The Z-invariant Ising model on isoradial graphs

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Outline

The Ising model

The Ising model via dimers

Z-invariance

Z-invariant Ising model out of criticality
The Ising model
The Ising model

- (planar) graph $G$
- spin configurations: $\sigma : G \to \pm 1$
- parameters: coupling constants $(J_e)_{e \in E(G)} > 0$
- Energy of a configuration:
  \[ H(\sigma) = - \sum_{e=xy} J_e \sigma_x \sigma_y \]
- Probability of a configuration:
  \[ \mathbb{P}(\sigma) = \frac{1}{Z(G, (J_e))} \times \exp \left( -H(\sigma) \right) \]
The Ising model on the square lattice

- A single parameter to study possible phase transitions: \( \beta \rightarrow J(e, \beta) \) increasing
- On a regular graph: \( J(e, \beta) = \beta J \quad (\beta = 1/T) \)

\( \beta > \beta_c \) (low T)  \hspace{2cm} \beta = \beta_c \hspace{2cm} \beta < \beta_c \) (high T)

Simulation pictures: Raphaël Cerf
The Ising model via dimers
The Ising model is free fermionic

Physics folklore: the Ising model is a model of free fermions

Kasteleyn: the partition of the Ising model on any planar graph can be written as a Pfaffian, in connection with dimers
dimer configurations = perfect matchings = 1-factors
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**Fisher**: another explicit correspondence with dimers on a decorated graph
Fisher’s bijection

Ising spins ↔ contours (separating spins) ↔ dimers

This version (Dubédat) is not a bijection: 2 choices for each decoration of a vertex
Kasteleyn’s theory of dimer models

Let $\mathcal{G}$ a finite planar graph.

- weights $\nu_e$ on edges of $\mathcal{G}$
- probability of a dimer conf. $\mathcal{C} \propto \prod_{e \in \mathcal{C}} \nu_e$

**Theorem (Kasteleyn)**

Let $K$ be the weighted oriented adjacency matrix of $G$ for an admissible orientation. Then:

- The partition function $Z_{\text{dimers}} := \sum_{\mathcal{C}} \prod_{e \in \mathcal{C}} \nu_e$ is $\pm \text{Pfaff } K$,
- The probability that $e_1 = (v_{i_1}, v_{i_2}), \ldots, e_k = (v_{i_{2k-1}}, v_{i_{2k}})$ occur in a random dimer configuration is

$$\left( \prod_{j} K(v_{i_{2j-1}}, v_{i_{2j}}) \right)^{\text{Pfaff}1 \leq p, q \leq 2k} K^{-1}(v_{ip}, v_{iq})^T$$

*Pfaffian process*
Z-invariance
Star-triangle transformation

$G$ and $G'$: planar graphs differing by a $Y - \nabla$ transformation

Coupling constants so that the Ising models are equivalent?
Star-triangle transformation

$G$ and $G'$: planar graphs differing by a $Y - \nabla$ transformation

Coupling constants so that the Ising models are equivalent?

<table>
<thead>
<tr>
<th>$\sigma_1 \sigma_2 \sigma_3$</th>
<th>$G$</th>
<th>$G'$</th>
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<tbody>
<tr>
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Isoradial graphs

- quad graph: projection of a surface in $\mathbb{Z}^d$
- star-triangle transformation: natural flip operation

- Each edge $e$ has a natural parameter $\theta_e = \frac{\beta - \alpha}{2}$
If we require that for isoradial graphs:

- for any edge $e$, $J(e) = J(\theta_e)$
- invariance under star-triangle transformations

1-parameter family of coupling constants:

\[
\sinh(2J(\theta|k)) = \text{sc}(\theta \frac{2K(k)}{\pi} | k) = \frac{\text{sn}(\theta \frac{2K(k)}{\pi} | k)}{\text{cn}(\theta \frac{2K(k)}{\pi} | k)} \quad [\text{Baxter}]
\]

The Ising model is then said to be **$Z$-invariant**
Z-invariant coupling constants

\[
\sinh(2J(\theta|k)) = \frac{2K(k)}{\pi} = \frac{\sin(\theta \frac{2K(k)}{\pi} |k)}{\cos(\theta \frac{2K(k)}{\pi} |k)}
\]

\(k\): elliptic modulus  \hspace{1cm} \hspace{1cm} k' = \sqrt{1 - k^2} \in (0, \infty) \leftrightarrow \text{temperature}

\[K(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2 \cos^2(t)}}\]  \hspace{1cm} \text{elliptic integral of 1st kind}

\(\text{sn}(\cdot|k), \text{cn}(\cdot|k), \text{sc}(\cdot|k)\) Jacobi elliptic functions functions:

\hspace{1cm} \text{generalization of } \sin, \cos, \tan \text{ respectively.}

Bonus: Kramers-Wannier duality built-in

\[
\sinh(2J(\theta|k)) \times \sinh(2J(\frac{\pi}{2} - \theta|k^*)) = 1 \hspace{1cm} \text{with } k' \times (k^*)' = 1
\]
Critical Z-invariant Ising model \((k = 0)\)

**Self-duality:** \(k^* = k \iff k = 0\)

- Elliptic functions \(\sim\) trigonometric: \(\sinh(2J(\theta|0)) = \tan(\theta)\)
- really critical [Li, Cimasoni–Duminil-Copin]
- discrete harmonic fermionic observable
- conformally invariant scaling limit [Mercat, Chelkak-Smirnov...]
- construction of probability measure in infinite volume on isoradial graphs (dimers, Fisher correspondence) [B.–de Tilière]
- locality of dimers (and thus spin) correlations
- related to local expression for Green function on isoradial graphs for conductances [Kenyon]

**Question:** does locality still hold out of criticality?
Critical $Z$-invariant Ising model ($k = 0$)

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Z-invariant Ising model out of criticality
Consider the dimer model on the Fisher graph $G$ coming from a $Z$-invariant Ising model on an isoradial graph $G$:

$$\nu_e = \begin{cases} 
\frac{\text{sn}(\frac{2K\theta}{\pi}|k)}{1 + \text{cn}(\frac{2K\theta}{\pi}|k)} & \text{if } e \text{ is an edge coming from } G \\
1 & \text{otherwise}
\end{cases}$$

Let $K$ the corresponding (infinite) Kasteleyn matrix on $G$

**Theorem (B.–de Tilière – Raschel)**

- For $k \neq 0$, the Kasteleyn operator on the Fisher graph has a unique inverse with bounded coefficients $K_{x,y}^{-1}$.
- These coefficients have a local expression

$$K_{x,y}^{-1} = \frac{k'}{8\pi} \int_{\Gamma_{x,y}} f_x(u + 2K)f_y(u) \text{Exp}_{x,y}(u|k) du$$
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\]

**Definition (massive exponential functions)**

\[
\text{Exp}_{x,y}(u|k) = \prod_j i \sqrt{k'} \text{sc}\left(\frac{u - \alpha_j}{2}|k\right), \quad u \in T_k
\]

**Definition (function \(f\))**

- If \(x\) internal to a decoration \(f_x(u) = \pm \text{cn}\left(\frac{u - \alpha}{2}|k\right)^{-1}\), where \(e^{i\alpha}\) edge of the quad-graph
- If \(x\) connected to an edge of \(G\), \(f_x\) is the sum of two such terms
\[ K_{x,y}^{-1} = \frac{k'}{8\pi} \int_{\Gamma_{x,y}} f_x(u+2K)f_y(u) \text{Exp}_{x,y}(u|k)du \]

- This expression is local: \( K_{x,y}^{-1} \) depends on the geometry of the graph only along a path from \( x \) to \( y \)
- It can be used to define a Gibbs measure on dimer configurations of the Fisher graph, and thus on Ising contours (without assumption on periodicity of the graph)
- Dimer statistics are local
On periodic isoradial graphs: spectral curve

- If $G$ is periodic, the Kasteleyn operator $K$ is also periodic
- $K(z, w)$ Fourier transform of $K$: matrix with rows/columns indexed by vertices in a fund. domain with extra $z^{\pm 1}$ or $w^{\pm 1}$ weight for edges crossing its boundary
- $P(z, w) = \det K(z, w)$ characteristic polynomial
- Fourier formula for $K^{-1}$:

$$K_{x,y}^{-1}(m,n) = \int \int_{|z|=|w|=1} z^{-m} w^{-n} \frac{Q_{x,y}(z,w)}{P(z,w)} \frac{dz}{2i\pi z} \frac{dw}{2i\pi w}$$

where $Q_{x,y}$ cofactor of $K(z, w)$.

Asymptotics depends on the zeros of $P$.

$C = \{(z, w) : P(z, w) = 0\}$ is called the spectral curve
Theorem (B–de Tilière–Raschel)

**spectral curve of a Z-invariant Ising model on isoradial graph**

\[ (z, w) \leftrightarrow \left( \frac{1}{z}, \frac{1}{w} \right) \]

- Parametrization: \( u \mapsto (\text{Exp}_{x,x+(1,0)}(u|k), \text{Exp}_{x,x+(0,1)}(u|k)) \)
- Area of the hole as a function of \( k \) and the local geom. of \( G \)
- Same curve for the Ising model with param. \( k \) and \( k^* \)
On periodic isoradial graphs: free energy

**free energy** $F_{\text{Ising}}$: normalized log of the partition function

**Theorem**

$$F_{\text{Ising}}(k) = -\frac{\log 2}{2} |V_1| - |V_1| \int_0^K 2H'(\theta) \log \text{sc}(\theta) d\theta + \sum_{e \in E_1} \left( -H(2\theta) \log \text{sc}(\theta) + \int_0^{\theta_e} 2H'(\theta) \log \text{sc}(\theta) d\theta \right).$$

As $k$ goes to 0,

$$F_{\text{Ising}}(k) = F_{\text{Ising}}(0) - \frac{|V_1|}{2} k^2 \log k^{-1} + O(k^2)$$
Z-invariant Ising model and rooted spanning forests

- This free energy is half the free energy of rooted spanning forests, “counted” by the determinant of a massive Laplacian on isoradial graphs, with conductances $sc(2K\theta/\pi|k)$ we introduced.

- Same phase transition in Ising as from spanning forests to spanning trees

- Massive exponential functions: harmonic for this massive Laplacian (elliptic generalisation of Mercat’s harmonic exponential functions)
Phase transition in the Ising model