Exact “combinatorial” simulation of continuous random variables

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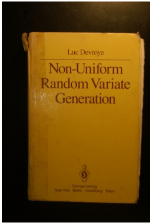
- **Exact** simulation algorithms are known for many distributions, usually assuming *exact computations over the reals*
- **The** reference : Devroye (1986)



Basic simulation tricks

- **Distribution function inversion :** if U is uniform and $F(x)$ is the (continuous, strictly increasing) distribution function ($F(x) = \mathbb{P}(X \leq x)$) for some distribution, $X = F^{-1}(U)$ has repartition function F

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- **Rejection :** if g is the density of some distribution (that one knows how to simulate), f is some other density with $f(x) \leq c.g(x)$ for some c and all x, the following rejection algorithm loops, on average, $1/c$ times, and simulates density f :
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  - draw X (g-distributed)
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- Rejection can be used when densities are only *proportional* to functions f and g with, say, $f \leq g$, without identifying/computing the multiplicative constant

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  (conditioned on X, the acceptance probability is $\exp(-(X-1)^2/2) = \exp(-x^2/2)/\exp(-x))$

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  - **flip()** (Bernoulli with parameter $1/2$; "coin flips")
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- Can we do the same for a variety of continuous distributions ? In a more or less systematic way ?

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- describes an **exact** algorithm for the exponential distribution, using only
  - independent uniforms on $[0, 1]$
  - comparisons of reals
  - (small) integer counters

The algorithm

1. Initialize counter K to 0
2. Draw a sequence $X_1, X_2, \dots X_n$ of independent uniforms on $[0, 1]$, until the **first ascent** $(X_n > X_{n-1})$
3. If n is odd : failure ; increment failure counter K, and go to 2.
4. (Otherwise) n is even : success, return $K + X_1$

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Proposition (von Neumann) : This algorithm terminates with probability 1, and its output follows the exponential distribution (density $f(x) = \exp(-x)\mathbf{1}_{x>0}$). The expected number of uniforms used is $\frac{e+e^2}{e-1} \simeq 5.88$.

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- Second attempt : $0.01 < 0.83$: even length, stop
- The output value is $1 + 0.01 = 1.01$

Proof of the algorithm (sketch)

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- For the algorithm : the final value of K follows the geometric distribution with parameter $1 - e^{-1}$, and the (independent) value of X_1 conditioned on success is distributed as an exponential, conditioned on being ≤ 1 ; the sum is exponentially distributed.

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- **[Karney, 2013]** describes such a product-less algorithm.

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- Today : description of an exact simulation method that is slightly more involved, but does not require the evaluation of any integrals or transcendental functions not in g.

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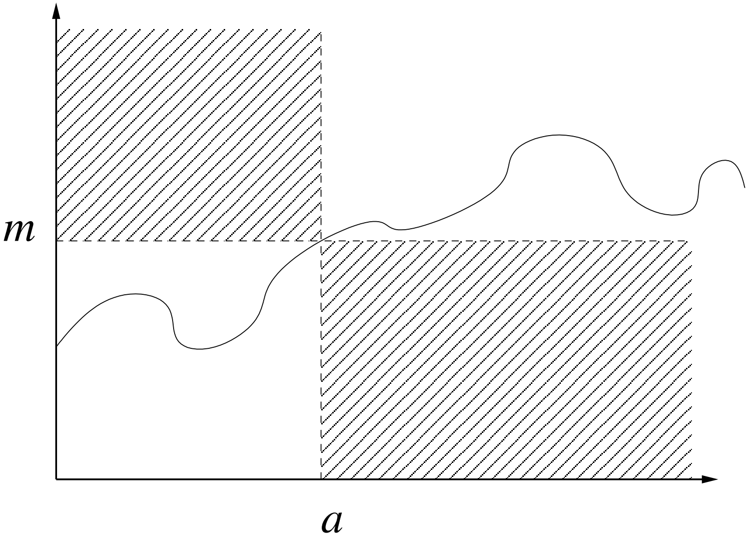
- **Then** we provide an exact simulation algorithm, using only uniform reals, additions, division by m, comparisons, and evaluations of g and h
- (Notice that the conditions reduce to g as a black box if g is known to be nondecreasing)

The "quadrant condition"



The differential equation

- The differential equation has solutions $f(t) = f(t_0)e^{-\int_{t_0}^{t} g(u)du}$; initial condition $f(t_0)$ would be determined by condition $\int_0^\infty f(t)dt = 1$ (but we will be proceeding by rejection and thus need not compute them)

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- We could try a rejection scheme : simulate an exponential E (using the von Neumann algorithm) and set $X = E/m$, then return X with appropriate probability, or restart.
- **Only**, the acceptance probability is not something we are allowed to compute :
$$\exp\left(-\int_a^X g(t)dt + m(X-a)\right) = \exp\left(-\int_a^X (g(t)-m)dt\right)$$

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- **Hypothesis :** we can draw uniforms, and have access to a Bernoulli generator with parameter p, for some *unknown* $0 < p < 1$, **Bern()** (*i.e.*, **Bern()** returns 1 with probability p and 0 with probability $1 - p$ on each call, with calls being independent)

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- **Then** we have a von Neumann-like algorithm for a Bernoulli with parameter e^{-p}

- Draw a sequence of independent **pairs** (X_i, B_i) with X_i uniform on $[0, 1]$, and B_i an independent Bernoulli with parameter p
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(proof along the same line as for von Neumann's algorithm, with powers of p addded, hence the e^{-p} instead of e^{-1})

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- Under suitable conditions, an integral can be interpreted as a probability for an easy-to-simulate event (that a random point falls into some domain)
- If needed, the integral can be written as a sum of integrals on smaller intervals (and the exponential becomes a product of exponentials ; the Bernoulli variable becomes a product of Bernoulli variables).

Picking intervals

Assume $X > a$; the case $X < a$ is treated analogously)

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- Stop at the first K such that $a_k \geq X$; instead set $a_K = X$

Rejection probability

- Now the wanted integral is

$$\int_a^X (g(t) - m)dt = \sum_{i=0}^{K} \int_{A_i} (g(t) - m)dt = \sum_i P_i$$

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- **Thus** we can apply the "exponential Buffon" construction to obtain a Bernoulli with parameter $\exp(-P_i)$
- **and in turn**, obtain the wanted Bernoulli with parameter $\exp(-\sum_i P_i)$, by taking the product (conjunction) of each individual Bernoulli for each smaller interval : this completes the algorithm.

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- (We simulate the absolute value, then add a random sign)

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- (In practice, large values of X are very likely to be rejected ; the rejection part should be run after each increment of the K counter for the exponential after $K = 1$, so as to allow early rejection)

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- In our general differential equation algorithm, we need a bit more than just a black box function g (unless g is known to be increasing)

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- No analysis of the expected (bit) complexity yet (will depend on the quality of upper bound h in the general method)
- The method is unlikely to be competitive with numerical methods (possibly paired with certified floating point calculations), **unless** one needs very high precision on their random variables

Thank you for your attention