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- LMP are not self-intersecting \(\Rightarrow\) they reach the outer face
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- LMP are not self-intersecting \(\Rightarrow\) they reach the outer face
- On the left of a LMP, corner labels decrease exactly by one.
LMP are almost geodesic

Leftmost path
Another path: can it be shorter?
LMP are almost geodesic

Euler Formula:
\[ |E(T_q)| = 3|V(T_q)| - 3 - (\ell_p + \ell_q) \]

3-orientation + LMP:
\[ |E(T_q)| \geq 3|V(T_q)| - 2\ell_q - 2 \]

\[ \implies \ell_q \geq \ell_p + 1 \]

Leftmost path
Another path: can it be shorter?
LMP are almost geodesic

Leftmost path

Another path: can it be shorter?

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LMP are almost geodesic

Leftmost path

Another path: can it be shorter? YES

\[ \ell_q \geq \ell_p + 1 \]

\[ \ell_q \geq \ell_p \]

\[ \ell_q \geq \ell_p + 3 \]

\[ \ell_q \geq \ell_p - 2 \]

with possible equality
LMP are almost geodesic

Leftmost path

Another path: can it be shorter? YES ... but not too often

Bad configuration = too many windings around the LMP

But w.h.p a winding cannot be too short.

\[ \Rightarrow \text{w.h.p the number of windings is } o(n^{1/4}). \]
**LMP are almost geodesic**

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too many **windings** around the LMP

But w.h.p a winding cannot be too short.

$$\implies \text{w.h.p the number of windings is } o(n^{1/4}).$$

---

**Proposition:**

For $\varepsilon > 0$, let $A_{n,\varepsilon}$ be the event that there exists $u \in M_n$ such that

Label of $u \geq d_{M_n}(u, \text{root}) + \varepsilon n^{1/4}$.

Then under the uniform law on $M_n$, for all $\varepsilon > 0$:

$$\mathbb{P}(A_{n,\varepsilon}) \to 0.$$
The result

**Theorem:** [Addario-Berry, A.]

$\{M_n\} =$ sequence of random **simple** triangulations, then:

$$\left( M_n, \left( \frac{3}{4n} \right)^{1/4} d_{M_n} \right) \xrightarrow{(d)} \text{Brownian map}$$

for the distance of Gromov-Hausdorff on the isometry classes of compact metric spaces.

Simulation by J.F Marckert
Beyond the universality

Simple triangulations converge to the Brownian map
⇒ properties of the Brownian map from the simple triangulations?
Beyond the universality

Simple triangulations converge to the Brownian map
⇒ properties of the Brownian map from the simple triangulations?

One motivation: Circle-packing theorem

Each simple triangulation $M$ has a unique (up to Möbius transformations and reflections) circle packing whose tangency graph is $M$.

[Koebe-Andreev-Thurston]

Gives a canonical embedding of simple triangulations in the sphere and possibly of their limit.
Random circle packing

Random circle packing = canonical embedding of random simple triangulation in the sphere.

Gives a way to define a canonical embedding of their limit?

Team effort: code by Kenneth Stephenson, Eric Fusy and our own.
Perspectives

Same approach works also for simple quadrangulations.

Can we make this approach work for the general setting of bijections developed in [A., Poulalhon] and in [Bernardi, Fusy]?

Can we say something about a random circle packing?
Perspectives

Same approach works also for simple quadrangulations.

Can we make this approach work for the general setting of bijections developed in [A., Poulalhon] and in [Bernardi, Fusy]?

Can we say something about a random circle packing?

Thank you!
Brownian snake \((e_t, Z_t)_{0 \leq t \leq 1}\)

1st step: the Brownian tree [Aldous]

\[ C_n^T \text{ (or } C_n) = \text{contour process} \]

\[ i \text{ and } j = \text{same vertex of } T \]

\[ \text{iff } C_n(i) = C_n(j) = \min_{i \leq k \leq j} C_n(k) \]
Brownian snake \((e_t, Z_t)_{0 \leq t \leq 1}\)

1st step: the Brownian tree [Aldous]

\[ C^T_n \text{ (or } C_n \text{)} = \text{contour process} \]

\[ (e_t)_{0 \leq t \leq 1} = \text{Brownian excursion} \]
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1st step: the Brownian tree [Aldous]

\[ T \]

\[ C_n^T \text{ (or } C_n) = \text{ contour process} \]

\[ \mathcal{T}_e \]

\[(e_t)_{0 \leq t \leq 1} = \text{ Brownian excursion} \]

\[ d_e(u, v) = e_u + e_v - 2 \min_{u \leq s \leq v} e_s \]

\[ \mathcal{T}_e = [0, 1]/ \sim_e \]

\[ u \sim_e v \text{ iff } d_e(u, v) = 0 \]
Brownian snake \((e_t, Z_t)_{0\leq t\leq 1}\)

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2nd step: Brownian labels

Conditional on \(T_e\), \(Z\) a centered Gaussian process with \(Z_0 = 0\) and

\[ E[(Z_s - Z_t)^2] = d_e(s, t) \]

\[ Z \sim \text{Brownian motion on the tree} \]
The Brownian map

\[ d_e(u, v) = e_u + e_v - 2 \min_{u \leq s \leq v} e_s \]
\[ \mathcal{T}_e = [0, 1] / \sim e \]
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Conditional on \( \mathcal{T}_e \), \( Z \) a centered Gaussian process with \( Z_\rho = 0 \) and
\[ E[(Z_s - Z_t)^2] = d_e(s, t) \]
\[ Z \sim \text{Brownian motion on the tree} \]
The Brownian map

\[ \bar{u} \]

\[ 0 \quad u \quad v \quad 1 \]

\[ d_e(u, v) = e_u + e_v - 2 \min_{u \leq s \leq v} e_s \]

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\[ D^\circ(s, t) = Z_s + Z_t - 2 \max \left( \inf_{s \leq u \leq t} Z_u, \inf_{t \leq u \leq s} Z_u \right), \quad s, t \in [0, 1]. \]
The Brownian map

\[ d_e(u, v) = e_u + e_v - 2 \min_{u \leq s \leq v} e_s \]

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\[ D^*(a, b) = \inf \left\{ \sum_{i=1}^{k-1} D^\circ(a_i, a_{i+1}) : k \geq 1, a = a_1, a_2, \ldots, a_{k-1}, a_k = b \right\}, \]
The Brownian map

\(d_e(u, v) = e_u + e_v - 2 \min_{u \leq s \leq v} e_s\)

\(\mathcal{T}_e = [0, 1]/\sim_e\)

\(u \sim_e v \iff d_e(u, v) = 0\)

Conditional on \(\mathcal{T}_e\), \(Z\) a centered Gaussian process with \(Z_0 = 0\) and

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Then \(M = (\mathcal{T}_e/\sim_{D^*}, D^*)\) is the \text{Brownian map}.\]