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Joint work with

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Helly’s original theorem (1923)

Let $F$ be a finite family of convex sets in $\mathbb{R}^d$.
If every $G \subseteq F$ with $|G| \leq d + 1$ has non-empty intersection,
then $F$ has non-empty intersection.

Small-sized certificate of empty intersection

If $F$ has empty intersection, some subfamily of size $\leq d + 1$ has empty intersection.
Intersection patterns and simplicial complexes

Intersection patterns

- Let $F$ be a (finite) family of subsets of an arbitrary ground set.
- The nerve $N(F)$ of $F$ is $\{ G \subseteq F \mid \bigcap G \neq \emptyset \}$.
- It is a simplicial complex (stable under taking subsets / a.k.a. a monotone hypergraph / a monotone set system).
**Definition**

**Missing face** $F'$ of $N(F)$:

\[
\begin{cases}
    F' \not\in N(F), \\
    \forall F'' \subset F', \ F'' \in N(F).
\end{cases}
\]

**Definition**

$F$ is **$k$-Helly** if all the missing faces of $N(F)$ have size $\leq k$.

**Small-sized certificate of empty intersection**

If $F$ has empty intersection, then some subfamily of $F$ of size $\leq k$ has empty intersection.
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**Helly-type theorems**

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For each $G \subseteq F$, if $G$ has empty intersection, then some subfamily of $G$ of size $\leq k$ has empty intersection.
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In this talk...

Results

- A new topological Helly-type theorem for families of disconnected geometric objects
- based on a generalization of the nerve theorem from topological combinatorics
- with applications to geometric transversal theory.
Warm-Up
**Topological Helly theorem**

**Wanted!**

Every “convex-like” family in $\mathbb{R}^d$ is $(d + 1)$-Helly.

**Wrong statement**

Replace “convex-like” with “contractible” (“without hole”; e.g., homeomorphic to a convex set).

**Definition**

A (finite) family $F$ of (open) geometric objects is **acyclic** (a.k.a. a good cover) if: For every $G \subseteq F$, $\bigcap_G$ is either empty or contractible.

**Topological Helly theorem**

Every acyclic family in $\mathbb{R}^d$ is $(d + 1)$-Helly [Helly, 1930].
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Nerve theorem

If $F$ is acyclic, then $\bigcup F \simeq N(F)$: they have “holes” in the same dimensions [Borsuk, 1948].

Proof(s)

- Follows “trivially” from algebraic topology arguments;
- more “hands-on” (homotopic) combinatorial proof [Björner, 2003].
Nerve theorem

Nerves as topological spaces
- Vertices in general position in $\mathbb{R}^d$, $d$ large;
- attach segments, triangles, tetrahedra, ...

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Topological interlude: “holes”

Holes in a topological space $S$

- Formally, $S$ has a $k$-hole if the $k$th dimensional reduced homology of $S$ is nonzero: $\dim \tilde{H}_k(S, \mathbb{Q}) > 0$.

- Intuitively, $S$ has a $k$-hole if some $k$-dimensional “closed part” of $S$ is the boundary of no $(k + 1)$-dimensional subset of $S$.

- Examples:
  - $S$ has a 0-hole if it is not connected;
  - $S$ has a 1-hole if it contains a closed curve that is not the boundary of a surface in $S$;
  - $S$ has a 2-hole if it contains a “bubble”…

- **Contractible** means “without hole”.

Intuitively, $S$ has a $k$-hole if some $k$-dimensional “closed part” of $S$ is the boundary of no $(k + 1)$-dimensional subset of $S$. **Contractible** means “without hole”.
Let $F$ be an acyclic family in $\mathbb{R}^d$.
Let $G$ be a missing face of $N(F)$.
- $N(G)$ has a $(|G| - 2)$-hole.
- On the other hand, we have $N(G) \simeq \bigcup G \ldots$
- and $\bigcup G \subseteq \mathbb{R}^d$, so $\bigcup G$ has no hole in dimension $\geq d$.
- So $|G| - 2 < d$, i.e., $|G| \leq d + 1$. \(\square\)
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Results
**Definition**

A family $F$ of sets in $\mathbb{R}^d$ is $r$-acyclic if $\forall G \subseteq F$, $\bigcap G$ is the disjoint union of at most $r$ contractible sets.

**Topological Helly theorem**

Let $F$ be a 1-acyclic family in $\mathbb{R}^d$. Then $F$ is $(d + 1)$-Helly.

**Remarks**

- The value $(d + 1)r$ cannot be lowered;
- strengthens a result by [Kalai and Meshulam, 2008] on $r$-families of acyclic families (also [Amenta, 1996]);
  - $r$-family $F$ of a “ground” family $G$: The intersection of a subfamily of $F$ is the disjoint union of at most $r$ elements in $G$.
- [Matoušek, 1997] had proved that $F$ is $k$-Helly for some (large) $k$. 
Generalized version

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A family $F$ of sets in $\mathbb{R}^d$ is $r$-acyclic if $\forall G \subseteq F$, $\bigcap G$ is the disjoint union of at most $r$ contractible sets.

**New topological Helly-type theorem:** Let $r \geq 1$

Let $F$ be an $r$-acyclic family in $\mathbb{R}^d$. Then $F$ is $(d + 1) \times r$-Helly.

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Comparison with other results

convex sets in $\mathbb{R}^d$
[Helly, 1923]
$d + 1$

acyclic families in $\mathbb{R}^d$
[Helly, 1930]
$d + 1$

$r$-family of convex sets in $\mathbb{R}^d$
[Amenta, 1996]
$(d + 1)r$

topological condition
[Matoušek, 1997]
no explicit bound

$r$-family of an acyclic family in $\mathbb{R}^d$
[Kalai and Meshulam, 2008]
$(d + 1)r$

$r$-family of a non-additive family $G$ closed under $\bigcap$
[Eckhoff and Nischke, 2009]
$r \times h(G)$

$r$-acyclic family
[CdV, G, and G]
$(d + 1)r$
Application to geometric transversal theory

- Let $C_1, \ldots, C_n$ be disjoint convex sets in $\mathbb{R}^d$.
- For each $i$, let $F_i$ be the set of lines meeting $C_i$.
- Let $F := \{F_1, \ldots, F_n\}$.

In which cases is $F$ $k$-Helly?

Central question in geometric transversal theory.

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<tr>
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Why does our result apply?

New topological Helly-type theorem

Let $F$ be a family of sets in $\mathbb{R}^d$ such that $\forall G \subseteq F$, $\bigcap G$ is the disjoint union of at most $r$ contractible sets. Then $F$ is $((d + 1)r)$-Helly.

Idea: Apply the main result in the space of lines of $\mathbb{R}^d$

- **Good**: Often, if $G \subseteq F$, each connected component of $\bigcap G$ corresponds to a geometric permutation of the objects $C_i$.
- **Bad**: The space of lines in $\mathbb{R}^d$ is a $(2d - 2)$-manifold. → Extension to arbitrary topological spaces ($d$=dimension of vanishing homology of open sets).
- **Bad**: Some components of $\bigcap G$ are not contractible. → For small $G$, allow $\bigcap G$ to have holes in low dimension.
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Sketch of Proof
The multinerve $M(F)$ of a family $F$ of sets is a blown-up version of the nerve $N(F)$: (roughly,) order the connected components of the intersecting subfamilies by inclusion.

- $M(F)$ is a more general simplicial poset [Björner, Stanley, ...];
- every “lower interval” is a simplex.
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**Multinerve theorem**

Let $F$ be a family of sets in $\mathbb{R}^d$ such that $\forall G \subseteq F$, $\bigcap_G$ is the disjoint union of finitely many contractible sets. Then $M(F)$ and $\bigcup_F$ have holes in the same dimensions.

**Proof**

- Spectral sequences with Leray’s acyclic cover theorem;
- alternatively, variation on [Björner, 2003].
We know that $M(F)$ has no hole in dimension $\geq d$; we want to infer that $N(F)$ has no hole in dimension $\geq (d + 1)r - 1$.

**Theorem** [Kalai and Meshulam, 2008]

- Let $M$ and $N$ be simplicial complexes.
- Let $\pi : M \to N$ be simplicial, size-preserving, at most $r$-to-one, and onto.
- Assume (roughly) that $M$ has no hole in dim. $\geq d$.
  Assume that some suitably defined subcomplexes of $sd(M)$ have no hole in dim. $\geq d$.

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New topological Helly theorem: proof sketch

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**Theorem (generalizing [Kalai and Meshulam, 2008])**

- Let $M$ be a simplicial poset and $N$ a simplicial complex.
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**Tools**

Algebraic topology (spectral sequences, multiple point set, etc.).
Definition
If $\sigma$ is a simplex of a simplicial poset $X$, then $\text{barylink}_X(\sigma)$ is the subcomplex of $\text{sd}_X$ that is the order complex of $(\sigma, \cdot]$ in $X$.

Lemma
For any acyclic family $F$ in $\mathbb{R}^d$, $\text{barylink}_{M(F)}(\sigma)$ has no hole in dimension $\geq d$. 
Proof sketch (end)

Multiple point set

\[ M_k := \{ m_1, \ldots, m_k \in |M|^k \mid \pi(m_1) = \ldots = \pi(m_k) \}. \]

Consequence of [Goryunov and Mond, 1993]

Some spectral sequence \((E_{p,q}^\bullet)\) converging to \(H_\ast(N)\) satisfies:

If, for all \(q\), for all \(p \leq r - 1\), and for all \(p + q \geq (d + 1)r - 1\), we have \(H_q(M_{p+1}) = 0\),

then \(E_{p,q}^1 = 0\) (and therefore \(H_k(N) = 0\) for all \(k \geq (d + 1)r - 1\)).

Rephrasing [Kalai and Meshulam, 2008]

Some spectral sequence \((E'_{p,q}^\bullet)\) converging to \(H_\ast(M_{p+1})\) satisfies

\[
E'_{p,q}^1 \cong \bigoplus_{(\sigma_2, \ldots, \sigma_k) \in S_p} \bigoplus_{i_1, \ldots, i_k \geq 0} \bigcap_{i=2}^k \tilde{\sigma}_i \left( M \left[ \bigcap_{i=2}^k \tilde{\sigma}_i \right] \right) \otimes \bigotimes_{j=2}^k \tilde{H}_{i_j-1} \left( \text{barylink}_M(\sigma_j) \right)
\]

Thus in our setting \(H_q(M_{p+1}) = 0\)
Conclusion
Fractional Helly-type theorems

**Definition**

$F$ is *$k$-fractional Helly* if the following holds: If “many” $k$-tuples of $F$ have non-empty intersection, then there exists a “large” subfamily of $F$ that has non-empty intersection.

More precisely: If a fraction $x$ of the $k$-tuples have non empty intersection, then a fraction $f(x)$ of the elements in $F$ have non-empty intersection, where $f(x)$ tends to one as $x$ tends to one.

**More theorems for free!**

Using [Alon, Kalai, Matoušek, Meshulam, 2002], we obtain immediately such fractional Helly theorems for $r$-acyclic families.
Conclusion: Get rid of topology?

Another proof without topology?

[Eckhoff and Nischke, 2009] reproves [Kalai and Meshulam, 2008] in a purely combinatorial way ("generalized pigeonhole principle"). Can we use that proof technique instead?

Core of their proof

- Let $M, N$ be simplicial complexes.
- Let $\pi : M \to N$ be simplicial, size-preserving, at most $r$-to-one, and onto.
- If $N$ contains all the strict subfamilies of a set $S$ of size $k + 1$, then $\pi^{-1}(2^S)$ contains all the subfamilies of size $\leq \left\lfloor \frac{k}{r} \right\rfloor$ of a set of size $k + 1$.

Can we allow $M$ to be a simplicial poset? Under which conditions?
Thanks!
Most general results

Common hypotheses

- Let $\Gamma$ be a locally arcwise connected topological space.
- Let $F$ be a finite family of open subsets of $\Gamma$ that is $r$-acyclic with slack $d$: for every subfamily $G \subseteq F$, $G \neq \emptyset$,
  - if $|G| \geq d$, then $G$ intersects in at most $r$ connected components.
  - for every $i \geq \max\{1, d - |G|\}$, we have $\tilde{H}_i(\bigcap G, \mathbb{Q}) = 0$.

General multinerve theorem

For every $i \geq d$, $\tilde{H}_i(M(F), \mathbb{Q}) \simeq \tilde{H}_i(\bigcup F, \mathbb{Q})$.

General topological Helly theorem

Assume moreover that every open set of $\Gamma$ has trivial homology in dimension $\geq d$. Then $F$ is $((d + 1)r)$-Helly.