321-avoiding affine permutations, heaps, and periodic parallelogram polyominoes

Riccardo Biagioli (Université Lyon 1)

joint work with
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Séminaire Philippe Flajolet
7 décembre 2017
Introduction and motivations

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321-avoiding affine permutations, heaps, and periodic parallelogram polyominoes
A permutation $\sigma \in S_n$ is **321-avoiding** if no integers $i < j < k$ are such that $\sigma(i) > \sigma(j) > \sigma(k)$.

In $S_6$, $\sigma = 513624$ is not 321-avoiding while $\sigma = 231564$ is.

They are counted by Catalan numbers $\frac{1}{n+1} \binom{2n}{n}$.

The **inversion number** $\text{inv}(\sigma)$ is the number of inversions of the permutation $\sigma$ i.e.

$$
\text{inv}(\sigma) = |\{(i,j) \in [n]^2 \mid i < j \text{ and } \sigma(i) > \sigma(j)\}|.
$$

For example $\sigma = 513624$ has $4+0+1+2+0=6$ inversions.
Affine permutations

**Definition (Affine permutations)**

The group $\tilde{S}_n$ is the set of permutations $\sigma$ of $\mathbb{Z}$ satisfying

$$\sigma(i + n) = \sigma(i) + n \text{ and } \sum_{i=1}^{n} \sigma(i) = \sum_{i=1}^{n} i.$$ 

Note that $\sigma(i) \equiv \sigma(j) \pmod{n}$ if and only if $i \equiv j \pmod{n}$.

An element of $\tilde{S}_4$ is

$$\ldots | 2, -7, -5, 4, | 6, -3, -1, 8, | 10, 1, 3, 12, | 14, 5, 7, 16 | \ldots$$

denoted simply by $[6, -3, -1, 8] = [\sigma(1), \sigma(2), \sigma(3), \sigma(4)]$. 

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An element of \( \tilde{S}_4 \) is

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\ldots \mid 2, -7, -5, 4, \mid 6, -3, -1, 8, \mid 10, 1, 3, 12, \mid 14, 5, 7, 16 \mid \ldots
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denoted simply by \([6, -3, -1, 8] = [\sigma(1), \sigma(2), \sigma(3), \sigma(4)]\).
**Definition**

An affine permutation is 321-avoiding if there are not \( i < j < k \) in \( \mathbb{Z} \) such that \( \sigma(i) > \sigma(j) > \sigma(k) \). We write \( \sigma \in \tilde{S}_n(321) \).

For example

\[ \ldots, | 2, -7, -5, 4, | 6, -3, -1, 8, | 10, 1, 3, 12, | 14, 5, 7, 16 | \ldots \in \tilde{S}_4(321) \]

\[ \ldots | -3, -5, 5 | 0, -2, 8 | 3, 1, 11 | 6, 4, 14 | 9, 7, 17 | \ldots \notin \tilde{S}_3(321) \]

**Definition (Affine inversions)**

\[ \text{inv}(\sigma) = |\{(i, j) \in [n] \times \mathbb{P} | i < j \text{ and } \sigma(i) > \sigma(j)\}|. \]

For example \( \text{inv}([6, -3, -1, 8]) = 9. \)
321-avoiding affine permutations

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321-avoiding affine permutations, heaps, and periodic parallelogram polyominoes
Generating function for permutations in $S_n(321)$

$$A_{n-1}(q) := \sum_{\sigma \in S_n(321)} q^{\text{inv}(\sigma)} \quad \text{and} \quad A(x, q) = \sum_{n \geq 0} A_n(q)x^n.$$ 

**Theorem (Barcucci, Del Lungo, Pergola, Pinzani, 2001)**

We have

$$A(x, q) = \frac{1}{1 - xq} \times \frac{J(xq)}{J(x)},$$

where

$$J(x) := \sum_{n \geq 0} \frac{(-x)^n q^n}{(q)_n(xq)_n}.$$

Here $(a)_0 := 1$ and $(a)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$, $n \geq 1$. 

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321-avoiding affine permutations, heaps, and periodic parallelogram polyominoes
Generating function for affine permutations in $\tilde{S}_n(321)$

$$\tilde{A}_{n-1}(q) := \sum_{\sigma \in \tilde{S}_n(321)} q^{\text{inv}(\sigma)} \quad \text{and} \quad \tilde{A}(x, q) := \sum_{n \geq 1} \tilde{A}_{n-1}(q) x^n$$

Theorem (B., Bousquet-Mélou, Jouhet, Nadeau, 2016)

$$\tilde{A}(x, q) = -x \frac{J'(x)}{J(x)} - \sum_{n \geq 1} \frac{x^n q^n}{1 - q^n}$$

- Hanusa and Jones [2009] found a complicated expression for $\tilde{A}(x, q)$ and showed that the coefficients of the series $\tilde{A}_{n-1}(q)$ are ultimately periodic of period dividing $n$.
- BJN [2013] characterized the series $\tilde{A}_{n-1}(q)$ by a systems of non-linear $q$-equations.
- BBJN [2016] found the previous formula for $\tilde{A}(x, q)$, manipulating such $q$-equations.
Generating function for affine permutations in $\tilde{S}_n(321)$

$$\tilde{\mathcal{A}}_{n-1}(q) := \sum_{\sigma \in \tilde{S}_n(321)} q^{\text{inv}(\sigma)} \quad \text{and} \quad \tilde{\mathcal{A}}(x, q) := \sum_{n \geq 1} \tilde{\mathcal{A}}_{n-1}(q)x^n$$

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- Hanusa and Jones [2009] found a complicated expression for $\tilde{\mathcal{A}}(x, q)$ and showed that the coefficients of the series $\tilde{\mathcal{A}}_{n-1}(q)$ are ultimately periodic of period dividing $n$.
- Bijn [2013] characterized the series $\tilde{\mathcal{A}}_{n-1}(q)$ by a systems of non-linear $q$-equations.
- Bijn [2016] found the previous formula for $\tilde{\mathcal{A}}(x, q)$, manipulating such $q$-equations.
This computational approach does not explain the simplicity of $\tilde{A}(x, q)$ and $A(x, q)$. In this talk, we provide two bijective explanations of them.

**Today's combinatorial methods.**

Encode 321-avoiding (affine) permutations by:

- (affine) alternating diagrams, then by
  - (periodic) parallelogram polyominoes, and
  - (marked) heaps of segments;

or by

- Motzkin type paths, and
- (marked) pyramids of monomers and dimers.
Fully commutative elements

The original motivation was the computation of the series

$$\sum_{w \in W^{FC}} q^{\ell(w)}$$

where $W^{FC}$ denotes the set of fully commutative elements in the Coxeter group $W$, and $\ell$ the Coxeter length.

### Coxeter group

$(W, S)$ Coxeter group $W$ given by Coxeter matrix $(m_{st})_{s, t \in S}$.

Relations:

$$s^2 = 1$$

$$sts \ldots = tst \ldots$$  \hspace{1cm} (called Braid relations)

(if $m_{st} = 2$ commutation relations)
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Relations:

$$\begin{cases} s^2 = 1 \\ sts \cdots = tst \cdots \end{cases} \quad (\text{called Braid relations})$$

(if $m_{st} = 2$ commutation relations)
**Reduced decompositions**

**Definition (Length)**

\[ \ell(w) = \text{minimal } l \text{ such that } w = s_1 s_2 \cdots s_l \text{ with } s_i \in S \]

Such a minimal word is a **reduced decomposition** of \( w \).

**Proposition (Matsumoto-Tits property)**

*Given two reduced decompositions of \( w \), there is a sequence of braid or commutation relations which can be applied to transform one into the other.*

**Definition**

An element \( w \) is **fully commutative** if given two reduced decompositions of \( w \), there is a sequence of **commutation relations** which can be applied to transform one into the other.
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The symmetric group

Example

The symmetric group $S_n$ is generated as a Coxeter group by the set $S$ of simple transpositions $s_i = (i, i + 1)$ with

$$s_i^2 = 1$$

Relations:

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ (braid relations)}$$

$$s_i s_j = s_j s_i \text{ if } j \neq i \pm 1 \text{ (commutation relations)}$$

All elements of $S_3$ are FC except $321 = s_1 s_2 s_1 = s_2 s_1 s_2$.

Note that $\ell(\sigma) = \text{inv}(\sigma)$. 
Fully commutative elements

Theorem (Billey-Jockush-Stanley, 1993)

A permutation in $S_n$ is fully commutative if and only if it is 321-avoiding.

Theorem (Green, 2001)

An affine permutation in $\tilde{S}_n$ is fully commutative if and only if it is 321-avoiding.

Theorem (Lusztig, 1983)

\begin{itemize}
  \item $\tilde{S}_n$ is a Coxeter group of type $\tilde{A}_{n-1}$;
  \item $s_0 = \ldots (-1-n, -n)(-1, 0)(-1+n, n) \ldots$.
  \item $s_i = \ldots (i, i+1)(i+n, i+1+n) \ldots$.
\end{itemize}

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Fully commutative elements

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Affine alternating diagrams
From line diagrams to alternating diagrams

Take the 321-avoiding permutation $\sigma = 461279358 \in S_9$. 

\[
\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}
\]
Take the 321-avoiding permutation $\sigma = 461279358 \in S_9$. 

![Diagram of 321-avoiding permutation](image)
Take the 321-avoiding permutation $\sigma = 461279358 \in S_9$. 

\begin{center}
\begin{tikzpicture}
\draw (1,0) -- (9,0);
\fill (1,0) circle (2pt) (2,0) circle (2pt) (3,0) circle (2pt) (4,0) circle (2pt) (5,0) circle (2pt) (6,0) circle (2pt) (7,0) circle (2pt) (8,0) circle (2pt) (9,0) circle (2pt);
\end{tikzpicture}
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From line diagrams to alternating diagrams

The alternating diagram of $\sigma = 461279358 \in S_9 : \text{inv}(\sigma) = 12.$

**Definition (Alternating diagram)**

An alternating diagram of rank $n$ is a poset:
- its elements are labeled by the generators $\{s_1, \ldots, s_{n-1}\}$ of $S_n$
- $\forall i$, elements with labels $s_i, s_{i+1}$ form an alternating chain;
- the ordering is given by the transitive closure of these chains.
From line diagrams to alternating diagrams

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From line diagrams to affine alternating diagrams

\( \sigma = \ldots \mid 2, -7, -5, 4, \mid 6, -3, -1, 8, \mid 10, 1, 3, 12, \mid 14, 5, 7, 16 \mid \ldots \)

Here \( \sigma \in \tilde{S}_4(321) \) and \( inv(\sigma) = 9 \).
From line diagrams to affine alternating diagrams

\[ \sigma = \ldots | 2, -7, -5, 4, | 6, -3, -1, 8, | 10, 1, 3, 12, | 14, 5, 7, 16 | \ldots \]

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Proposition (Characterization of affine alternating diagrams)

- Same number of occurrences of \( s_0 \) in the first and last column.
From line diagrams to affine alternating diagrams

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Proposition (Characterization of affine alternating diagrams)

- **Same** number of occurrences of \( s_0 \) in the first and last column.
Representations of alternating diagrams on a cylinder

Two affine alternating diagrams of $\tilde{S}_8$.

The second is self-dual.

Two excluded diagrams!
(rectangular shape)

They do not represent posets.
Representations of alternating diagrams on a cylinder

Two affine alternating diagrams of $\tilde{S}_8$.

The second is self-dual.

Two excluded diagrams! (rectangular shape)

They do not represent posets.
The map $\Delta$ between $\tilde{S}_n(321)$ and affine alternating diagrams is a bijection such that:

- $\sigma \in S_n(321)$ if and only if $\Delta(\sigma)$ do not contain any $s_0$
- $\sigma$ is an involution if and only if $\Delta(\sigma)$ is self-dual.
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Periodic parallelogram polyominoes
From alternating diagrams to parallelogram polyominoes

Theorem (Viennot, 1992)
There is a bijection between alternating diagrams and parallelogram polyominoes (PP).

A PP is a convex polyomino enclosed by two paths consisting of unit horizontal and vertical steps, both starting in the same point and ending in the same point and non-intersecting elsewhere.

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A PP is a convex polyomino enclosed by two paths consisting of unit horizontal and vertical steps, both starting in the same point, ending in the same point, and non-intersecting elsewhere.
Parallelogram polyominoes are coded by sequences $(a_i, b_i)_{1 \leq i \leq n}$ with $a_1 = 1$, where:

- $b_i$ is the **height** of the column $C_i$;
- $a_i$ is the **number of common rows** between $C_{i-1}$ and $C_i$.

Any of such finite sequences $(a_i, b_i)_{1 \leq i \leq n}$ satisfies:

\[ 1 \leq b_1 \geq a_2 \leq b_2 \geq \ldots \leq b_{n-1} \geq a_n \leq b_n. \]
Classical case (Bousquet-Mélou–Viennot, 1992)

\[(1, 5), (5, 5), (4, 4), (1, 1), (1, 3), (2, 3)\]
From parallelogram polyominoes to heaps of segments

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Theorem (Bousquet-Mélou–Viennot, 1992)

The map $f$ is a bijection between the set of parallelogram polyominoes and the set of half pyramids of segments.
Let $S$ be the set of finite sequences $(a_i, b_i)_{1 \leq i \leq n}$ satisfying

$$a_1 \leq b_1 \geq a_2 \leq b_2 \geq \ldots \leq b_{n-1} \geq a_n \leq b_n.$$ 

Let $S$ be the set of finite sequences $(a_i, b_i)_{1 \leq i \leq n}$ satisfying

$$(4, 5), (5, 5), (4, 4), (1, 1), (1, 3), (2, 3)$$

The map $f$ extends to a bijection between the set of $(PP)$ marked in their first column and the set of heaps of segments $\mathcal{H}$. 

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321-avoiding affine permutations, heaps, and periodic parallelogram polyominoes
Definition (PPP)

A periodic parallelogram polyomino is a couple \((P, c)\), where \(P\) is a marked PP and \(c\) is an integer between 1 and the height of the last column of \(P\).

Periodic parallelogram polyominoes as sequences

These naturally correspond to sequences \((a_i, b_i)_{1 \leq i \leq n} \in S\) such that \(b_n \geq a_1\), i.e.

\[ b_n \geq a_1 \leq b_1 \geq a_2 \leq b_2 \geq \ldots \leq b_{n-1} \geq a_n \leq b_n. \]
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These naturally correspond to sequences \((a_i, b_i)_{1 \leq i \leq n} \in S\) such that \(b_n \geq a_1\), i.e.

\[
b_n \geq a_1 \leq b_1 \geq a_2 \leq b_2 \geq \ldots \leq b_{n-1} \geq a_n \leq b_n.
\]
From PPP to heaps of segments

Let \( \tilde{\mathcal{H}} \) be the set of heaps satisfying condition (\( \sim \)) i.e. the beginning of the rightmost maximal segment should be on the left of the end of the leftmost minimal segment.

**Proposition**

The map \( f \) induces a bijection between the set \( PPP \) and \( \tilde{\mathcal{H}} \).
Summary

\[ S_n(321) \quad \leftrightarrow \quad \text{Parallelogram Polyominoes} \quad \leftrightarrow \quad (S, a_1 = 1) \quad \leftrightarrow \quad \text{Half pyramids} \]

\[ (S, a_1 \leq b_n) \quad \leftrightarrow \quad \text{Heaps of segments in } \tilde{\mathcal{H}} \]

Recent papers (2016) on PPP also by Aval, Boussicault, Laborde–Zubieta, and Pétréolle.
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Summary

\[ S_n(321) \quad \leftrightarrow \quad \text{Parallelogram} \quad \leftrightarrow \quad (S, a_1 = 1) \quad \leftrightarrow \quad \text{Half pyramids} \]

\[ S_n(321) \quad ? \quad \leftrightarrow \quad (S, a_1 \leq b_n) \quad \leftrightarrow \quad \text{Heaps of segments in } \tilde{H} \]

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321-avoiding affine permutations, heaps, and periodic parallelogram polyominoes
Connection between \( \tilde{S}_n(321) \) and PPP
Back to 321-avoiding affine permutations
From 321-avoiding affine permutations to $PPP^*$
From 321-avoiding affine permutations to \( PPP^* \)

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321-avoiding affine permutations, heaps, and periodic parallelogram polyominoes
From 321-avoiding affine permutations to \( PPP^* \)
From 321-avoiding affine permutations to $PPP^*$

A mark in $[a_1, b_1]$ to recover $s_0$

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321-avoiding affine permutations, heaps, and periodic parallelogram polyominoes
From 321-avoiding affine permutations to PPP*
From $PPP^*$ to 321-avoiding affine permutations

Theorem (B, Jouhet, Nadeau, (2016))

The previous application is a bijection between 321-avoiding affine permutations and marked $PPP$ of non-rectangular shape.
Why the rectangular shape?

It is not a PPP.

It is a PPP of rectangular shape.

It is not a PPP.

It is a PPP of rectangular shape.
Marked PPP

(2, 4), (3, 5), (5, 5), (4, 4), (1, 1), (1, 3)

Mark in \([a_1, b_1]\]

**Definition**

\(PPP^* = PPP\) with a mark between \(a_1\) and \(b_1\); 
\(\tilde{\mathcal{H}}^* = \) heaps in \(\tilde{\mathcal{H}}\) with a mark in their rightmost maximal segment.

**Corollary**

The map \(f\) induces a bijection between \(PPP^*\) and \(\tilde{\mathcal{H}}^*\).
Introduction and motivations

Alternating diagrams

Parallelograms polyominoes

Generating functions

Summary

S_n(321) ↔ PP ↔ Half pyramids

Parallelogram polyominoes

PPP ↔ Heaps of segments in \( \tilde{H} \)

Periodic parallelogram polyominoes

PPP* ↔ Marked heaps of segments in \( \tilde{H}^* \)

\( \tilde{S}_n(321) \) ↔ PPP* ↔ Marked periodic parallelogram polyominoes (non rectangular)

Recent papers (2016) on parallelogram polyominoes also by Aval, Boussicault, Laborde–Zubieta, and Pérèolle.

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321-avoiding affine permutations, heaps, and periodic parallelogram polyominoes
Generating functions
Statistics on $PPP$

- $\text{width}(P) = |H|$ the number of segments in $H$
- $\text{height}(P) = \ell(H)$ the sum of the lengths of the segments of $H$
- $\text{area}(P) = e(H)$ sum of right endpoints of the segments of $H$

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321-avoiding affine permutations, heaps, and periodic parallelogram polyominoes
Statistics on \textit{PPP}

\[ e(H) - |H| \quad \text{area}(P) - \text{width}(P) \quad \text{ell}(H) + |H| \quad \text{half perimeter}(P) \quad \text{inv}(\sigma) \quad n = \text{number of vertical columns : } \sigma \in \tilde{S}_n(321) \]

Hence we have that

\[ \tilde{A}(x, q) = \sum_{n \geq 1} \left( \sum_{\sigma \in \tilde{S}_{n+1}(321)} q^{\text{inv}(\sigma)} \right) x^n = \sum_{H \in \tilde{H}^*} x^{\ell(H) + |H|} q^{e(H) - |H|} - \sum_{n \geq 1} \frac{x^n q^n}{1 - q^n}. \]
Generating functions for \textit{PPP}

\textbf{Goal: to compute the series}

\[
\tilde{\mathcal{H}}^*(x, y, q) = \sum_{H \in \tilde{\mathcal{H}}^*} x^{\ell(H)} y^{|H|} q^{e(H)}.
\]

\textbf{Theorem (Inversion Lemma - Viennot, 1985)}

\[
\mathcal{H}(x, y, q) = \frac{1}{T(x, y, q)} \quad \text{and} \quad \mathcal{H}\mathcal{P}(x, y, q) = \frac{T^c(x, y, q)}{T(x, y, q)}
\]

where $T$ (resp. $T^c$) is the signed GF for trivial heaps (resp. not touching abscissa 1), and $\mathcal{H}\mathcal{P}$ denotes the half pyramids.

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321-avoiding affine permutations, heaps, and periodic parallelogram polyominoes
Generating functions for PPP

Goal: to compute the series

\[ \tilde{\mathcal{H}}^*(x, y, q) = \sum_{H \in \tilde{\mathcal{H}}^*} x^{\ell(H)} y^{|H|} q^{e(H)}. \]

Theorem (Inversion Lemma - Viennot, 1985)

\[ \mathcal{H}(x, y, q) = \frac{1}{\mathcal{T}(x, y, q)} \quad \text{and} \quad \mathcal{H}\mathcal{P}(x, y, q) = \frac{\mathcal{T}^c(x, y, q)}{\mathcal{T}(x, y, q)} \]

where \( \mathcal{T} \) (resp. \( \mathcal{T}^c \)) is the signed GF for trivial heaps (resp. not touching abscissa 1), and \( \mathcal{H}\mathcal{P} \) denotes the half pyramids.

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Trivial heaps

A trivial heap $T \in \mathcal{T}$ has no two pieces in concurrence.

$$T = \begin{array}{cccccccccccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}$$

$$v(T) = x^5 y^3 q^{17}$$

Signed generating function for trivial heaps

$$\mathcal{T} = \mathcal{T}(x, y, q) = \sum_{T \in \mathcal{T}} (-1)^{|T|} x^{\ell(T)} y^{|T|} q^{e(T)}.$$
Generating functions for heaps of segments

**Theorem (Bousquet-Mélou, Viennot, 1992)**

\[
\mathcal{T} = \sum_{n \geq 0} \frac{(-y)^n q^{\binom{n+1}{2}}}{(q)_n (xq)_n} \quad \text{and} \quad \mathcal{T}^c = \sum_{n \geq 1} \frac{(-y)^n q^{\binom{n+1}{2}}}{(q)_{n-1} (xq)_n}.
\]

Since **321-avoiding permutations** are in bijection with **half pyramids**, we obtain back the result of Barcucci et al, by setting \( y \rightarrow y/q \) (recall that we added a box in each column), and then \( y \rightarrow x \), in Viennot formula. Note that \( \text{inv}(\sigma) = e(H) \).

**Theorem (Barcucci et al.)**

\[
A(x, q) = \frac{1}{1 - xq} \times \frac{J(xq)}{J(x)} \quad \text{where} \quad J(x) := \sum_{n \geq 0} \frac{(-x)^n q^{\binom{n}{2}}}{(q)_n (xq)_n}.
\]
Adaptation to our special heaps of segments in $\tilde{\mathcal{H}}^*$

We can adapt the Viennot’s technique to $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{H}}^*$, but condition ($\sim$) is very complicated to handle.

**Theorem (B, Jouhet, Nadeau, 2016)**

$$PPP(x, y, q) = -y \frac{\partial_y T}{T} \quad PPP^*(x, y, q) = -x \frac{\partial_x T}{T}.$$ 

Since marked $PPP^*$ (minus those of rectangular shape) of half-perimeter $n$ are in bijection with 321-avoiding affine permutations of size $n$, we obtain (after taking care about the weight, $y \to y/q$, and $y \to x$) that

**Theorem (B., Bousquet-Mélou, Jouhet, Nadeau)**

$$\tilde{A}(x) = -x \frac{J''(x)}{J(x)} - \sum_{n \geq 1} \frac{x^n q^n}{1 - q^n}.$$
Adaptation to our special heaps of segments in $\tilde{H}^*$

We can adapt the Viennot’s technique to $\tilde{H}$ and $\tilde{H}^*$, but condition ($\sim$) is very complicated to handle.

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A different encoding
A different bijection

Theorem (BJN, 2013)

The map $\varphi'$ is a bijection between:

1. $\tilde{S}_n(321)$ and

2. $O_n^* \setminus \{\text{paths at constant height } h > 0 \text{ with all steps having the same label } L \text{ or } R\}$, where $O_n^*$ is the set of length $n$ paths with starting and ending point at the same height, with steps in $(1, 1)$, $(1, -1)$ and $(1, 0)$ satisfying condition $(\ast)$. 

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A different bijection

**Theorem (BJN, 2013)**

The map $\varphi'$ is a bijection between:

1. $\tilde{S}_n(321)$ and
2. $\mathcal{O}^*_n \setminus \{\text{paths at constant height } h > 0 \text{ with all steps having the same label } L \text{ or } R\}$, where $\mathcal{O}^*_n$ is the set of length $n$ paths with starting and ending point at the same height, with steps in $(1,1)$, $(1,-1)$ and $(1,0)$ satisfying condition (*)&.

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Corollary

\[ \tilde{A}_{n-1}^{FC}(q) = O_n^*(q) - \frac{2q^n}{1 - q^n} = \frac{q^n(\tilde{O}_n(q) - 2)}{1 - q^n} + \tilde{O}_n^*(q), \]

from which the periodicity follows.

Corollary (Hanusa and Jones, 2010)

The coefficients of \( \tilde{A}_{n-1}^{FC}(q) \) are ultimately periodic of period dividing \( n \).
Encoding by heaps of monomers and dimers

A marked pyramid (Pm) of dimers and monomers L, R with condition (*)

(*) only in column 0

Monomers: weight $x q_i$

Dimers: weight $x 2q_i^2 + 1$

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Encoding by heaps of monomers and dimers

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Encoding by heaps of monomers and dimers

A marked pyramid $(P_m)$ of dimers and monomers $L, R$ with condition (*)

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Monomers: weight $xq_i$

Dimers: weight $x^{2q_i+1}$

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GF of heaps of monomers and dimers

For a marked heap $E$, the weight is

$$
\nu(E) := \prod_{\text{monomers } [i]} xq^i \prod_{\text{dimers } [i,i+1]} x^2 q^{2i+1}.
$$

We need to compute the GF of marked pyramids $\Pi_m(x)$. If the GF of heaps is $E(x)$, the GF for marked heaps is $xE'(x)$.

**Proposition (Viennot)**

$$
xE'(x) = \Pi_m(X) \times E(x).
$$
Once again we conclude using the Inversion Lemma.

**Theorem (Inversion Lemma - Viennot, 1985)**

$$E(x) = \frac{1}{T^*(x)},$$

where $T^*$ is the signed GF for trivial heaps satisfying condition $(\ast)$. A computation shows that $T^*(x) = (xq; q)_\infty J(x)$ from which we obtain the previous result

$$\tilde{A}(x) = -x \frac{J'(x)}{J(x)} - \sum_{n \geq 1} \frac{x^n q^n}{1 - q^n}.$$
### 321-avoiding involutions in $S_n$ and $\tilde{S}_n$

\[ \mathcal{A} = \sum_{n \geq 0} A_n^{\text{Invo}}(q)x^n \quad \text{and} \quad \tilde{\mathcal{A}} = \sum_{n \geq 1} \tilde{A}_{n-1}^{\text{Invo}}(q)x^n. \]

**Theorem (B., Bousquet-Mélou, Jouhet, Nadeau, 2016)**

We have

\[ \mathcal{A} = \frac{\mathcal{J}(-xq)}{\mathcal{J}(x)} \quad \text{and} \quad \tilde{\mathcal{A}} = -x \frac{\mathcal{J}'(x)}{\mathcal{J}(x)}, \quad \text{where} \]

\[ \mathcal{J}(x) = \sum_{n \geq 0} (-1)^{\lceil n/2 \rceil} x^n q^{n(2)} \frac{n}{(q^2)^{\lfloor n/2 \rfloor}}. \]

Give a proof of this results using $PPP^*$. 

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321-avoiding affine permutations, heaps, and periodic parallelogram polyominoes
Open problem : pyramids

Denote by $\Pi$ the set of pyramids (heaps with a unique maximal element).

By using the bijection $\phi$ we find

$$
\sum_{H \in \Pi} x^{\ell(H)} y^{|H|} q^{e(H)} = -y \frac{\partial y}{\partial T} = \sum_{H \in \tilde{\mathcal{H}}} x^{\ell(H)} y^{|H|} q^{e(H)}
$$

A bijection between the set $\tilde{\mathcal{H}}$ and the set of pyramids $\Pi$ would be nice, as would be a direct way of encoding periodic parallelogram polyominoes as pyramids.
The end