When orientability makes a difference?

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Uniwersytet Wrocławski

joint work with

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Valentin Féray, Universität Zürich

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I. Maps

Maps

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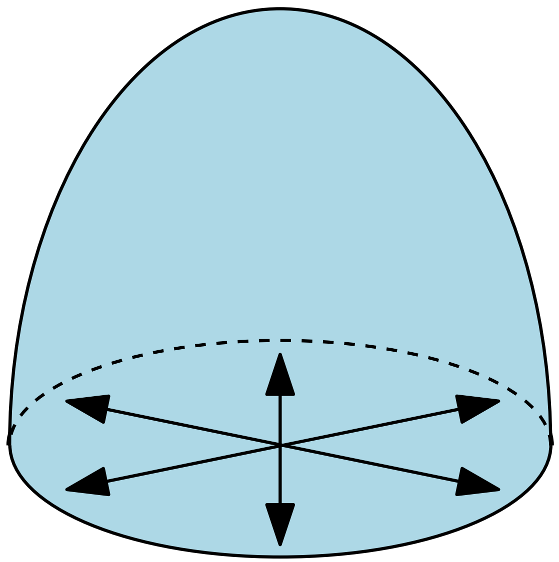
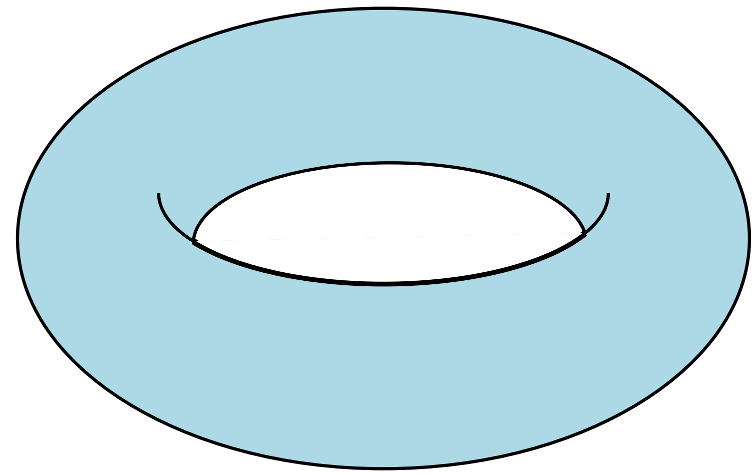


Projective plane



Torus

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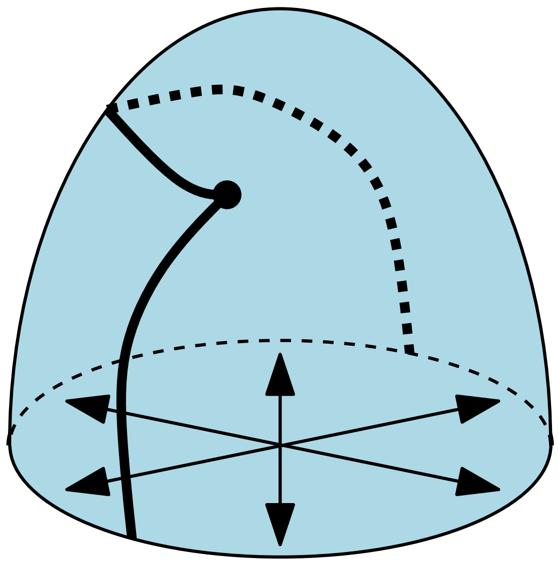
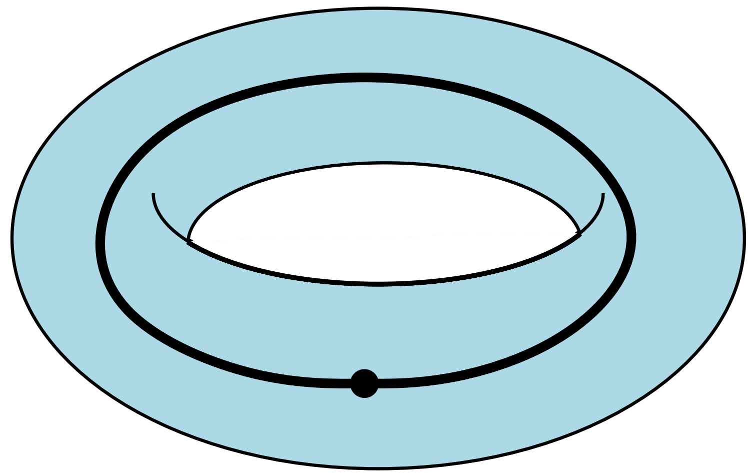


This is a map



This is not a map!

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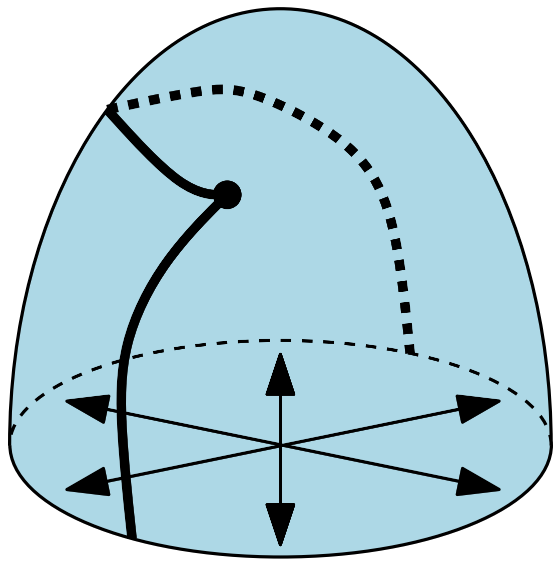

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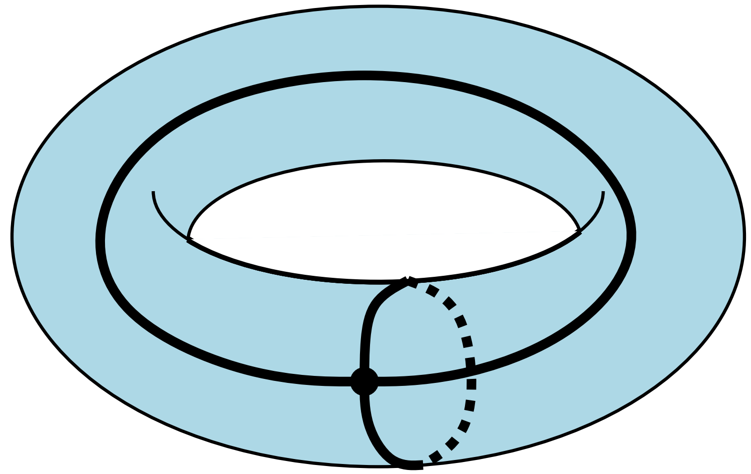

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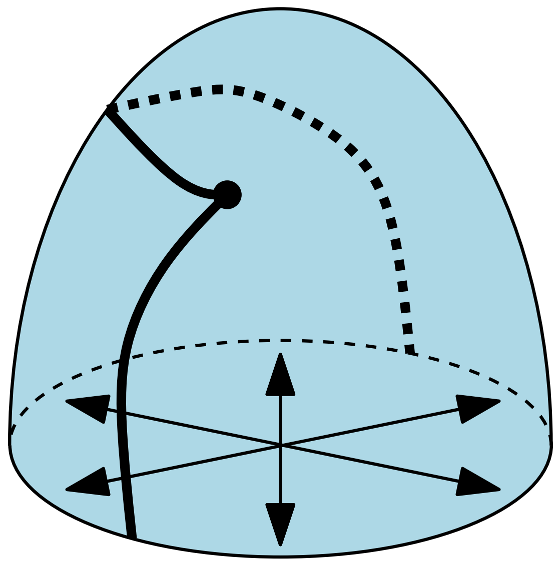

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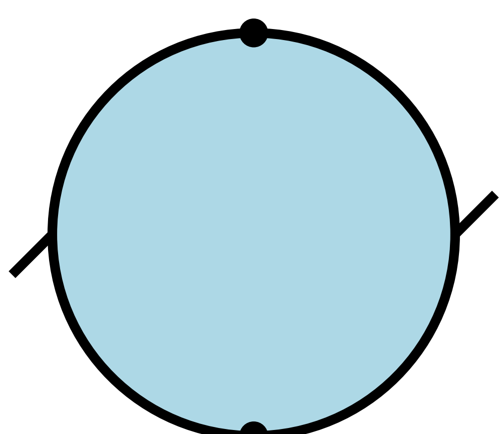

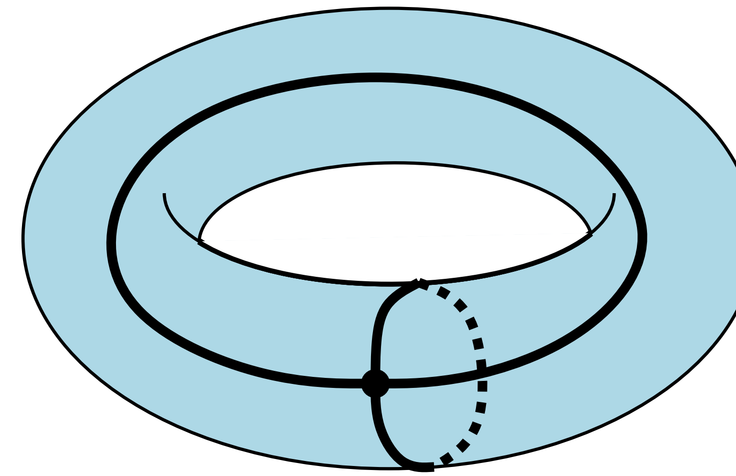

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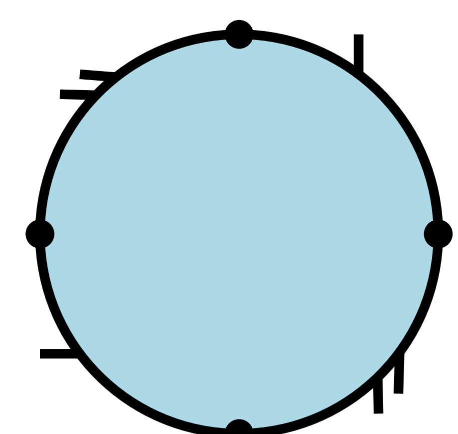

Orientable vs. non-orientable

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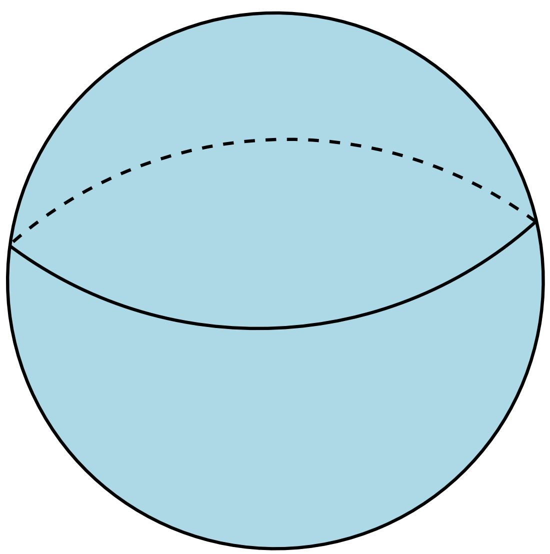
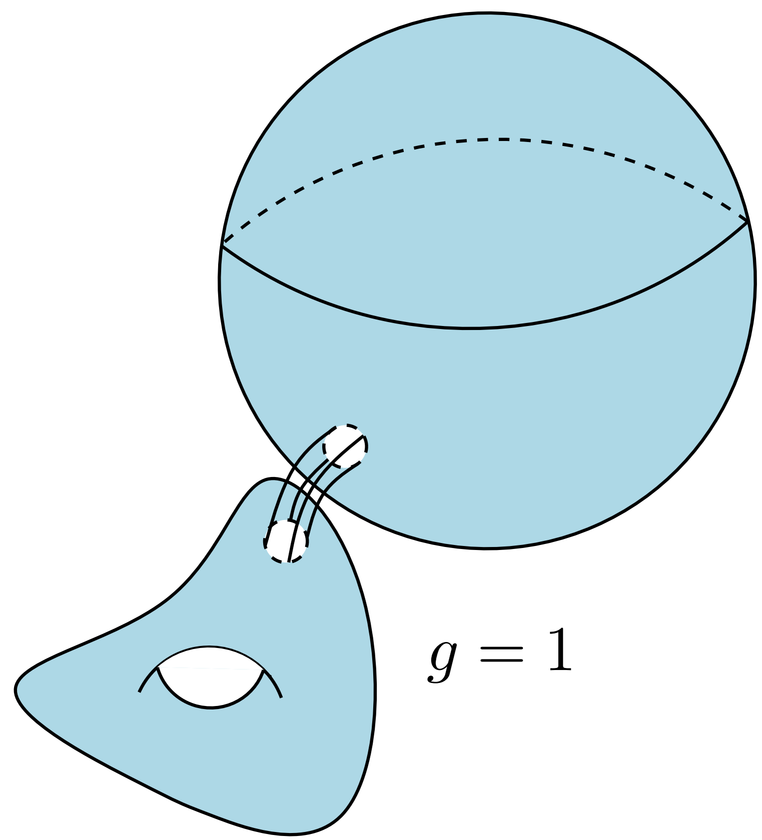
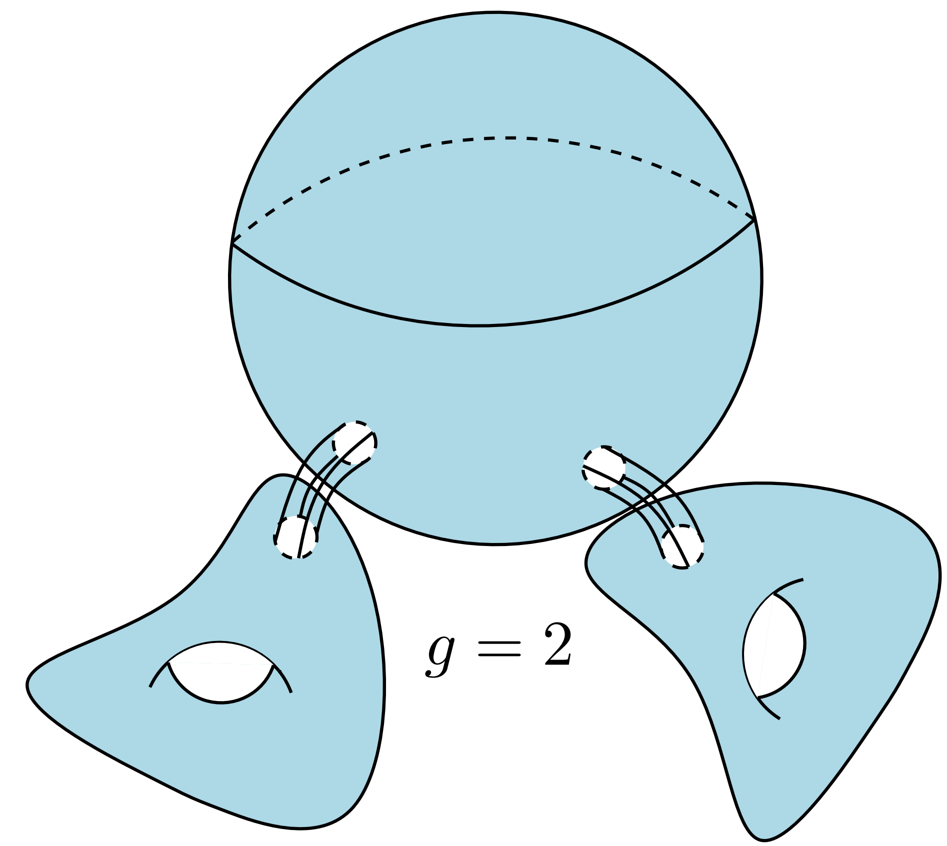


$g = 0$



$g = 1$



$g = 2$

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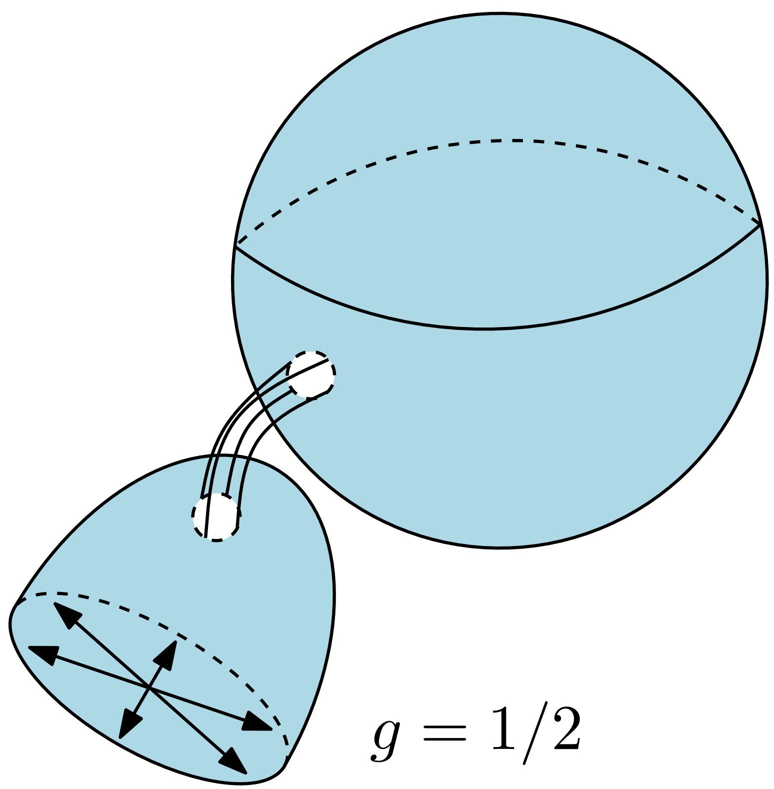

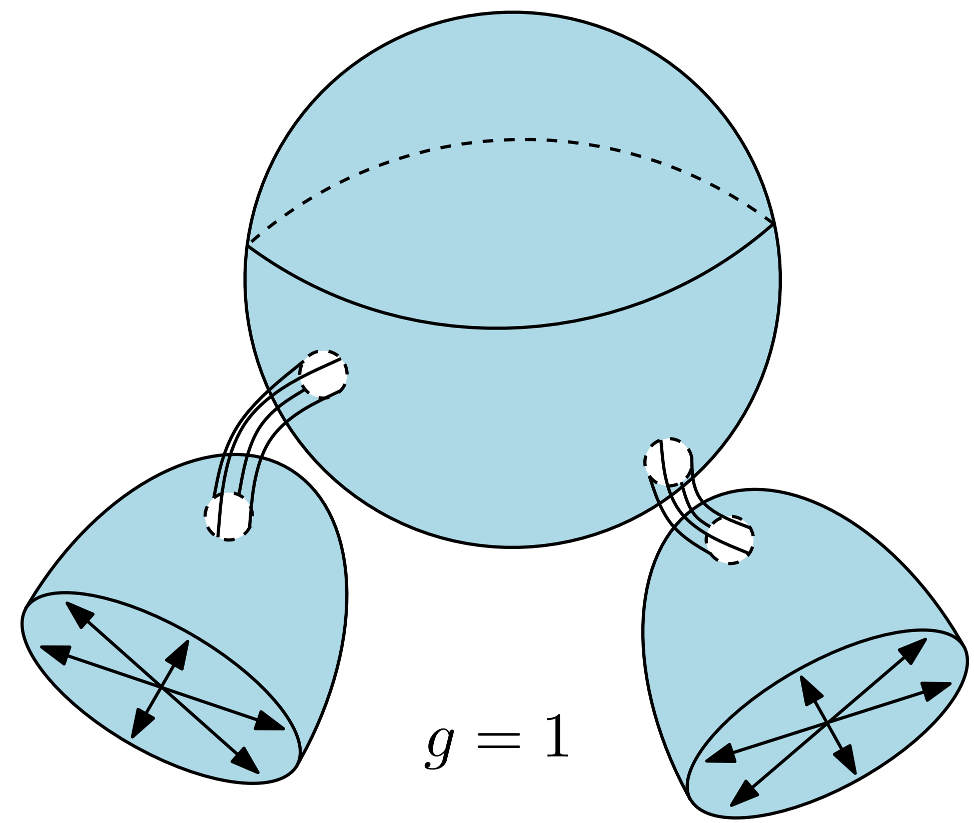

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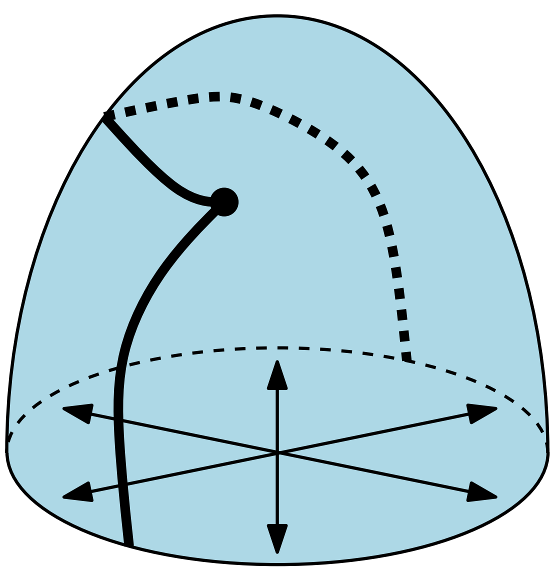
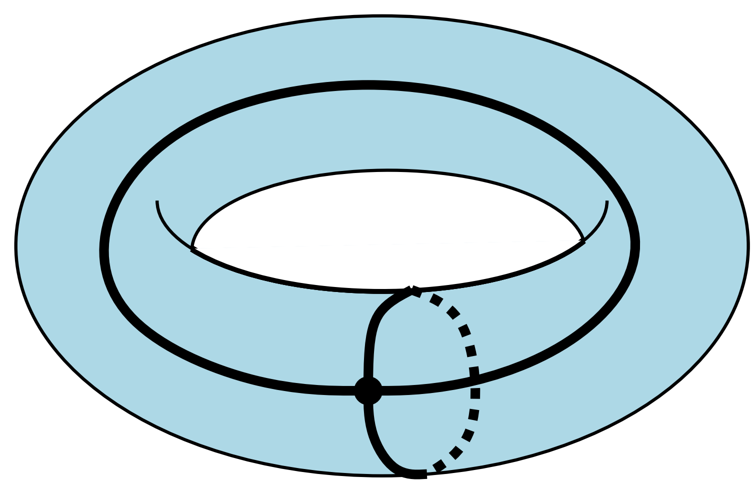


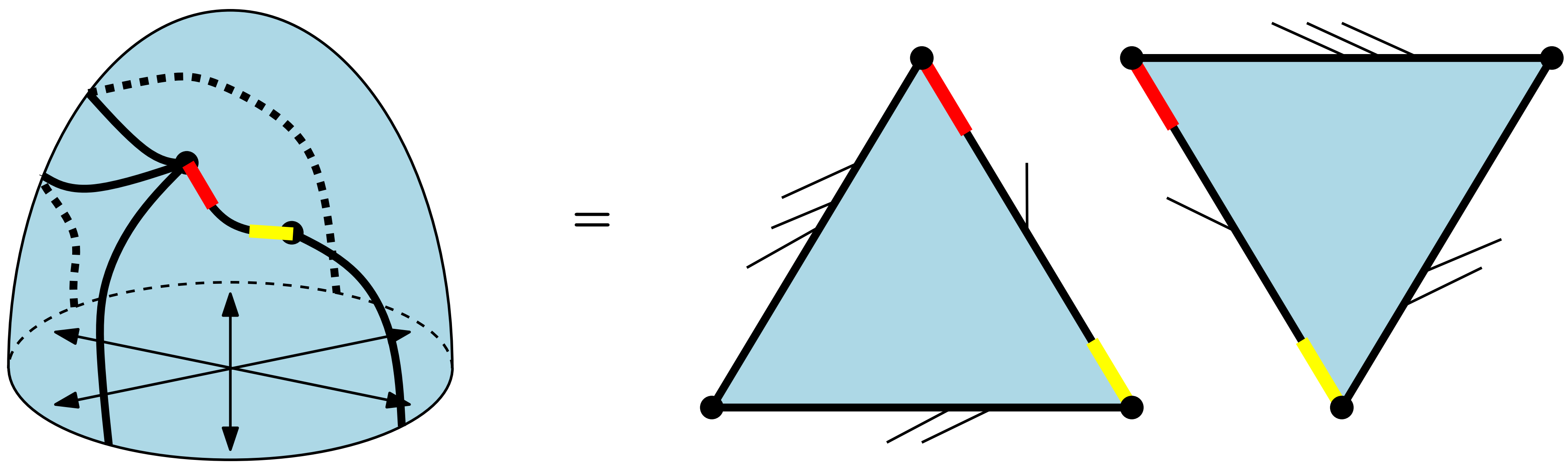
Non-orientable map of type $1/2$



Orientable map of type 1

Rooted maps

Each edge consists of two half-edges.



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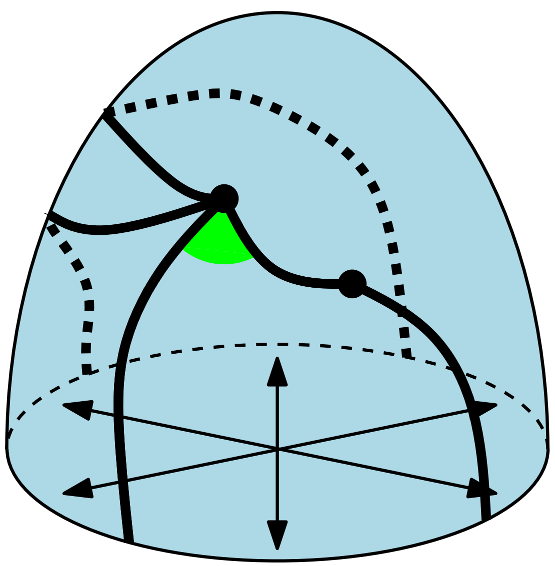

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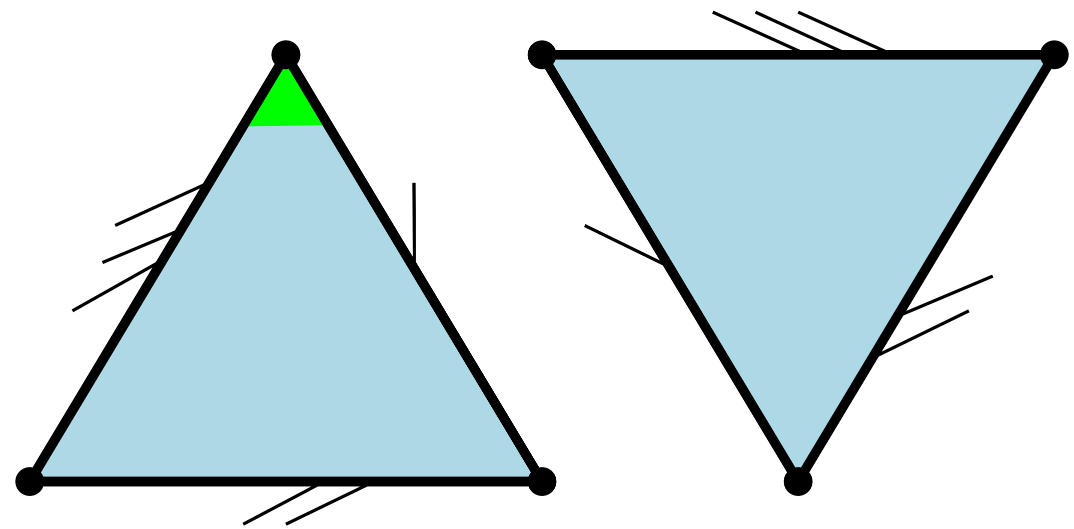

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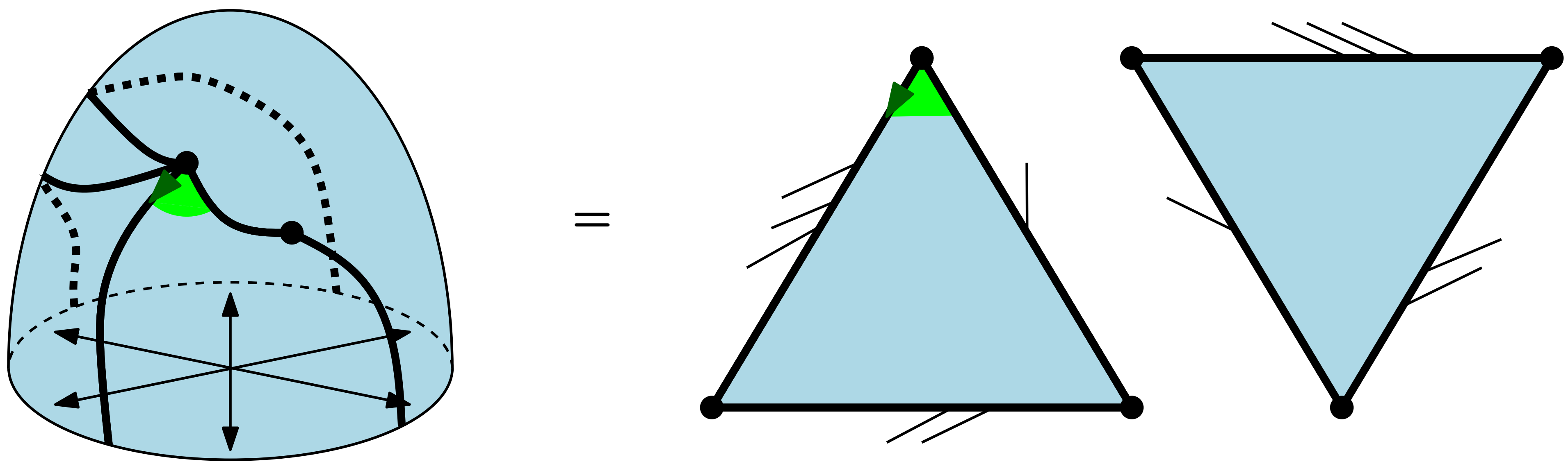

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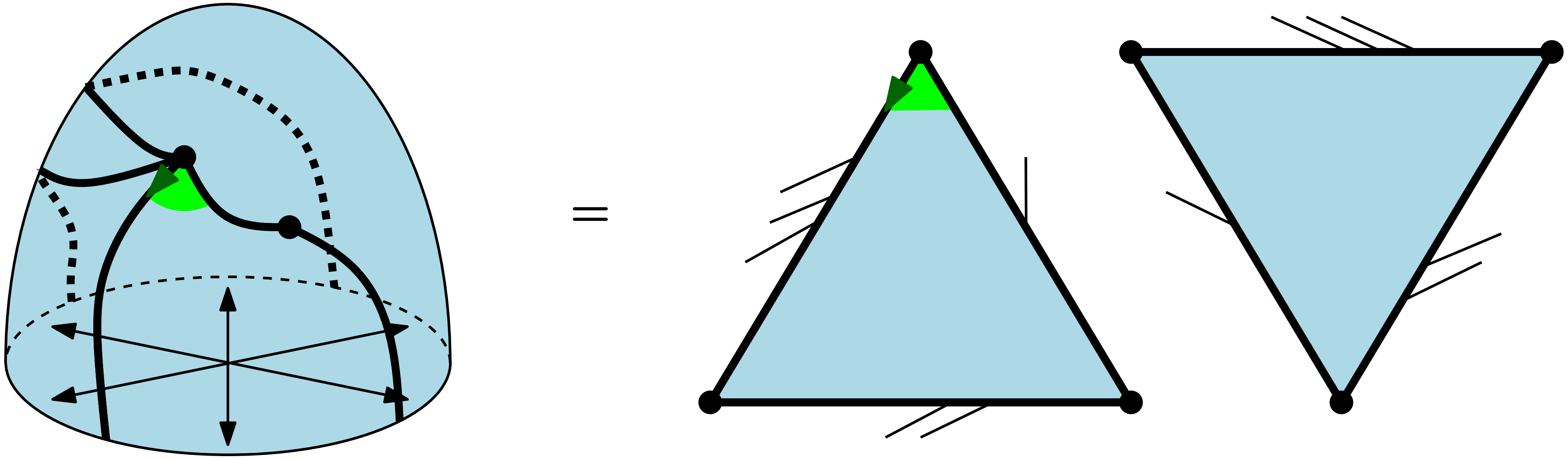

Remark:
Tutte noticed that maps are much simpler to enumerate, when rooted, because of the lack of symmetry. From now on, all maps will be rooted!

II. Enumeration of maps

Number of maps with n edges

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Combinatorial explanation:

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- When $\chi(\mathbb{S}) = 2 - 2g$, and $\mathbb{S}$ is NON-ORIENTABLE: no combinatorial interpretation was known.

Maps with n edges vs. bipartite quadrangulations with n faces

Map M is bipartite if vertices can be colored by two different colors ($V_\bullet(M)$ - set of black vertices, $V_\circ(M)$ - set of white vertices, root vertex is black by convention) such that each edge connects two vertices of different colors.

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Theorem [Tutte 1960]
There is a bijection between

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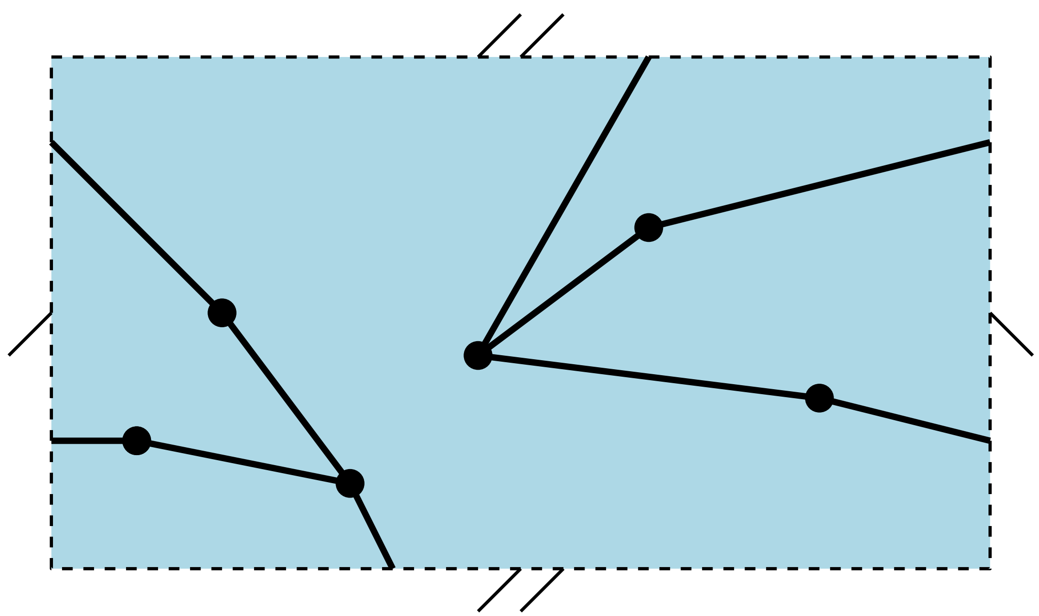

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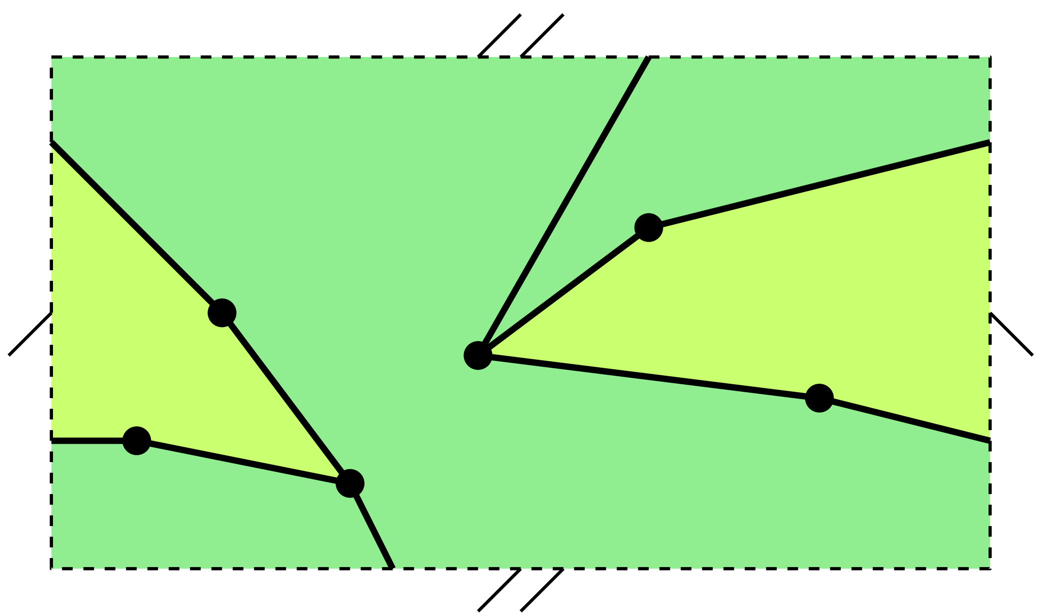

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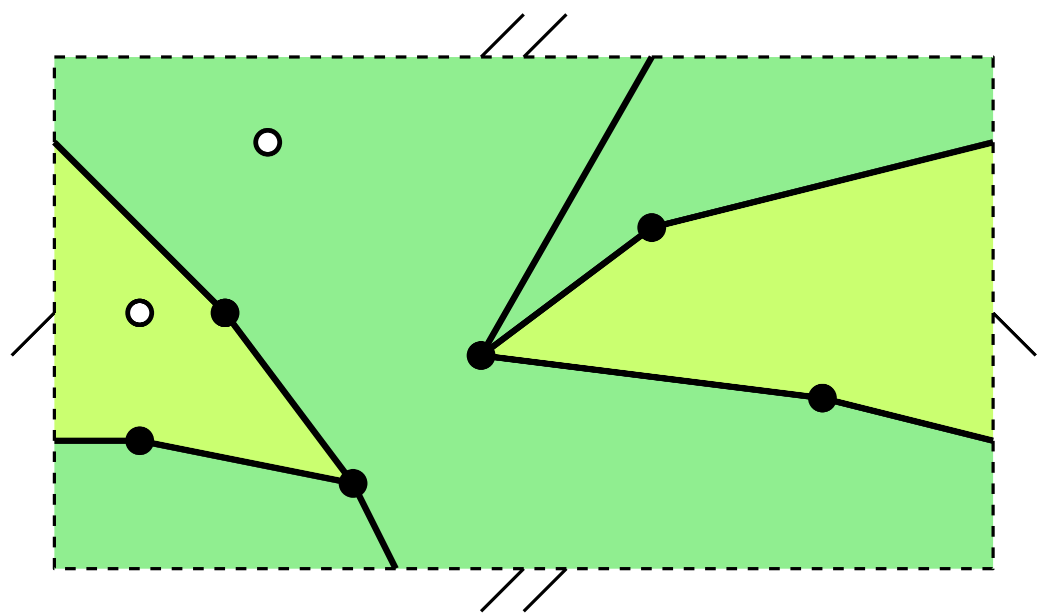

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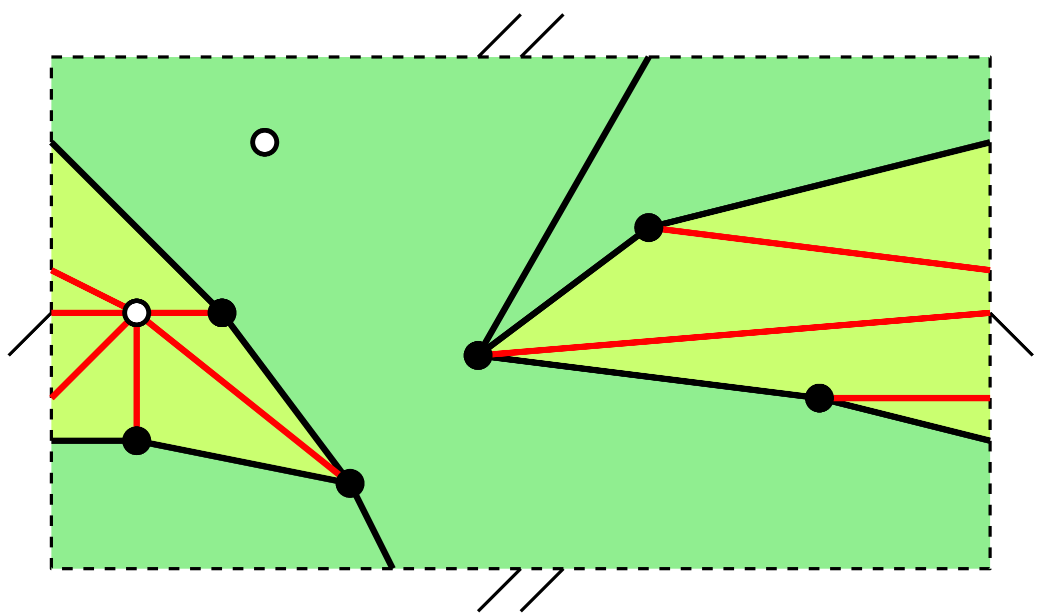

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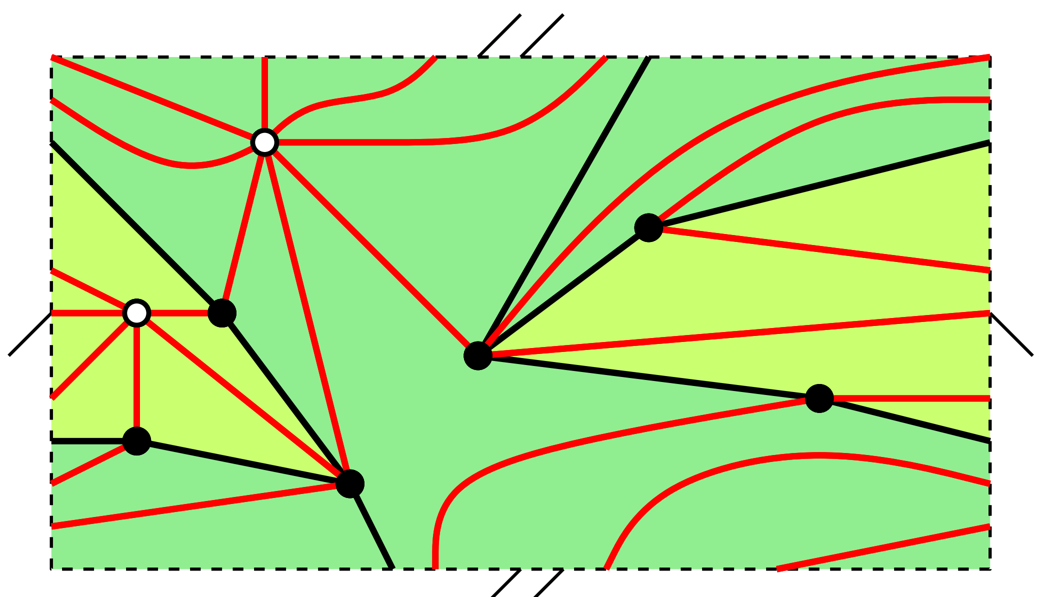

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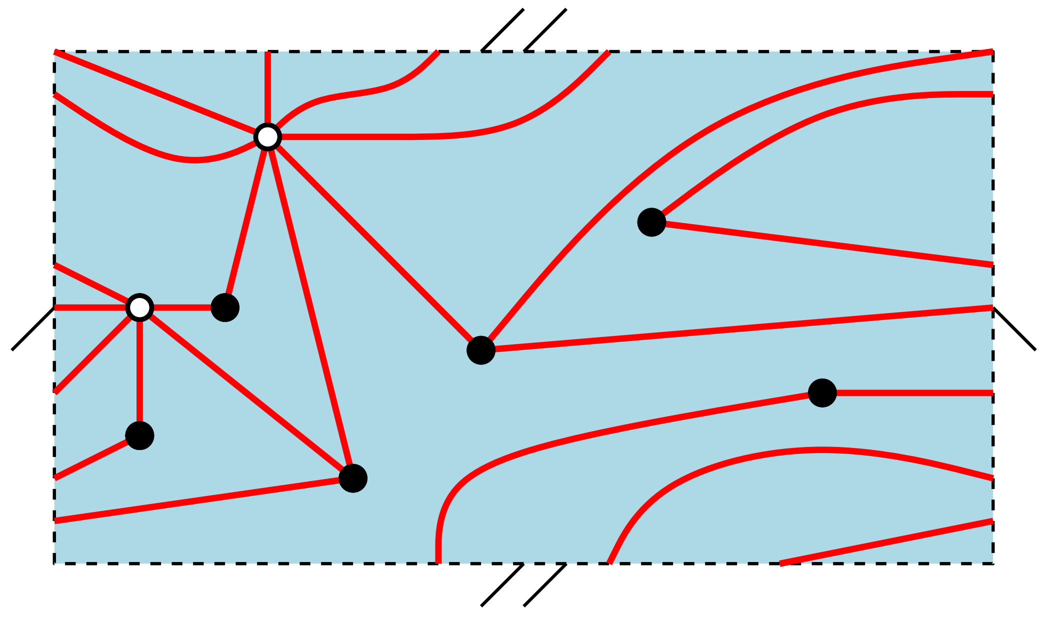

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Number of maps with n edges on $\mathbb{S}$	=	Number of bipartite quadrangulations with n faces on $\mathbb{S}$

Labeled and well-labeled maps

A map is called labeled if its vertices are labeled by integers such that:

- the root vertex has label 1;
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If in addition we have:

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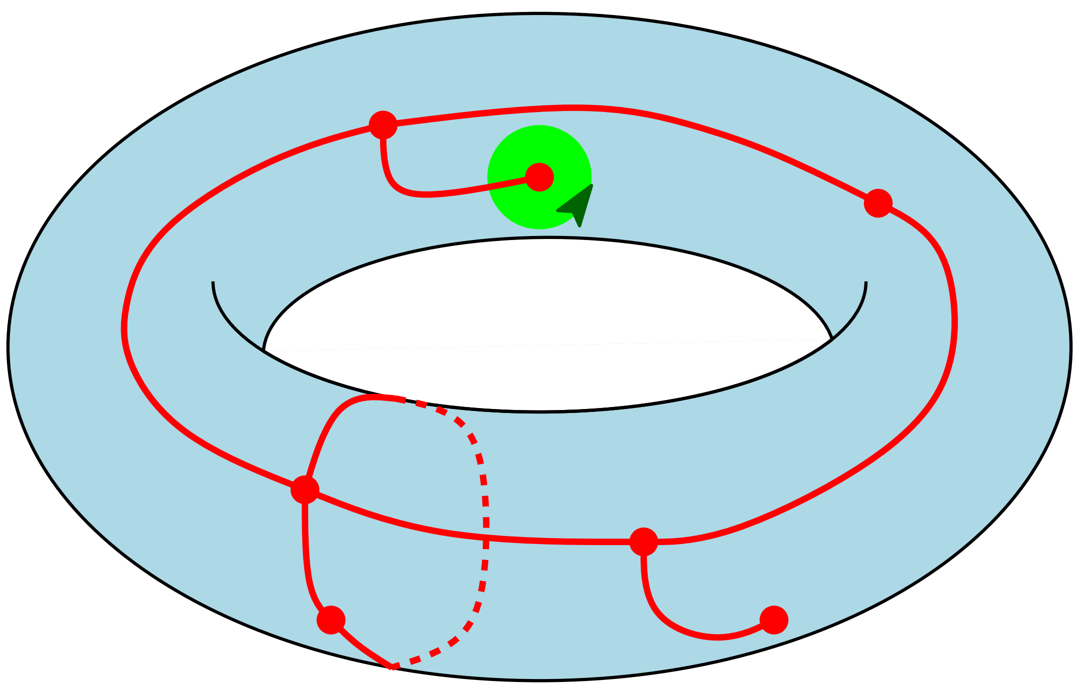

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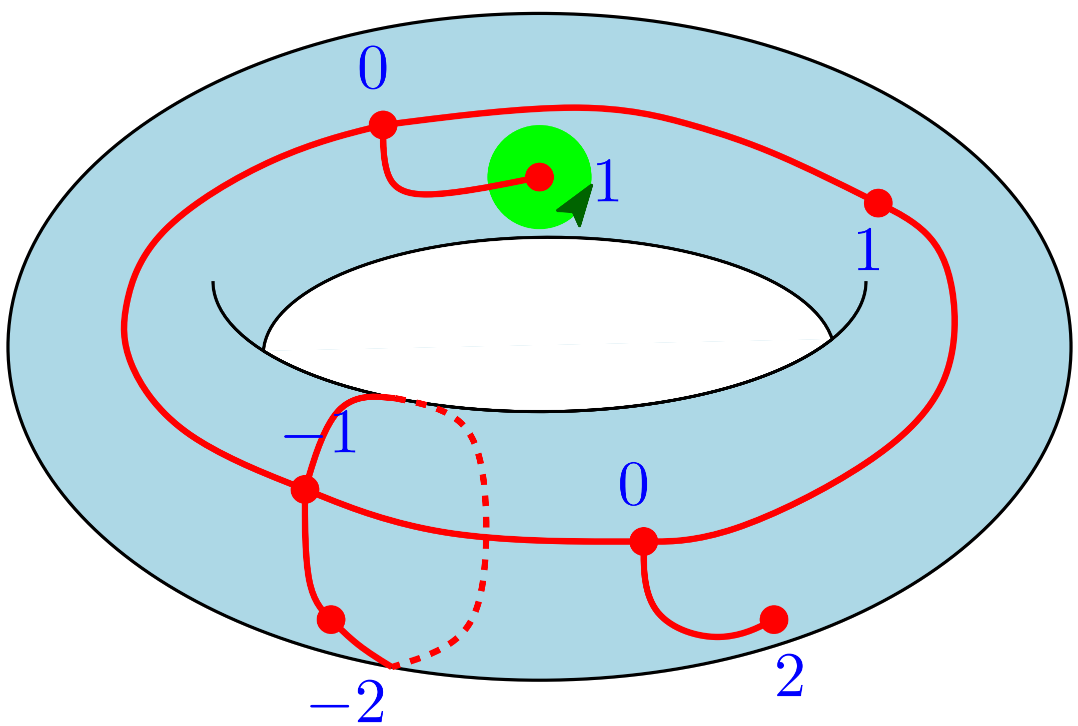

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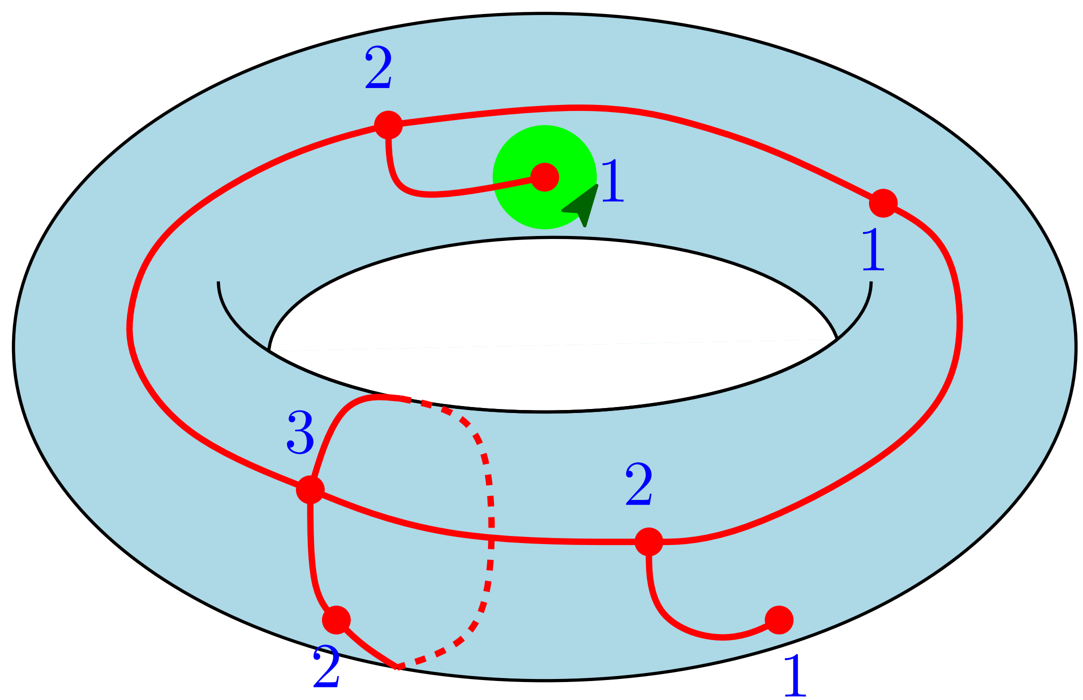

this map is labeled and well-labeled as well

Orientable case

Theorem [Marcus, Schaeffer 1996]
There exists a bijection between:

- rooted, bipartite quadrangulations on ORIENTABLE surface $\mathbb{S}$ with n faces and N_i vertices at distance i from the root vertex $(i \geq 1)$;
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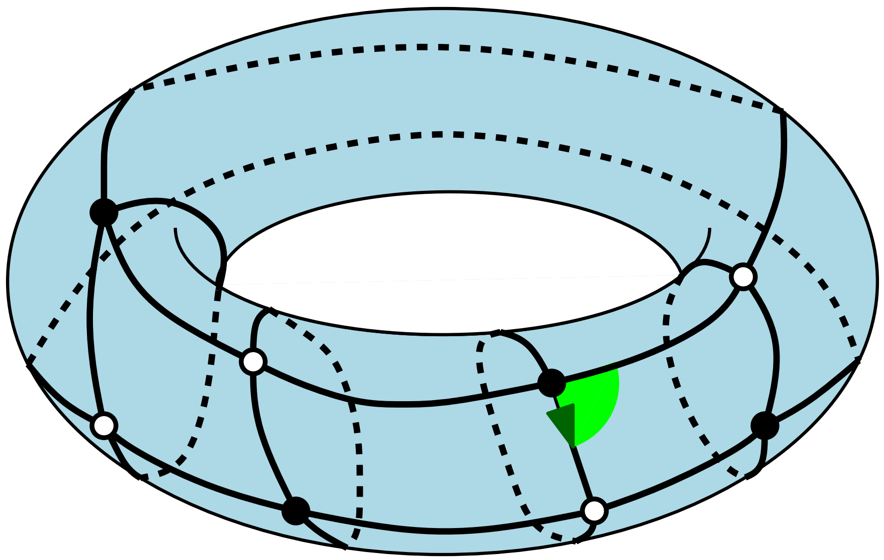

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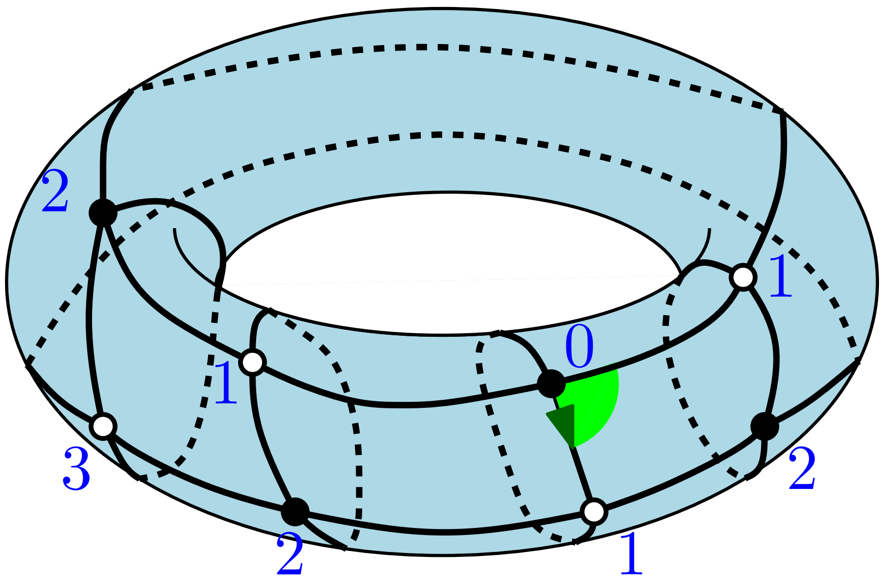

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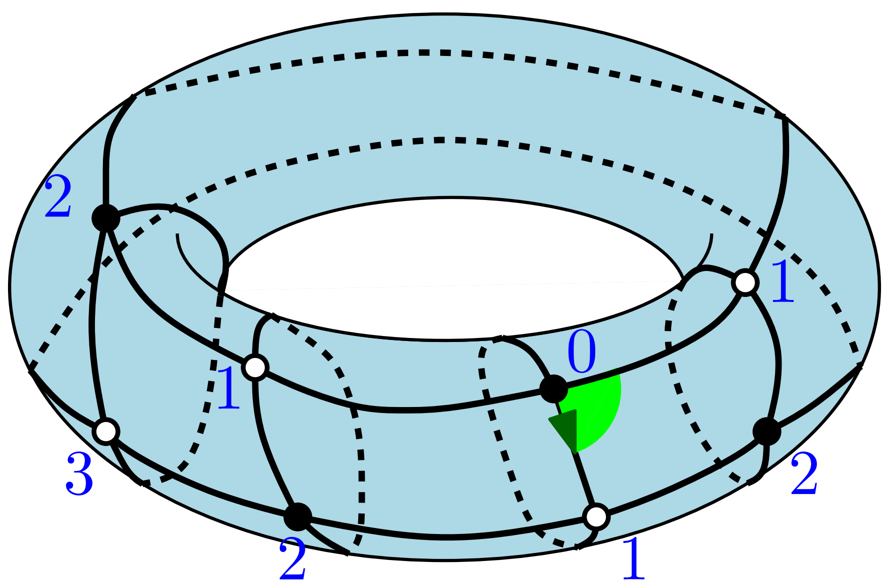

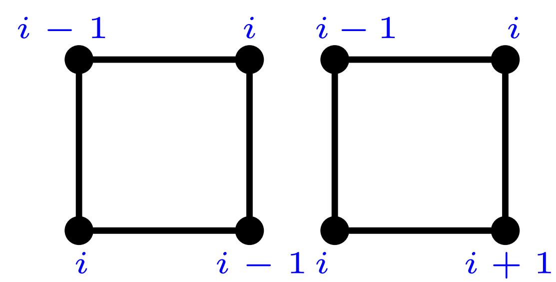

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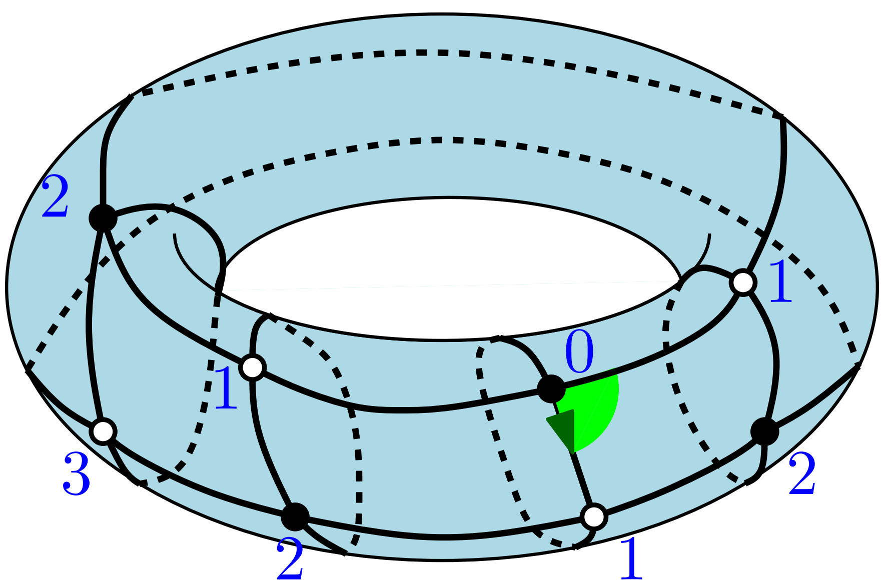

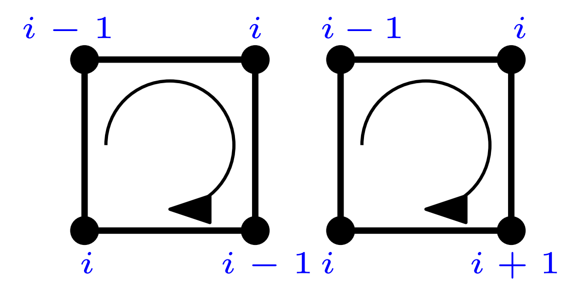

Orientable case

Theorem [Marcus, Schaeffer 1996]

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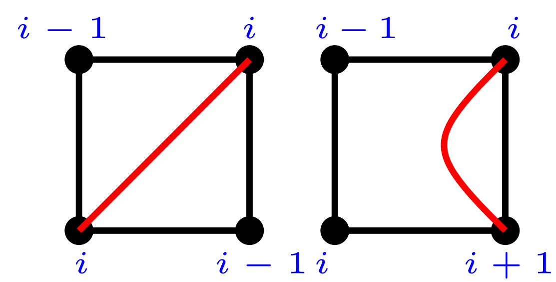

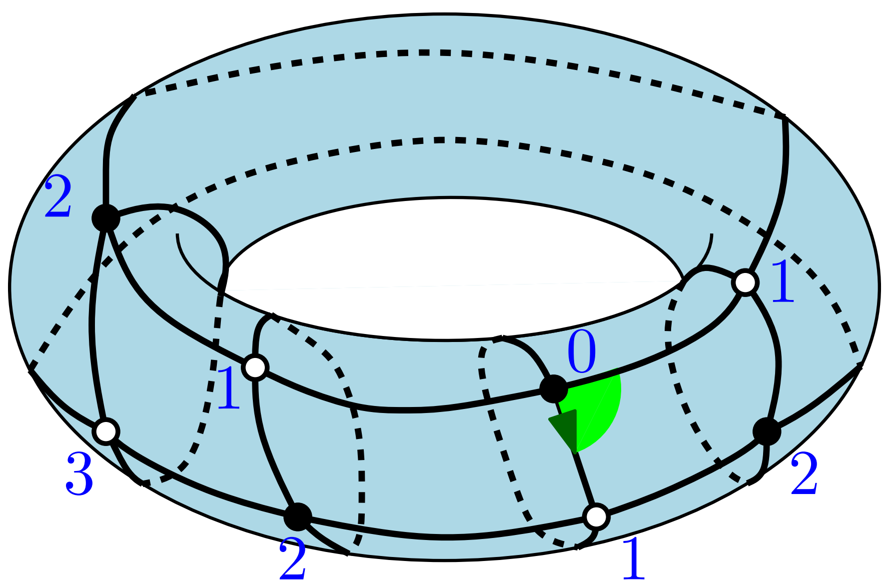

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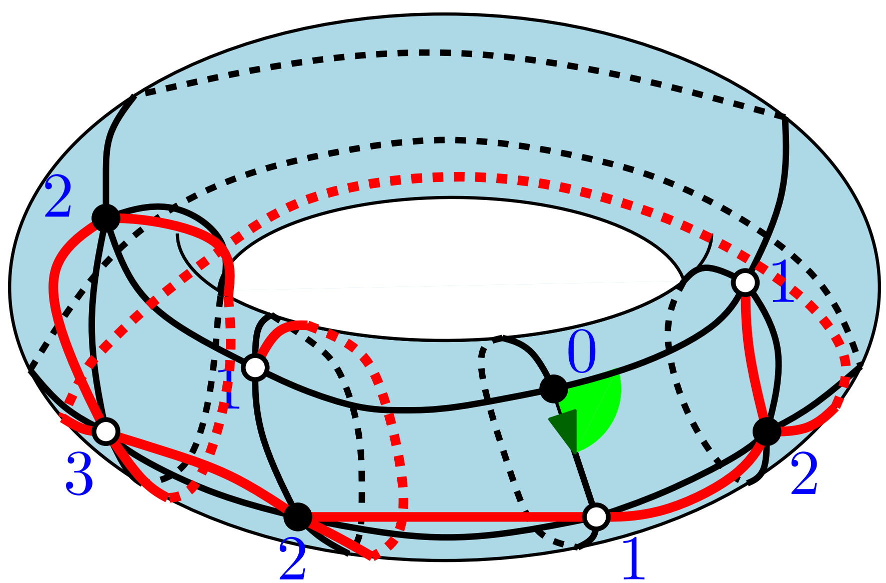

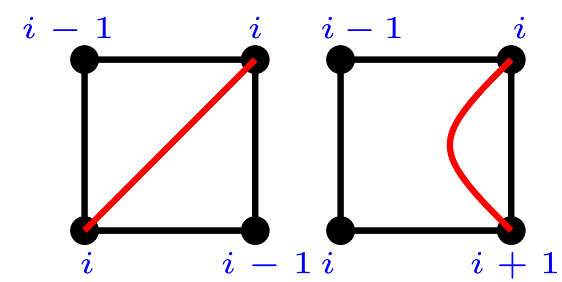

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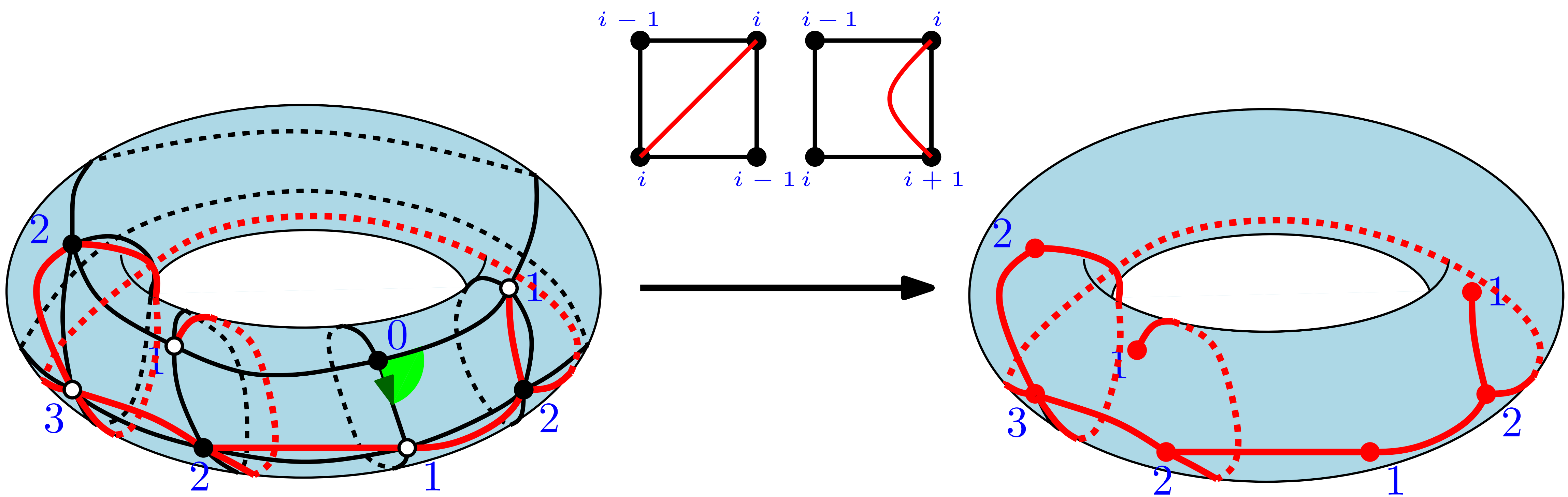

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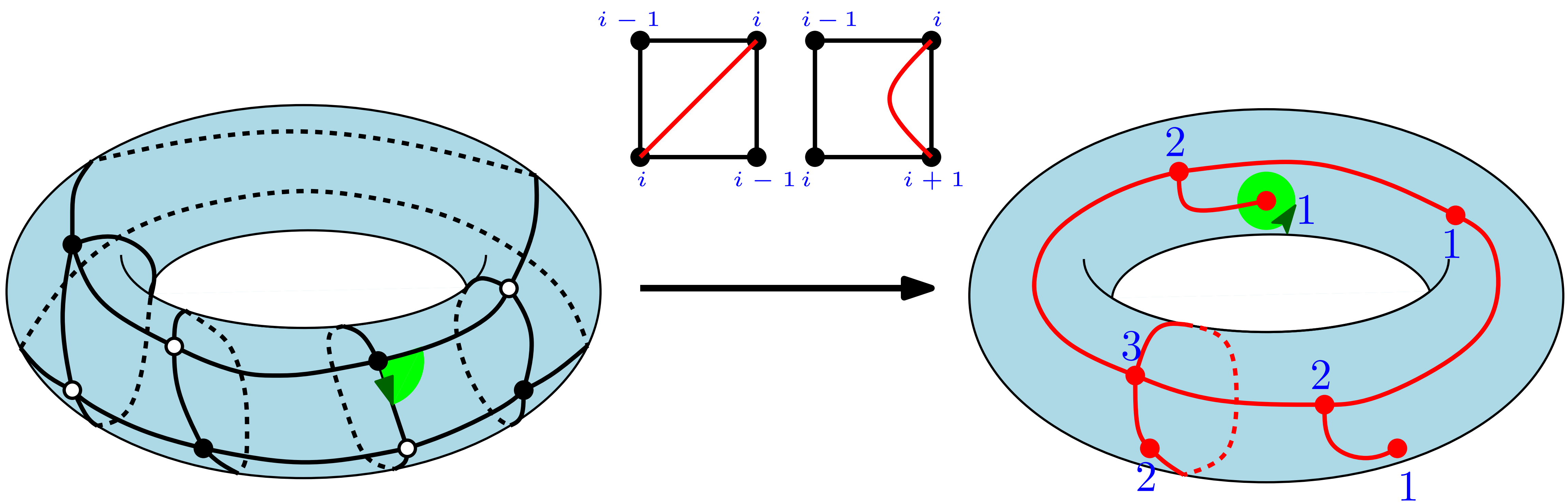

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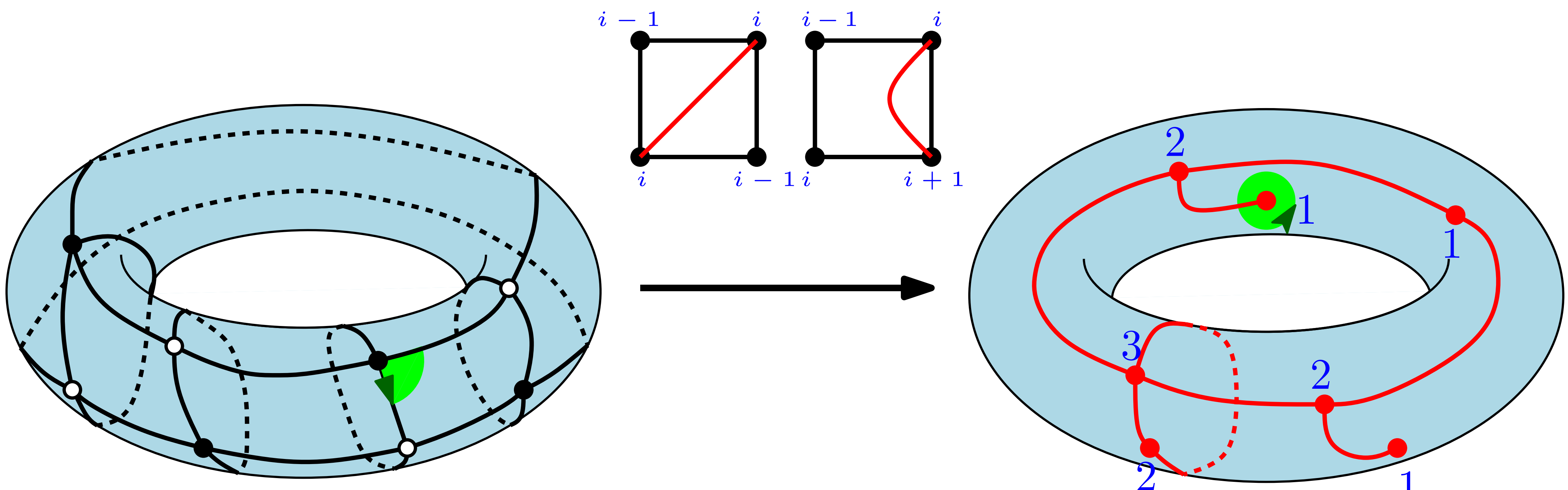

Are **non-orientable** maps different?

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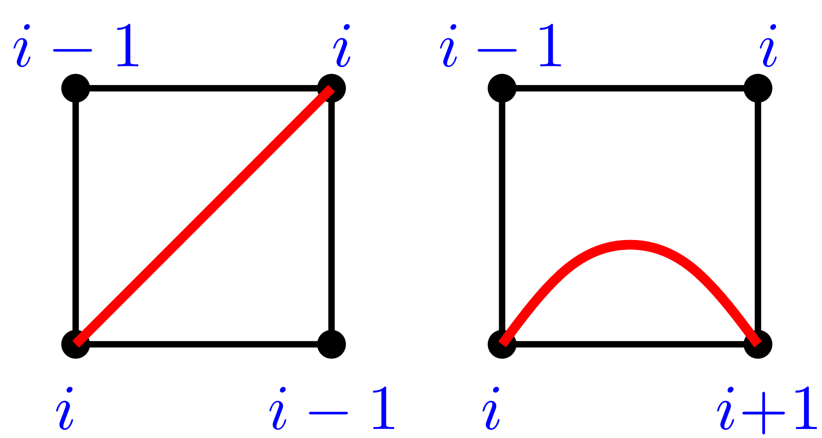

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- local rules are the same,
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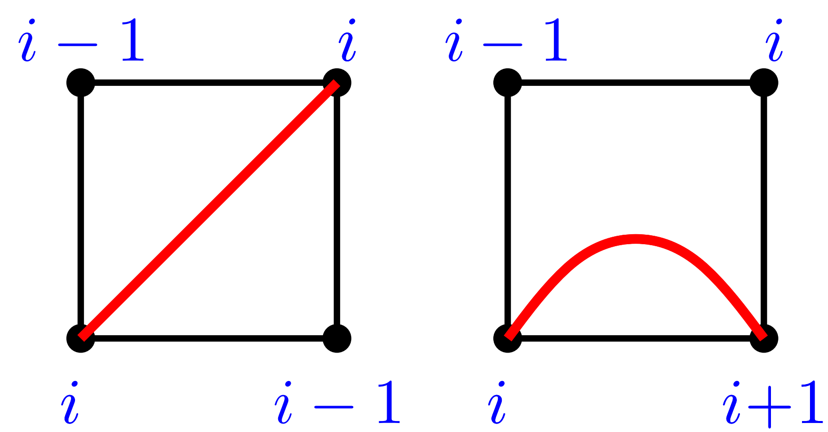

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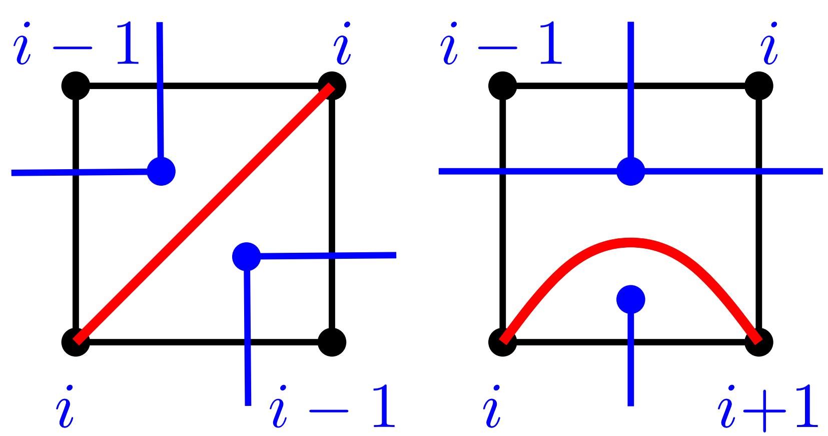

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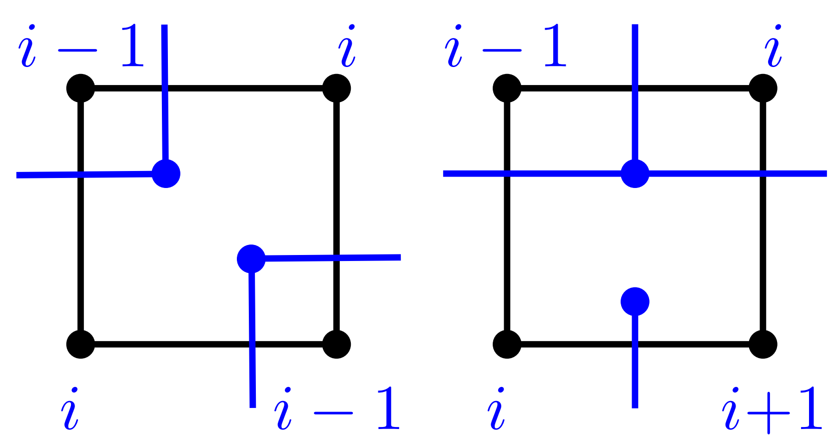

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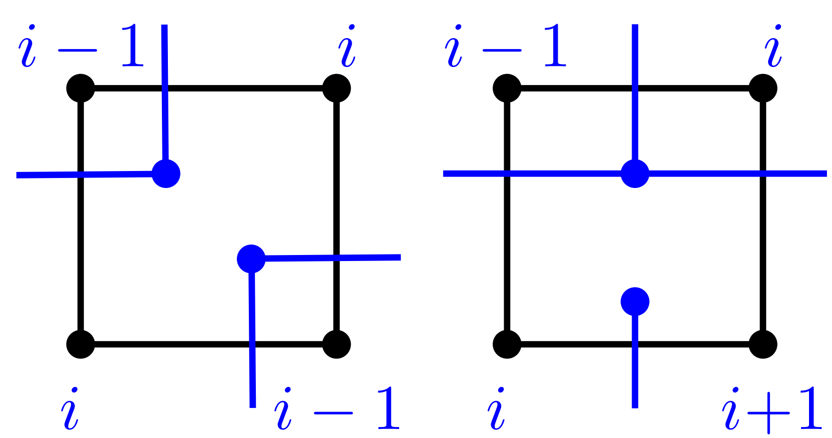

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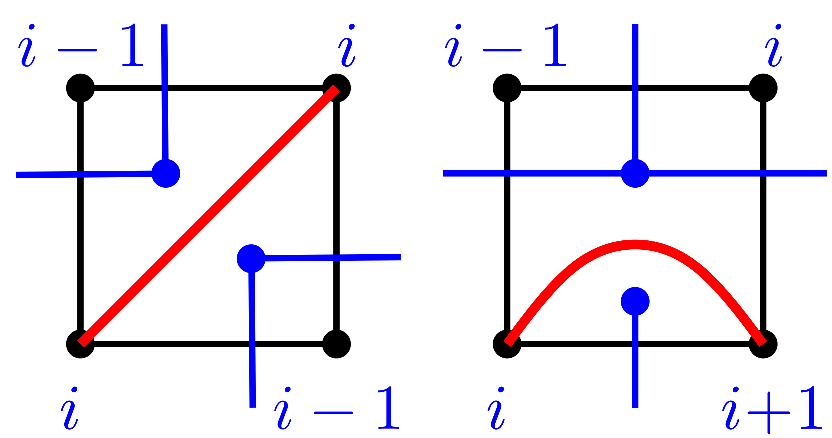

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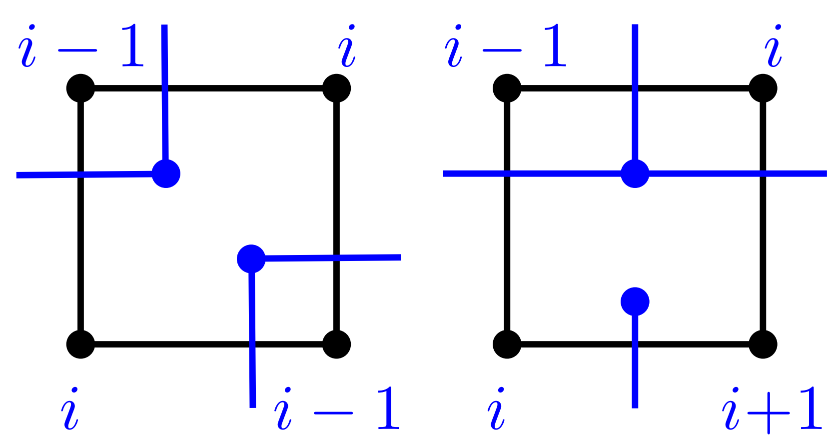
- If the construction of blue graph is local then it is invertible and it leads to a **BIJECTION!**



General case (II)

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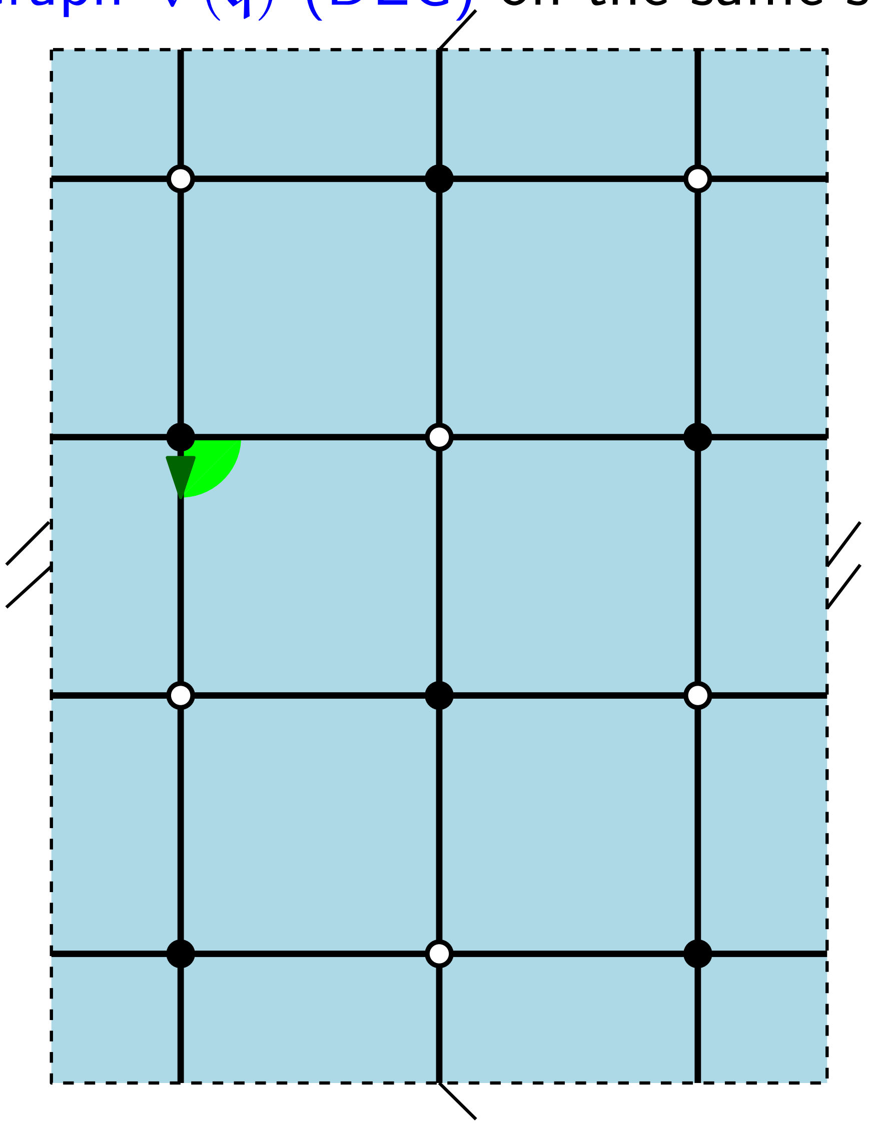

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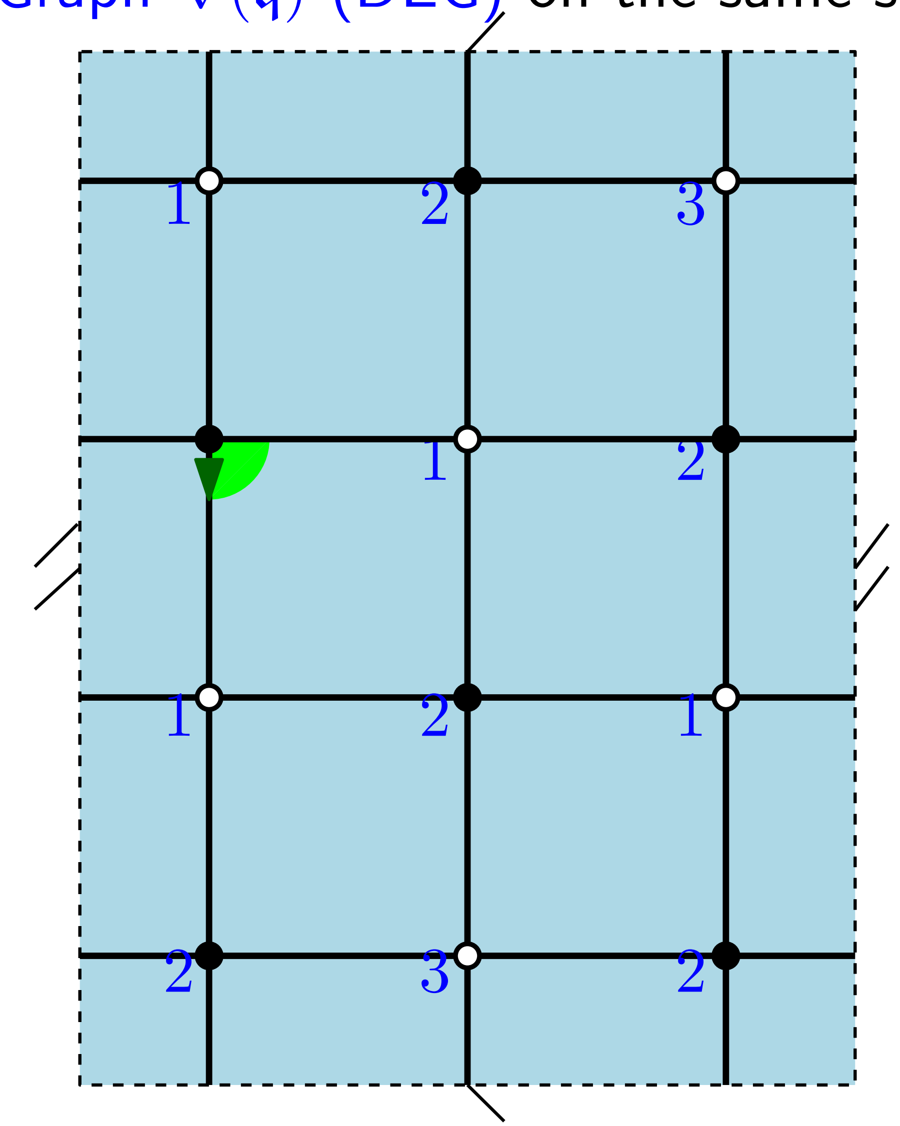

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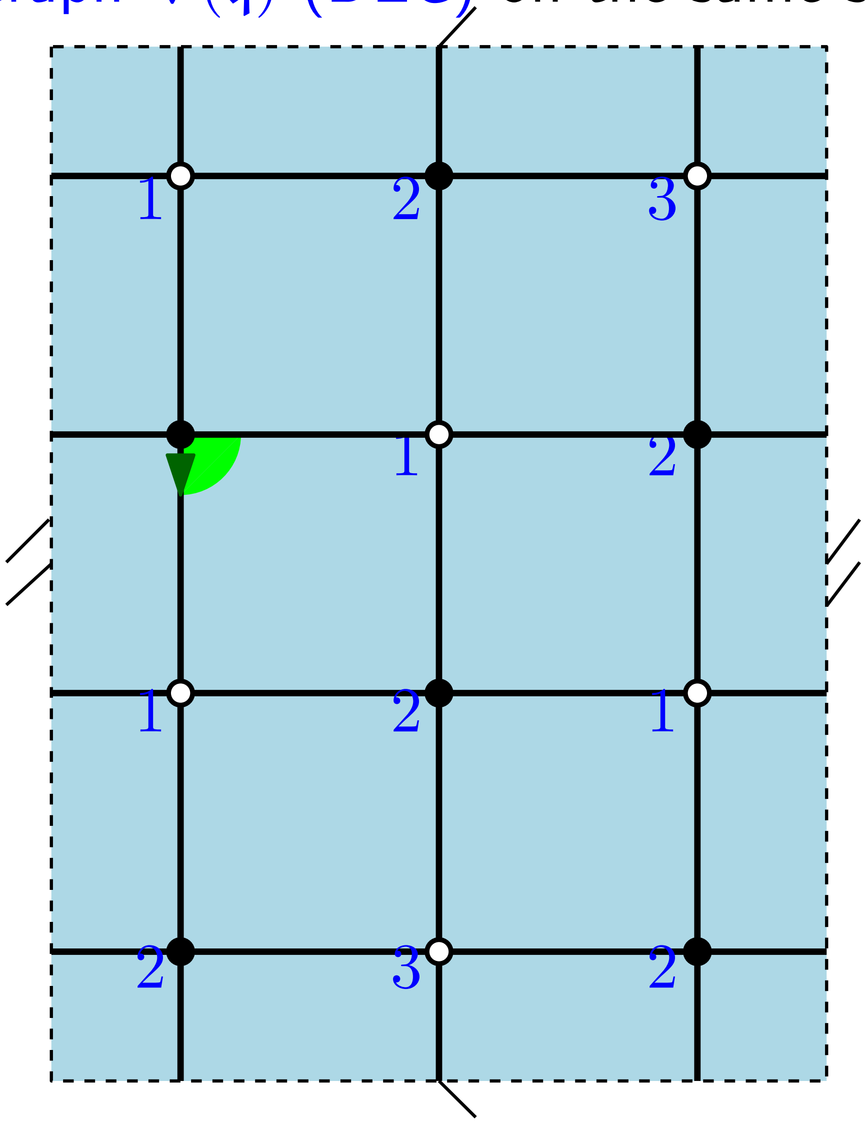

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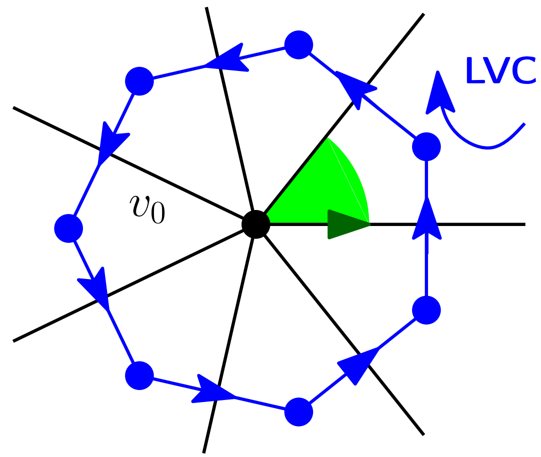

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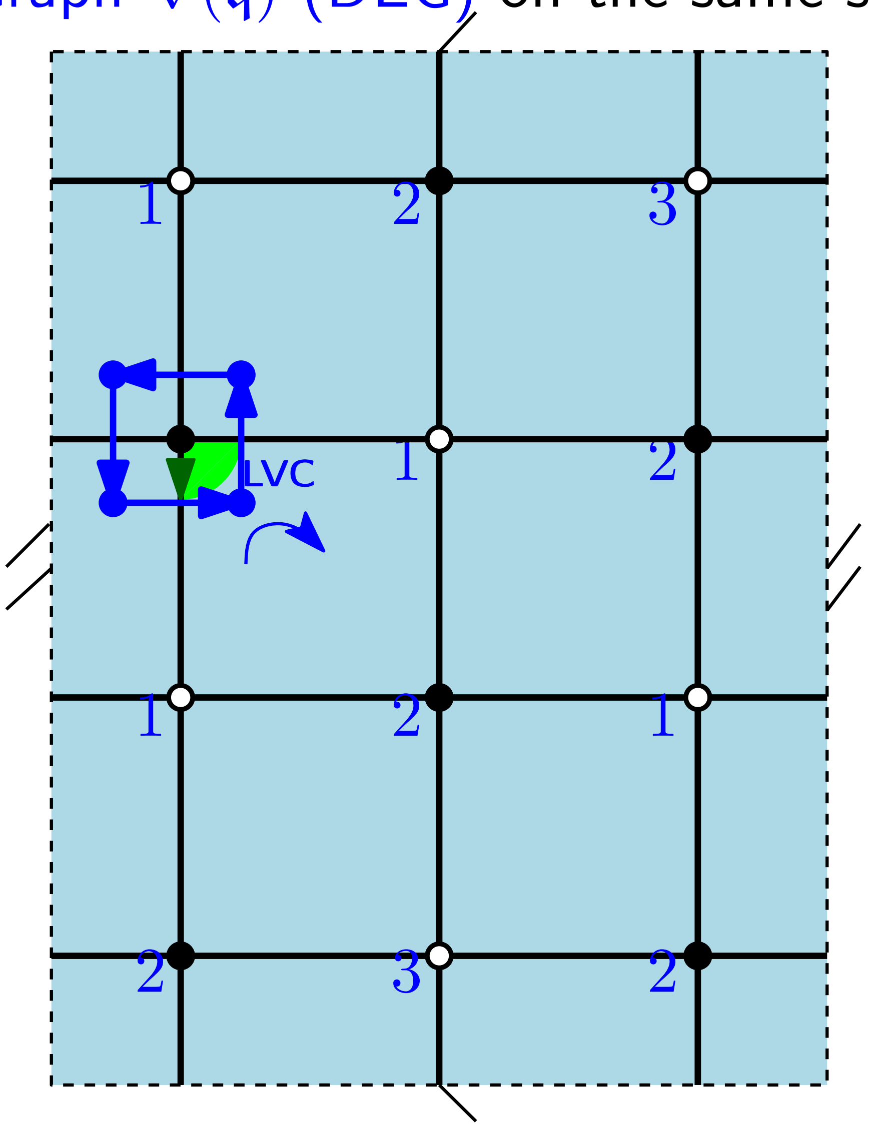

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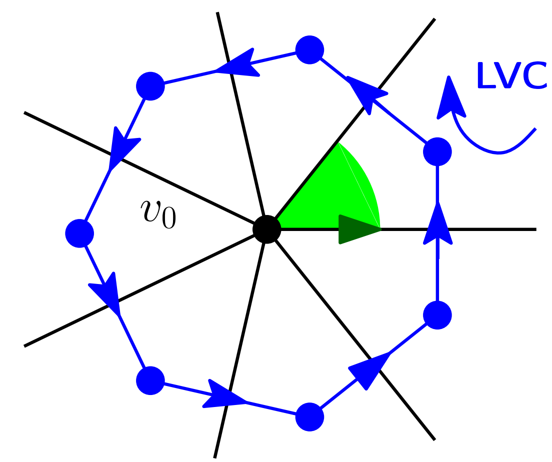

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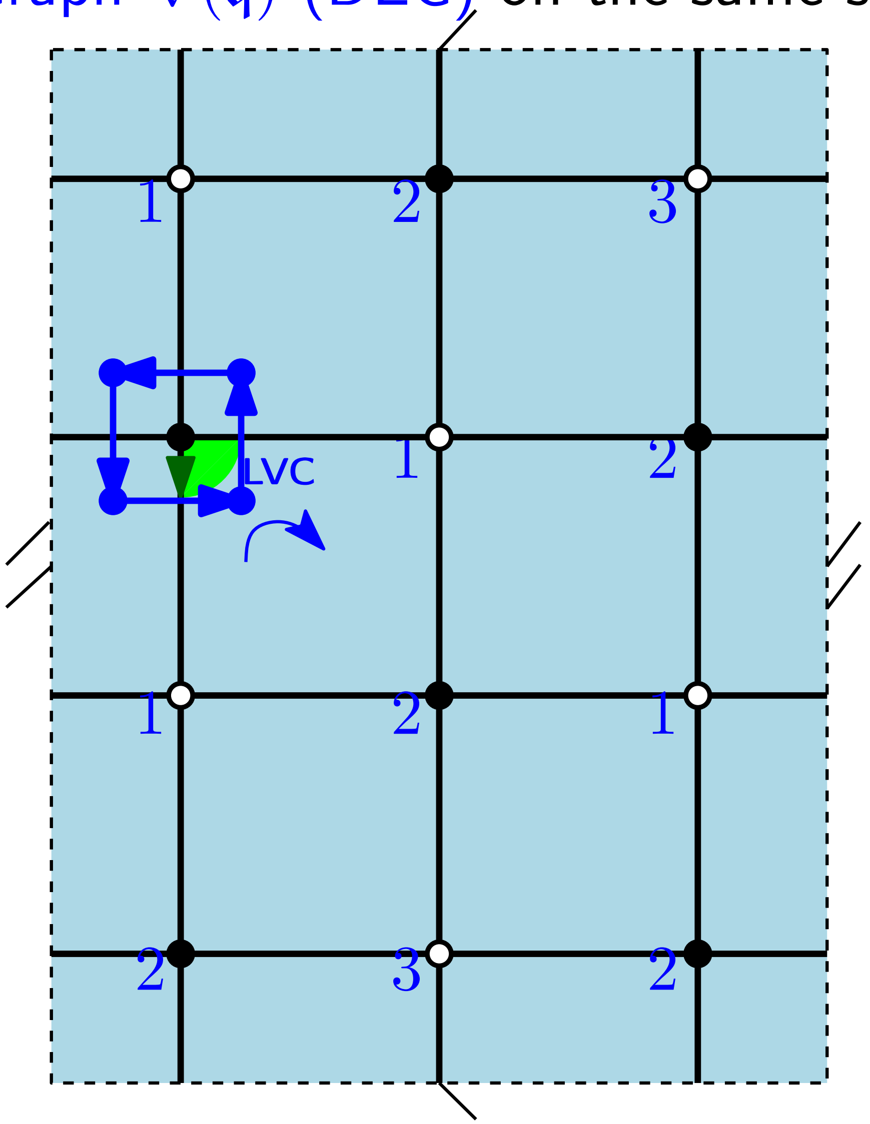

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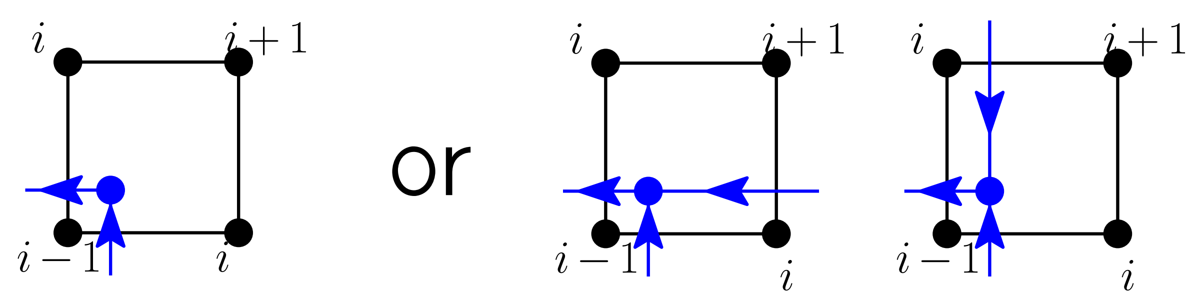

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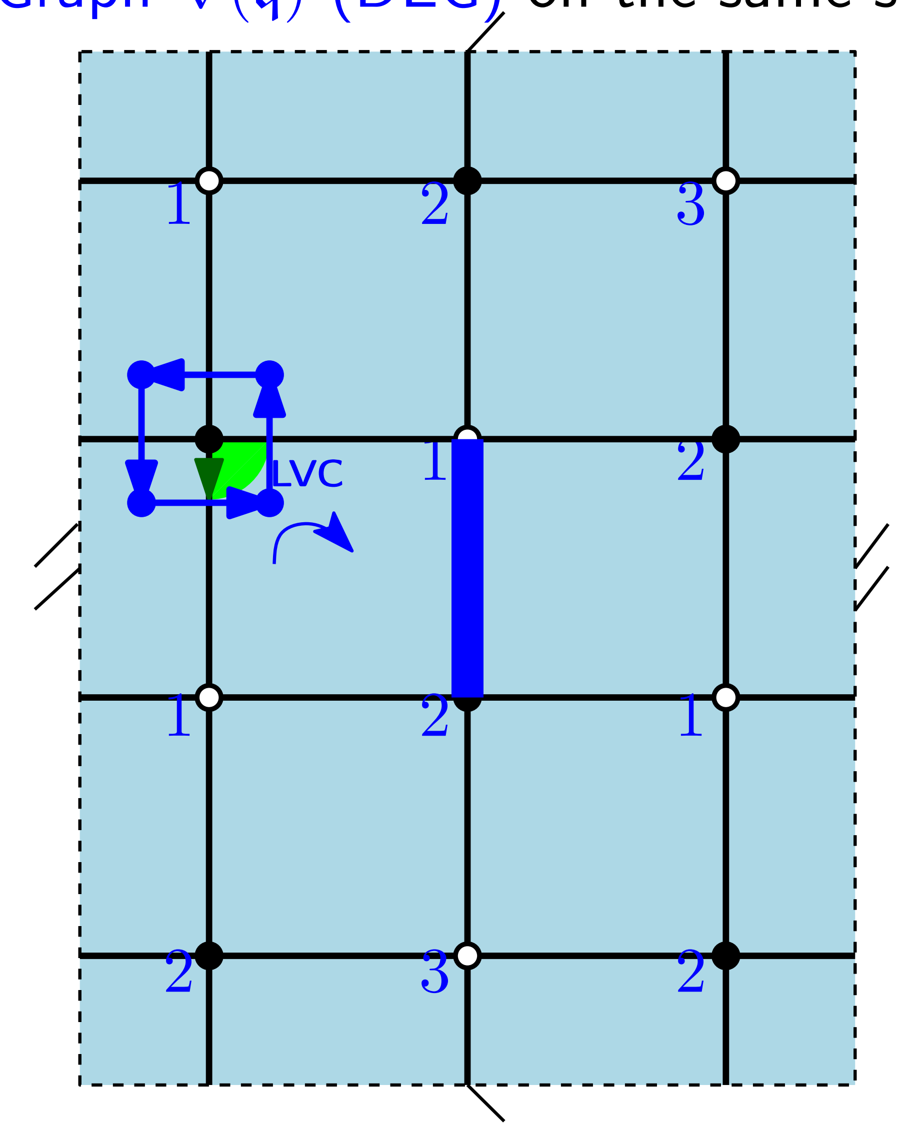

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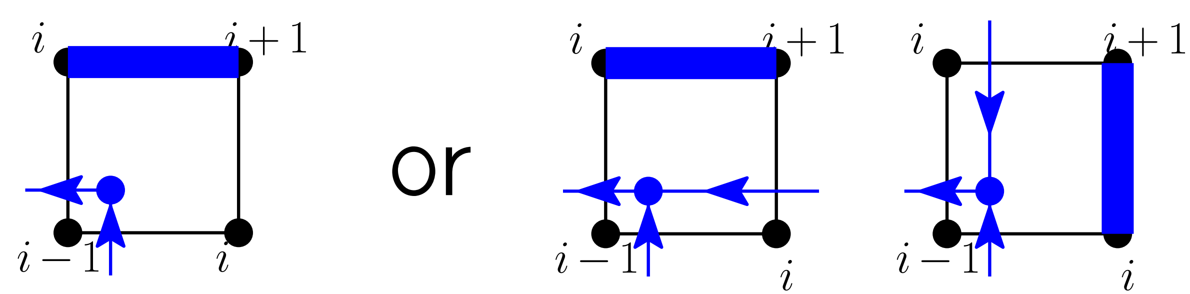

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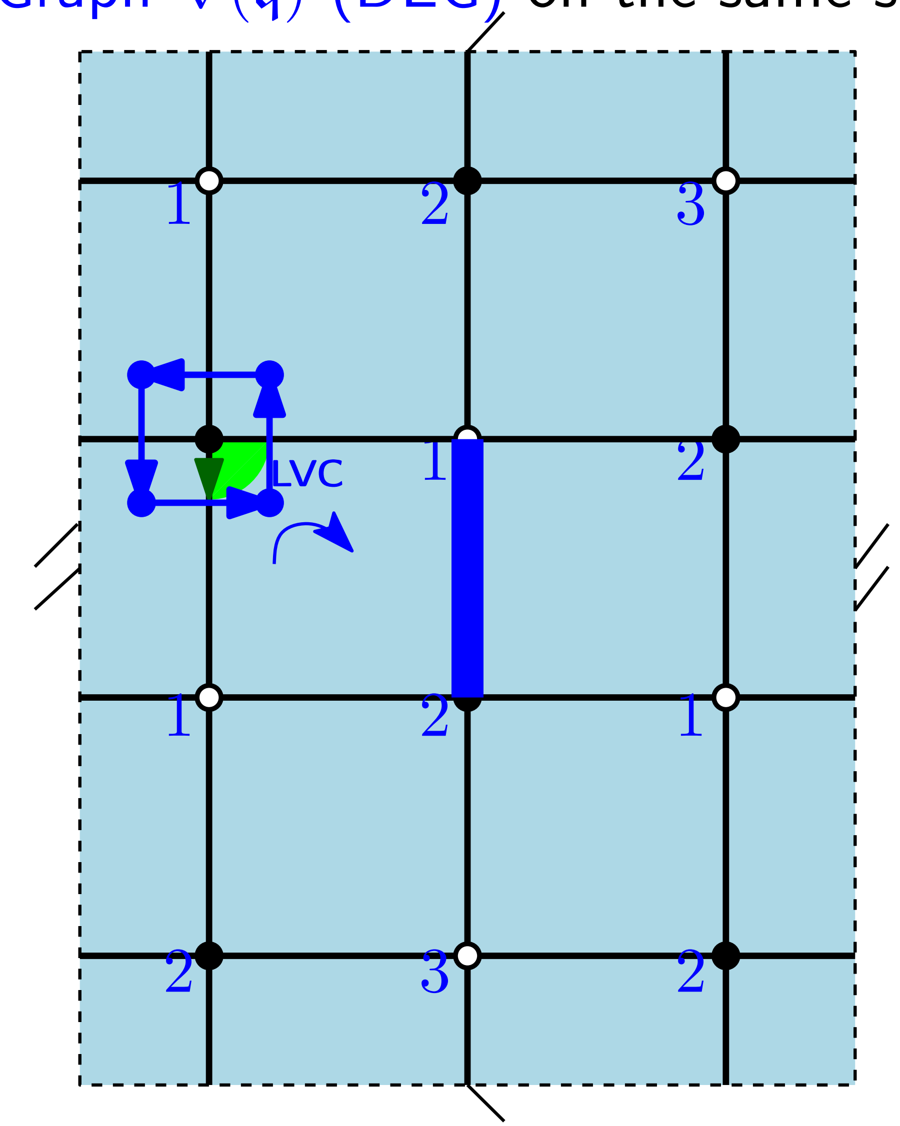
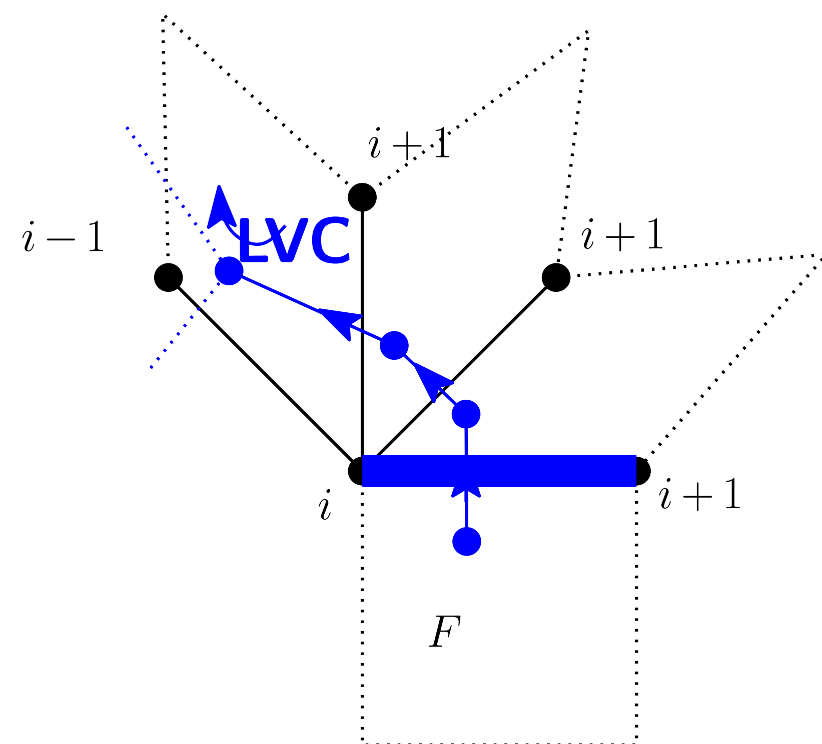


Step 2: Attaching a new branch of blue edges labeled by i starting across e



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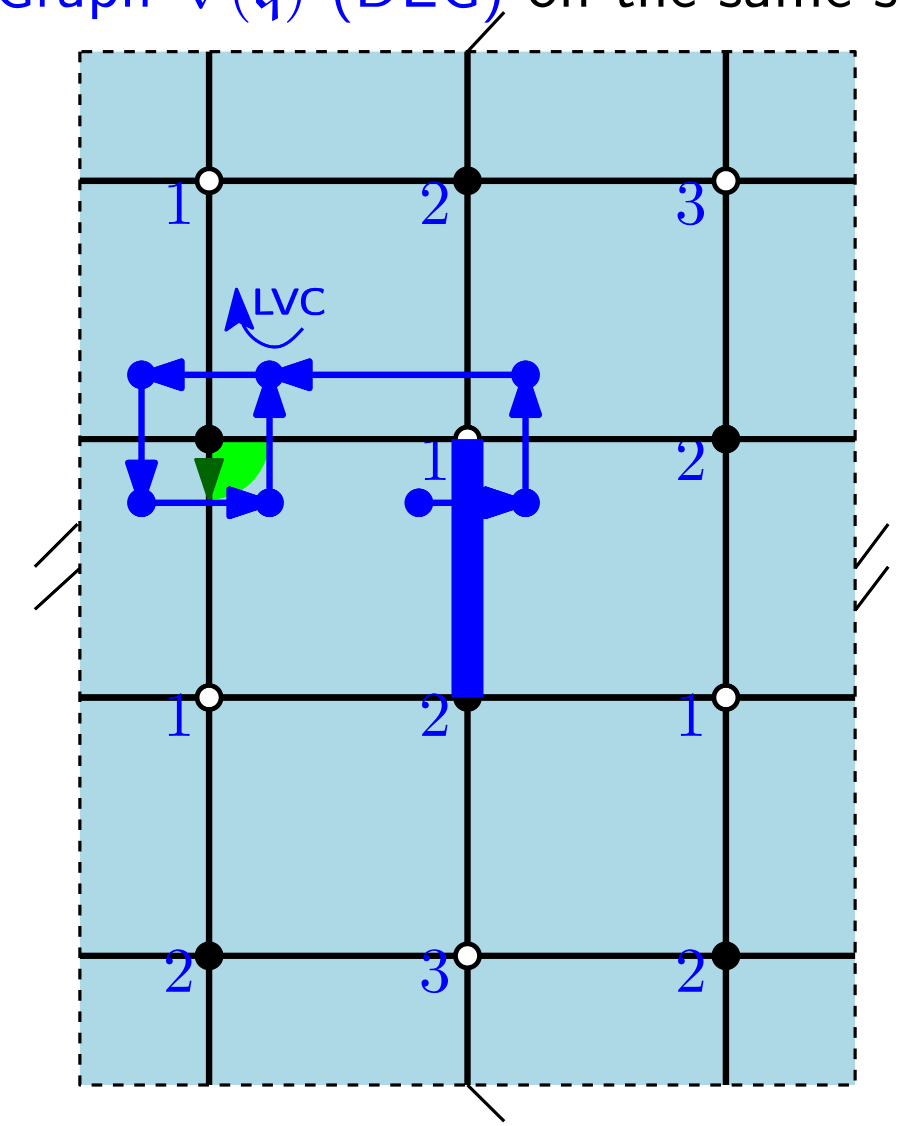

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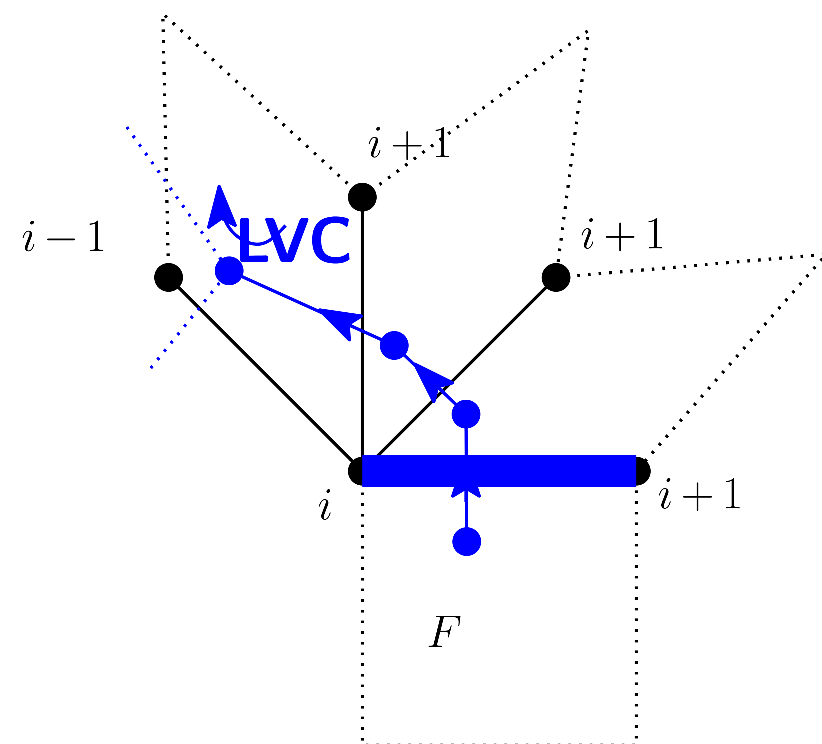

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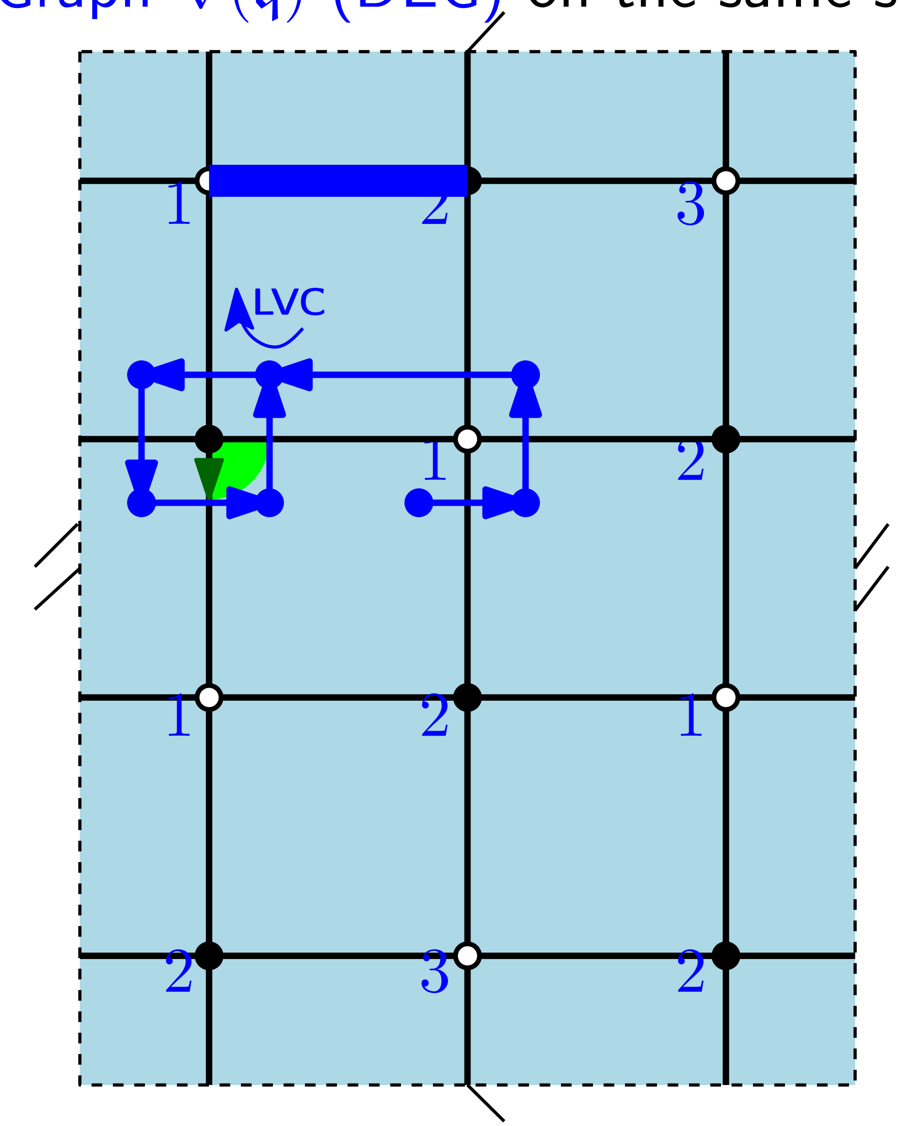

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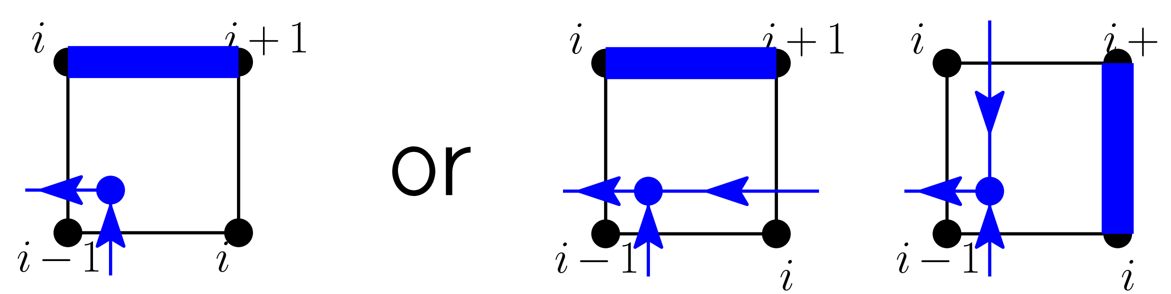

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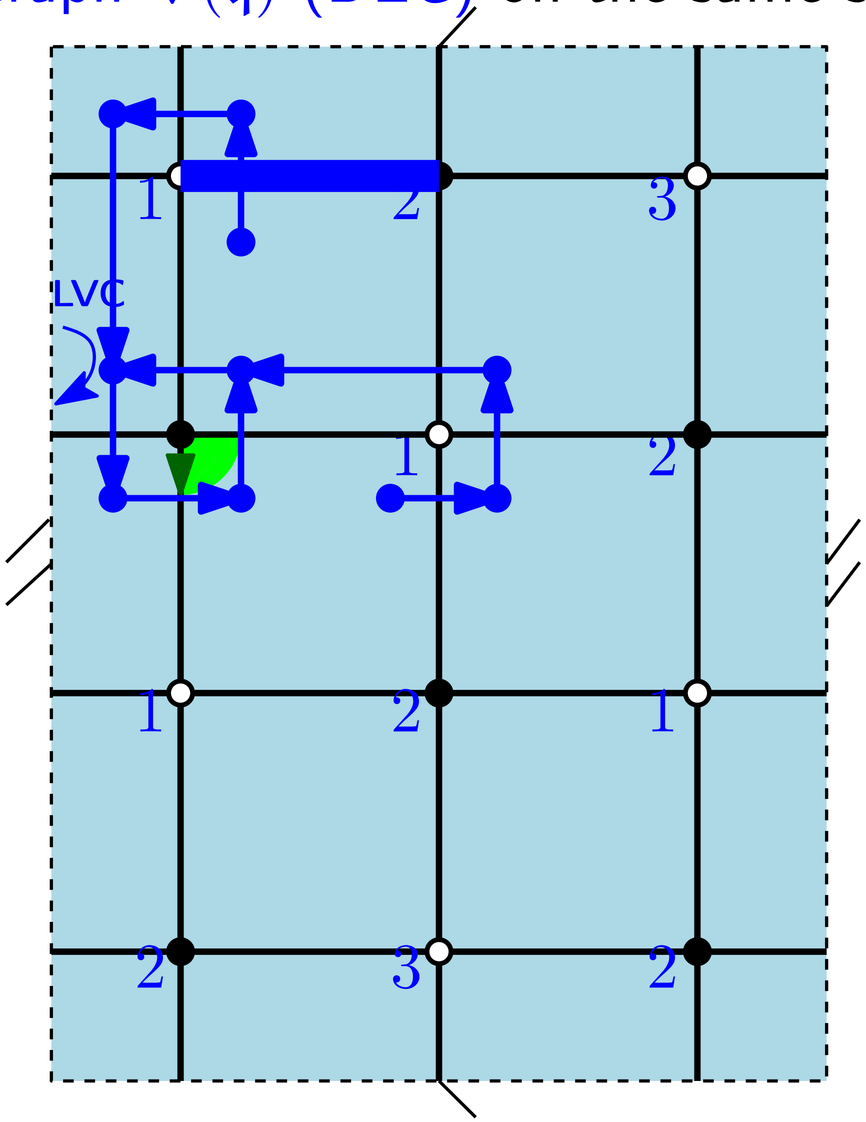

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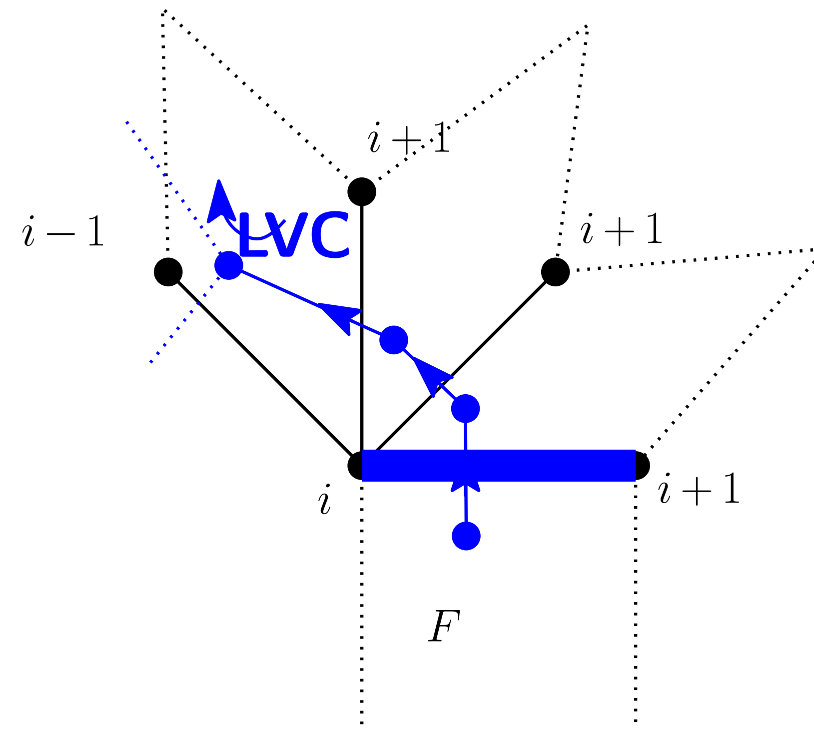

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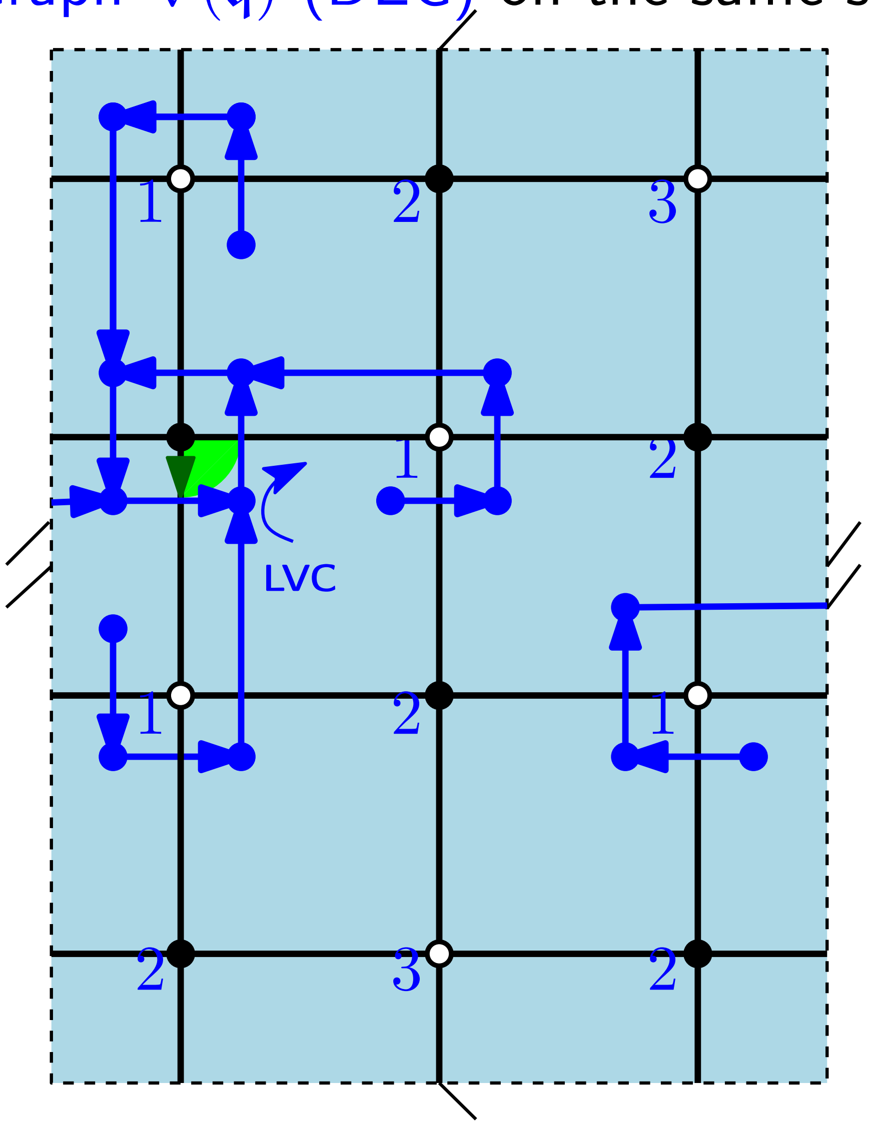

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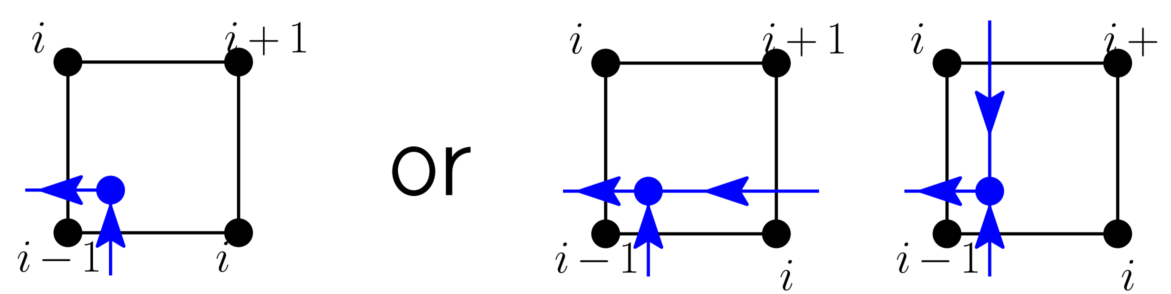

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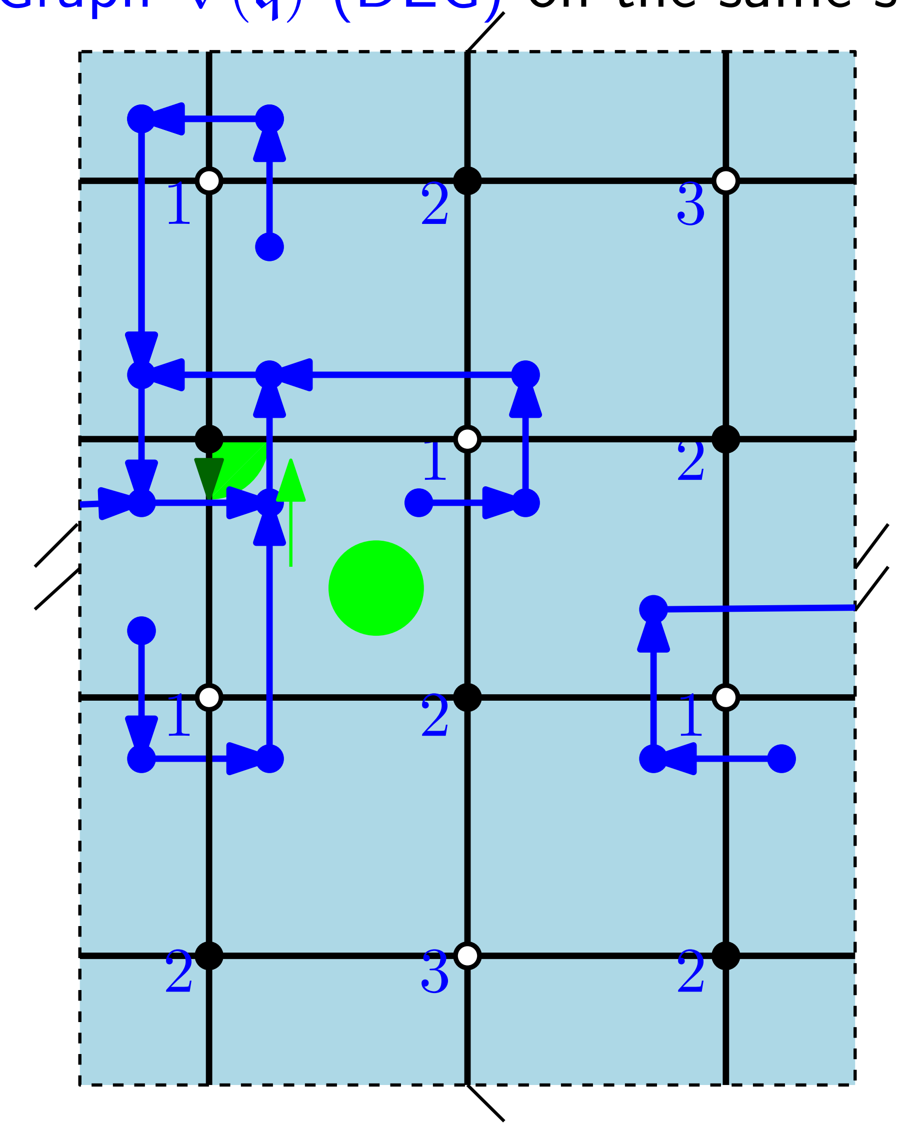

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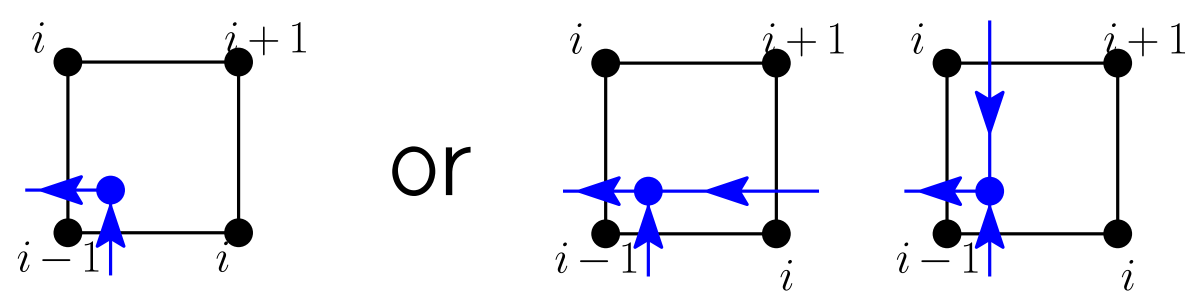

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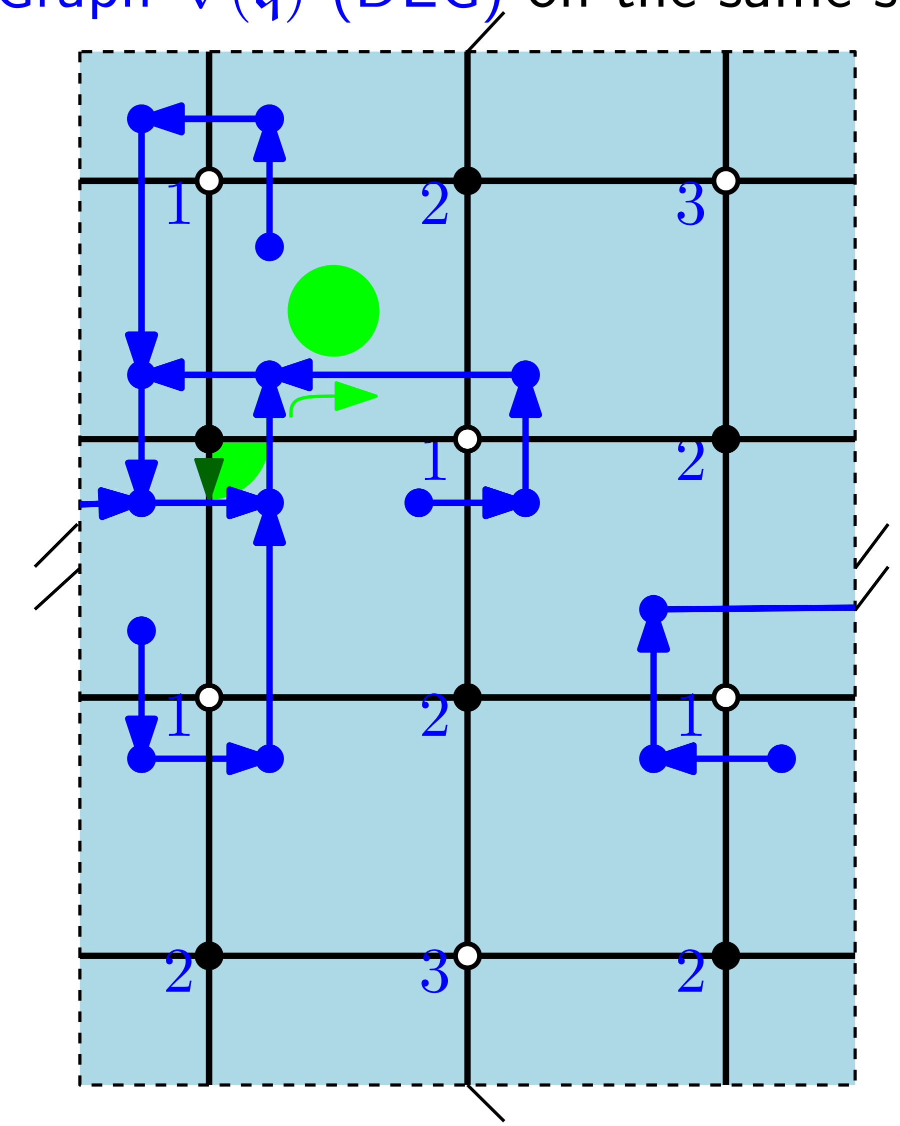

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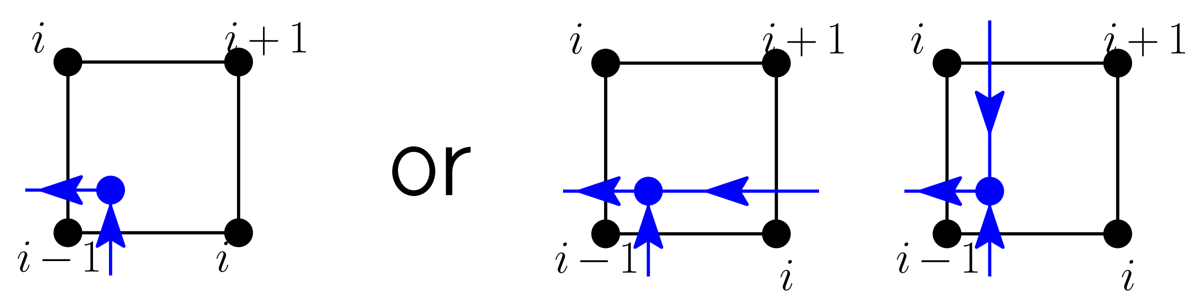

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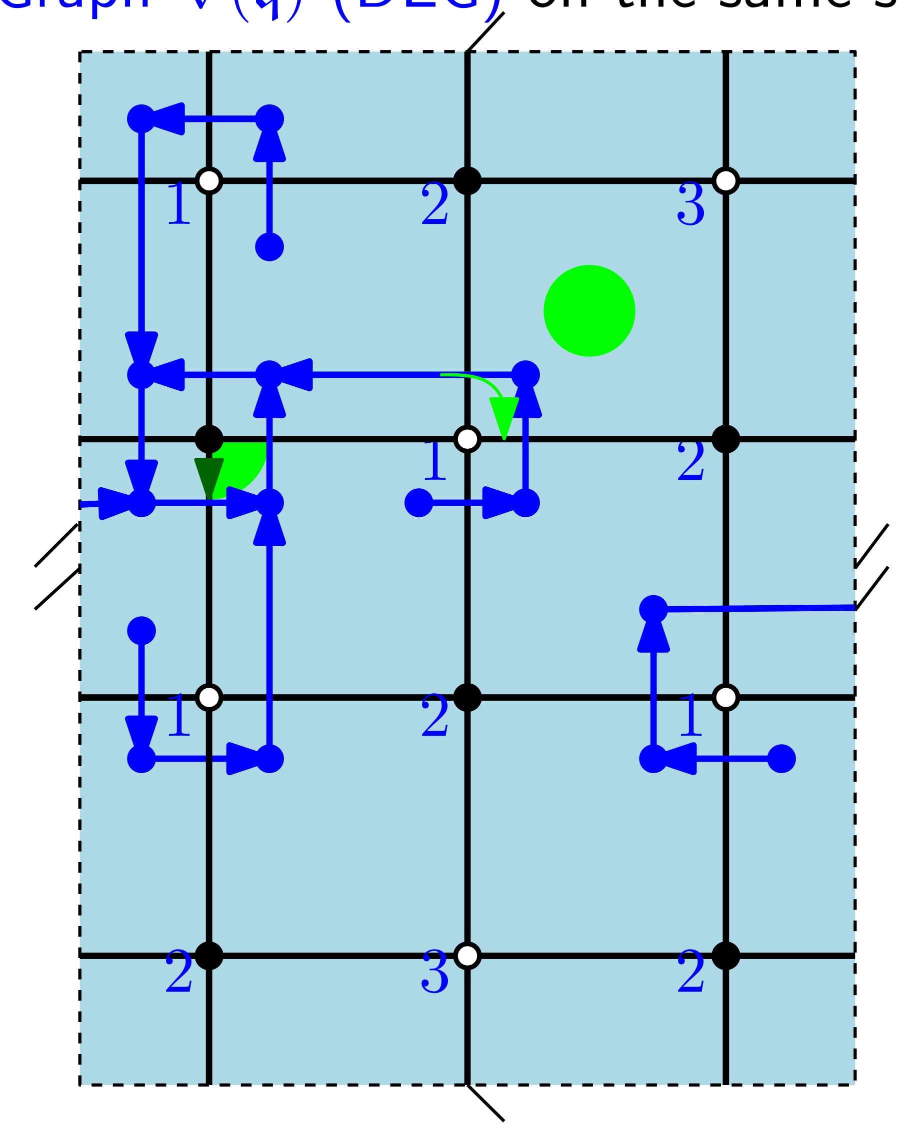

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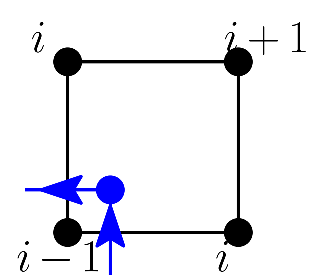

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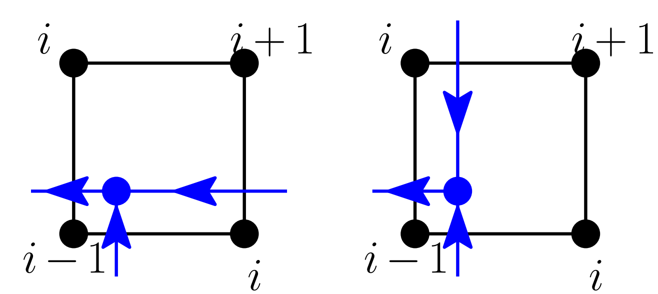

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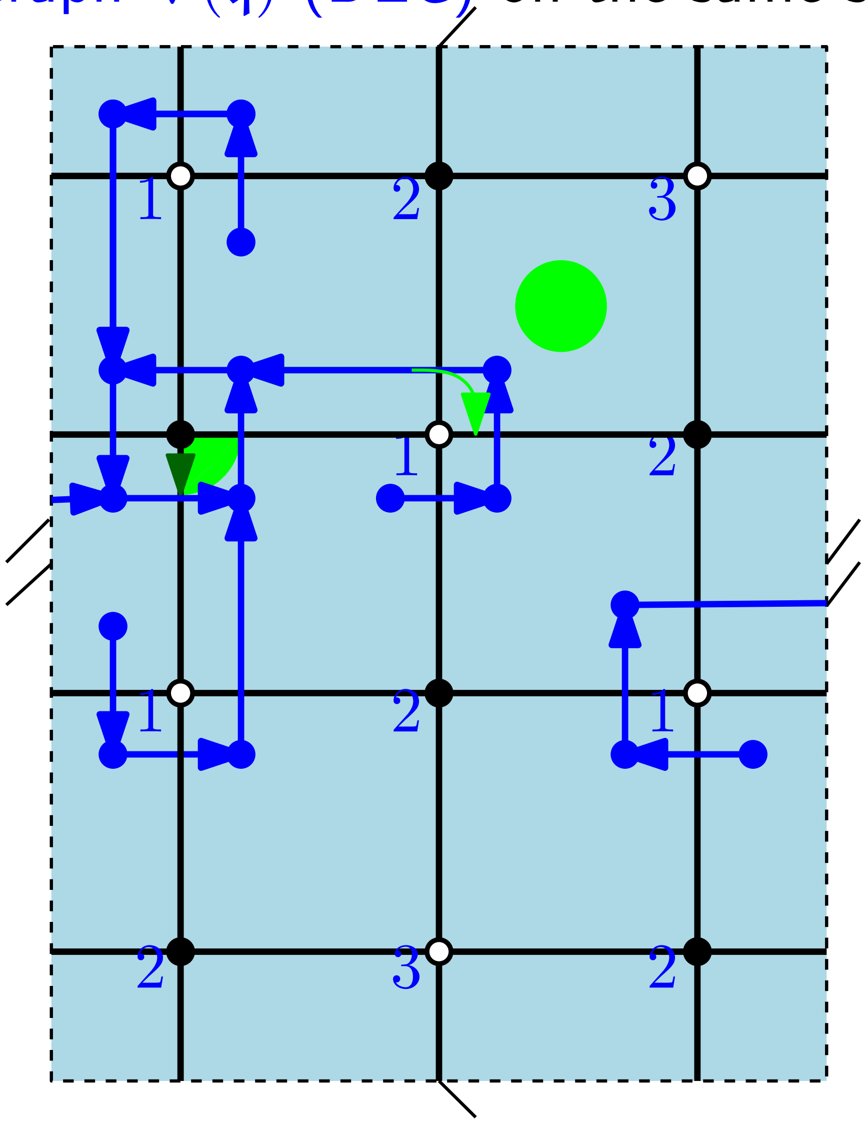

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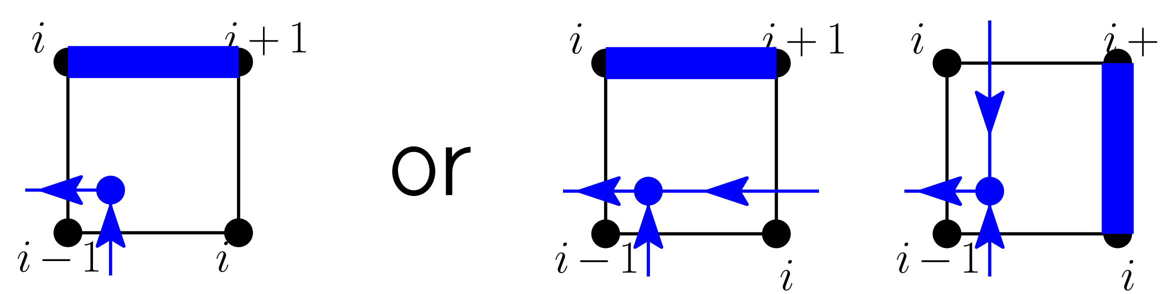

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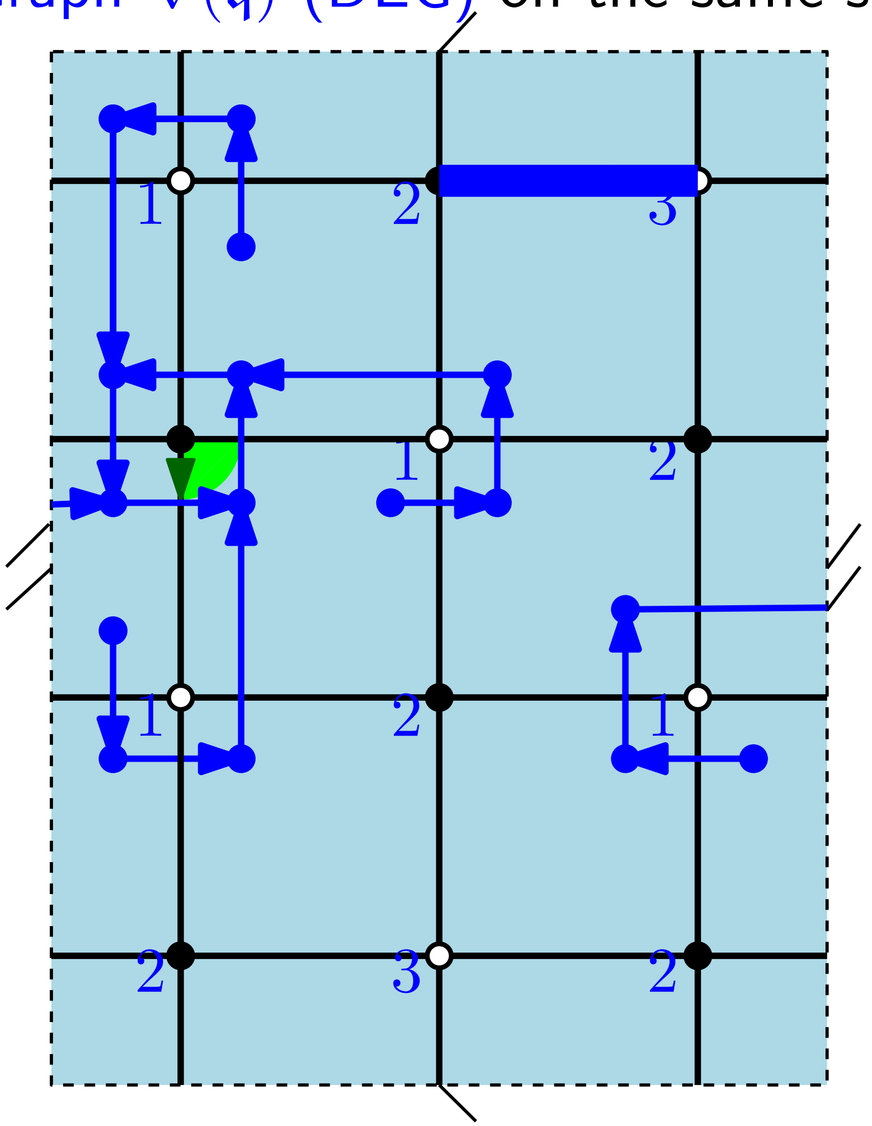
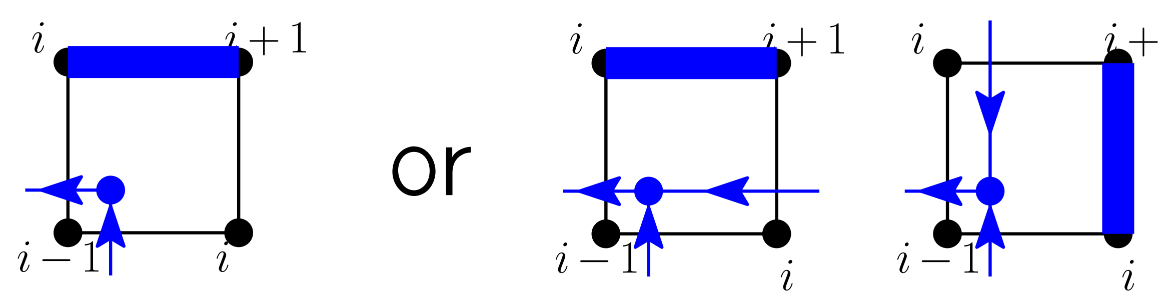


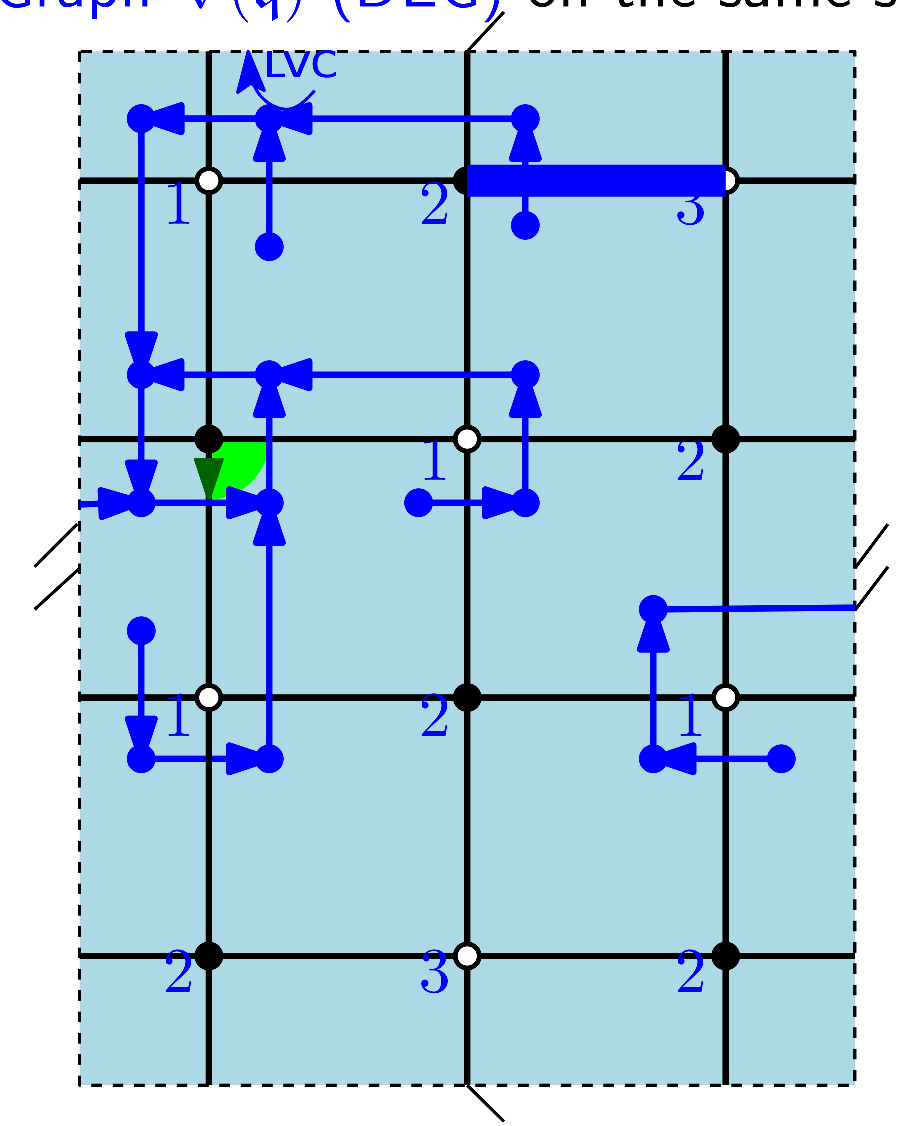
Step 1: Choosing where to start

- we perform the tour of the blue graph, starting from the LVC. We stop as soon as we visit a face F having the following properties: F is of type $(i-1, i, i+1, i)$, and F has exactly one blue vertex already placed inside it.
- we choose an edge e in F by the following rule:

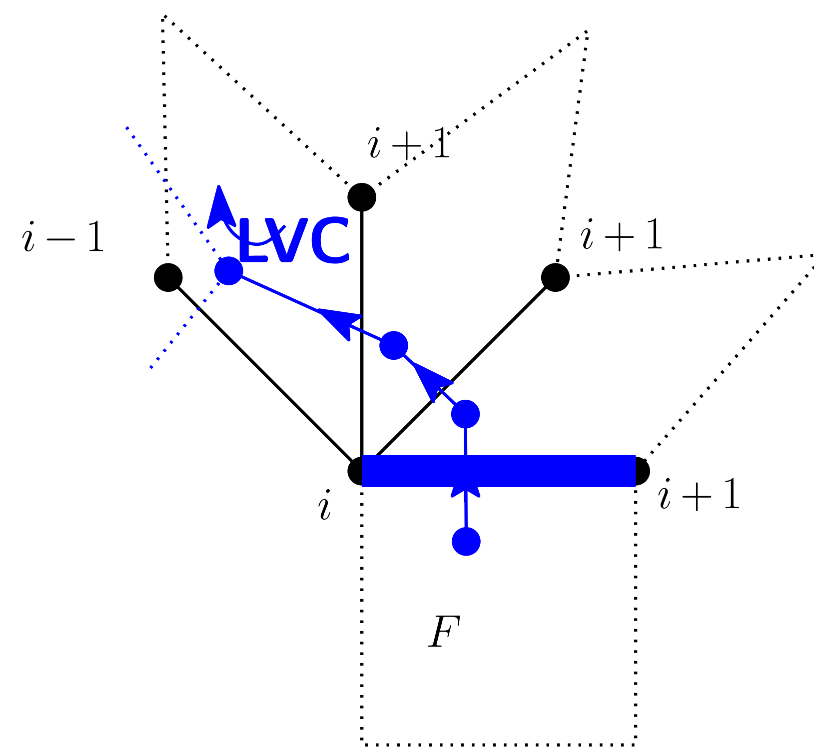


General case (II)

For a given quadrangulation $\mathfrak{q}$ we construct recursively a Dual Exploration Graph $\nabla(\mathfrak{q})$ (DEG) on the same surface:



Step 2: Attaching a new branch of blue edges labeled by i starting across e



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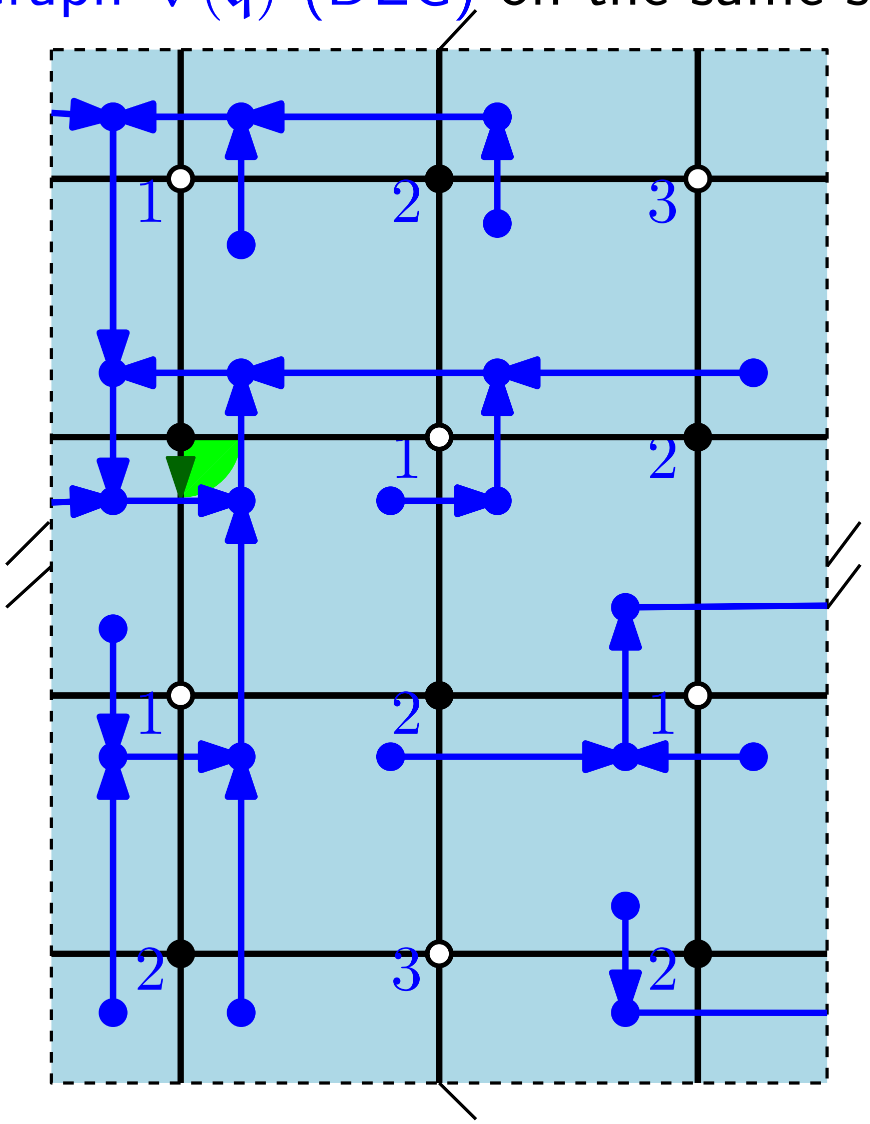

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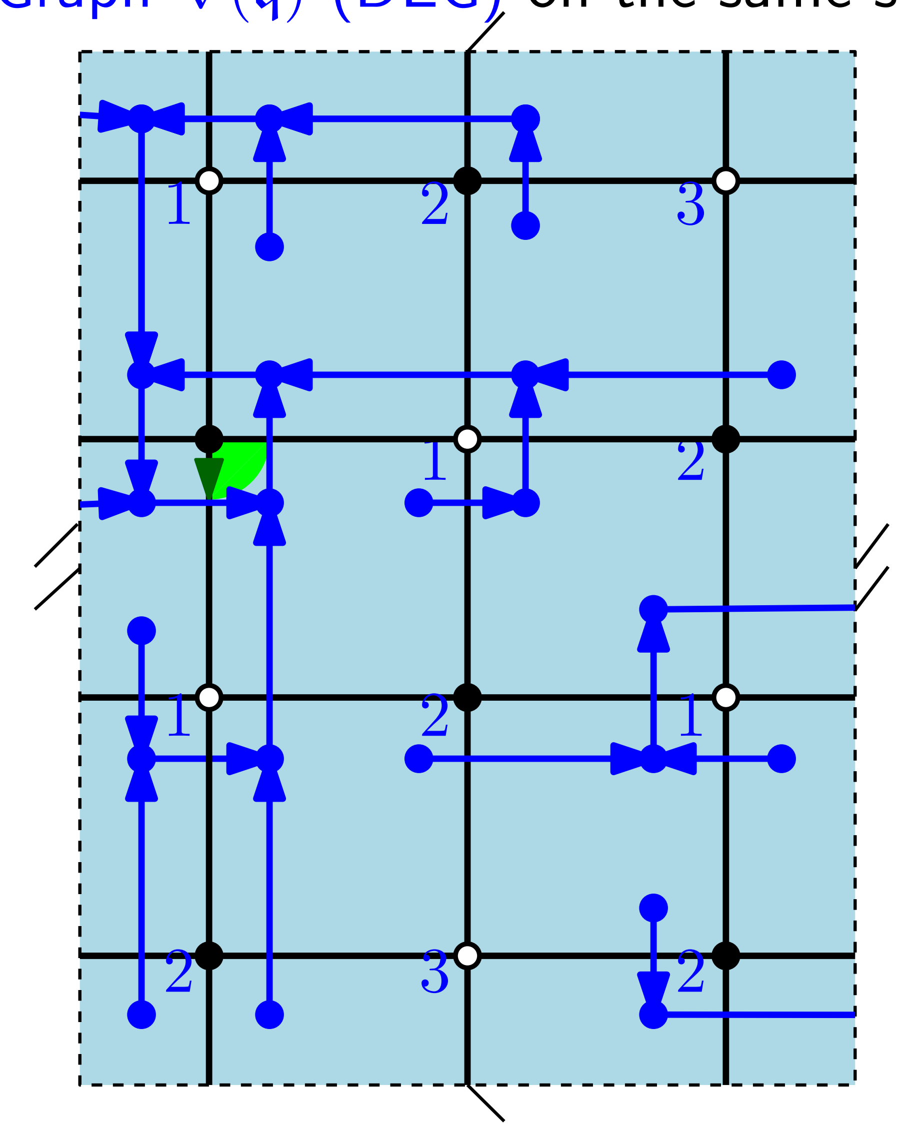
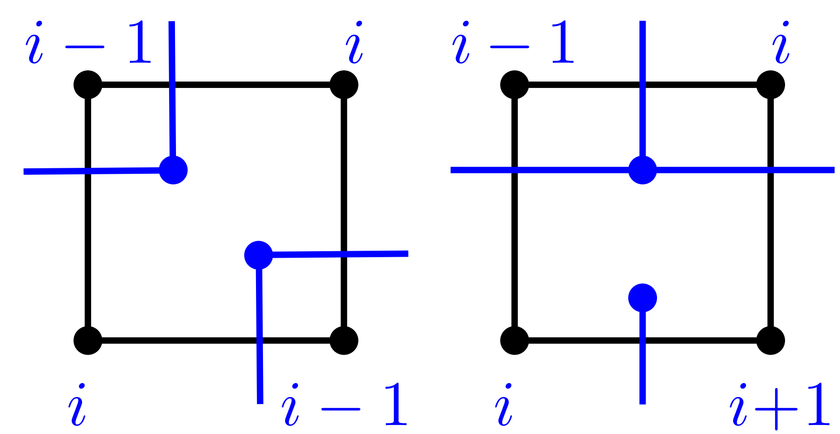


Proposition:
DEG $\nabla(\mathfrak{q})$ is formed by a unique oriented cycle encircling root vertex v_0, to which oriented trees are attached. After the construction of $\nabla(\mathfrak{q})$ is complete, each face of $\mathfrak{q}$ is of one of the two types:



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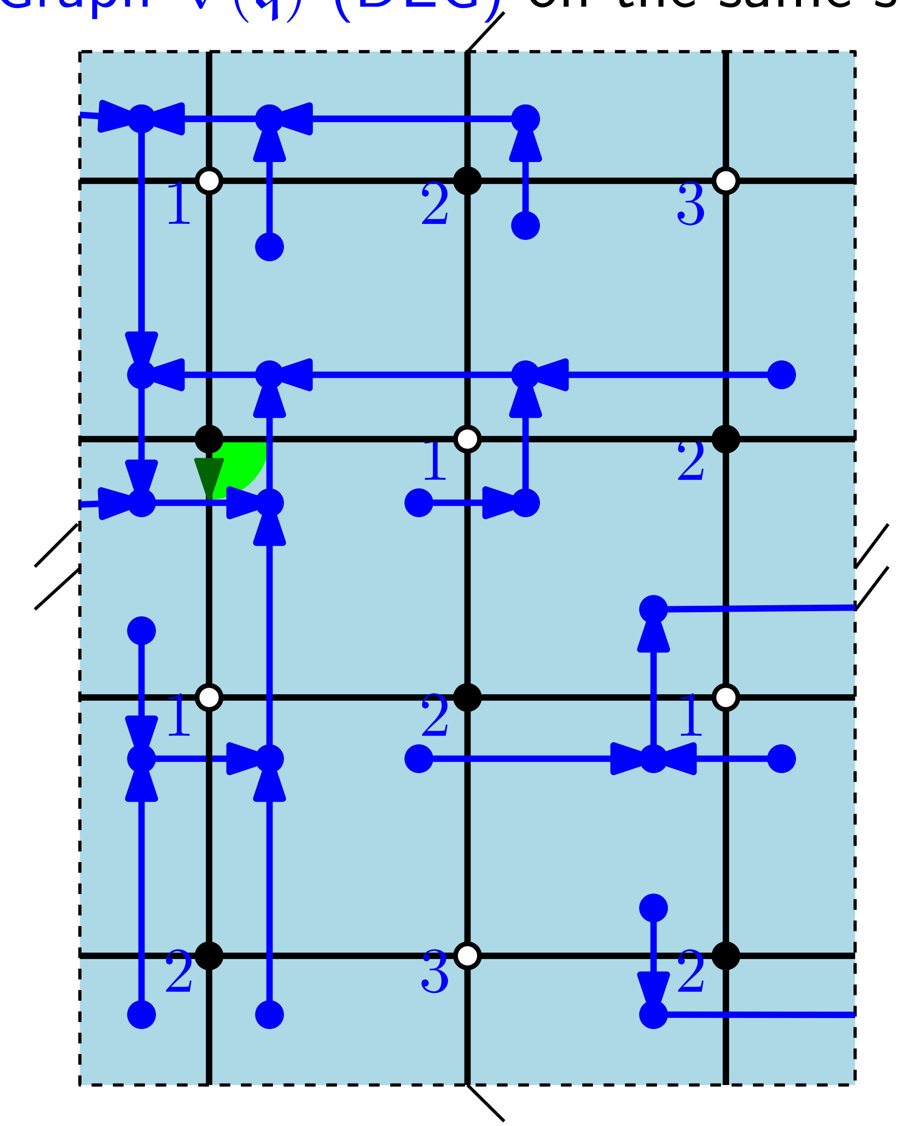

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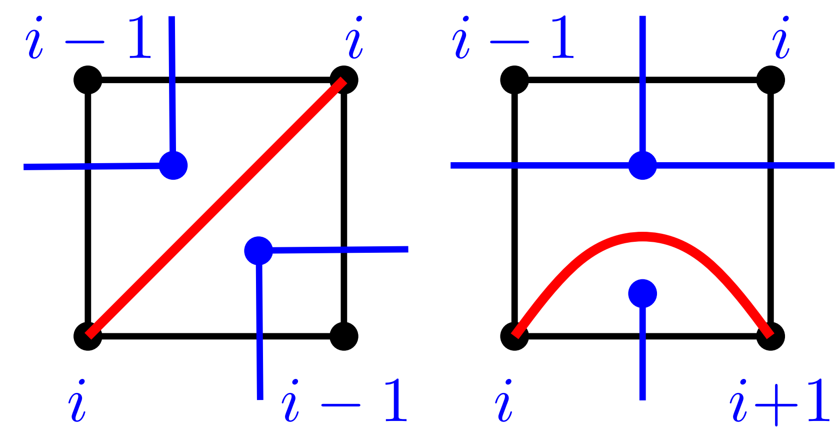

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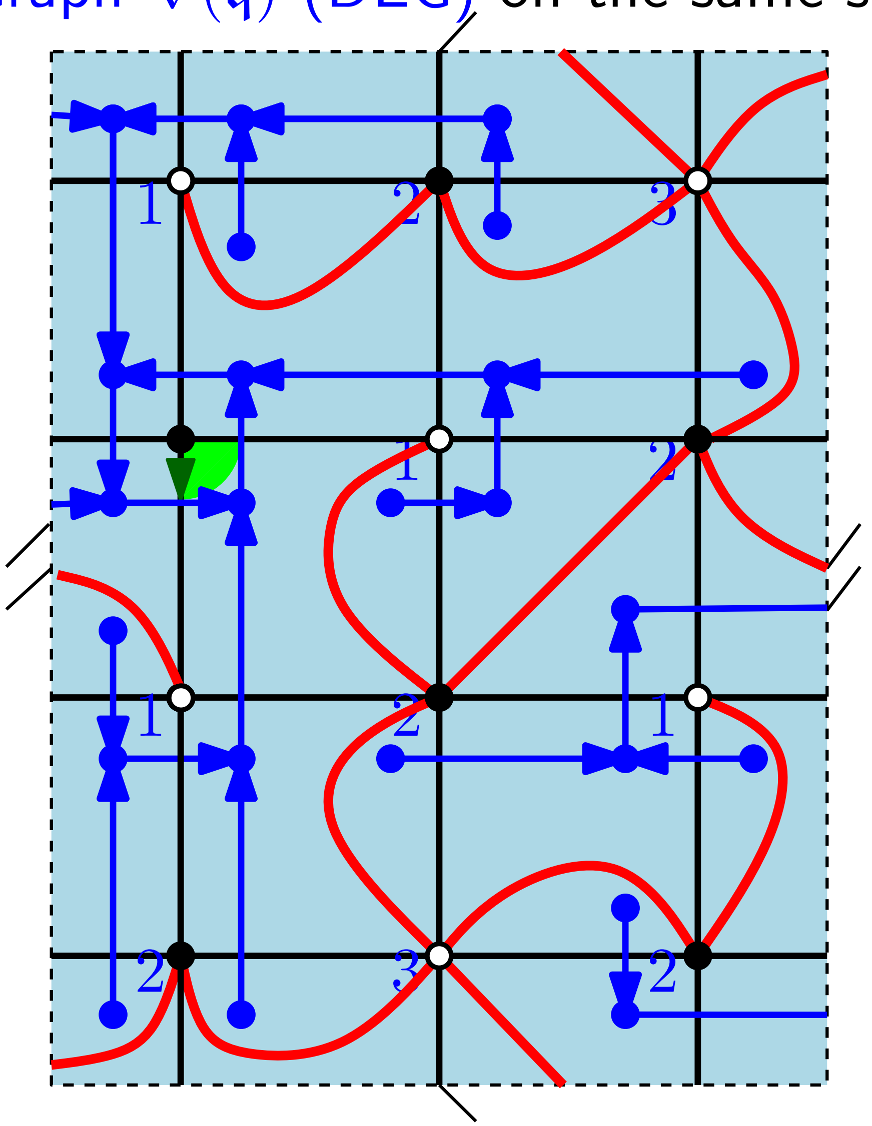

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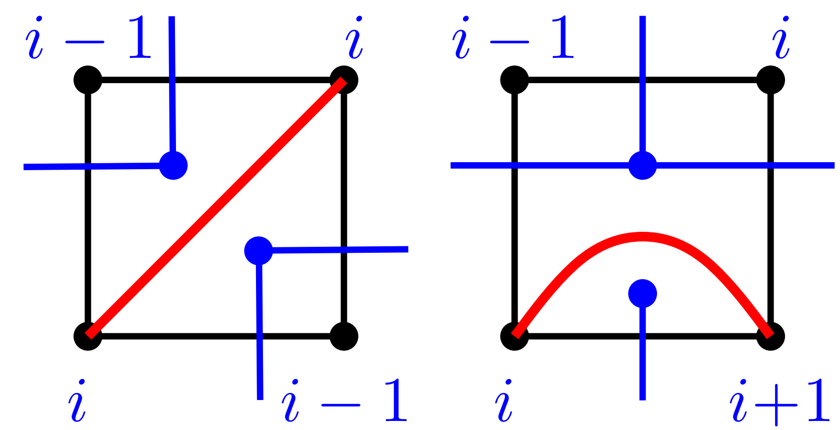

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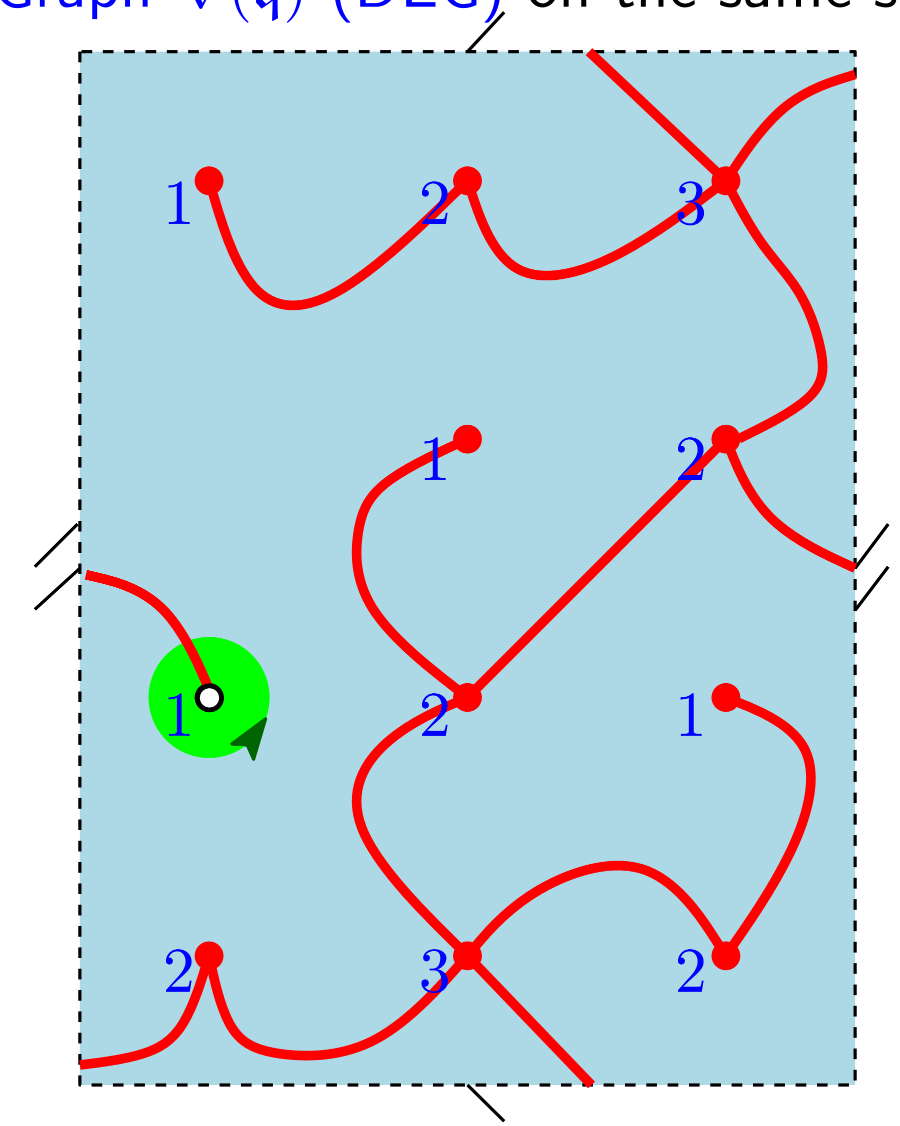

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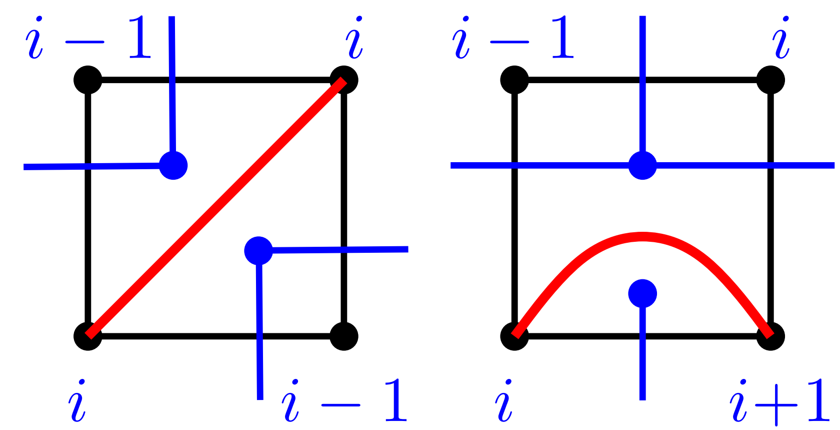

Corollary:
Red map $\phi(\mathfrak{q})$ is a one-face well-labeled rooted map with n edges, where n is the number of faces of $\mathfrak{q}$.

General case (III)

{rooted, bipartite quadrangulations on $\mathbb{S}$ with n faces and N_i vertices at distance i from the root vertex $(i \geq 1)$}

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Double rooting trick and Hall's marriage theorem!

Applications - enumeration

Theorem [Bender, Canfield 1986]
Let

$$Q_{\mathbb{S}}(t) := \sum_{n\geq 0} \vec{q}_{\mathbb{S},\bullet} t^n = \sum_{n\geq 0} (n+2-2h)\vec{q}_{\mathbb{S}}(n) t^n$$

be the generating function of rooted maps of type g pointed at a vertex or a face, by the number of edges. Moreover let $U \equiv U(t)$ and $T \equiv T(t)$ be the two formal power series defined by: $T = 1 + 3tT^2, \qquad U = tT^2(1 + U + U^2)$.
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Corollary [Bender, Canfield 1986]
When $\chi(\mathbb{S}) = 2-2g$, then there exists a constant $c(\mathbb{S})$ such that the number $m_{\mathbb{S}}(n)$ of rooted maps with n edges on $\mathbb{S}$ satisfies:
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Remark
Our main theorem allows us to recover Bender and Canfield results (that was already recovered using combinatorial methods in the orientable case [Chapuy, Marcus, Schaeffer 2009]). In particular we can give some explicit (but very complicated) formula for the constant $c(\mathbb{S})$.

Applications - random maps

Let $(\mathcal{M}, v)$ be a map with distinguished vertex v. We define:

- radius of a map $\mathcal{M}$ centered at v by the quantity
  $$R(\mathcal{M}, v) = \max_{u \in V(\mathcal{M})} d_{\mathcal{M}}(v, u);$$
- profile of distances from the distinguished point v (for any $r > 0$) by:
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Theorem [Chapuy, D. 2015]

Let q_n be uniformly distributed over the set of rooted, bipartite quadrangulations with n faces on $\mathbb{S}$, let v_0 be a root vertex of q_n and let v_* be uniformly chosen vertex of q_n. Then, there exists a continuous, stochastic process $L^{\mathbb{S}} = (L_t^{\mathbb{S}}, 0 \leq t \leq 1)$ such that:

- $\frac{9}{8n}^{1/4} R(q_n, v_*) \to \sup L^{\mathbb{S}} - \inf L^{\mathbb{S}};$
- $\frac{9}{8n}^{1/4} d_{q_n}(v_0, v_*) \to \sup L^{\mathbb{S}};$
- $\frac{I_{(q_n, v_*)}\left((8n/9)^{1/4} \cdot\right)}{n+2-2h} \to \mathcal{I}^{\mathbb{S}},$

where $\mathcal{I}^{\mathbb{S}}$ is defined as follows: for every non-negative, measurable $g : \mathbb{R}_+ \to \mathbb{R}_+$,
$$\langle \mathcal{I}^{\mathbb{S}}, g \rangle = \int_0^1 dt g(L_t^{\mathbb{S}} - \inf L^{\mathbb{S}}).$$

Further directions

- Generalization of the Bouttier-Di Francesco-Guitter bijection for non-orientable maps (bijection between bipartite $2p$-angulations, or, more generally bipartite maps with n faces of prescribed degrees and some kind of non-orientable mobiles?)

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• Generalization of the Bouttier-Di Francesco-Guitter bijection for non-orientable maps (bijection between bipartite $2p$-angulations, or, more generally bipartite maps with n faces of prescribed degrees and some kind of non-orientable mobiles?)

• Studying random maps on ANY surface in Gromov-Hausdorff topology (using our bijection and already established methods we (Bettinelli, Chapuy, D.) can prove a convergence of bipartite quadrangulations up to extraction of SUBSEQUENCE - what about full convergance)?).

III. Enumeration - different approach

Enumeration via symmetric functions (I)

Let $\mathcal{M}$ be a bipartite map with n edges.

- Degrees of white vertices gives a partition μ of n;
- Degrees of black vertices gives a partition ν of n;
- Degree of faces are even and sum up to $2n$, hence degrees of faces divided by 2 gives a partition τ of n.

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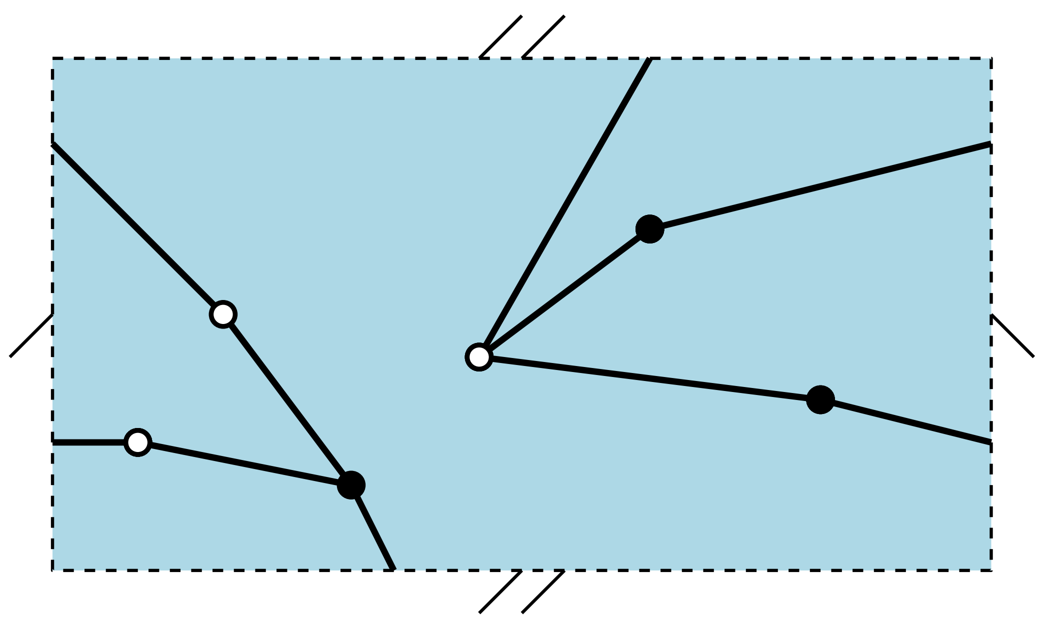

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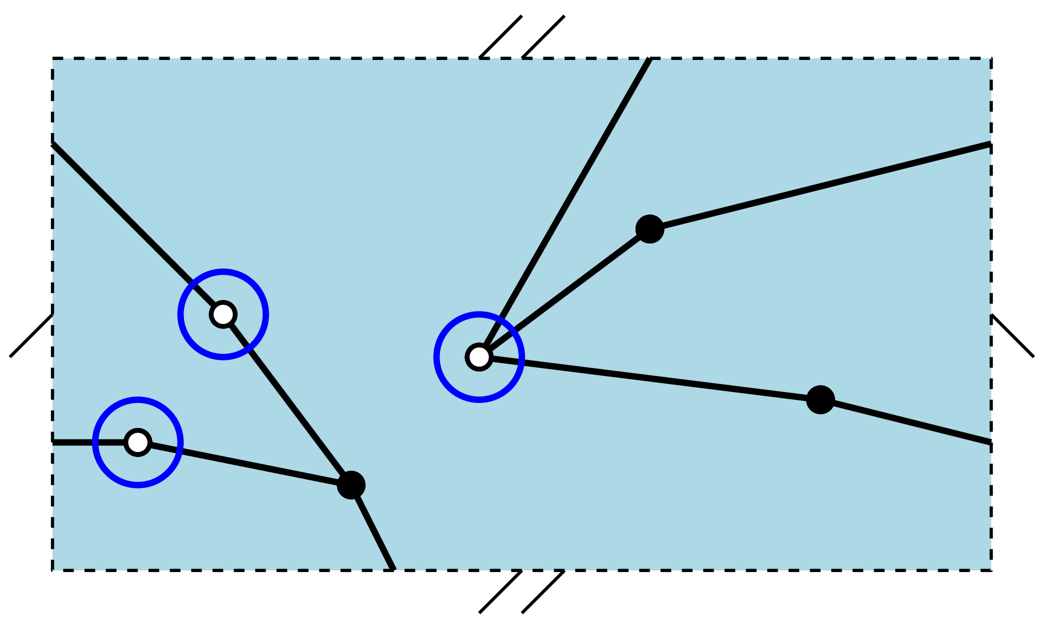

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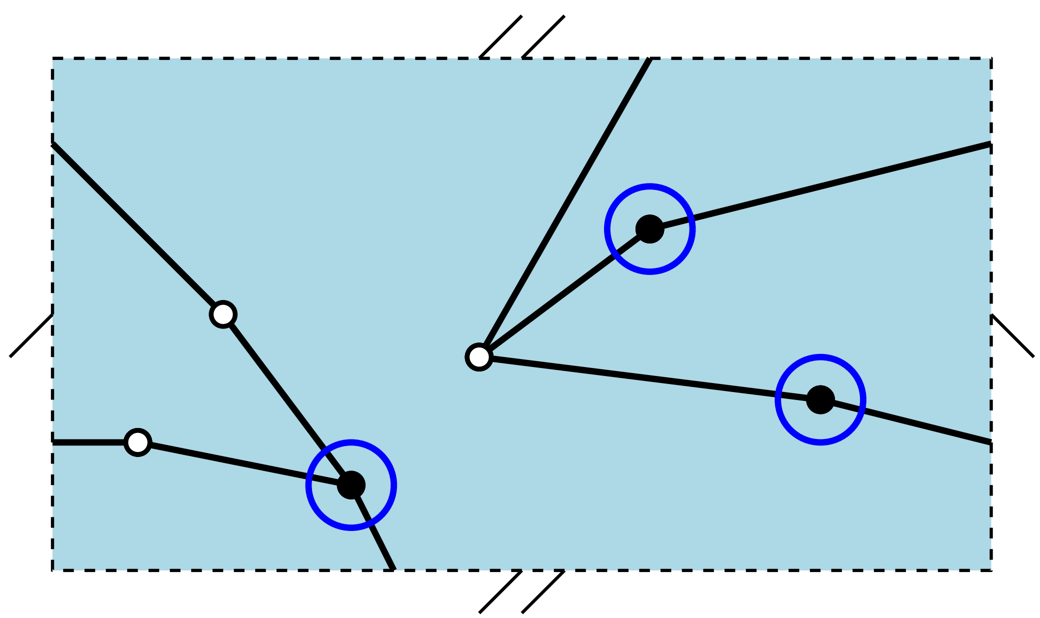

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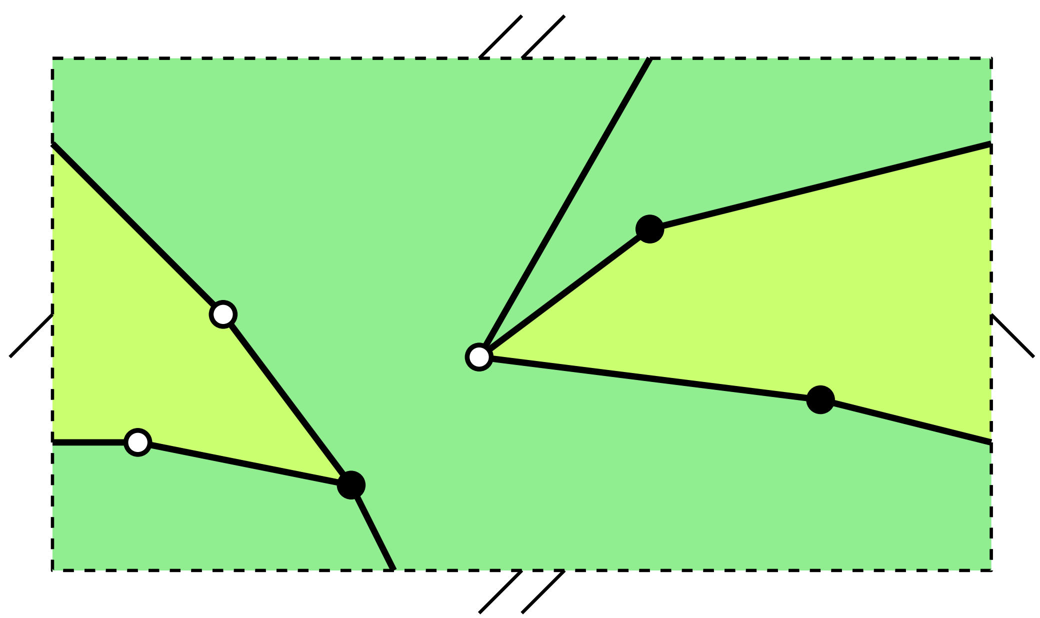

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- We define two generating functions:
  - $\phi(x, y, z) := \sum_{n \geq 1} t^n \sum_{\mu,\nu\ \tau \vdash n} \sum_{m \in \mathcal{M}_{(\mu,\nu,\tau)}} p_\mu(x) p_\nu(y) p_\tau(z);$
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- $\phi(x,y,z) = t\frac{\partial}{\partial_t}\log\left(\sum_{n\geq 0}\sum_{\lambda\vdash n} H_\lambda s_\lambda(x)s_\lambda(y)s_\lambda(z)t^n\right)$ [Jackson, Visentin 1990],
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Schur polynomials and Zonal polynomials are special cases of Jack polynomials $J_\lambda^\alpha(x)$ (for special values of α).

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Let us define

$$\psi(x,y,z,\alpha) := \alpha t \frac{\partial}{\partial_t} \log\left(\sum_{n\geq 0} \sum_{\lambda\vdash n} \frac{J_\lambda^{(\alpha)}(x) J_\lambda^{(\alpha)}(y) J_\lambda^{(\alpha)}(z)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle} t^n \right) = \sum_{n\geq 1} t^n \sum_{\mu,\nu,\tau\vdash n} h_{\mu,\nu\tau}(\beta) p_\mu(x) p_\nu(y) p_\tau(z),$$

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$$\psi(x,y,z,\alpha) := \alpha t \frac{\partial}{\partial_t} \log\left(\sum_{n\geq 0}\sum_{\lambda\vdash n} \frac{J_\lambda^{(\alpha)}(x)J_\lambda^{(\alpha)}(y)J_\lambda^{(\alpha)}(z)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle} t^n\right) = \sum_{n\geq 1} t^n \sum_{\mu,\nu,\tau\vdash n} h_{\mu,\nu\tau}(\beta) p_\mu(x) p_\nu(y) p_\tau(z),$$

where $\beta = \alpha - 1$.

- $\psi(x,y,z,0) = \phi(x,y,z)$ hence $h_{\mu,\nu,\tau}(0) = |\mathcal{M}_{(\mu,\nu,\tau)}|$;
- $\psi(x,y,z,1) = \widetilde{\phi}(x,y,z)$ hence $h_{\mu,\nu,\tau}(1) = |\widetilde{\mathcal{M}}_{(\mu,\nu,\tau)}|$;

Conjecture (β-conjecture) [Goulden, Jackson 1996]

Let $\mu,\nu,\tau \vdash n$. Then $h_{\mu,\nu,\tau}(\beta)$ is a polynomial in β with positive, integer coefficients. Moreover, there exists a statistic $\eta : \widetilde{\mathcal{M}}_{(\mu,\nu,\tau)} \to \mathbb{N}$ such that:

$$h_{\mu,\nu,\tau}(\beta) = \sum_{m\in\widetilde{\mathcal{M}}_{(\mu,\nu,\tau)}} \beta^{\eta(m)}$$

and $\eta\left(\mathcal{M}_{(\mu,\nu,\tau)}\right) = 0, \eta\left(\widetilde{\mathcal{M}}_{(\mu,\nu,\tau)} \setminus \mathcal{M}_{(\mu,\nu,\tau)}\right) > 0.$

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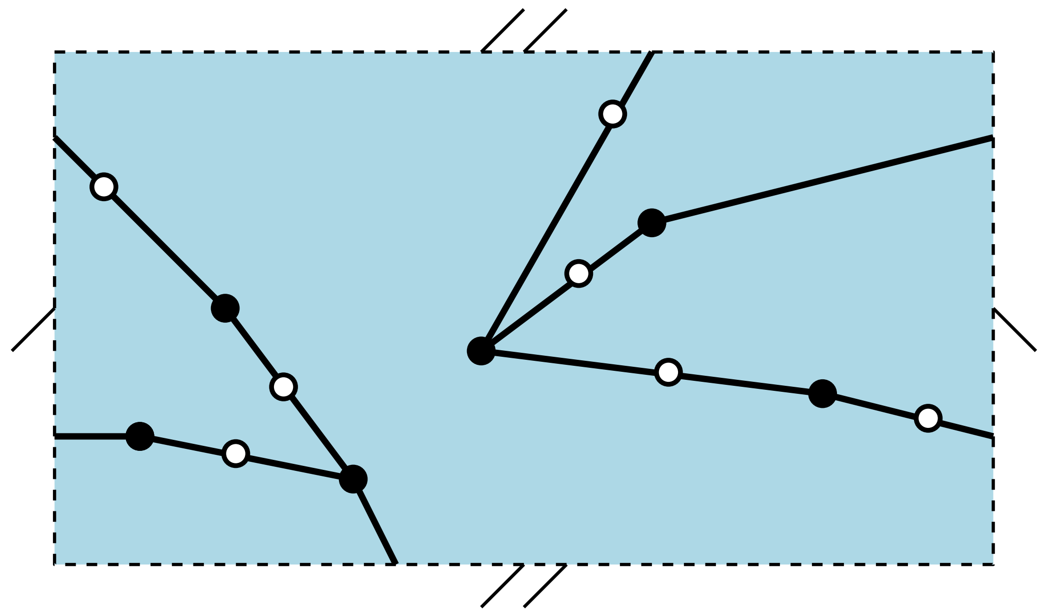

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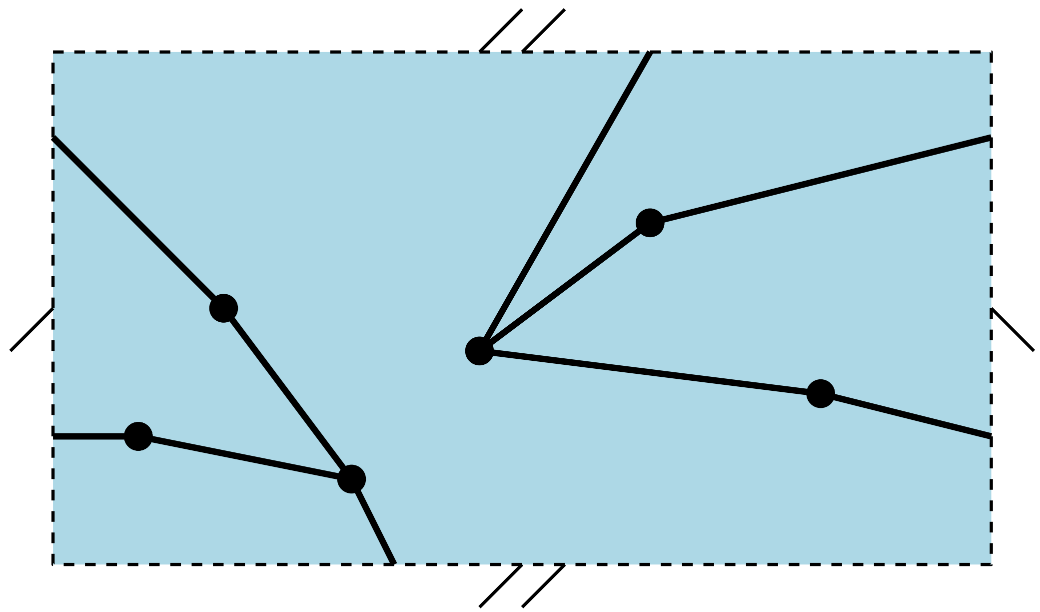

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$\sim$ set of maps with n edges, vertex distribution ν and fixed number of faces v.

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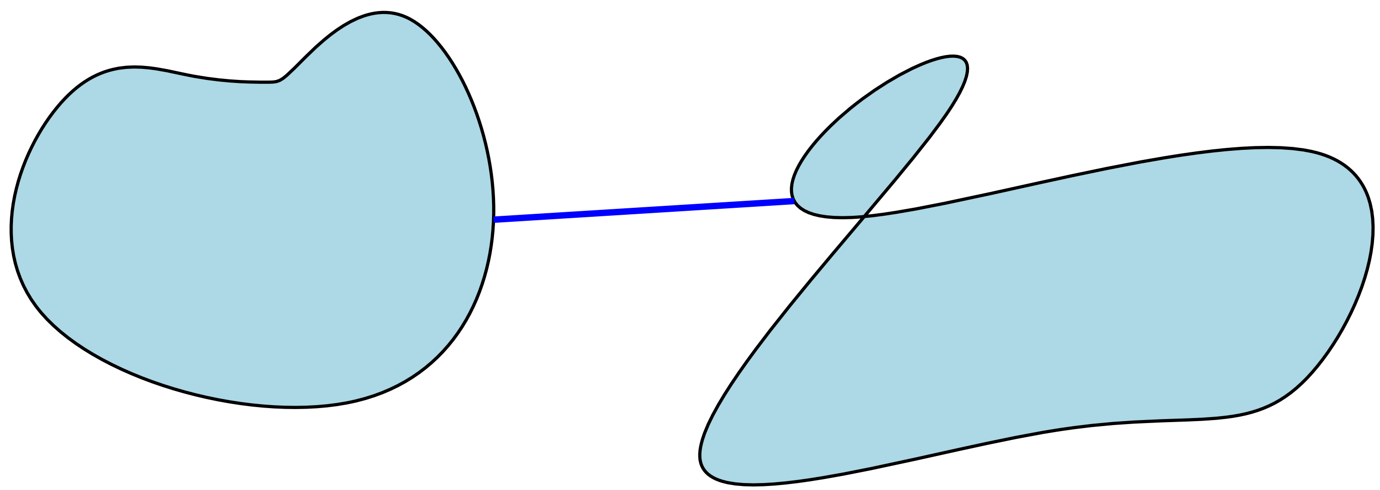

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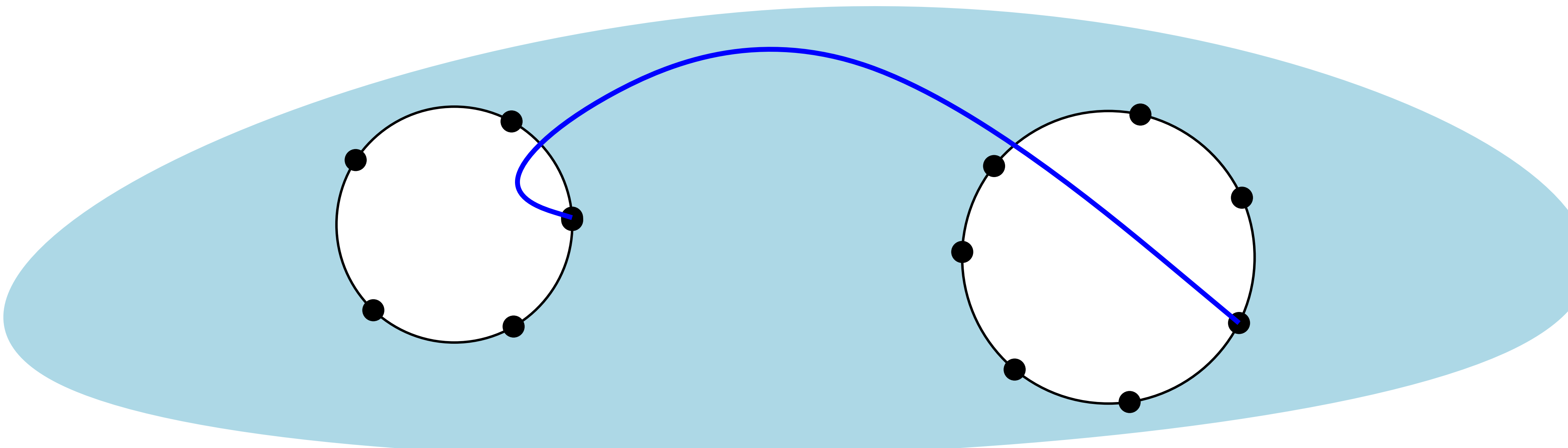

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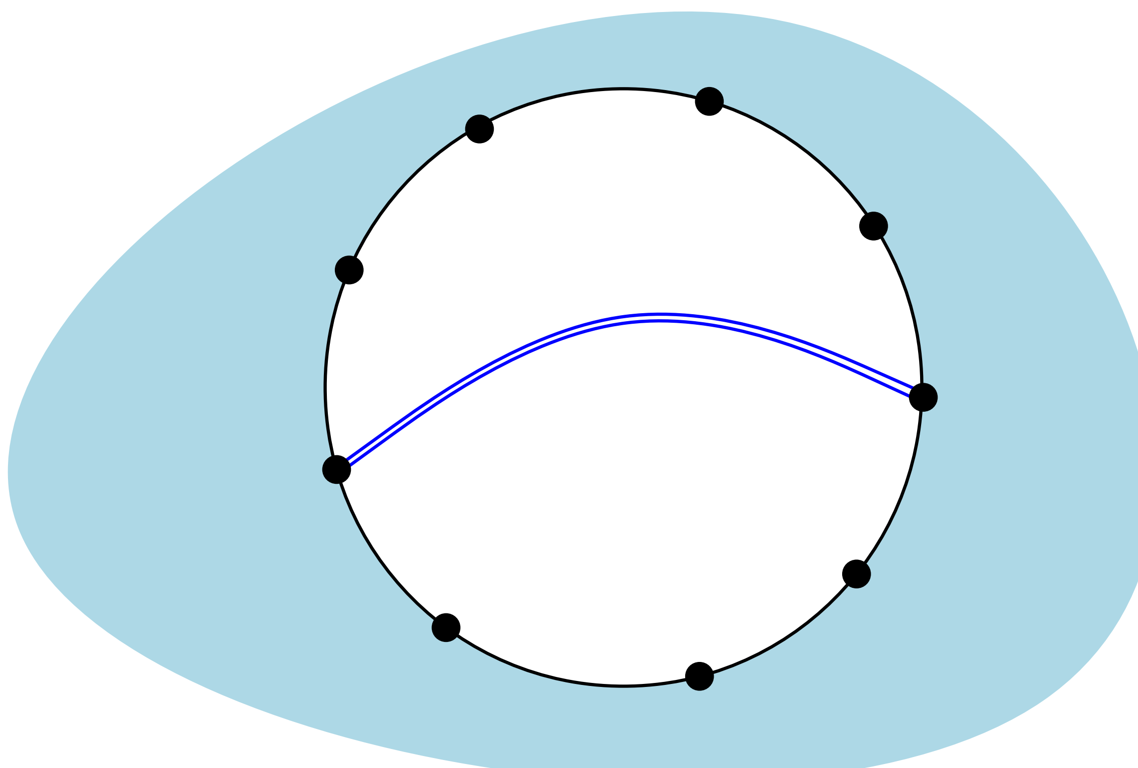

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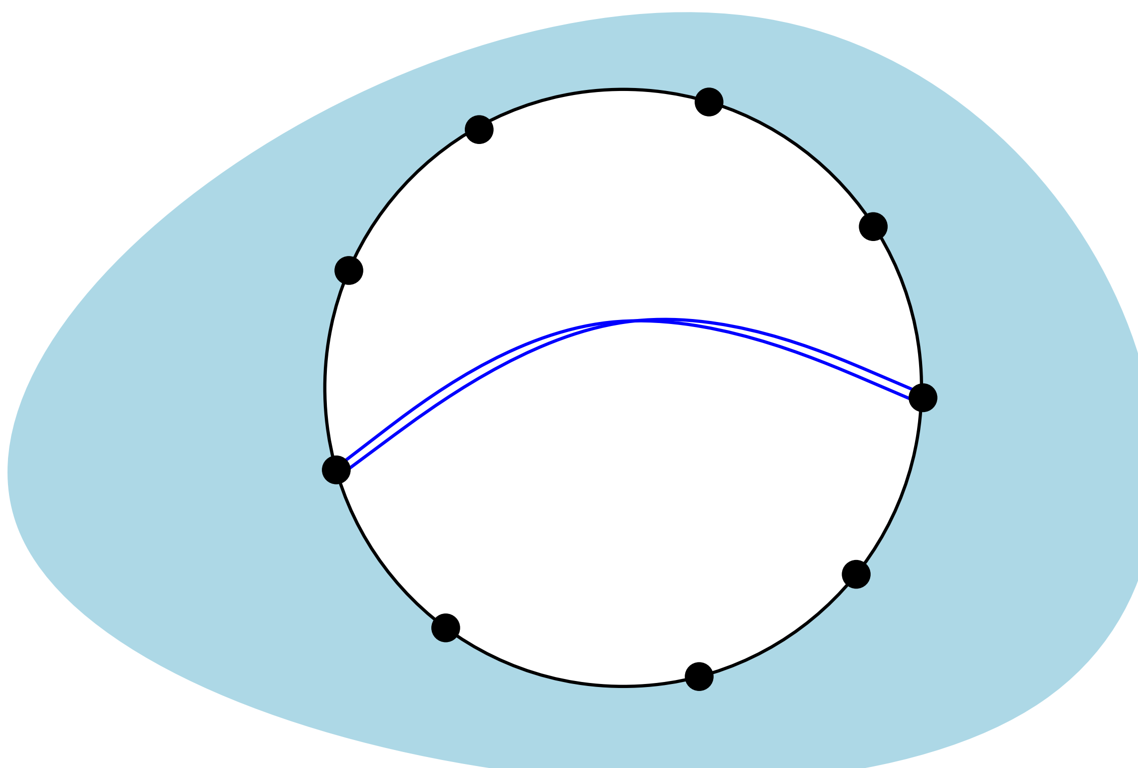

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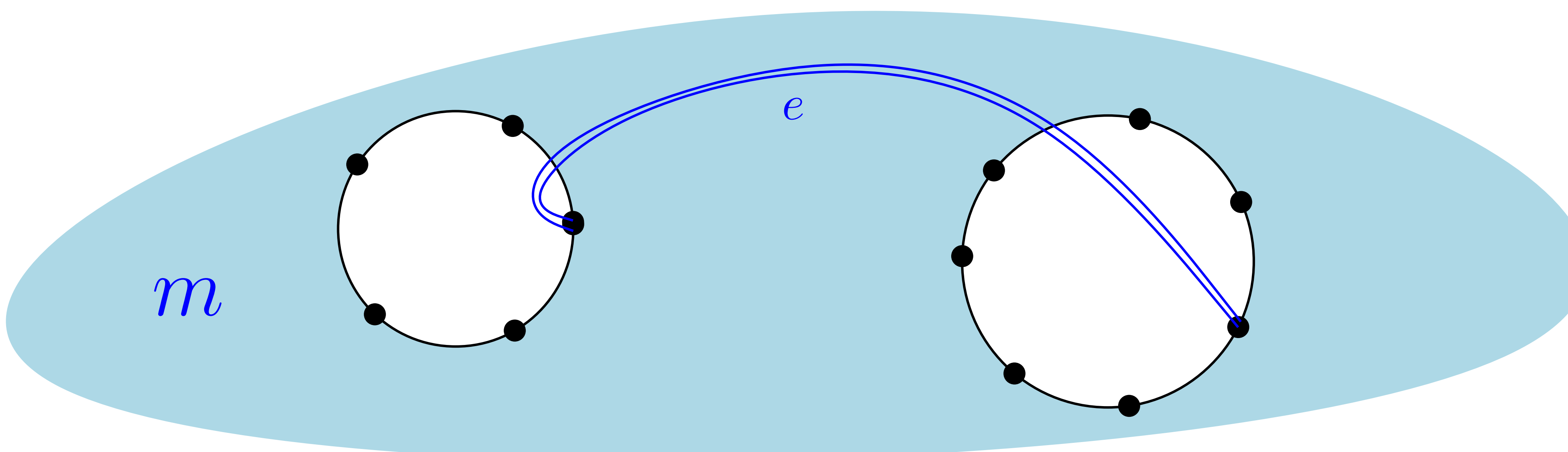

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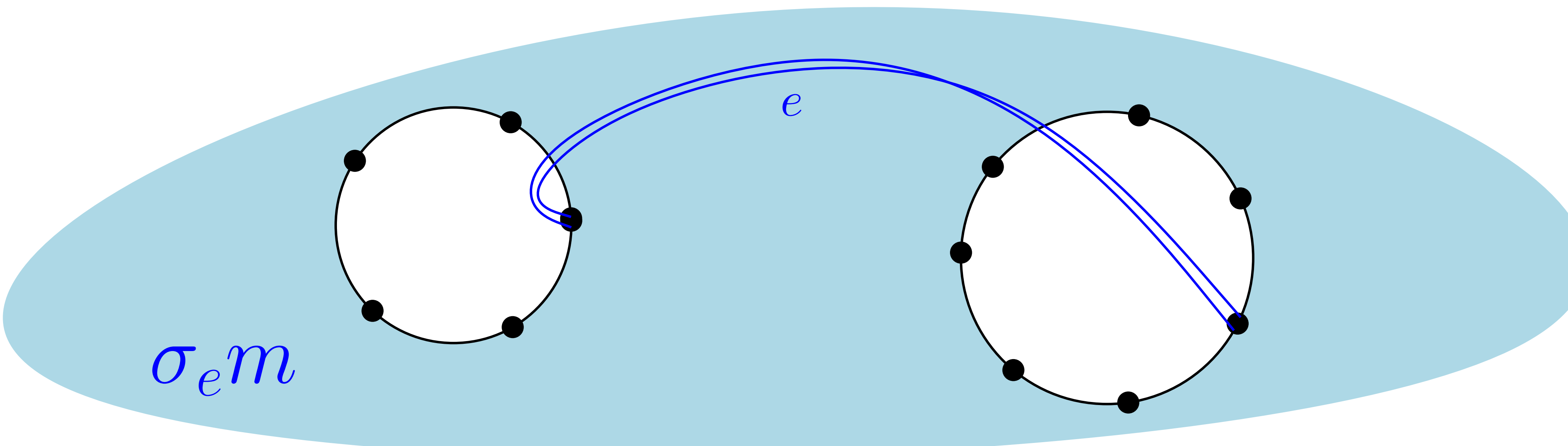

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Top-degree coeffcient of $h_{\mu,\nu,\tau}(\beta)$ is given by $(-1)^{n+2-\ell(\mu)-\ell(\nu)-\ell(\tau)} h_{\mu,\nu,\tau}(-1)$

Unhandled one-face maps

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One-face unhandled maps (II)

Lemma [D., Féray 2015]
There is a bijection between unhandled maps of type $(\mu, \nu, (n))$ and orientable maps of type (μ, ν, τ) for some $\tau \vdash n$. Moreover, for any unhandled one-face map m, an associated orientable map $f(m)$ is obtained by twisting some edges $e_1, \dots, e_l$ of m, that is $f(m)$ is of the form $\sigma_{e_l} \cdots \sigma_{e_1} m$.

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Proof:

- Induction on the number of edges n;
- m - one-face unhandled map. Its root e might be:
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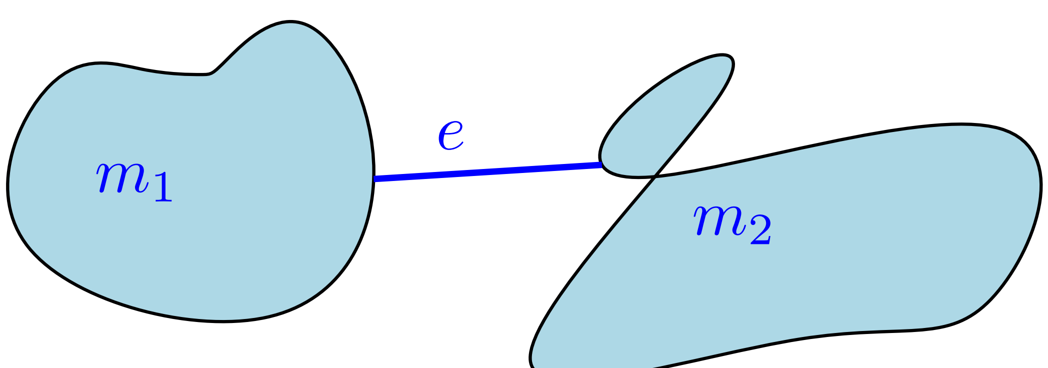
  Then $m \setminus e$ decompose into two disjoint unhandled one-face maps m_1, m_2. Let $f(m_1) = \sigma_{e_l} \cdots \sigma_{e_1} m_1$ and $f(m_2) = \sigma_{\tilde{e}_k} \cdots \sigma_{\tilde{e}_1} m_2$. Then we define $f(m) = \sigma_{e_l} \cdots \sigma_{e_1} \sigma_{\tilde{e}_k} \cdots \sigma_{\tilde{e}_1} m$.

$m =$ 

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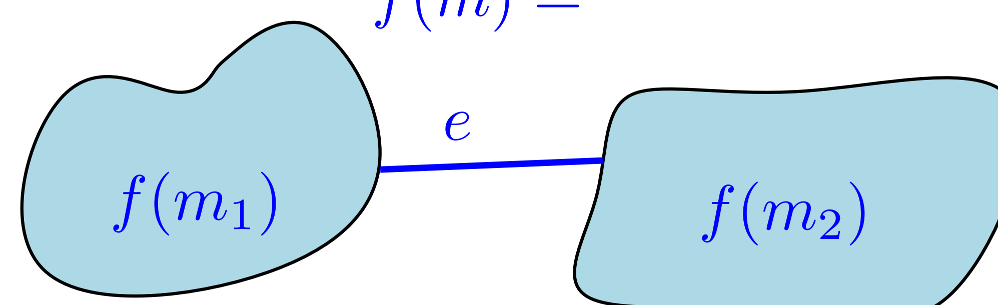

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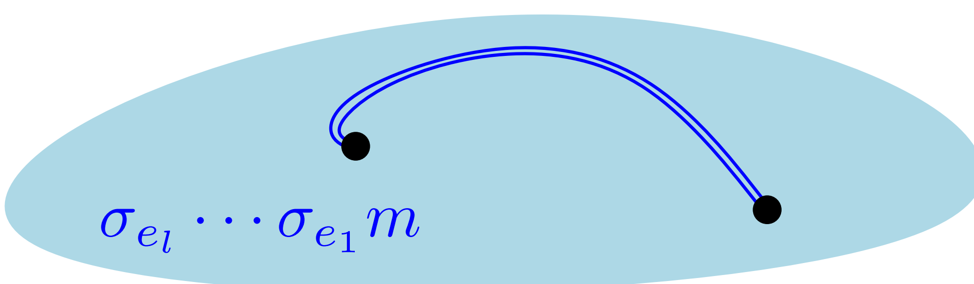

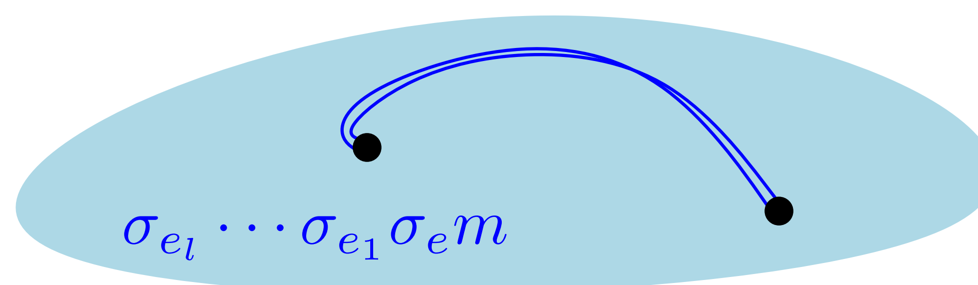

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Question:

What can we say about the class of unhandled maps with arbitrary face distribution? Are they in a bijection with some class of face-colored orientable maps? Is η introduced by La Croix is a correct invariant in general?

THANK

YOU!