Dynamique euclidienne: une approche symbolique

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Séminaire de combinatoire Philippe Flajolet
Euclid algorithm...

and...

- continued fractions
- dynamical analysis, costs
- symbolic dynamics: the Sturmian case
- higher-dimensional generalizations
Analysis of algorithms

An algorithm

Euclid algorithm

According to Knuth

‘the granddaddy of all algorithms, because it is the oldest nontrivial algorithm that has survived to the present day’

Euclidean dynamics

An algorithm

Euclid algorithm

together with a dynamical system

Gauss map

\[ T : [0, 1] \rightarrow [0, 1], \ x \mapsto \{1/x\} \]
Euclid algorithm

We start with two nonnegative integers $u_0$ and $u_1$

\[
\begin{align*}
    u_0 &= u_1 \left\lfloor \frac{u_0}{u_1} \right\rfloor + u_2 \\
    u_1 &= u_2 \left\lfloor \frac{u_1}{u_2} \right\rfloor + u_3 \\
    &\vdots \\
    u_{m-1} &= u_m \left\lfloor \frac{u_{m-1}}{u_m} \right\rfloor + u_{m+1} \\
    u_{m+1} &= \gcd(u_0, u_1) \\
    u_{m+2} &= 0
\end{align*}
\]
We start with two coprime integers $u_0$ and $u_1$

$$u_0 = u_1 a_1 + u_2$$

$$\vdots$$

$$u_{m-1} = u_m a_m + u_{m+1}$$

$$u_m = u_{m+1} a_{m+1} + 0$$

$$u_{m+1} = 1 = \gcd(u_0, u_1)$$
Euclid algorithm and continued fractions

We start with two coprime integers $u_0$ and $u_1$

$$u_0 = u_1 a_1 + u_2$$

$$u_{m-1} = u_m a_m + u_{m+1}$$

$$u_m = u_{m+1} a_{m+1} + 0$$

$$u_{m+1} = 1 = \gcd(u_0, u_1)$$

Euclid’s algorithm yields the digits for the continued fraction expansion of $\frac{u_1}{u_0}$
Euclid algorithm and continued fractions

We start with two coprime integers $u_0$ and $u_1$

\[ u_0 = u_1 a_1 + u_2 \]

\[ \vdots \]

\[ u_{m-1} = u_m a_m + u_{m+1} \]

\[ u_m = u_{m+1} a_{m+1} + 0 \]

\[ u_{m+1} = 1 = \gcd(u_0, u_1) \]

\[
\frac{u_1}{u_0} = \frac{1}{a_1 + \frac{u_2}{u_1}} \quad \implies \quad \frac{u_1}{u_0} = \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_m + \frac{1}{a_{m+1}}}}} \]
Continued fractions and dynamical systems

Consider the Gauss map

\[ T : [0, 1] \to [0, 1], \ x \mapsto \{1/x\} \]

\[ x_1 = T(x) = \{1/x\} = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor = \frac{1}{x} - a_1 \]

\[ x = \frac{1}{a_1 + x_1} \quad \quad a_n = \left\lfloor \frac{1}{T^{n-1}x} \right\rfloor \]

\[ x = \frac{1}{a_1 + \frac{1}{\frac{a_2 + \frac{1}{a_3 + \cdots}}{a_1}}} \]
Continued fractions and dynamical systems

Consider the Gauss map

\[ T : [0, 1] \rightarrow [0, 1], \ x \mapsto \{1/x\} \]

Figure 1:

\[ T(x) = \{1/x\} = \frac{1}{x} - \left[ \frac{1}{x} \right] = \frac{1}{x} - a_1 \]

\[ \frac{1}{k+1} < x \leq \frac{1}{k} \implies a_1 = k \]
Discrete dynamical system

We are given a dynamical system

\[ T : X \rightarrow X \]

Discrete stands for discrete time

We consider orbits/trajectories of points of $X$ under the action of the map $T$

\[ \{ T^n x \mid n \in \mathbb{N} \} \]

How well are the orbits distributed?

According to which measure?
Continued fractions and ergodicity

Ergodicity has to do with the long term statistical behaviour of orbits
Continued fractions and ergodicity

Ergodicity has to do with the long term statistical behaviour of orbits

The Gauss map is ergodic with respect to the Gauss measure

\[ \mu(B) = \frac{1}{\log 2} \int_B \frac{1}{1 + x} \, dx \]

\[ \mu(B) = \mu(T^{-1}B) \quad T\text{-invariance} \]

\[ T^{-1}B = B \implies \mu(B) = 0 \text{ or } 1 \quad \text{ergodicity} \]

\[ \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \int f \, d\mu \quad \text{ergodic theorem} \]

The mean behaviour along an orbit= the mean value of \( f \) with respect to \( \mu \).
Measure-theoretic results

- Gauss measure

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{dx}{1 + x}$$

- Convergents

For a.e. $x$, $$\lim_{n \to \infty} \frac{\log q_n}{n} = \frac{\pi^2}{12 \log 2}$$

- Densities of partial quotients

For a.e. $x$ and $a \geq 1$

$$\lim_{N \to \infty} \frac{1}{N} \{k \leq N; \ a_k = a\} = \frac{1}{\log 2} \log \frac{(a + 1)^2}{a(a + 2)}$$
Rational vs. irrational parameters

Euclid algorithm ⇾ gcd ⇾ rational parameters
Continued fractions ⇾ irrational parameters
Rational vs. irrational parameters

Euclid algorithm $\leadsto$ gcd $\leadsto$ rational parameters
Continued fractions $\leadsto$ irrational parameters

- When computing a gcd, we work with integer/rational parameters
- This set has zero measure
- Ergodic methods produce results that hold only almost everywhere

Is it relevant to compare generic orbits and orbits for integer parameters?
Dynamical analysis of Euclid algorithm
Number of steps $\ell(u, v)$

$\ell(u, v)$ : number of steps in Euclid algorithm $0 < v < u$

- **Worst case**

  \[
  \ell(u, v) = O(\log v) \quad (\leq 5 \log_{10} v, \text{ Lamé 1844})
  \]

  Reynaud 1821 [$\ell(u, v) < v/2$], see Shallit’s survey
Number of steps $\ell(u, v)$

$\ell(u, v)$: number of steps in Euclid algorithm $0 < v < u$

- **Worst case**
  
  $\ell(u, v) = O(\log v)$ \quad ($\leq 5 \log_{10} v$, Lamé 1844)

- **Mean case**
  
  $0 < v < u \leq N \quad \gcd(u, v) = 1$

  $E_N(\ell) \sim \frac{12 \log 2}{\pi^2} \cdot \log N + \eta$

[see Knuth, Vol. 2]
Number of steps $\ell(u, \nu)$

$\ell(u, \nu)$ : number of steps in Euclid algorithm $0 < \nu < u$

- **Worst case**

  $\ell(u, \nu) = O(\log \nu)$  
  ($\leq 5 \log_{10} \nu$, Lamé 1844)

- **Mean case** $0 < \nu < u \leq N$  
  $\gcd(u, \nu) = 1$

  \[
  \frac{12 \log 2}{\pi^2} \cdot \log N + \eta + O(N^{-\gamma})
  \]

  asymptotically normal distribution

  [Heilbronn’69, Dixon’70, Porter’75, Hensley’94, Baladi-Vallée’05...]
Distributional dynamical analysis

\[ \gcd(u_0, u_1) = 1 \quad N \geq u_0 > u_1 > \cdots \quad u_{k-1} = a_k u_k + u_{k+1} \]

\textbf{Cost} of moderate growth \( c(a) = O(\log a) \)

- Number of \textbf{steps} in Euclid algorithm \( c \equiv 1 \)
- Number of \textbf{occurrences} of a quotient \( c = 1_a \)
- \textbf{Binary length} of a quotient \( c(a) = \log_2(a) \)

\[ \text{Theorem} \quad \left[ \text{Baladi-Vallée'05} \right] \quad \text{Cost} = 12 \log 2 \pi 2 \hat{\mu}(\text{Cost}) \log N + O(1) \]

The distribution is asymptotically Gaussian (CLT)
Distributional dynamical analysis

\[ \gcd(u_0, u_1) = 1 \quad N \geq u_0 > u_1 > \cdots \quad u_{k-1} = a_k u_k + u_{k+1} \]

Cost of moderate growth \( c(a) = O(\log a) \)

- Number of steps in Euclid algorithm \( c \equiv 1 \)
- Number of occurrences of a quotient \( c = 1_a \)
- Binary length of a quotient \( c(a) = \log_2(a) \)

**Theorem** [Baladi-Vallée’05]

\[
\mathbb{E}_N[\text{Cost}] = \frac{12 \log 2}{\pi^2} \cdot \hat{\mu}(\text{Cost}) \cdot \log N + O(1)
\]

The distribution is asymptotically Gaussian (CLT)

Discrete framework-Euclid algorithm
Ergodic theorem

**Theorem** [Baladi-Vallée’05]

\[ E_N[\text{Cost}] = \frac{12 \log 2}{\pi^2} \cdot \hat{\mu}(\text{Cost}) \cdot \log N + O(1) \]
Ergodic theorem

**Theorem [Baladi-Vallée’05]**

\[ \mathbb{E}_N[\text{Cost}] = \frac{12 \log 2}{\pi^2} \cdot \hat{\mu}(\text{Cost}) \cdot \log N + O(1) \]

\[ \mathbb{E}_N[c] = \frac{\text{dimension}}{\text{entropy}} \cdot \hat{\mu}(c) \cdot \log N + O(1) \]

\[ \hat{\mu}(c) = \int_0^1 c([1/x]) \frac{1}{\log 2} \frac{1}{1 + x} dx \]

Continuous framework-truncated trajectories
Cost of truncated trajectories

Cost of moderate growth

\[ c(a_i) = O(\log a_i) \text{ for } a_i \text{ partial quotient} \]

\[ x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} \]
Cost of truncated trajectories

Cost of moderate growth

\[ c(a_i) = O(\log a_i) \text{ for } a_i \text{ partial quotient} \]

Cost of a truncated trajectory

\[ C_n(x) = \sum_{i=1}^{n} c(a_i(x)) \quad a_i = \left[ \frac{1}{T^{i-1}(x)} \right] \]

According to the ergodic theorem, for a.e. \( x \in [0, 1] \)

\[ C_n(x)/n \to \hat{\mu}(x) \]

\[ \hat{\mu}(C) = \int_{0}^{1} c\left( \left[ \frac{1}{x} \right] \right) \cdot \frac{1}{\log 2} \frac{1}{1 + x} \cdot dx \]

\[ \mathbb{E}_N[C] = \frac{2}{\pi^2/(6 \log 2)} \cdot \hat{\mu}(C) \cdot \log N \]
Dynamical analysis of algorithms [Vallée]

It belongs to the area of

- Analysis of algorithms [Knuth’63]
  probabilistic, combinatorial, and analytic methods

- Analytic combinatorics [Flajolet-Sedgewick]
  generating functions and complex analysis, analytic functions, analysis of the singularities
Dynamical analysis of algorithms [Vallée]

It mixes tools from

- **dynamical systems** (transfer operators, density transformers, Ruelle-Perron-Frobenius operators)
- **analytic combinatorics** (generating functions of Dirichlet type)

the **singularities** of (Dirichlet) generating functions are expressed in terms of **transfer** operators
Euclidean dynamics [Vallée]

One starts with a **discrete** algorithm

- This algorithm is extended into a **continuous** one in terms of a **dynamical system**
  
  Orbits/trajectories = executions

- Main parameters of the algorithm are studied in the continuous framework

  **rational** trajectories $\leftrightarrow$ **generic** trajectories

- One comes back to the discrete algorithm

  A transfer from continuous to discrete

  ‘The probabilistic behaviour of gcd algorithms is quite similar to the behaviour of their continuous counterparts’
Rational vs. irrational parameters

Euclid algorithm $\leadsto$ gcd $\leadsto$ rational parameters
Continued fractions $\leadsto$ irrational parameters

Is it relevant to compare generic orbits and orbits for integer parameters?
Rational vs. irrational parameters

Euclid algorithm \(\leadsto\) gcd \(\leadsto\) rational parameters

Continued fractions \(\leadsto\) irrational parameters

Is it relevant to compare generic orbits and orbits for integer parameters?

Average-case analysis vs. a.e. results

Fact Orbits of rational points tend to behave like generic orbits

And their probabilistic behaviour can be captured thanks to the methods of dynamical analysis of algorithms
Gauss map

&

symbolic dynamics
We are given a dynamical system

$$T : X \rightarrow X$$
We are given a dynamical system

\[ T : X \rightarrow X \]

We consider orbits/trajectories of points of \( X \) under the action of the map \( T \)

\[ \{ T^n x \mid n \in \mathbb{N} \} \]
Discrete dynamical system

We are given a dynamical system

\[ T : X \rightarrow X \]

We partition \( X \) in to a finite number of subsets \( X = \bigcup_{i=1}^{d} X_i \)

We code the trajectory of a point \( x \) with respect to \((X_i)\)

\[ \{ T^n x \mid n \in \mathbb{N} \} \mapsto (u_n)_{n \in \mathbb{N}} \in \{1, 2, \ldots, d\}^\mathbb{N} \]
Discrete dynamical system

We are given a dynamical system

\[ T : X \to X \]

We code the trajectory of a point \( x \) with respect to \((X_i)\)

\[ \{ T^n x \mid n \in \mathbb{N} \} \leadsto (u_n)_{n \in \mathbb{N}} \in \{1, 2, \cdots, d\}^\mathbb{N} \]

The map acting on \( \{1, 2, \cdots, d\}^\mathbb{N} \) is the shift \( S \)

\[ S((u_n)_n) = (u_{n+1})_n \]

\((X, T) \leadsto (Y, S)\) with \( Y \subset \{1, 2, \cdots, d\}^\mathbb{N} \)

From geometric dynamical systems to symbolic dynamical systems and backwards
Arithmetic dynamics

Arithmetic dynamics [Sidorov-Vershik] arithmetic codings of dynamical systems that preserve their arithmetic structure
Arithmetic dynamics

Arithmetic dynamics [Sidorov-Vershik] arithmetic codings of dynamical systems that preserve their arithmetic structure

Example Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $x \mapsto x + \alpha \mod 1$

One codes trajectories according to the finite partition

$$\{I_0 = [0, 1 - \alpha[, \ I_1 = [1 - \alpha, 1]\}$$
Sturmian dynamical systems

Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, \ x \mapsto x + \alpha \mod 1$

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Sturmian dynamical systems

Let $R_\alpha : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}, \ x \mapsto x + \alpha \mod 1$

One codes trajectories according to the finite partition

$$\{ l_0 = [0, 1 - \alpha[, \ l_1 = [1 - \alpha, 1]\}$$

This yields a measure-theoretic isomorphism

$$(R_\alpha, \mathbb{R}/\mathbb{Z}) \sim (X_\alpha, S)$$

where $S$ is the shift and $X_\alpha \subset \{0, 1\}^\mathbb{N}$
Sturmian dynamical systems

Let $R_\alpha : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$, $x \mapsto x + \alpha \mod 1$

One codes trajectories according to the finite partition

$$\{ l_0 = [0, 1 - \alpha[, \ l_1 = [1 - \alpha, 1] \}$$

One has a measure-theoretic isomorphism

$$(R_\alpha, \mathbb{R}/\mathbb{Z}) \sim (X_\alpha, S)$$

\[
\begin{array}{ccc}
\mathbb{R}/\mathbb{Z} & \xrightarrow{R_\alpha} & \mathbb{R}/\mathbb{Z} \\
\downarrow & & \downarrow \\
X_\alpha & \xrightarrow{S} & X_\alpha
\end{array}
\]
Sturmian dynamical systems

Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, \ x \mapsto x + \alpha \ mod \ 1$

One codes trajectories according to the finite partition

$$\{I_0 = [0, 1 - \alpha[, \ I_1 = [1 - \alpha, 1]\}$$

[Lothaire, Algebraic combinatorics on words, N. Pytheas Fogg, Substitutions in dynamics, arithmetics and combinatorics CANT Combinatorics, Automata and Number theory]
Sturmian words and continued fractions

0110110101101101
Sturmian words and continued fractions

0110110101101101

11 and 00 cannot occur simultaneously
Sturmian words and continued fractions

0110110101101101101

One considers the substitutions

\[ \sigma_0 : 0 \mapsto 0, \quad \sigma_0 : 1 \mapsto 10 \]

\[ \sigma_1 : 0 \mapsto 01, \quad \sigma_1 : 1 \mapsto 1 \]

One has

\[ 0110110101101101101 = \sigma_1(0101001010) \]

\[ 0101001010 = \sigma_0(011011) \]

\[ 011011 = \sigma_1(0101) \]

\[ 0101 = \sigma_1(00) \]
Sturmian words and continued fractions

\[
0110110101101101
\]

One considers the substitutions

\[
\sigma_0: 0 \mapsto 0, \quad \sigma_0: 1 \mapsto 10
\]

\[
\sigma_1: 0 \mapsto 01, \quad \sigma_1: 1 \mapsto 1
\]

The Sturmian words of slope \( \alpha \) are provided by an infinite composition of substitutions

\[
\lim_{n \to +\infty} \sigma_0^{a_1} \sigma_1^{a_2} \cdots \sigma_{2n}^{a_{2n}} \sigma_{2n+1}^{a_{2n+1}}(0)
\]

where the \( a_i \) are produced by the continued fraction expansion of \( \alpha \)
Sturmian words and continued fractions

0110110101101101

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![Diagram of Sturmian words and continued fractions](image-url)
Euclid algorithm and discrete segments

\[
\begin{align*}
11 &= 2 \cdot 4 + 3 \\
4 &= 1 \cdot 3 + 1 \\
3 &= 3 \cdot 1 + 0
\end{align*}
\]

\[
\frac{4}{11} = \frac{1}{2 + \frac{1}{1 + \frac{1}{3}}}
\]

\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3
\]

\[
\begin{array}{c}
(11, 4) & (3, 4) & (3, 1) & (0, 1) \\
& a & a & a & a \\
& b & aab & b & aaab \\
\end{array}
\]

\[
w = w_0 \longrightarrow w_1 \longrightarrow w_2 \longrightarrow w_3 = b
\]
Euclid algorithm and discrete segments

\[ 11 = 2 \cdot 4 + 3 \]
\[ 4 = 1 \cdot 3 + 1 \]
\[ 3 = 3 \cdot 1 + 0 \]

\[ \frac{4}{11} = \frac{1}{2} + \frac{1}{1 + \frac{1}{3}} \]

\[ w = aaabaaabaaabaab \]

\[ w = w_0 \leftarrow w_1 \leftarrow w_2 \leftarrow w_3 = b \]

\[ (11, 4) \xleftarrow{} (3, 4) \xleftarrow{} (3, 1) \xleftarrow{} (0, 1) \]

\[ a \mapsto a \quad b \mapsto aaab \]

\[ a \mapsto ab \quad a \mapsto a \quad b \mapsto b \quad b \mapsto aaab \]
Higher-dimensional framework

- How to discretize a line in the space?
- How to compute the gcd of three or more numbers?
- How to compare gcd/cf algorithms?
- Integer parameters vs. rational parameters
- Can we generalize the Sturmian framework to translations on $\mathbb{T}^d$?
The Tribonacci fractal

The Tribonacci substitution \( \sigma : 1 \mapsto 12, \ 2 \mapsto 13, \ 3 \mapsto 1 \)

\[
\sigma^\infty(1) = 121312112 \cdots
\]

One represents \( \sigma^\infty(1) \) as a broken line

\[
1 \mapsto \vec{e}_1, \ 2 \mapsto \vec{e}_2, \ 3 \mapsto \vec{e}_3,
\]

that we will be projected according to the eigenspaces of

\[
M_\sigma = \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{bmatrix}
\]
Rauzy fractal and dynamics

One first defines an *exchange of pieces* acting on the Rauzy fractal
Rauzy fractal and dynamics

One first defines an exchange of pieces acting on the Rauzy fractal. This due to the fact that the subtiles are disjoint in measure.

This exchange of pieces factorizes into a translation of $\mathbb{T}^2$. This due to the fact that the Rauzy fractal tiles periodically the plane.
Rauzy fractal and codings

\[ \sigma : 1 \mapsto 12, \quad 2 \mapsto 1312, \quad 3 \mapsto 112 \]

\[ \sigma^\infty(1) = 12131212112 \ldots \]
Rauzy fractal and codings

\[ \sigma: 1 \mapsto 12, \ 2 \mapsto 1312, \ 3 \mapsto 112 \]

\[ \sigma^\infty(1) = 12131212112 \ldots \]
Rauzy fractal and codings

\[ \sigma : 1 \mapsto 12, \quad 2 \mapsto 1312, \quad 3 \mapsto 112 \]

\[ \sigma^\infty(1) = 12131212112 \ldots \]

Trajectories are coded according to the partition
Rauzy fractal and codings

\( \sigma : 1 \mapsto 12, \ 2 \mapsto 1312, \ 3 \mapsto 112 \)

\( \sigma^\infty(1) = 12131212112 \ldots \)

Trajectory : 2
Rauzy fractal and codings

\[ \sigma : 1 \mapsto 12, \ 2 \mapsto 1312, \ 3 \mapsto 112 \]

\[ \sigma^\infty(1) = 12131212112\ldots \]

Trajectory : 21
Rauzy fractal and codings

\[ \sigma : 1 \mapsto 12, \ 2 \mapsto 1312, \ 3 \mapsto 112 \]

\[ \sigma^\infty(1) = 12131212112 \ldots \]

Trajectory : 213
Rauzy fractal and codings

\[ \sigma : 1 \mapsto 12, \ 2 \mapsto 1312, \ 3 \mapsto 112 \]

\[ \sigma^\infty(1) = 12131212112 \ldots \]

Trajectory : 2131
Rauzy fractal and codings

\[ \sigma : 1 \mapsto 12, \quad 2 \mapsto 1312, \quad 3 \mapsto 112 \]

\[ \sigma^\infty(1) = 12131212112 \ldots \]

Trajectory : 21312
Rauzy fractal and codings

\[ \sigma : 1 \mapsto 12, \quad 2 \mapsto 1312, \quad 3 \mapsto 112 \]

\[ \sigma^\infty(1) = 12131212112 \ldots \]

Trajectory 2 : 213121
Rauzy fractal and codings

$$\sigma : 1 \mapsto 12, \ 2 \mapsto 1312, \ 3 \mapsto 112$$

$$\sigma^\infty(1) = 12131212112 \ldots$$

Trajectory : 2131212
Rauzy fractal and codings

\[ \sigma : 1 \mapsto 12, \ 2 \mapsto 1312, \ 3 \mapsto 112 \]

\[ \sigma^\infty(1) = 12131212112\ldots \]

Density and even equidistribution of orbits
Theorem [Rauzy, Chekhovaya-Hubert-Messaoudi]

- \((X_\sigma, S)\) is measure-theoretically isomorphic with a two-dimensional translation and is equal to the codings of the orbits under the action of the translation

\[R_\beta : \mathbb{T}^2 \to \mathbb{T}^2, \ x \mapsto x + \left( \frac{1}{\beta}, \frac{1}{\beta^2} \right)\]

with respect to the pieces of the Rauzy fractal
**Triboonacci rotation** \( \sigma : 1 \mapsto 12, \ 2 \mapsto 13, \ 3 \mapsto 1 \)

**Theorem** [Rauzy, Chekhovaya-Hubert-Messaoudi]

- \((X_\sigma, S)\) is measure-theoretically isomorphic with a two-dimensional translation and is equal to the codings of the orbits under the action of the translation

\[ R_\beta : \mathbb{T}^2 \to \mathbb{T}^2, \ x \mapsto x + (1/\beta, 1/\beta^2) \]

with respect to the pieces of the Rauzy fractal

- The points of the broken line corresponding to \( \sigma^n(1), \ n \in \mathbb{N} \), produce the sequence of best approximations for the vector \( (1/\beta, 1/\beta^2) \) for a given norm associated with the incidence matrix \( M_\sigma \)
We want to find

- ‘good’ symbolic codings for \( d \)-dimensional translations
  \[ R(\alpha_1, \ldots, \alpha_d) : \mathbb{T}^d \rightarrow \mathbb{T}^d \]

- ‘good’ partitions of the torus \( \mathbb{T}^d \)

Take a multidimensional continued fraction algorithm and transform it into substitutions

[B.-Steiner-Thuswaldner, B.-Jolivet-Siegel, Arnoux-B.-Labbé]
Comparing Euclid/cf algorithms

- Number of steps and costs functions for algorithms defined on rational entries
  - worst-case, mean behavior, average-case analysis
- Convergence properties
- Ergodic properties
  - ergodic invariant measure, natural extension
- Arithmetic properties
  - cubic numbers and periodic expansions, Diophantine approximation
Multidimensional Euclid’s algorithms

- **Jacobi-Perron** We subtract the first one to the two other ones with $0 \leq u_1, u_2 \leq u_3$
  \[(u_1, u_2, u_3) \mapsto (u_2 - \left\lfloor \frac{u_2}{u_1} \right\rfloor u_1, u_3 - \left\lfloor \frac{u_3}{u_1} \right\rfloor u_1, u_1)\]

- **Brun** We subtract the second largest entry and we reorder. If $u_1 \leq u_2 \leq u_3$
  \[(u_1, u_2, u_3) \mapsto (u_1, u_2, u_3 - u_2)\]

- **Poincaré** We subtract the previous entry and we reorder
  \[(u_1, u_2, u_3) \mapsto (u_1, u_2 - u_1, u_3 - u_2)\]

- **Selmer** We subtract the smallest to the largest and we reorder
  \[(u_1, u_2, u_3) \mapsto (u_1, u_2, u_3 - u_1)\]

- **Fully subtractive** We subtract the smallest one to the other ones and we reorder
  \[(u_1, u_2, u_3) \mapsto (u_1, u_2 - u_1, u_3 - u_1)\]
Number of steps

Consider parameters \((u_1, \cdots, u_d)\) with \(0 \leq u_1, \cdots, u_d \leq N\)

**Thm**   
Expectation of the number of steps = \(\frac{\text{dimension}}{\text{Entropy}} \times \log N\)

**Dimension**

- \(d = \) Number of parameters
Number of steps

Consider parameters \((u_1, \cdots, u_d)\) with \(0 \leq u_1, \cdots, u_d \leq N\).

**Thm** Expectation of the number of steps \(= \frac{\text{dimension}}{\text{Entropy}} \times \log N\)

- Euclid algorithm

\[
\frac{2}{\pi^2/(6 \log 2)} \log N
\]

[Heilbronn’69, Dixon’70, Hensley’94, Baladi-Vallée’03, Lhote-Vallée’08, ...]
Number of steps

Consider parameters \((u_1, \cdots, u_d)\) with \(0 \leq u_1, \cdots, u_d \leq N\)

**Thm** Expectation of the number of steps \(= \frac{\text{dimension}}{\text{Entropy}} \times \log N\)

- Jacobi-Perron  
  [Fischer-Schweiger’75]
- Brun  
  [B.-Lhote-Vallée, work in progress]
Number of steps

Consider parameters \((u_1, \cdots, u_d)\) with \(0 \leq u_1, \cdots, u_d \leq N\)

Thm  Expectation of the number of steps = \(\frac{\text{dimension}}{\text{Entropy}} \times \log N\)

- Formal power series with coefficients in a finite field and polynomials with degree less than \(m\)

\[
\frac{2}{2^\frac{q}{q-1}} m = \frac{q-1}{q} m
\]

[Knopfmacher-Knopfmacher’88, Friesen-Hensley’96, Lhote-Vallée’06’08, B.-Nakada-Natsui-Vallée’12]
Formal power series

Let $q$ be a power of a prime number $p$

We have the correspondence

- $\mathbb{Z} \sim \mathbb{F}_q[X]$
- $\mathbb{Q} \sim \mathbb{F}_q(X)$
- $\mathbb{R} \sim \mathbb{F}_q((X^{-1}))$

\[ f = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_0 + a_{-1}X^{-1} + \cdots \]

Laurent formal power series
Formal power series

Let $f \in \mathbb{F}_q((X^{-1})) \quad f \neq 0$

$$f = a_n X^n + a_{n-1} X^{n-1} + \cdots \quad a_n \neq 0$$

- **Degree** \quad \deg f = n
- **Distance** \quad |f| = q^{\deg f}

Ultrametric space

$$|f + g| \leq \max(|f|, |g|)$$

No carry propagation!
Continued fractions

One can expand series \( f \) into continued fractions

\[
f = a_0(X) + \cfrac{1}{a_1(X) + \cfrac{1}{a_2(X) + \cdots}} := [a_0(X); a_1(X), a_2(X), \ldots],
\]

The digits \( a_i(X) \) are polynomials of positive degree

\[
a_k \geq 1 \iff \deg a_k(X) \geq 1
\]

- Unique expansion even if \( f \) does not belong to \( \mathbb{F}_q(X) \)
- Finite expansion iff \( f \in \mathbb{F}_q(X) \)
- But there exist explicit examples of algebraic series with bounded partial quotients [Baum-Sweet]
- Roth’s theorem does not hold for algebraic series (see e.g. [Lasjaunias-de Mathan])

[B.-Nakada, Expositiones Mathematicae]
Why is everything simpler?

Ultrametric space!

- Digits are equidistributed: the Haar measure is invariant
- Hence, understanding the ‘polynomial case’ can help the understanding of the ‘integer case’
And now..

- Numeration dynamics $T_\beta : x \mapsto \{\beta x\}$
- Discrete lines and planes
- Invariant measures