Inhomogeneous Multispecies TASEP on a ring

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The Asymmetric Simple exclusion Process

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Multispecies ASEP general framework

We consider a periodic lattice $\mathbb{Z}/L\mathbb{Z}$ on which we have for $1 \leq \alpha \leq N$, $m_\alpha$ particles of species $\alpha$, $\sum_{\alpha=1}^{N} m_\alpha = L$

The rates $p_{\alpha,\beta}$ for a local exchange $\alpha \leftrightarrow \beta$ depends on the species involved.

The case that we are interested in is

$$p_{\alpha,\beta} = \begin{cases} 
0 & \text{for } \alpha \geq \beta \\
\tau_\alpha + \nu_\beta & \text{for } \alpha < \beta 
\end{cases}$$

We’ll see later where this choice comes from.
The homogeneous TASEP process ($\nu_\beta = 0$ and $\tau_\alpha = \tau$) has appeared recently in a work by Lam.

He was interested in certain infinite random reduced words in affine Weyl groups that can be defined as random walks on the affine Coxeter arrangement, conditioned never to cross the same hyperplane twice.
Random walk in an affine Weyl group [Lam]
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Among other remarkable results, he proved that the walk almost surely gets stuck in a Weyl chamber and that the walk will almost surely tend to a certain direction in that chamber.

For the case of $\tilde{A}_n$ the probability of getting stuck in the Weyl chamber $C_\sigma$ is

$$P(C_\sigma) = P_{\sigma^{-1}\sigma_0}$$

where $P_\sigma$ is the stationary probability of being in the state $\{\sigma(1), \ldots, \sigma(N)\}$ for the homogeneous TASEP with $N$ species on a $\mathbb{Z}/N\mathbb{Z}$
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Multispecies TASEP on a ring

We consider the M-TASEP on a ring $\mathbb{Z}/L\mathbb{Z}$. A state of this system is just a periodic word $w$ of length $L(w) = L$, $w_i = w_{i+L}$.

$$w = \{2, 2, 5, 3, 2, 5\}$$

The dynamics conserves the total number of particles of a given species. We denote the species content of a word $w$ by

$$m(w) = \{ \ldots, m_\alpha(w), m_{\alpha+1}(w), \ldots \} \in \mathbb{N}^\mathbb{Z}$$

which means that we have $m_\alpha(w)$ particles of species $\alpha$

$$\sum_{\alpha=\mathbb{Z}} m_\alpha(w) = L(w)$$

$$m(w) = \{ m_{<2} = 0, m_2 = 3, m_3 = 1, m_4 = 0, m_5 = 2, m_{>5} = 0 \}, \quad L = 6$$
For a given (periodic) word $w$ we define the **descents number**

$$d(w) = \# \{ 1 \leq i \leq L | w_i > w_{i+1} \}$$

Normalizing the “probability” of a state $w^*$ with the minimal descent number ($d(w^*) = 1$) as

$$\psi_{w^*} = \chi_m(\tau, \nu) := \prod_{\alpha < \beta} (\tau_\alpha + \nu_\beta)^{(\beta - \alpha - 1)(m_\alpha + m_\beta - 1)}$$
Positivity conjectures [Lam-Williams, L.C]

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\textbf{Positivity Conjecture}

The polynomials $\psi_w(\tau, \nu)$ have positive integer coefficients
Let $t = t_1, t_2, \ldots$ and $v = v_1, v_2 \ldots$ two infinite sets of commuting variables.

**Definition: double Schubert polynomials**

For the longest permutation $w_0 \in S_n$

$$\mathcal{G}_{w_0}(t, v) := \prod_{i+j \leq n} (t_i - v_j)$$

for generic $w \in S_n$

$$\mathcal{G}_w(t, v) = \partial_{w^{-1}w_0} \mathcal{G}_{w_0}(t, v)$$

where $\partial_{w^{-1}w_0} = \partial_{s_{i_1}} \partial_{s_{i_2}} \cdots \partial_{s_{i_\ell}}$, $(s_{i_1} \cdot s_{i_2} \cdots s_{i_\ell}$ is a reduced decomposition of $w^{-1}w_0$) and

$$\partial_{s_{i_1}} = \frac{1 - s_i^t}{t_i - t_{i+1}}, \quad s_i^t : t_i \leftrightarrow t_{i+1}.$$
Conjecture

- The functions $\psi_w(\tau, \nu)$ can be expressed as polynomials of double Schubert polynomials with positive integer coefficients.

- The double Schubert polynomials appearing in the expression of $\psi_w(\tau, \nu)$ correspond to permutations in $S_{L(w)}$ and the variables $t, v$ are chosen as

$$t = \underbrace{\tau_{\min(m)}, \ldots, \tau_{\min(m)}}_{m_{\min(m)}}, \underbrace{\tau_{\min(m)} + 1, \ldots, \tau_{\min(m)} + 1}_{m_{\min(m)} + 1}, \ldots$$

$$v = \underbrace{-\nu_{\max(m)}, \ldots, -\nu_{\max(m)}}_{m_{\max(m)}}, \underbrace{-\nu_{\max(m)} - 1, \ldots, -\nu_{\max(m)} - 1}_{m_{\max(m)} - 1}, \ldots$$
The case $\nu_\alpha = 0$ and multiline queues [Ayyer-Linusson, Arita-Mallick]

The positivity conjecture has been settled by Arita and Mallick in the case $\nu_\alpha = 0$ in terms of *multiline queues* as conjectured by Ayyer and Linusson.

Multiline queues have been introduced by Ferrari and collaborators. A multiline queue of type $\mathbf{m}$ is a $\mathbb{Z} \times L$ array ($L = \sum m_i$), which has $\sum_{j \leq i} m_j$ particles on the $i$-th row.

```
  . . . . . . O O O O  
O O . . . . O . . . .  
O O . . . . O O O O O  
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![Multiline queue diagram]

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Multispecies TASEP
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![Multiline Queue Diagram](image_url)
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The case $\nu_\alpha = 0$ and multiline queues

To a multiline queue $q$ one can associate a M-TASEP state of content $m$ through the Bully Path (BP) algorithm.

**Theorem** [Arita Mallick]

\[
\psi_w \propto \sum_{q | BP(q) = w} \prod_{\alpha < \beta} \left( \frac{\tau_\beta}{\tau_\alpha} \right)^{z_{\alpha,\beta}(q)}
\]

where $z_{\alpha,\beta}(q)$ is the number of vacancies on row $j$ that are covered by a $i$ Bully Path.

**Open question**

Does such a construction extend to the general case $\nu_\alpha \neq 0$?
Integrability

The master equation for the time evolution of the probability of a configuration is

\[
\frac{d}{dt} P_w(t) = \sum_{w' | w' \rightarrow w} M_{w,w'} P_w(t) - \sum_{w | w \rightarrow w'} M_{w',w} P_w(t)
\]

\[
\frac{d}{dt} P(t) = \mathcal{M} P(t)
\]

The important point to remark here is that the Markov matrix $\mathcal{M}$ is the sum of local terms acting on $V_m$, the vector space with a basis labeled by periodic words of content $m$.

\[
\mathcal{M} = \sum_{i=1}^{L} M^{(i)}, \quad M^{(i)} = \sum_{1 \leq \alpha \neq \beta \leq N} p_{\alpha,\beta} M^{(i)}_{\alpha,\beta}
\]
Integrability

The master equation for the time evolution of the probability of a configuration is

$$\frac{d}{dt} P_w(t) = \sum_{w' \leftarrow w} M_{w,w'} P_{w'}(t) - \sum_{w \rightarrow w'} M_{w',w} P_w(t)$$

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Integrability

Now suppose that we have a matrix \( \mathcal{R}(x, y) \) depending on two formal commuting variables, such that

\[
\mathcal{R}(x, x) = 1, \quad \frac{d}{dx} \mathcal{R}(x, y)|_{x=y=0} \propto \sum_{1 \leq \alpha \neq \beta} p_{\alpha, \beta} M_{\alpha, \beta}
\]

and a vector

\[
\psi(z) \in V_m \otimes \mathbb{Z}[z], \quad z = \{z_1, \ldots, z_{L(m)}\}
\]

such that the following Exchange equations

\[
\mathcal{R}_i(z_i, z_{i+1})\psi(z) = s_i \circ \psi(z)
\]

where \( s_i \) acts on \( \mathbb{Z}[z] \) by the exchange \( z_i \leftrightarrow z_{i+1} \).
Integrability

Then I claim that $\psi(0)$ is proportional to the TASEP stationary probability

$$\mathcal{M}\psi(0) = 0$$

The consistency of the exchange equations is ensured by the unitarity relation

$$\tilde{R}_i(x, y)\tilde{R}_i(y, x) = 1$$

and the braid Yang-Baxter equation

$$\tilde{R}_i(y, z)\tilde{R}_{i+1}(x, z)\tilde{R}_i(x, y) = \tilde{R}_{i+1}(x, y)\tilde{R}_i(x, z)\tilde{R}_{i+1}(y, z)$$
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Multispecies TASEP: baxterized form of R-matrix

**Theorem**

The most general solution of the unitarity and Yang Baxter equations of the Baxterized form

\[ \tilde{R}(x, y) = 1 + \sum_{1 \leq \alpha < \beta \leq N} g_{\alpha, \beta}(x, y) M_{\alpha, \beta} \]

is given by

\[ g_{\alpha, \beta}(x, y) = \frac{(y - x)(\tau_\alpha + \nu_\beta)}{(\tau_\alpha y - 1)(\nu_\beta x + 1)} \rightarrow p_{\alpha < \beta} = \tau_\alpha + \nu_\beta \]

**Theorem**

The exchange equations corresponding the the \( \tilde{R} \) matrix of the Multispecies TASEP admit a polynomial solution, unique up to multiplication of a completely symmetric polynomial in the \( z \).
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Exchange equations in components

Once expanded in components, the exchange equations read as follows

\[ \psi_{\ldots, w_i = w_{i+1}, \ldots}(z) = s_i \circ \psi_{\ldots, w_i = w_{i+1}, \ldots}(z) \]

\[ \psi_{\ldots, w_i > w_{i+1}, \ldots}(z) = \hat{\pi}_i(w_i, w_{i+1}) \psi_{\ldots, w_{i+1}, w_i, \ldots}(z) \]

and

\[ \hat{\pi}_i(\alpha, \beta) = \frac{(\tau_\alpha z_{i+1} - 1)(\nu_\beta z_i + 1)}{\tau_\alpha + \nu_\beta} \frac{1 - s_i}{z_i - z_{i+1}} \]

This system of equations is cyclic: given \( \psi_w(z) \) for a word \( w \) one can obtain \( \psi_{w'}(z) \) for any other \( w' \) by acting with the \( \hat{\pi} \) operators.
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This system of equation is cyclic: given \( \psi_w(z) \) for a word \( w \) one can obtain \( \psi_{w'}(z) \) for any other \( w' \) by acting with the \( \hat{\pi} \) operators.
The operators $\hat{\pi}_i(\alpha, \beta)$ satisfy a spectral parameter deformation (not baxterization!) of the 0-Hecke algebra (recovered for $t_\alpha$ and $\nu_\alpha$ independent of $\alpha$)

\[
\hat{\pi}_i(\alpha, \beta) = -\hat{\pi}_i(\alpha, \beta)
\]

\[
\hat{\pi}_i(\beta, \gamma)\hat{\pi}_{i+1}(\alpha, \gamma)\hat{\pi}_i(\alpha, \beta) = \hat{\pi}_{i+1}(\alpha, \beta)\hat{\pi}_i(\alpha, \gamma)\hat{\pi}_{i+1}(\beta, \gamma)
\]

\[
[\hat{\pi}_i(\alpha, \beta), \hat{\pi}_j(\gamma, \delta)] = 0 \quad |i - j| > 2
\]
Simple consequences of the exchange equations

• If the word \( w \) has a sub-sequence \( w_\ell \leq w_{\ell+1} \leq \cdots \leq w_{k-1} \leq w_k \) then

\[
\psi_w(z) = \prod_{i=\ell}^{k} \left( \prod_{\alpha \in w_{\ell,k}}^{\alpha < w_i} (\tau_\alpha z_i - 1) \prod_{\alpha \in w_{\ell,k}}^{\alpha > w_i} (\nu_\beta z_i + 1) \right) \tilde{\psi}_w(z)
\]

where \( \tilde{\psi}_w(z) \) is symmetric in the variable \( \{z_\ell, \ldots, z_k\} \)

• In particular if \( w = w^* \) has minimum number of descents \( w_\ell \leq w_\ell \leq \cdots \leq w_{\ell-2} \leq w_{\ell-1} \) then \( \tilde{\psi}_{w^*}(z) \) is symmetric in the whole set of variables \( z \) and by cyclicity is a common factor of all the \( \psi_w(z) \).
Simple consequences of the exchange equations

- If the word $w$ has a sub-sequence $w_\ell \leq w_{\ell+1} \leq \cdots \leq w_{k-1} \leq w_k$ then

$$
\psi_w(z) = \prod_{i=\ell}^{k} \left( \prod_{\alpha \in w_{\ell,k}} (\tau_{\alpha} z_i - 1) \prod_{\alpha \in w_{\ell,k}} (\nu_{\beta} z_i + 1) \right) \tilde{\psi}_w(z)
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where $\tilde{\psi}_w(z)$ is symmetric in the variable $\{z_\ell, \ldots, z_k\}$

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Simple consequences of the exchange equations

- The solution of the exchange equation of minimal degree in the sector \( m \) has degree

\[
\deg_{z_i} \psi^{(m)}(z) = \#\{\alpha | m_\alpha \neq 0\} - 1
\]

- Normalization choice

\[
\psi_w^*(z) = \chi_m(\tau, \nu) \prod_{i=1}^{L} \left( \prod_{\alpha < w_i^*} (1 - \tau_\alpha z_i) \prod_{\beta > w_i^*} (1 + \nu_\beta z_i) \right)
\]

**Conjecture**

With the normalization given above, the components \( \psi_w \) are polynomials in all their variables \((z, \tau, \nu)\) with no common factors.
Simple consequences of the exchange equations

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- Normalization choice

- Conjecture
  With the normalization given above, the components $\psi_w$ are polynomials in all their variables $(z, \tau, \nu)$ with no common factors.
Recursions

**Proposition**

By specializing $z_L = \tau^{-1}_{\min(m)}$ or $z_L = -\nu^{-1}_{\max(m)}$ we have the following recursion

$$
\psi_w(z)|_{z_L = \tau^{-1}_{\min(m)}} = \begin{cases} 
0 & w_L \neq \min(m) \\
K^-(z \setminus z_L) \psi_{w \setminus w_L}(z \setminus z_L) & w_L = \min(m)
\end{cases}
$$

$$
\psi_w(z)|_{z_L = -\nu^{-1}_{\max(m)}} = \begin{cases} 
0 & w_L \neq \max(m) \\
K^+(z \setminus z_L) \psi_{w \setminus w_L}(z \setminus z_L) & w_L = \max(m)
\end{cases}
$$

where the factors $K^\pm(z \setminus z_L)$ can be easily computed by inspection of $\psi_{w^*}(z)$. 
Simplest non trivial component

Let \( w^{(\alpha)} \) such that for \( i \leq j \leq L - m_\alpha \)

\[
  w_i \neq \alpha \quad \text{and} \quad w_i \leq w_j
\]

For example

\[
  w^{(6)} = 2 2 3 5 5 5 7 9 9 6 6 6
\]

Then

\[
  \psi_{w^{(\alpha)}}^{(m)}(z) = (\text{Trivial Factors}) \times \phi^{(m)}_{\alpha}(z_1, \ldots, z_{L-m_\alpha})
\]

where \( \phi^{(m)}_{\alpha}(z_1, \ldots, z_{L-m_\alpha}) \) is a symmetric polynomial in \( z_1, \ldots, z_{L-m_\alpha} \) of degree 1 in each variable separately.

- These polynomials turn out to be the building blocks of more general components
- Thanks to the recursion relations they can be computed explicitly
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Simplest non trivial component

For any \( n > 0 \) and \( 1 \leq \beta \leq n \) define the following polynomials

\[
\Phi^n_\beta(z; t; v) := \Delta(t, v) \int_t^{-} \frac{dw}{2\pi i} \prod_{1 \leq \rho \leq \beta} (w - t_\rho) \prod_{1 \leq \sigma \leq n - \beta + 1} (w - v_\sigma) \prod_{i=1}^{n-1} (1 - wz_i)
\]

Notice that these specialize to double Schubert Polynomials

\[
\Phi^n_\beta(0; t; v) = S_{1, \beta+1, \beta+2, ..., n, 2, 3, ..., \beta}(t; v)
\]
Simplest non trivial component

For any $n > 0$ and $1 \leq \beta \leq n$ define the following polynomials

$$\Phi_{\beta}^n(z; t; v) := \Delta(t, v) \int_{t} \frac{dw}{2\pi i} \frac{\prod_{i=1}^{n-1}(1 - wz_i)}{\prod_{1 \leq \rho \leq \beta}(w - t_{\rho}) \prod_{1 \leq \sigma \leq n-\beta+1}(w - v_{\sigma})}$$

Notice that these specialize to double Schubert Polynomials

$$\Phi_{\beta}^n(0; t; v) = \mathcal{S}_{1, \beta+1, \beta+2, \ldots, n, 2, 3, \ldots, \beta}(t; v)$$
Proposition

\[ \phi_{\alpha}^{(m)}(z_1, \ldots, z_{L-m}) = \Phi_{\beta}^{L-m\alpha}(z; t; v) \]

with \( \beta = 1 + \sum_{\gamma < \alpha} m_{\gamma} \), and

\[ t = \{ \ldots, \tau_{\gamma}, \ldots, \tau_{\gamma}, \ldots, \tau_{\alpha-1}, \ldots, \tau_{\alpha-1}, \tau_{\alpha} \} \]

\[ v = \{ -\nu_{\alpha}, -\nu_{\alpha+1}, \ldots, -\nu_{\alpha+1}, \ldots, -\nu_{\gamma}, \ldots, -\nu_{\gamma}, \ldots \} \]
Factorization of components with least ascending

We have seen that to each “ascent” in a word \( w \) one has a bunch of trivial factors, therefore the intuition is that the more ascents \( w \) has the “simpler” is its component \( \psi_w \).

Actually the words \( \tilde{w} \) which have minimal number of ascent are also computable

Exm

\[
\tilde{w} = 9 9 7 6 6 6 5 5 5 3 2 2
\]

Conjecture

Calling \( z_\alpha = \{ z_i | w_i = \alpha \} \)

\[
\psi_{\tilde{w}} = \prod_{\alpha} \phi_{\alpha}^{(m)}(z \setminus z_\alpha)
\]

This conjecture has passed several non-trivial tests but at the moment unfortunately is still unproven.
Factorization of components with least ascending

We have seen that to each “ascent” in a word $w$ one has a bunch of trivial factors, therefore the intuition is that the more ascents $w$ has the “simpler” is its component $\psi_w$. Actually the words $\tilde{w}$ which have minimal number of ascent are also computable

Exm

$$\tilde{w} = 9 \ 9 \ 7 \ 6 \ 6 \ 6 \ 5 \ 5 \ 5 \ 3 \ 2 \ 2$$

Conjecture

Calling $z_\alpha = \{z_i|w_i = \alpha\}$

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This conjecture has passed several non-trivial tests but at the moment unfortunately is still unproven.
Factorization of components with least ascending: corollaries

**Corollary I**

The formula for the least ascending component implies and generalizes a formula conjectured by Lam and Williams which expresses $\psi_{\tilde{w}}$ in the case $m = \{\ldots, 0, 1, 1 \ldots, 1, 0 \ldots \}$ as a product of double-Schubert Polynomials of $\tau, \nu$

$$\psi_{L, L-1, \ldots, 1} = S_{1, 2, 3, \ldots, L} S_{1, 3, 4, \ldots, L, 2} S_{1, 4, 5, \ldots, L, 2, 3} S_{1, \ldots, L-1}$$

**Corollary II**

Suppose that we condition $w$ to split as $w^{(k)} w^{(k-1)} \ldots w^{(2)} w^{(1)}$, with $w^{(j)}$ of fixed length $L_j$ (possibly 0) and

$$w_{i}^{(r)} < w_{j}^{(s)} \quad \text{for} \quad r < s$$

then the events $w^{(j)}$ are independent.
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The formula for the least ascending component implies and generalizes a formula conjectured by Lam and Williams which expresses $\psi_{\tilde{w}}$ in the case $m = \{\ldots, 0, 1, 1 \ldots, 1, 0 \ldots \}$ as a product of double-Schubert Polynomials of $\tau, \nu$

$$\psi_{L_1, L_1-1, \ldots, 1} = \mathfrak{S}_1, 2, 3 \ldots, L \mathfrak{S}_1, 3, 4 \ldots, L, 2 \mathfrak{S}_1, 4, 5 \ldots, L, 2, 3 \mathfrak{S}_1, L, 2, 3 \ldots, L-1$$

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Suppose that we condition $w$ to split as $w^{(k)}w^{(k-1)} \ldots w^{(2)}w^{(1)}$, with $w^{(j)}$ of fixed length $L_j$ (possibly 0) and

$$w^{(j)}_{i(r)} < w^{(j)}_{i(s)} \quad \text{for} \quad r < s$$

then the events $w^{(j)}$ are independent.
Normalization

In order to compute actual probabilities we need the normalization

$$Z^{(m)}(z) = \sum_{w | m(w) = m} \psi_w(z)$$

Thanks to the exchange relations this polynomial turns out to be symmetric in $z$ and satisfies the recursion relation induced by $\psi(z)$ itself.

Unfortunately in the general case we are not able to provide a formula for $Z^{(m)}(z)$.

What we can solve is the case $\nu_\alpha = \nu$ for $\alpha \leq \gamma$, $\tau_\alpha = \tau$ for $\alpha \geq \gamma$ for some $\gamma$. 

Luigi Cantini
Multispecies TASEP
Let us define the idempotent operators $\Pi_{\alpha_1, \ldots, \alpha_k}^\beta$ (by convention I assume $\beta \notin \{\alpha_1, \ldots, \alpha_k\}$)

$$\Pi_{\alpha_1, \ldots, \alpha_k}^\beta(w)_i = \begin{cases} w_i & \text{if } w_i \notin \{\alpha_1, \ldots, \alpha_k\} \\ \beta & \text{if } w_i \in \{\alpha_1, \ldots, \alpha_k\} \end{cases}$$

and extend it by linearity. We get projectors on the Hilbert space of the configurations of the M-TASEP.

\begin{lemma}
Suppose that $(\tau_\alpha, \nu_\alpha) = (\tau_{\alpha+1}, \nu_{\alpha+1})$, then we have that the $\hat{R}$ matrix commutes with the projection $\Pi_{\alpha+1}^\alpha$

$$\hat{R}_i(x, y)\Pi_{\alpha+1}^\alpha = \Pi_{\alpha+1}^\alpha \hat{R}_i(x, y)$$
\end{lemma}
Projections

Let us define the idempotent operators $\Pi_{\beta, \alpha_1, \ldots, \alpha_k}$ (by convention I assume $\beta \notin \{\alpha_1, \ldots, \alpha_k\}$)

$$\Pi_{\beta, \alpha_1, \ldots, \alpha_k}(w)_i = \begin{cases} w_i & \text{if } w_i \notin \{\alpha_1, \ldots, \alpha_k\} \\ \beta & \text{if } w_i \in \{\alpha_1, \ldots, \alpha_k\} \end{cases}$$

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Proposition

This means that if \((\tau_\alpha, \nu_\alpha) = (\tau_{\alpha+1}, \nu_{\alpha+1})\), then \(\Pi_{\alpha+1}^\alpha \psi^{(m)}(z)\) is again solution of the exchange equations and therefore

\[
\Pi_{\alpha+1}^\alpha \psi^{(m)}(z) = \rho^{(m,\alpha)}(z) \psi^{(\Pi_{\alpha+1}^\alpha(m))}(z)
\]

where

\[
\Pi_{\gamma}^\alpha(m)_{\beta} = \begin{cases} 
  m_{\beta} & \text{for } \beta \neq \alpha, \gamma \\
  m_\alpha + m_\gamma & \text{for } \beta = \alpha \\
  0 & \text{for } \beta = \gamma 
\end{cases}
\]

and \(\rho^{(m',\alpha)}(z)\) is a symmetric function of degree 1 in each \(z_i\) which is given by a specialization of \(\Phi_{\beta}^n(z; t; v)\).
Factorization of the sum rule

If for some $\gamma$ we have

\[ \nu_\alpha = \nu \quad \text{for} \quad \alpha \leq \gamma \]
\[ \tau_\alpha = \tau \quad \text{for} \quad \alpha \geq \gamma \]

(and $m_\alpha > 0$ for $\min(m) \leq \alpha \leq \max(m)$), by projecting “downward” from $\max(m)$ and “upward” from $\min(m)$ until $\gamma$

**Theorem**

\[ Z^{(m)}(z) = \prod_{\alpha = \min(m)}^{\gamma - 1} \phi_{m_\alpha}^{(m)}(z) \prod_{\alpha = \gamma + 1}^{\max(m)} \phi_{m_\alpha}^{(m)}(z) \]

where

\[ m_\downarrow_\alpha = \prod_{\alpha, \alpha + 1}^{\alpha - 1} m, \quad m_\uparrow_\alpha = \prod_{\alpha + 1, \alpha - 1}^{\alpha} m \]
Some open questions

- Correlation functions, currents, etc.
  - Do the components $\psi_w(z)$ have a combinatorial expression?
  - What is the “right” context for the 0-Hecke algebra with spectral parameters?
  - The operators $\hat{\pi}(\alpha, \beta)$ can be used for example to define a family of deformed Grothendieck “polynomials” which depend on the parameters $\tau, \nu$. Do they have any geometric meaning?
  - Deal with others Weyl groups.
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