Explicit Generating Series for Small-Step Walks in the Quarter Plane

Frédéric Chyzak

October 8, 2015
Based on work with A. Bostan, M. van Hoeij, M. Kauers, and L. Pech
Lattice Walks, Why?

Applications in many areas of science

- discrete mathematics (permutations, trees, words, urns, . . .)
- statistical physics (Ising model, . . .)
- probability theory (branching processes, games of chance, . . .)
- operations research (queueing theory, . . .)

A history and a survey of lattice path enumeration

Katherine Humphreys

Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, FL 33431, USA

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Lattice path
Reflection principle
Method of images

ABSTRACT

In celebration of the Sixth International Conference on Lattice Path Counting and Applications, it is befitting to review the history of lattice path enumeration and to survey how the topic has progressed thus far.

We start the history with early games of chance specifically the ruin problem which later appears as the ballot problem. We discuss André's Reflection Principle and its misnomer, its relation with the method of images and possible origins from physics and Brownian motion, and the earliest evidence of lattice path techniques and solutions.

In the survey, we give representative articles on lattice path enumeration found in the literature in the last 35 years by the lattice, step set, boundary, characteristics counted, and solution method. Some of this work appears in the author's 2005 dissertation.

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Applications in many areas of science

- discrete mathematics (permutations, trees, words, urns, …)
- statistical physics (Ising model, …)
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This talk:

Computer Algebra applied to Combinatorics
Nearest-neighbor walks in the quarter plane = walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a fixed subset $\mathcal{S}$ of

\[ \{\searrow, \leftarrow, \swarrow, \uparrow, \nearrow, \rightarrow, \nwarrow, \downarrow\}. \]

Example with $n = 45, i = 14, j = 2$ for:

$\mathcal{S} = \begin{array}{c}
\bullet \\
\bullet \\
\end{array}\begin{array}{c}
\bullet \\
\bullet \\
\end{array}\begin{array}{c}
\bullet \\
\bullet \\
\end{array}$
Nearest-neighbor walks in the quarter plane = walks in \( \mathbb{N}^2 \) starting at \((0,0)\) and using steps in a \textit{fixed} subset \( \mathcal{S} \) of

\[
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Example with \( n = 45 \), \( i = 14 \), \( j = 2 \) for:

Counting sequence: \( f_{n;i,j} \) = number of walks of length \( n \) ending at \((i,j)\).
Nearest-neighbor walks in the quarter plane = walks in $\mathbb{N}^2$ starting at $(0, 0)$ and using steps in a fixed subset $\mathcal{S}$ of

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Counting sequence: $f_{n;i,j}$ = number of walks of length $n$ ending at $(i, j)$.

Specializations:

- $f_{n;0,0}$ = number of walks of length $n$ returning to origin ("excursions");
- $f_n = \sum_{i,j \geq 0} f_{n;i,j}$ = number of walks with prescribed length $n$. 
Generating Series and Combinatorial Problems

Complete generating series: \( F(x, y; t) = \sum_{n=0}^{\infty} \left( \sum_{i,j=0}^{\infty} f_{n,i,j} x^i y^j \right) t^n \in \mathbb{Q}[x, y][[t]]. \)
Complete generating series: \( F(x, y; t) = \sum_{n=0}^{\infty} \left( \sum_{i,j=0}^{\infty} f_{n,i,j} x^i y^j \right) t^n \in \mathbb{Q}[x,y][[t]]. \)

Specializations:
- Walks returning to the origin ("excursions"): \( F(0, 0; t); \)
- Walks with prescribed length: \( F(1, 1; t) = \sum_{n \geq 0} f_n t^n. \)
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Combinatorial questions: Given \( \mathcal{G} \), what can be said about \( F(x, y; t) \), resp. \( f_{n;i,j} \), and their variants?

- **Algebraic nature** of \( F \): algebraic? transcendental?
- **Explicit form**: of \( F \) of \( f \)?
- **Asymptotics** of \( f \)?
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- **Asymptotics** of \( f \)?

Our goal: Use computer algebra to give computational answers.
Small-Step Models of Interest

From the $2^8$ step sets $\mathcal{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, some are:

- trivial,
- simple,
- intrinsic to the half plane,
- symmetrical.

One is left with 79 interesting distinct models. Is any further classification possible?
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One is left with 79 interesting distinct models.

Is any further classification possible?
Classification of Univariate Power Series

D-finite power series

- Algebraic: $S(t) \in \mathbb{Q}[[t]]$ root of a polynomial $P \in \mathbb{Q}[t, S(t)]$, i.e., $P(t, S(t)) = 0$.

- Hypergeometric: $S(t) = \sum_{n=0}^{\infty} s_n t^n$ such that $s_{n+1}/s_n \in \mathbb{Q}(n)$. E.g., $2F1(a \mid b \mid c \mid t) = \sum_{n=0}^{\infty} \binom{a}{n} \binom{b}{n} \binom{c}{n} t^n / n!$, where $\binom{x}{y} = x(x+1)\cdots(x+y-1)$.

- Small-Step Walks

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Classification of Univariate Power Series

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- **D-finite**: $S(t) \in \mathbb{Q}[[t]]$ satisfying a linear differential equation with polynomial coefficients $c_r(t)S^{(r)}(t) + \cdots + c_0(t)S(t) = 0$.

- **Hypergeometric**: $S(t) = \sum_{n=0}^{\infty} s_n t^n$ such that $s_{n+1} / s_n \in \mathbb{Q}(n)$.

E.g.,

\[
\begin{align*}
&2F1(a \mid b \mid c \mid t) = \\
&\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} t^n,
\end{align*}
\]

where $(a)_n = a(a+1)\cdots(a+n-1)$.
Classification of Univariate Power Series

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- **Hypergeometric**: \( S(t) = \sum_{n=0}^{\infty} s_n t^n \) such that \( \frac{s_{n+1}}{s_n} \in \mathbb{Q}(n) \). E.g.,

  \[
  _2F_1 \left( \begin{array}{c} a \ b \\ c \end{array} \mid t \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad (a)_n = a(a+1) \cdots (a+n-1),
  \]

  \[
  t(1-t)S''(t) + (c - (a + b + 1)t)S'(t) - abS(t) = 0.
  \]
### Table of All Conjectured D-Finite $F(1, 1; t)$ [Bostan & Kauers, 2009]

<table>
<thead>
<tr>
<th>OEIS</th>
<th>$\mathcal{G}$</th>
<th>alg ord</th>
<th>equiv</th>
<th>OEIS</th>
<th>$\mathcal{G}$</th>
<th>alg ord</th>
<th>equiv</th>
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<tr>
<td>A005566</td>
<td>$\uparrow$</td>
<td>N 3</td>
<td>$\frac{4}{\pi} \frac{4^n}{n}$</td>
<td>A151275</td>
<td>$\uparrow$</td>
<td>N 5</td>
<td>$\frac{12\sqrt{30}}{\pi} \frac{(2\sqrt{6})^n}{n^2}$</td>
</tr>
<tr>
<td>A018224</td>
<td>$\downarrow$</td>
<td>N 3</td>
<td>$\frac{2}{\pi} \frac{4^n}{n}$</td>
<td>A151314</td>
<td>$\uparrow$</td>
<td>N 5</td>
<td>$\frac{\sqrt{6\lambda\mu C^{5/2}}}{5\pi} \frac{(2C)^n}{n^2}$</td>
</tr>
<tr>
<td>A151312</td>
<td>$\downarrow$</td>
<td>N 3</td>
<td>$\frac{6}{\pi} \frac{6^n}{n}$</td>
<td>A151255</td>
<td>$\uparrow$</td>
<td>N 5</td>
<td>$\frac{24\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^2}$</td>
</tr>
<tr>
<td>A151331</td>
<td>$\downarrow$</td>
<td>N 3</td>
<td>$\frac{8}{3\pi} \frac{8^n}{n}$</td>
<td>A151287</td>
<td>$\uparrow$</td>
<td>N 5</td>
<td>$\frac{2\sqrt{2}A^{7/2}}{\pi} \frac{(2A)^n}{n^2}$</td>
</tr>
<tr>
<td>A151266</td>
<td>$\downarrow$</td>
<td>N 5</td>
<td>$\frac{1}{2} \sqrt{\frac{3}{\pi} \frac{3^n}{n^{1/2}}}$</td>
<td>A001006</td>
<td>$\downarrow$</td>
<td>Y 3</td>
<td>$\frac{3}{2} \sqrt{\frac{3}{\pi} \frac{3^n}{n^{3/2}}}$</td>
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<td>A151307</td>
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<td>A129400</td>
<td>$\uparrow$</td>
<td>Y 3</td>
<td>$\frac{3}{2} \sqrt{\frac{3}{\pi} \frac{6^n}{n^{3/2}}}$</td>
</tr>
<tr>
<td>A151291</td>
<td>$\downarrow$</td>
<td>N 5</td>
<td>$\frac{4}{3\sqrt{\pi}} \frac{4^n}{n^{1/2}}$</td>
<td>A005558</td>
<td>$\uparrow$</td>
<td>N 4</td>
<td>$\frac{8}{\pi} \frac{4^n}{n^2}$</td>
</tr>
<tr>
<td>A151326</td>
<td>$\downarrow$</td>
<td>N 5</td>
<td>$\frac{2}{3\pi} \frac{6^n}{n^{1/2}}$</td>
<td></td>
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<td></td>
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<tr>
<td>A151302</td>
<td>$\downarrow$</td>
<td>N 5</td>
<td>$\frac{1}{3} \sqrt{\frac{5}{2\pi} \frac{5^n}{n^{1/2}}}$</td>
<td>A151265</td>
<td>$\downarrow$</td>
<td>Y</td>
<td>$\frac{2\sqrt{2}}{\Gamma(1/4)} \frac{3^n}{n^{3/4}}$</td>
</tr>
<tr>
<td>A151329</td>
<td>$\downarrow$</td>
<td>N 5</td>
<td>$\frac{1}{3} \sqrt{\frac{7}{3\pi} \frac{7^n}{n^{1/2}}}$</td>
<td>A151278</td>
<td>$\downarrow$</td>
<td>Y</td>
<td>$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)} \frac{3^n}{n^{3/4}}$</td>
</tr>
<tr>
<td>A151261</td>
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<td>$\frac{12\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^2}$</td>
<td>A151323</td>
<td>$\downarrow$</td>
<td>Y</td>
<td>$\frac{\sqrt{23^{3/4}}}{\Gamma(1/4)} \frac{6^n}{n^{3/4}}$</td>
</tr>
<tr>
<td>A151297</td>
<td>$\uparrow$</td>
<td>N 5</td>
<td>$\frac{\sqrt{3}B^{7/2}}{2\pi} \frac{(2B)^n}{n^2}$</td>
<td>A060900</td>
<td>$\uparrow$</td>
<td>Y</td>
<td>$\frac{4\sqrt{3}}{3\Gamma(1/3)} \frac{4^n}{n^{2/3}}$</td>
</tr>
</tbody>
</table>

$A = 1 + \sqrt{2}, \ B = 1 + \sqrt{3}, \ C = 1 + \sqrt{6}, \ \lambda = 7 + 3\sqrt{6}, \ \mu = \sqrt{\frac{4\sqrt{6} - 1}{19}}$

▶ Computerized discovery of ODE/poly. by enumeration + Hermite–Padé.

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Small-Step Walks
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<tr>
<td>A005566</td>
<td>N</td>
<td>3</td>
<td>$\frac{4^4}{\pi n}$</td>
<td>A151275</td>
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<td>5</td>
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<td>A018224</td>
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<td>A151314</td>
<td>N</td>
<td>5</td>
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<td>A001006</td>
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<td>A005558</td>
<td>N</td>
<td>4</td>
<td>$\frac{8}{\pi n^2}$</td>
</tr>
<tr>
<td>A151326</td>
<td>N</td>
<td>5</td>
<td>$\frac{2^2}{\sqrt{3^3 6^n}{\pi n^{1/2}}}$</td>
<td></td>
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</tr>
<tr>
<td>A151302</td>
<td>N</td>
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<td>$\frac{1}{3} \sqrt{5^5 2^n}{\pi n^{1/2}}$</td>
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<td>$\frac{1}{3} \sqrt{7^7 7^n}{\pi n^{1/2}}$</td>
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<tr>
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<td>N</td>
<td>$\frac{12\sqrt{3} (2\sqrt{3})^n}{\pi n^2}$</td>
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<td>$\frac{4^3}{3\pi n^2}$</td>
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$A = 1 + \sqrt{2}$, $B = 1 + \sqrt{3}$, $C = 1 + \sqrt{6}$, $\lambda = 7 + 3\sqrt{6}$, $\mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$

> Computerized discovery of asymptotics by enumeration + LLL/PSLQ.
Further Previous Work

Confirmation of D-finiteness

▷ Human proofs for cases 1–22 in [Bousquet-Mélou & Mishna, 2010], but method not adapted to exhibit ODEs.
▷ Computer proof for case 23 in [Bostan & Kauers, 2010].

Fix of asymptotic formulas (first observed/proved by Melczer)

In fact:

<table>
<thead>
<tr>
<th>OEIS</th>
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<th>equiv</th>
</tr>
</thead>
<tbody>
<tr>
<td>A151261</td>
<td>$\frac{12\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^2}$ (n = 2p)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{18}{\pi} \frac{(2\sqrt{3})^n}{n^2}$ (n = 2p + 1)</td>
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<tr>
<td>A151275</td>
<td>$\frac{12\sqrt{30}}{\pi} \frac{(2\sqrt{6})^n}{n^2}$ (n = 2p)</td>
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</tr>
<tr>
<td></td>
<td>$\frac{144}{\sqrt{5}\pi} \frac{(2\sqrt{6})^n}{n^2}$ (n = 2p + 1)</td>
<td></td>
</tr>
<tr>
<td>A151255</td>
<td>$\frac{24\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^2}$ (n = 2p)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{32}{\pi} \frac{(2\sqrt{2})^n}{n^2}$ (n = 2p + 1)</td>
<td></td>
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</tbody>
</table>
Proof of formerly guessed linear differential operators for $F(1,1;t)$.
Contributions

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- Discovery and proof of explicit hypergeometric expressions for $F(x, y; t)$.
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- Similar proofs for $F(0,0; t)$, $F(0,1; t)$, and $F(1,0; t)$. 
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- Discovery and proof of explicit hypergeometric expressions for $F(x,y;t)$.
- Proof of algebricity, resp. transcendence, of those series.
- Similar proofs for $F(0,0;t), F(0,1;t)$, and $F(1,0;t)$.

- Similar conjectured asymptotic formulas for $F(0,0;t), F(0,1;t), F(1,0;t)$.
### Table of D-Finite $F(x,y;t)$ at $x = y = 0$ [This work]

<table>
<thead>
<tr>
<th>OEIS</th>
<th>$\mathfrak{S}$ alg</th>
<th>conj’d equiv</th>
<th>OEIS</th>
<th>$\mathfrak{S}$ alg</th>
<th>conj’d equiv</th>
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<td>1</td>
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<td>$\left{ \begin{array}{l} 32 \pi^4 n^3 \ 0 \end{array} \right. \quad (n = 2p)$</td>
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<td>3</td>
<td>A151362</td>
<td>$\left{ \begin{array}{l} 3\sqrt{6} 6^n \ 0 \end{array} \right. \quad (n = 2p)$</td>
<td>15</td>
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<td>$\left{ \begin{array}{l} 16\sqrt{2} \ \pi \end{array} \right. \quad (2\sqrt{2})^n$</td>
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<tr>
<td>4</td>
<td>A172361</td>
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<td>16</td>
<td>A151357</td>
<td>$\left{ \begin{array}{l} 2A^{3/2} \ \pi \end{array} \right. \quad (2A)^n$</td>
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<td>$\left{ \begin{array}{l} 16\pi^4 \ 0 \end{array} \right. \quad (n = 4p)$</td>
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<td>$\left{ \begin{array}{l} 81/8 \pi^4 \ n^4 \end{array} \right. \quad (2\sqrt{3})^n$</td>
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<td>$\left{ \begin{array}{l} 2\pi^4 n^3 \ (2\pi)^n \end{array} \right. \quad (n = 4p)\quad (n = 2p + 1)$</td>
<td>18</td>
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<td>$\left{ \begin{array}{l} 27\sqrt{3} \ \pi \end{array} \right. \quad (6^n)$</td>
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<td>$\left{ \begin{array}{l} 12\pi^4 \ 0 \end{array} \right. \quad (n = 2p)$</td>
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<td>$\left{ \begin{array}{l} 768 \pi^4 \ 4^n \end{array} \right. \quad (n = 2p)$</td>
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<td>8</td>
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<td>$\left{ \begin{array}{l} 2B^{3/2} \pi \ (2B)^n \end{array} \right. \quad (n = 2p)\quad (n = 2p + 1)$</td>
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<td>$\left{ \begin{array}{l} 2\mu^3 C^{3/2} \ \pi \end{array} \right. \quad (2C)^n$</td>
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# Table of D-Finite $F(x, y; t)$ at $x = 0, y = 1$ [This work]

<table>
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<tr>
<th>OEIS</th>
<th>$G$ alg</th>
<th>conjd equiv</th>
<th>OEIS</th>
<th>$G$ alg</th>
<th>conjd equiv</th>
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<td>N</td>
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<td>$\frac{3b^{7/2}}{2\pi} \frac{(2B)^n}{n^3}$</td>
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<td>$\begin{cases} \frac{4}{\pi} \frac{4^n}{n^2} &amp; (n = 2p) \ 0 &amp; (n = 2p + 1) \end{cases}$</td>
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<td>N</td>
<td>$\frac{3\sqrt{6}}{2\pi} \frac{6^n}{n^2}$</td>
<td>A151492</td>
<td>N</td>
<td>$\frac{6\lambda^5 c^{5/2}}{5\pi} \frac{(2c)^n}{n^3}$</td>
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<td>N</td>
<td>$\frac{32}{9\pi} \frac{8^n}{n^2}$</td>
<td>A151375</td>
<td>N</td>
<td>$\begin{cases} 448\sqrt{2} \frac{(2\sqrt{2})^n}{9\pi} \frac{n^3}{n^3} &amp; (n = 4p) \ 640 \frac{(2\sqrt{2})^n}{9\pi} \frac{n^3}{n^3} &amp; (n = 4p + 1) \ 416\sqrt{2} \frac{(2\sqrt{2})^n}{9\pi} \frac{n^3}{n^3} &amp; (n = 4p + 2) \ 512 \frac{(2\sqrt{2})^n}{9\pi} \frac{n^3}{n^3} &amp; (n = 4p + 3) \end{cases}$</td>
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<td>Y</td>
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The Kernel Equation [≤ Knuth, 1968]: an Example, ✨

walk of length $n + 1 =$
walk of length $n$ followed by a step from $\{←, ↑, →, ↓\}$
The Kernel Equation [≤ Knuth, 1968]: an Example, ☞

walk of length \( n + 1 = \)
walk of length \( n \) followed by a step from \( \{ \leftarrow, \uparrow, \rightarrow, \downarrow \} \),

provided this remains in the quarter plane!
walk of length $n + 1 =$
walk of length $n$ followed by a step from $\{←, ↑, →, ↓\}$,
provided this remains in the quarter plane!

Recurrence relation:

$$f_{n+1;i,j} = f_{n;i+1,j} + \begin{cases} 0 < j \end{cases} f_{n;i,j-1} + \begin{cases} 0 < i \end{cases} f_{n;i-1,j} + f_{n;i,j+1}.$$
walk of length \( n + 1 = \) walk of length \( n \) followed by a step from \( \{←, ↑, →, ↓\} \), provided this remains in the quarter plane!

Recurrence relation:

\[
f_{n+1;i,j} = f_{n;i+1,j} + \left[ 0 < j \right] f_{n;i,j-1} + \left[ 0 < i \right] f_{n;i-1,j} + f_{n;i,j+1}.
\]

\[
f_{n+1;i,j} x^i y^j t^{n+1} = \left( f_{n;i+1,j} x^{i+1} y^j t^n \right) \times \bar{x} t + \left[ 0 < j \right] \left( f_{n;i,j-1} x^i y^{j-1} t^n \right) \times y t + \left[ 0 < i \right] \left( f_{n;i-1,j} x^{i-1} y^j t^n \right) \times x t + \left( f_{n;i,j+1} x^i y^{j+1} t^n \right) \times \bar{y} t,
\]

Notation: \( \bar{x} = \frac{1}{x}, \bar{y} = \frac{1}{y} \).
walk of length $n + 1 = \text{walk of length } n \text{ followed by a step from } \{←, ↑, →, ↓\},$

provided this remains in the quarter plane!

Recurrence relation:

$$f_{n+1;i,j} = f_{n;i+1,j} + \left[0 < j\right] f_{n;i,j-1} + \left[0 < i\right] f_{n;i-1,j} + f_{n;i,j+1}.$$ 

$$f_{n+1;i,j} x^i y^j t^{n+1} = \left( f_{n;i+1,j} x^{i+1} y^j t^n \right) \times \bar{x} t + \left[0 < j\right] \left( f_{n;i,j-1} x^i y^{j-1} t^n \right) \times y t + \left[0 < i\right] \left( f_{n;i-1,j} x^{i-1} y^j t^n \right) \times x t + \left( f_{n;i,j+1} x^i y^{j+1} t^n \right) \times \bar{y} t, \quad F(x, y; t) - 1 = \left( F(x, y; t) - F(0, y; t) \right) \times \bar{x} t + F(x, y; t) \times y t + F(x, y; t) \times x t + \left( F(x, y; t) - F(x, 0; t) \right) \times \bar{y} t,$$

Notation: \(\bar{x} = \frac{1}{x}, \bar{y} = \frac{1}{y}\).
The Kernel Equation \[\leq\text{Knuth, 1968}\]: an Example, \(\Diamond\)

walk of length \(n + 1 =\)
walk of length \(n\) followed by a step from \(\{\leftarrow, \uparrow, \rightarrow, \downarrow\}\),
provided this remains in the quarter plane!

Recurrence relation:

\[
f_{n+1;i,j} = f_{n;i+1,j} + \begin{cases} 0 < j \end{cases} f_{n;i,j-1} + \begin{cases} 0 < i \end{cases} f_{n;i-1,j} + f_{n;i,j+1}.
\]

Functional (“kernel”) equation:

\[
(1 - t(x + \bar{x} + y + \bar{y})) F(x, y; t) = -\bar{y}tF(x, 0; t) - \bar{x}tF(0, y; t) + 1.
\]
The Kernel Equation \([<\text{Knuth, 1968}]\): an Example, \(\square\)

walk of length \(n + 1 = \) walk of length \(n\) followed by a step from \(\{←, ↑, →, ↓\}\),

provided this remains in the quarter plane!

Recurrence relation:

\[
f_{n+1;i,j} = f_{n;i+1,j} + \mathbb{1}_{0 < j} f_{n;i,j-1} + \mathbb{1}_{0 < i} f_{n;i-1,j} + f_{n;i,j+1}.\]

Functional (“kernel”) equation:

\[
(1 - t (x + \bar{x} + y + \bar{y})) F(x, y; t) = -\bar{y}t F(x, 0; t) - \bar{x}t F(0, y; t) + 1.
\]

Remarks:

- Erasing the constraint leads to a rational generating series.
- Direct attempt to solve leads to tautologies.
$J = 1 - t \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y})$ is \textit{invariant} under the change of $(x, y)$ into, respectively:

$$(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y}).$$
$J = 1 - t \sum_{(i,j) \in S} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y})$ is invariant under the change of $(x, y)$ into, respectively:

$(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})$.

Kernel equation:

$$J(x, y; t)xyF(x, y; t) = -txF(x, 0; t) - tyF(0, y; t) + xy,$$
\( J = 1 - t \sum_{(i,j) \in \mathcal{G}} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y}) \) is invariant under the change of \((x, y)\) into, respectively:

\((\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})\).

Kernel equation:

\[
J(x, y; t)xyF(x, y; t) = -txF(x, 0; t) - tyF(0, y; t) + xy,
- J(x, y; t)\bar{x}yF(\bar{x}, y; t) = t\bar{x}F(\bar{x}, 0; t) + tyF(0, y; t) - \bar{x}y,
\]
$J = 1 - t \sum_{(i,j) \in S} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y})$ is invariant under the change of $(x, y)$ into, respectively:

$(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})$.

Kernel equation:

$J(x, y; t)xyF(x, y; t) = -txF(x, 0; t) - tyF(0, y; t) + xy,$

$-J(x, y; t)\bar{x}yF(\bar{x}, y; t) = t\bar{x}F(\bar{x}, 0; t) + tyF(0, y; t) - \bar{x}y,$

$J(x, y; t)\bar{x}\bar{y}F(\bar{x}, \bar{y}; t) = -t\bar{x}F(\bar{x}, 0; t) - t\bar{y}F(0, \bar{y}; t) + \bar{x}\bar{y},$
D-Finiteness via the Finite Group: an Example, ✡

\[ J = 1 - t \sum_{(i,j) \in \mathcal{G}} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y}) \] is invariant under the change of \((x,y)\) into, respectively:

\((\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})\).

Kernel equation:

\[
\begin{align*}
J(x, y; t)xyF(x, y; t) &= -txF(x, 0; t) - tF(0, y; t) + xy, \\
-J(x, y; t)\bar{x}yF(\bar{x}, y; t) &= t\bar{x}F(\bar{x}, 0; t) + t\bar{y}F(0, \bar{y}; t) - \bar{x}y, \\
J(x, y; t)\bar{x}\bar{y}F(\bar{x}, \bar{y}; t) &= -t\bar{x}F(\bar{x}, 0; t) - t\bar{y}F(0, \bar{y}; t) + \bar{x}\bar{y}, \\
-J(x, y; t)x\bar{y}F(x, \bar{y}; t) &= txF(x, 0; t) + t\bar{y}F(0, \bar{y}; t) - x\bar{y}.
\end{align*}
\]
\[ J = 1 - t \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y}) \] is invariant under the change of \((x, y)\) into, respectively:
\[(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y}).\]

Kernel equation:
\[ J(x, y; t) xyF(x, y; t) = -txF(x, 0; t) - tyF(0, y; t) + xy, \]
\[ - J(x, y; t) \bar{x} yF(\bar{x}, y; t) = t\bar{x}F(\bar{x}, 0; t) + tyF(0, y; t) - \bar{x} y, \]
\[ J(x, y; t) \bar{y} F(\bar{x}, \bar{y}; t) = -t\bar{x}F(\bar{x}, 0; t) - t\bar{y}F(0, \bar{y}; t) + x \bar{y}, \]
\[ - J(x, y; t) x \bar{y} F(x, \bar{y}; t) = txF(x, 0; t) + t\bar{y}F(0, \bar{y}; t) - x \bar{y}. \]

Summing up yields:
\[ J(x, y; t) \sum_{g \in \mathcal{G}} \text{sign}(g) g(xy F(x, y; t)) = xy - \bar{x} y + x \bar{y} - x \bar{y}. \]
$J = 1 - t \sum_{(i,j) \in \mathcal{G}} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y})$ is invariant under the change of $(x, y)$ into, respectively:

$$(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y}).$$

Kernel equation:

$$J(x, y; t)xyF(x, y; t) = -txF(x, 0; t) - tyF(0, y; t) + xy,$$

$$- J(x, y; t)\bar{x}yF(\bar{x}, y; t) = t\bar{x}F(\bar{x}, 0; t) + tyF(0, y; t) - \bar{x}y,$$

$$J(x, y; t)\bar{y}F(\bar{x}, \bar{y}; t) = - t\bar{x}F(\bar{x}, 0; t) - t\bar{y}F(0, \bar{y}; t) + \bar{x}\bar{y},$$

$$- J(x, y; t)x\bar{y}F(x, \bar{y}; t) = txF(x, 0; t) + t\bar{y}F(0, \bar{y}; t) - x\bar{y}.$$

Summing up yields:

$$\sum_{g \in \mathcal{G}} \text{sign}(g) g(xy F(x, y; t)) = \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{J(x, y; t)}.$$
D-Finiteness via the Finite Group: an Example, 

\[ J = 1 - t \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y}) \] is invariant under the change of \((x, y)\) into, respectively:

\[(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y}).\]

Kernel equation:

\[
\begin{align*}
J(x, y; t)xyF(x, y; t) &= -txF(x, 0; t) - tyF(0, y; t) + xy, \\
- J(x, y; t)\bar{x}yF(\bar{x}, y; t) &= t\bar{x}F(\bar{x}, 0; t) + tyF(0, y; t) - \bar{x}y, \\
J(x, y; t)\bar{y}F(\bar{x}, \bar{y}; t) &= -t\bar{x}F(\bar{x}, 0; t) - t\bar{y}F(0, \bar{y}; t) + \bar{x}\bar{y}, \\
- J(x, y; t)x\bar{y}F(x, \bar{y}; t) &= txF(x, 0; t) + t\bar{y}F(0, \bar{y}; t) - x\bar{y}.
\end{align*}
\]

Summing up yields:

\[
[x^>][y^>] \sum_{g \in \mathcal{G}} \text{sign}(g) g(xy F(x, y; t)) = [x^>][y^>] \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{J(x, y; t)}.
\]
\[ J = 1 - t \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y}) \text{ is invariant under the change of } (x, y) \text{ into, respectively:} \]

\[ (\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y}). \]

Kernel equation:

\[ J(x, y; t)xyF(x, y; t) = -txF(x, 0; t) - tyF(0, y; t) + xy, \]
\[ -J(x, y; t)\bar{x}yF(\bar{x}, y; t) = t\bar{x}F(\bar{x}, 0; t) + tyF(0, y; t) - \bar{x}y, \]
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\[ -J(x, y; t)x\bar{y}F(x, \bar{y}; t) = txF(x, 0; t) + t\bar{y}F(0, \bar{y}; t) - x\bar{y}. \]

Summing up yields:

\[ xy F(x, y; t) = [x^>][y^>] \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{J(x, y; t)}. \]
Cases 1–19 are D-Finite

\[ J = 1 - t \sum_{(i,j) \in \mathcal{S}} x^i y^j \quad \rightarrow \quad \text{a group } \mathcal{G} \text{ of birational transformations} \]

**Theorem** [Bousquet-Mélou & Mishna, 2010]

Let \( \mathcal{S} \) be one of the step sets 1–19. Then, the group \( \mathcal{G} \) is finite and:

\[
xy F(x, y; t) = [x^>] [y^>] \frac{\sum_{g \in \mathcal{G}} \text{sign}(g) g(xy)}{J(x, y; t)}.
\]

In particular, \( F(x, y; t) \) is D-finite.
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▷ Remark: The formula provides no direct information for \( x = y = 1 \).
Theorem [This work]

Let \( S \) be one of the step sets 1–19. Then, the generating series \( F(x, y; t) \) is expressible using iterated integrals of \( _2F_1 \) functions.
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Let $\mathcal{S}$ be one of the step sets 1–19. Then, the generating series $F(1, 1; t)$ is expressible using iterated integrals of $\,_{2}F_{1}$ functions.

Example: King walks in the quarter plane (A025595, [link])

$$F(1, 1; t) = \frac{1}{t} \int_{0}^{t} \frac{1}{(1 + 4x)^{3}} \cdot _{2}F_{1}\left(\frac{3}{2}, \frac{3}{2} \left| \frac{16x(1 + x)}{(1 + 4x)^{2}} \right. \right) \, dx$$

$$= 1 + 3t + 18t^{2} + 105t^{3} + 684t^{4} + 4550t^{5} + 31340t^{6} + 219555t^{7} + \cdots$$
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$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

Proved by deriving and solving:

$$t^2(4t + 1)(8t - 1)(2t - 1)(t + 1)y''' + t(576t^4 + 200t^3 - 252t^2 - 33t + 5)y'' +$$

$$(1152t^4 + 88t^3 - 468t^2 - 48t + 4)y' + (384t^3 - 72t^2 - 144t - 12)y = 0.$$
Theorem [This work]

Let $\mathcal{S}$ be one of the step sets 1–19. Then, the generating series $F(x, y; t)$ is expressible using iterated integrals of $\,_{2}F_{1}$ functions.

Proof uses Creative telescoping, ODE factorization, ODE solving:

1. If $R = \sum_{g} \frac{\text{sign}(g) g(xy)}{f(x,y;t)}$, then $F = \text{Res}_{u,v} H$, for $H = \frac{R(1/u,1/v;t)}{(1-xu)(1-yv)}$.

2. If $L \in \mathbb{Q}(x,y)[t]\langle \partial_t \rangle$ and $U, V \in \mathbb{Q}(x,y,u,v,t)$ such that $L(H) = \partial_u U + \partial_v V$, then $L(F(x,y;t)) = 0$.
   Use creative telescoping to find $L$ (as well as $U$ and $V$).

3. Factor $L$ as $L_2 \cdot P_1 \cdots P_t$, where $L_2$ has order $\leq 2$ and the $P_i$ have order 1.

4. Solve $L_2$ in terms of $\,_{2}F_{1}$s and deduce $F$. 

Frédéric Chyzak
Small-Step Walks
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Explicit Expressions for the Cases 1–19

**Theorem [This work]**

Let $\mathcal{S}$ be one of the step sets 1–19. Then, the generating series $F(x, y; t)$ is expressible using iterated integrals of $2F_1$ functions.

▷ **Proof uses** Creative telescoping, ODE factorization, ODE solving:

1. If $R = \sum g \frac{\text{sign}(g) g(xy)}{f(x,y;t)}$, then $F = \text{Res}_{u,v} H$, for $H = \frac{R(1/u,1/v;t)}{(1-xu)(1-yv)}$.
   Taking algebraic residues commutes with specializing $x$ and $y$!

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   Use creative telescoping to find $L$ (as well as $U$ and $V$).
   Works in practice with early evaluation $(x, y) = (1, 1)$, but not for symbolic $(x, y)$.
   Works also for $(0, 0)$, $(x, 0)$, and $(0, y)$!

3. Factor $L$ as $L_2 \cdot P_1 \cdots P_t$, where $L_2$ has order $\leq 2$ and the $P_i$ have order 1.

4. Solve $L_2$ in terms of $2F_1$s and deduce $F$.

5. For $F(x, y; t)$, run whole process for $F(0,0;t)$, $F(x,0;t)$, and $F(0,y;t)$, then substitute into kernel equation!
### Hypergeometric Series Occurring in Explicit Expressions for $F(x, y; t)$

<table>
<thead>
<tr>
<th>( \mathcal{S} ) occurring ( _2F_1 )</th>
<th>( w )</th>
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Hypergeometric Series Occurring in Explicit Expressions for $F(x, y; t)$

<table>
<thead>
<tr>
<th></th>
<th>$\mathcal{G}$ occurring $2F_1$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2F_1\left(\frac{1}{2} \frac{1}{2} \frac{1}{4} \frac{3}{4} \ w\right)$</td>
<td>$16t^2$</td>
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<td>$2F_1\left(\frac{1}{2} \frac{1}{2} \frac{1}{4} \frac{3}{4} \ w\right)$</td>
<td>$16t^2$</td>
<td>11</td>
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Observation: Related to complete elliptic integrals, $E(\sqrt{w})$ and $K(\sqrt{w})$. 

Frédéric Chyzak
Small-Step Walks
Well-studied algorithms

- Factorization of ODE: [Beke, 1894], [Schwarz, 1989], [Grigor’ev, 1990], [Singer, 1996], [van Hoeij, 1997]

Already combined for a simpler problem: Diagonal 3D Rook Paths
[Bostan, Chyzak, van Hoeij, & Pech, 2011]

Problem: Determine the number $a_n$ of paths from $(0, 0, 0)$ to $(n, n, n)$ that use positive multiples of $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

Solution: $G(x) = 1 + 6 \cdot \int_0^x \frac{2F_1 \left( \frac{1}{3}, \frac{2}{3} \mid 2 \right)}{(1 - 4w)(1 - 64w)} \frac{27w(2-3w)}{(1-4w)^3} \, dw$. 
Key Idea (Step 1): Encoding Positive Parts as Algebraic Residues

Problem: Definitions of residues and positive parts of rational functions?

\[ \cdots - \frac{1}{w^3} - \frac{1}{w^2} - \frac{1}{w} = \frac{1}{1 - w} = 1 + w + w^2 + \cdots \]
Problem: Definitions of residues and positive parts of rational functions?

\[ \cdots - \frac{1}{w^3} - \frac{1}{w^2} - \frac{1}{w} \equiv \frac{1}{1 - w} \equiv 1 + w + w^2 + \cdots \]

\[ -1 \equiv \text{Res}_w \frac{1}{1 - w} \equiv 0 \]
Key Idea (Step 1): Encoding Positive Parts as Algebraic Residues

Problem: Definitions of residues and positive parts of rational functions?

\[
\cdots - \frac{1}{w^3} - \frac{1}{w^2} - \frac{1}{w} \equiv \frac{1}{1-w} \equiv 1 + w + w^2 + \cdots \\
0 \equiv [w^>] \frac{1}{1-w} \equiv w + w^2 + \cdots
\]
Key Idea (Step 1): Encoding Positive Parts as Algebraic Residues

New formula

\[ F(a, b; t) = \text{Res}_{x,y} \left[ \frac{\bar{x}\bar{y}R(x, y; t)}{(x - a)(y - b)} \right]_{\Gamma_1} = \text{Res}_{x,y} \left[ \frac{R(\bar{x}, \bar{y}; t)}{(1 - ax)(1 - by)} \right]_{\Gamma_2}. \]

Interpretation [Aparicio-Monforte & Kauers, 2013]

- \( \text{Res}_{x,y} \) is linear on the vector space \( \mathbb{Q}^{\mathbb{Z}^2} \);
- the rational functions \( R(x, y; t) \) and \( (x - a)^{-1}(y - b)^{-1} \) are expanded as a series with support in the cone \( \Gamma_1 = \{x^i y^j t^n : i, |j| \leq n \geq 0\} \);
- the rational functions \( R(\bar{x}, \bar{y}; t) \) and \( (1 - ax)^{-1}(1 - by)^{-1} \) are expanded as a series with support the cone \( \Gamma_2 = \{x^i y^j t^n : -i, |j| \leq n \geq 0\} \);
- a theory of series with support in a cone legitimates the product.

Link with creative telescoping [This work]

\[ L(H) = \partial_u U + \partial_v V \implies L([H]_\Gamma) = 0 \]

provided \( H, U, V \) admit expansions with respect to the same cone \( \Gamma \).
Proofs of Algebraicity/Transcendence of $F(x, y; t)$ and $F(1, 1; t)$

**Theorem**

- In cases 1–19, $F(x, y; t)$ is transcendental since $F(0, 0; t)$ is.
- In cases 1–16 and 19, $F(1, 1; t)$ is transcendental.
- Specific simplifications prove algebraicity of $F(1, 1; t)$ in cases 17–18.

**Proof:** Define $G = (P_1 \cdots P_t)(F)$ so that $L_2(G) = 0$.

- $F$ is algebraic $\implies G$ is algebraic.
- Computing a few coefficients of $G$ shows that this is not 0 on all cases of interest.
- Applying Kovacic’s algorithm to $L_2$ (order 2) or just computing exponential solutions (order 1) decides whether $L_2$ has nonzero algebraic solutions.
ODE: \[ t^3(4t - 1)(12t^2 - 1)(4t^2 + 1)(576t^7 + \cdots - 3) \frac{d^5F}{dt^5} + \cdots = 0 \]

Recurrence: \[ 3(n + 11)(n + 12)(n + 13)(n + 14)^2u_{n+12} + \cdots = 0 \]
The Problem of Counting Excursions Asymptotically: an Example,

ODE: \[ t^3(4t - 1)(12t^2 - 1)(4t^2 + 1)(576t^7 + \cdots - 3)\frac{d^5 F}{dt^5} + \cdots = 0 \]

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<th>( u_1 )</th>
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<th>( u_{10} )</th>
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The Problem of Counting Excursions Asymptotically: an Example,

**ODE:** \[ t^3(4t - 1)(12t^2 - 1)(4t^2 + 1)(576t^7 + \cdots - 3) \frac{d^5 F}{dt^5} + \cdots = 0 \]

**Recurrence:** \[ 3(n + 11)(n + 12)(n + 13)(n + 14)^2 u_{n+12} + \cdots = 0 \]

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We need exact "global" information related to \( u_n \) to get the \( \kappa_i \) symbolically.

\( \kappa_1 = 0, \kappa_2 = \sqrt{3} + 8 \pi, \kappa_3 = \sqrt{3} - 8 \pi, \kappa_4 = (2p)^3, \kappa_5 = -(2p)^3 \) (7 other regimes)

Frédéric Chyzak
Small-Step Walks
The Problem of Counting Excursions Asymptotically: an Example,

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\[ u_0 = c_0 \quad u_n = \kappa_1 \left( \frac{4^n}{\sqrt{n}} + \cdots \right) + \kappa_2 \left( \frac{\sqrt{12}^n}{n^2} + \cdots \right) + \kappa_3 \left( \frac{(-\sqrt{12})^n}{n^2} + \cdots \right) + \kappa_4 \left( \frac{(2i)^n}{n^3} + \cdots \right) + \kappa_5 \left( \frac{(-2i)^n}{n^3} + \cdots \right) + (7 \text{ other regimes}) \]
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\[
\left( \begin{array}{c}
\text{connection matrix} \\
\end{array} \right) \left( \begin{array}{c}
c_0 \\
\vdots \\
c_{11} \\
\end{array} \right) = \left( \begin{array}{c}
\kappa_1 \\
\vdots \\
\kappa_{12} \\
\end{array} \right)
\]

\[
6.62 \frac{\sqrt{12^{2p}}}{(2p)^2} \quad 5.73 \frac{\sqrt{12^{2p+1}}}{(2p+1)^2} \quad 2.44 10^{-6} \quad \frac{4^n}{\sqrt{n}}
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\[ u_1 = \cdots \]
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We need exact “global” information related to \( u_n \) to get the \( \kappa_i \) symbolically.

\[ \text{e.g.: } \kappa_1 = 0, \kappa_2 = \frac{6\sqrt{3}+9}{\pi}, \kappa_3 = \frac{6\sqrt{3}-9}{\pi} \]
Singularity analysis [Flajolet & Odlyzko, 1990] =

Method to get the asymptotics of Taylor coefficients

\[ f(z) = \sum_{n=0}^{\infty} f_n z^n \quad \longrightarrow \quad f_n \sim \cdots \]

- Determine **dominant singularities** of the complex-analytic function \( f \).
- Find asymptotic expansion

\[ f(z) =_{z \to s} \sum_{\alpha, \gamma} c_{\alpha, \gamma} (s - z)^{\alpha} \left( \ln \frac{1}{s - z} \right)^{\gamma}. \]

- **Syntactic transfer** into an asymptotic expansion for \( f_n \). E.g., for \( \alpha > 0 \):

\[
\begin{aligned}
  f(z) =_{z \to s} c_0 (1 - \rho z)^{\alpha} + c_1 (1 - \rho z)^{\alpha+1} + O((1 - \rho z)^{\alpha+2}) & \longrightarrow \\
  f_n =_{n \to \infty} & \frac{c_0}{\Gamma(-\alpha)n^{\alpha+1}} + \frac{c_1}{\Gamma(-\alpha - 1)n^{\alpha+2}} + O \left( \frac{1}{n^{\alpha+3}} \right).
\end{aligned}
\]
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\]

**D-finite functions are in principle amenable to this method.**
Example of Asymptotic Behaviour Driven by the $\binom{\frac{3}{2}}{2} F_1$ at $(1, 1)$

$$F(1, 1; t) = \frac{1}{t} \int f \text{ for } f = (1 - 2t)(1 + 2t)^{-3/2}(1 + 6t)^{-3/2} \binom{\frac{3}{2}}{2} F_1\left(\frac{3}{2}, \frac{3}{2} \middle| w\right)$$

where $$w = \frac{16t}{(1 + 2t)(1 + 6t)} = 1 - \frac{(1 - 6t)(1 - 2t)}{(1 + 2t)(1 + 6t)}.$$  

Singularities: $\frac{1}{2}, -\frac{1}{2}, -\frac{1}{6}, w = 1, w = \infty \rightarrow$ Dominant singularities = $\pm \frac{1}{6}$.

$$f(t) \sim \begin{cases} t \to \frac{1}{6} & -\frac{\sqrt{6}}{\pi} (1 - 6t)^{-1} \rightarrow \frac{\sqrt{6}}{\pi} 6^n \\ t \to -\frac{1}{6} & +\frac{\sqrt{6}}{4\pi} \ln(1 + 6t) \rightarrow (-1)^n \frac{\sqrt{6}}{4\pi} \frac{6^n}{n} \end{cases}$$
Example of Asymptotic Behaviour Driven by the $2F_1$: at $(1, 1)$

\[ F(1, 1; t) = \frac{1}{t} \int f \quad \text{for} \quad f = (1 - 2t)(1 + 2t)^{-3/2}(1 + 6t)^{-3/2} \begin{pmatrix} \frac{3}{2} & \frac{3}{2} \\ 2 & 2 \end{pmatrix} w \]

where

\[ w = \frac{16t}{(1 + 2t)(1 + 6t)} = 1 - \frac{(1 - 6t)(1 - 2t)}{(1 + 2t)(1 + 6t)}. \]

Singularities: $\frac{1}{2}, -\frac{1}{2}, -\frac{1}{6}, w = 1, w = \infty \rightarrow$ Dominant singularities $= \pm \frac{1}{6}$.

\[ f(t) \sim_{t \to \frac{1}{6}^-} \frac{\sqrt{6}}{\pi} (1 - 6t)^{-1} \quad \rightarrow \quad \frac{\sqrt{6}}{\pi} 6^n \]

\[ f(t) \sim_{t \to -\frac{1}{6}^+} \frac{\sqrt{6}}{4\pi} \ln(1 + 6t) \quad \rightarrow \quad (-1)^n \frac{\sqrt{6}}{4\pi} \frac{6^n}{n} \]

\[ f \rightarrow f_n \sim \frac{\sqrt{6}}{\pi} 6^n \]
Example of Asymptotic Behaviour Driven by the $2F_1$: at $(1, 1)$

$$F(1, 1; t) = \frac{1}{t} \int f \quad \text{for} \quad f = (1 - 2t)(1 + 2t)^{-3/2}(1 + 6t)^{-3/2} 2F_1\left(\frac{3}{2}, \frac{3}{2} \mid w\right)$$

where \( w = \frac{16t}{(1 + 2t)(1 + 6t)} = 1 - \frac{(1 - 6t)(1 - 2t)}{(1 + 2t)(1 + 6t)}. \)

Singularities: $\frac{1}{2}, -\frac{1}{2}, -\frac{1}{6}$, $w = 1, w = \infty \rightarrow$ Dominant singularities $= \pm \frac{1}{6}$.

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$$f(t) \sim_{t \rightarrow -\frac{1}{6}^+} \frac{\sqrt{6}}{4\pi} \ln(1 + 6t) \quad \rightarrow \quad (-1)^n \frac{\sqrt{6}}{4\pi} \frac{6^n}{n}$$

$$\int f \quad \rightarrow \quad f_n \sim \frac{\sqrt{6}}{\pi} \frac{6^{n-1}}{n}$$
Example of Asymptotic Behaviour Driven by the $$_2F_1$$ at $(1, 1)$

$$F(1, 1; t) = \frac{1}{t} \int f \quad \text{for} \quad f = (1 - 2t)(1 + 2t)^{-3/2}(1 + 6t)^{-3/2} _2F_1 \left(\frac{3}{2}, \frac{3}{2} \Bigg| w \right)$$

where \( w = \frac{16t}{(1 + 2t)(1 + 6t)} = 1 - \frac{(1 - 6t)(1 - 2t)}{(1 + 2t)(1 + 6t)} \).

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\( f(t) \sim t \rightarrow \frac{\sqrt{6}}{\pi} (1 - 6t)^{-1} \quad \rightarrow \quad \frac{\sqrt{6}}{\pi} 6^n \)

\( f(t) \sim t \rightarrow -\frac{1}{6} + \frac{\sqrt{6}}{4\pi} \ln(1 + 6t) \quad \rightarrow \quad (-1)^n \frac{\sqrt{6}}{4\pi} \frac{6^n}{n} \)

\( \frac{1}{t} \int f \quad \rightarrow \quad f_n \sim \frac{\sqrt{6}}{\pi} \frac{6^n}{n + 1} \)
Example of Asymptotic Behaviour Driven by the $\text{}_2\text{F}_1$: at $(1, 1)$

\[ F(1, 1; t) = \frac{1}{t} \int f \quad \text{for} \quad f = (1 - 2t)(1 + 2t)^{-3/2}(1 + 6t)^{-3/2} \text{}_2\text{F}_1\left(\frac{3}{2}, \frac{3}{2} \left| w \right. \right) \]

where \( w = \frac{16t}{(1 + 2t)(1 + 6t)} = 1 - \frac{(1 - 6t)(1 - 2t)}{(1 + 2t)(1 + 6t)}. \)

Singularities: $\frac{1}{2}, -\frac{1}{2}, -\frac{1}{6}, w = 1, w = \infty \rightarrow$ Dominant singularities $= \pm \frac{1}{6}$.

\[
\begin{align*}
f(t) & \sim_{t \to \frac{1}{6}^-} \frac{\sqrt{6}}{\pi} (1 - 6t)^{-1} \quad \rightarrow \quad \frac{\sqrt{6}}{\pi} 6^n \\
f(t) & \sim_{t \to -\frac{1}{6}^+} \frac{\sqrt{6}}{4\pi} \ln(1 + 6t) \quad \rightarrow \quad (-1)^n \frac{\sqrt{6}}{4\pi} \frac{6^n}{n}
\end{align*}
\]

\[
\frac{1}{t} \int f \quad \rightarrow \quad f_n \sim \frac{\sqrt{6}}{\pi} \frac{6^n}{n}
\]
Example of Behaviour Not Driven by the $\binom{2}{1}$: at $(1, 1)$

$$F(1, 1; t) = \frac{1}{t(1-t)} \int \frac{t(4 + \int f)}{(1-4t)^{3/2}}$$ where

$$f = \frac{(1+2t)(1-4t)^{1/2}}{2t^2} \left(1 + \frac{1}{2t(1+2t)(1+4t^2)^{1/2}}h\right) = \frac{1}{t^2} + O(1),$$

$$h = (1 + t)(1 - 4t + 8t^2) \binom{2}{1} \binom{1}{1} \binom{1}{2} w - (1 - t) \binom{2}{1} \binom{3}{2} \binom{1}{2} w,$$

$$w = \frac{16t^2}{1 + 4t^2} = 1 - \frac{1 - 12t^2}{1 + 4t^2}.$$ 

Singularities: $\frac{1}{4}, -\frac{1}{2}, \pm \frac{i}{2}, 1, w = 1, w = \infty \rightarrow$ Dominant singularity $= \frac{1}{4}$. 
Example of Behaviour Not Driven by the $2F_1$: at $(1, 1)$

\[ F(1, 1; t) = \frac{1}{t(1 - t)} \int \frac{t(4 + \int f)}{(1 - 4t)^{3/2}} \]

where

\[ f = \frac{(1 + 2t)(1 - 4t)^{1/2}}{2t^2} \left( 1 + \frac{1}{2t(1 + 2t)(1 + 4t^2)^{1/2}} h \right) = \frac{1}{t^2} + O(1), \]

\[ h = (1 + t)(1 - 4t + 8t^2) \, _2F_1 \left( \begin{array}{c} 1 \ 1 \ \\ \frac{1}{2} \ \\
 \end{array} \right) w - (1 - t) \, _2F_1 \left( \begin{array}{c} 3/2 \ 1/2 \ \\ \frac{3}{2} \ \\
 \end{array} \right) w, \]

\[ w = \frac{16t^2}{1 + 4t^2} = 1 - \frac{1 - 12t^2}{1 + 4t^2}. \]

Singularities: $\frac{1}{4}, -\frac{1}{2}, \pm \frac{i}{2}, 1, w = 1, w = \infty \longrightarrow$ Dominant singularity = $\frac{1}{4}$.

\[ f_n \sim 4 \frac{4^n}{3 \sqrt{\pi \sqrt{n}}} \] holds under the conjecture \[ \int_0^{1/4} \left( f(t) - \frac{1}{t^2} \right) dt = 2. \]
Example of Behaviour Not Driven by the $\,_{2}F_{1}$ at $(1,1)$

\[
F(1,1; t) = \frac{1}{t(1-t)} \int \frac{t(4 + \int f)}{(1-4t)^{3/2}} \text{ where }
\]

\[
f = \frac{(1+2t)(1-4t)^{1/2}}{2t^2} \left( 1 + \frac{1}{2t(1+2t)(1+4t^2)^{1/2}}h \right) = \frac{1}{t^2} + O(1),
\]

\[
h = (1 + t)(1 - 4t + 8t^2)_{2F1} \left( \frac{1}{2} \left| \frac{1}{2} \right| w \right) - (1 - t)_{2F1} \left( \frac{3}{2} \left| \frac{1}{2} \right| w \right),
\]

\[
w = \frac{16t^2}{1 + 4t^2} = 1 - \frac{1 - 12t^2}{1 + 4t^2}.
\]

Singularities: $\frac{1}{4}, -\frac{1}{2}, \pm \frac{i}{2}, 1, w = 1, w = \infty \rightarrow$ Dominant singularity = $\frac{1}{4}$.

\[
f_n \sim \frac{4}{3} \sqrt{\frac{1}{\pi}} \frac{4^n}{\sqrt{n}} \text{ holds under the conjecture } \int_{0}^{\frac{1}{4}} \left( f(t) - \frac{1}{t^2} \right) dt = 2.
\]

Remark: Showing close enough to 2 already proves behaviour in $\frac{4^n}{\sqrt{n}}$. 
Further Examples with Added Difficulties (1/2): \( \Delta \) at (1, 1)

\[
Q(t) = \frac{1 - 2t}{4t^2} \left[ 1 - \frac{\sqrt{1 + t}}{\sqrt{1 - 3t}} \left( 1 - \int_0^t \frac{\phi(u)}{\sqrt{1 - 3u}} \, du \right) \right]
\]

for \( \phi(t) = \frac{(1 - 6t^2 - 8t^3)\, _2F_1\left(\frac{1}{4}, \frac{3}{4}; 1 | 64t^4\right) + 4t^3(1 - 7t + 4t^2)\, _2F_1\left(\frac{3}{4}, \frac{5}{4}; 2 | 64t^4\right)}{(1 - 2t)^2(1 + t)^{3/2}} \). 

Dominant singularity: \( \frac{1}{3} \)?
Further Examples with Added Difficulties (1/2): \[ \hat{Q} \] at (1, 1)

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for \( \phi(t) = (1 - 6t^2 - 8t^3) \frac{2}{1} \binom{1/4}{3/4} \left| \begin{array}{c} 64t^4 \\ 1 \end{array} \right\} + 4t^3(1 - 7t + 4t^2) \binom{3/4}{5/4} \left| \begin{array}{c} 64t^4 \\ 2 \end{array} \right\} \)

\[
= \frac{(1 - 2t)^2(1 + t)^{3/2}}{2} \binom{1}{2} \left| \begin{array}{c} 64t^4 \\ 1 \end{array} \right\} + 4t^3(1 - 7t + 4t^2) \binom{3/4}{5/4} \left| \begin{array}{c} 64t^4 \\ 2 \end{array} \right\}.
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Dominant singularity: \( \frac{1}{3} \)? No, because \( 1 = \int_0^{1/3} \frac{\phi(u)}{\sqrt{1 - 3u}} \, du \) (conj.).
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\[
\int_1^{t} \frac{\phi(u)}{\sqrt{1-3u}} \, du \sim K \left( \frac{1}{\sqrt{8}} - t \right) \ln \left( \frac{1}{\sqrt{8}} - t \right).
\]
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$$\frac{(1 - 6t^2 - 8t^3)_{2F1} \left( \frac{1}{4}, \frac{3}{4} \Big| \frac{64t^4}{1} \right)}{2} + \frac{4t^3(1 - 7t + 4t^2)_{2F1} \left( \frac{3}{4}, \frac{5}{4} \Big| \frac{64t^4}{2} \right)}{(1 - 2t)^2(1 + t)^{3/2}}.$$

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This explains behaviour in $\frac{\sqrt{8}^n}{n^2}$. 
Further Examples with Added Difficulties (2/2): at $(1,1)$

\[
\phi(t) = \frac{(1 - 24t^3) {}_2F_1 \left( \begin{array}{c} 1/2, 3/2 \\ 2 \end{array} \right | w(t)) + 18t^2(2t - 1) {}_2F_1 \left( \begin{array}{c} 1/2, 5/2 \\ 3 \end{array} \right | w(t))}{(1 - 2t)^2 \sqrt{4t^2 + 1}}.
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where

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w(t) = \frac{16t^2}{4t^2 + 1}.
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Two even series recombined in a non-symmetric way.
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Asymptotic behaviour in $\kappa(n \mod 2)\rho^n n^\alpha$. 

Frédéric Chyzak

Small-Step Walks
Conclusions

A succession of equations of several types:
rec. relation on $f_{n;i,j}$ → kernel equation on $F(x,y;t)$ → ODE on $F(1,1;t)$
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creative telescoping → ODE factorization → ODE solving
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Summary of contributions:

- Three kinds of conjectures now proved:
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  - asymptotics of coefficients via the new closed forms (in progress).

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Wanted: better understand the systematic emergence of elliptic integrals