Intervalles de Tamari généralisés et cartes planaires orientées

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Rotation operations on binary trees

The image shows diagrams of binary trees with nodes labeled A, B, and C. The left rotation is depicted by an arrow pointing to the left, and the right rotation is shown with an arrow pointing to the right. The diagrams illustrate how the tree structure changes with these rotations.

Left rotation:
- Node B is rotated to the left of node A, with node C remaining as the right child of B.

Right rotation:
- Node B is rotated to the right of node A, with node C remaining as the left child of B.

These rotations are fundamental operations in binary search trees and are used to maintain balance in self-balancing binary search trees such as AVL trees or red-black trees.
The Tamari lattice

$\mathcal{B}_n := \text{set of binary trees with } n \text{ nodes}$

The Tamari lattice $\text{Tam}_n$ is the partial order on $\mathcal{B}_n$ where the covering relation corresponds to right rotation.
Rotation ⇔ flip on triangulated dissections

cf the associahedron
Tamari intervals

An interval in a poset \((E, \leq)\) is a pair \(x, x' \in E\) such that \(x \leq x'\)

Let \(\mathcal{I}_n := \text{set of intervals in } \text{Tam}_n\)
Tamari intervals

An interval in a poset \((E, \leq)\) is a pair \(x, x' \in E\) such that \(x \leq x'\)

Let \(\mathcal{I}_n := \text{set of intervals in } \text{Tam}_n\)

**Theorem** [Chapoton’06]: \(|\mathcal{I}_n| = \frac{2}{n(n+1)} \binom{4n+1}{n-1}\)
Tamari intervals

An interval in a poset \((E, \leq)\) is a pair \(x, x' \in E\) such that \(x \leq x'\)

\[
I_n := \text{set of intervals in } \text{Tam}_n
\]

**Theorem** [Chapoton’06]: \( |I_n| = \frac{2}{n(n+1)} \left( \frac{4n+1}{n-1} \right) \)

Very active research domain over last 10 years:

- various extensions with nice counting formulas
  - \(m\)-Tamari
  - labelled \(m\)-Tamari
  - \(v\)-Tamari

- connections to algebra

- bijective links: planar maps
  - interval posets

References:

- [Bousquet-Mélou,F,Préville-Ratelle’11]
- [Bousquet-Mélou, Chapuy, Préville-Ratelle’12]
- [Préville-Ratelle, Viennot’14]
- [Bergeron, Préville-Ratelle’11]
- [Bernardi,Bonichon’07]
- [Fang, Préville-Ratelle’16]
- [Chatel,Pons’13]
The covering relation for Dyck walks

- Encoding by left-to-right postfix order (⇔ right-to-left prefix order)
The covering relation for Dyck walks

- Encoding by left-to-right postfix order (⇔ right-to-left prefix order)

- Effect of a rotation on the associated Dyck walk:
The covering relation for Dyck walks

- Encoding by left-to-right postfix order (⇔ right-to-left prefix order)

- Effect of a rotation on the associated Dyck walk:

Rk: If $\gamma \leq \gamma'$ in $\text{Tam}_n$, then $\gamma$ is below $\gamma'$
Bracket-vectors

Bracket-vector of a Dyck walk

\[ V(\gamma) = (5, 2, 4, 4, 5) \]
Bracket-vectors

Bracket-vector of a Dyck walk

\[ V(\gamma) = (5, 2, 4, 4, 5) \]

Property: \( \gamma \leq \gamma' \) in Tam\(_n\) iff \( V(\gamma) \leq V(\gamma') \)
Recursive decomposition of intervals

• Reduction of an interval $(\gamma, \gamma') \in \mathcal{I}_n$: 

\begin{align*}
\Leftrightarrow & \quad \text{size } n \quad \Leftrightarrow \quad \text{size } n-1 \quad \Leftrightarrow \\
& \quad ,
\end{align*}
Recursive decomposition of intervals

- Reduction of an interval \((\gamma, \gamma') \in \mathcal{I}_n\):

Let \(F(t, u)\) the GF, with \(t \leftrightarrow \text{size}\) and \(u \leftrightarrow \#(\text{bottom-contacts})\).

(Rk: \(|\mathcal{I}_n| = [t^n]F(t, 1)\))

Then: \[ F(t, u) = u + t \cdot u \frac{F(t, u) - F(t, 1)}{u - 1} \cdot F(t, u) \]
Recursive decomposition of intervals

- Reduction of an interval $(\gamma, \gamma') \in I_n$:

Let $F(t, u)$ the GF, with $t \leftrightarrow \text{size}$ and $u \leftrightarrow \#(\text{bottom-contacts})$

(Rk: $|I_n| = [t^n]F(t, 1)$)

Then: $F(t, u) = u + t \cdot u \frac{F(t, u) - F(t, 1)}{u - 1} \cdot F(t, u)$

Quadratic method (or guessing-checking) gives

[Chapoton’06]

[Brown, Tutte, Bousquet-Mélou Jehanne’06]
Planar maps, triangulations

Def. Planar map = connected graph embedded in the plane up to isotopy

rooted map = map + marked corner 
with the outer face on its left
Planar maps, triangulations

**Def.** Planar map = connected graph embedded in the plane up to isotopy

rooted map = map + marked corner with the outer face on its left

- Triangulation = simple planar map with all faces of degree 3

Let $\mathcal{T}_n := \text{set of rooted triangulations on } n + 3 \text{ vertices}$

[Tutte’62]: $|\mathcal{T}_n| = \frac{2}{n(n+1)} \binom{4n+1}{n-1}$ (bijective proof [Poulalhon,Schaeffer’06])
Schnyder woods

Local conditions:
Theo: Any triangulation admits a Schnyder wood

Property: the edges in each color form a tree
**Theo:** Any triangulation admits a Schnyder wood

**Property:** the edges in each color form a tree

- A Schnyder wood with no cw circuit is called **minimal**
Schnyder woods

**Theo:** Any triangulation admits a Schnyder wood

**Property:** the edges in each color form a tree

- A Schnyder wood with no cw circuit is called **minimal**

 Local conditions:

![Diagram of a Schnyder wood](image)
**Theo:** Any triangulation admits a Schnyder wood

**Property:** the edges in each color form a tree

- A Schnyder wood with no cw circuit is called **minimal**
**Theo:** Any triangulation admits a Schnyder wood

**Property:** the edges in each color form a tree

- A Schnyder wood with no cw circuit is called **minimal**

**Theo:** Any triangulation has a unique minimal Schnyder wood

(cf set of Schnyder woods on fixed triangulation is a distributive lattice)

[Schnyder’89, Ossona de Mendez’94, Brehm’03, Felsner’03]
The Bernardi-Bonichon bijection

Bijection between $\mathcal{T}_n$ and $\mathcal{I}_n$ via superfamilies

Schnyder woods on $n + 3$ vertices

non-crossing pairs of Dyck paths of lengths $2n$
The Bernardi-Bonichon bijection

Bijection between $T_n$ and $I_n$ via superfamilies

Schnyder woods on $n + 3$ vertices

non-crossing pairs of Dyck paths of lengths $2n$

\[
\begin{array}{c}
\begin{array}{c}
\text{not minimal}
\end{array}
\end{array}
\Rightarrow
\begin{array}{c}
\begin{array}{c}
\text{not minimal}
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\Rightarrow
\begin{array}{c}
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\text{minimal}
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\Rightarrow
\begin{array}{c}
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\text{minimal}
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\]

bracket-vectors

\[
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3 2 3 3
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\Rightarrow
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\Rightarrow
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2 2 4 4
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\]
The $\nu$-Tamari lattice

For $\nu$ any walk in $\{E, N\}^n$, let $\mathcal{W}_\nu := \{\text{walks above } \nu\}$.

$\nu$-Tamari lattice: poset $\text{Tam}_\nu$ on $\mathcal{W}_\nu$ for the covering relation

$p' = \text{next point after } p \text{ with same horizontal distance to } \nu$
Other realization from the canopy

\[ \gamma \]

\[ \text{north step} \quad E \text{ if followed by North step} \]

\[ N \text{ if followed by East step} \]

\[ \text{Can}(\gamma) = (E, E, N, N, E) \]
Other realization from the canopy

\[ \gamma \]

\[ \text{north step} \]

\[ E \text{ if followed by North step} \]

\[ N \text{ if followed by East step} \]

\[ \text{Can}(\gamma) = (E, E, N, N, E) \]

on binary tree
Other realization from the canopy

$\gamma$

north step

$E$ if followed by North step

$N$ if followed by East step

Can($\gamma$) = (E, E, N, N, E)

• Two types of covering relation

Hence $\gamma \leq \gamma'$ in Tam$_n$ $\Rightarrow$ Canopy($\gamma$) $\leq$ Canopy($\gamma'$) (with $N < E$)
Other realization from the canopy

[Préville-Ratelle, Viennot’16], [Fang, Préville-Ratelle’17]

covering relations commute under the bijection

\[ N^{\alpha_0} E^{\beta_0} N^{\alpha_1} E^{\beta_1} \ldots \]

\[ \gamma = E^{b_0} N E^{b_1} N \ldots \]

\[ \nu = E^{a_0} N E^{a_1} N \ldots \]

\[ \alpha_i = a_i + 1 \]

\[ \beta_i = b_i + 1 \]
Generalized Tamari intervals

\[ \mathcal{I}_\nu := \{ \gamma, \gamma' \mid \gamma \leq \gamma' \text{ in } \text{Tam}_\nu \} \]

\[ G_n := \bigcup_{\nu \in \{E,N\}^n} \mathcal{I}_\nu \]

\[ G_{i,j} := \bigcup_{\nu \in \mathcal{S}(E^i N^j)} \mathcal{I}_\nu \]

Rk: \( |G_{i,j}| = |G_{j,i}| \) from involution

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E & N & N \\
\end{array}
\]
Generalized Tamari intervals

\[ \mathcal{I}_\nu := \{\gamma, \gamma' \mid \gamma \leq \gamma' \text{ in } \text{Tam}_\nu\} \]

\[ \mathcal{G}_n := \bigcup_{\nu \in \{E,N\}^n} \mathcal{I}_\nu \]

\[ \mathcal{G}_{i,j} := \bigcup_{\nu \in \mathfrak{S}(E^iN^j)} \mathcal{I}_\nu \]

**Rk:** \[ |\mathcal{G}_{i,j}| = |\mathcal{G}_{j,i}| \] from involution

[Fang, Préville-Ratelle’17]

Let \( \mathcal{N}_n := \{\text{rooted non−sep. maps with } n+2 \text{ edges}\} \)

Let \( \mathcal{N}_{i,j} := \{\text{rooted non−sep. maps with } i+2 \text{ vertices and } j+2 \text{ faces}\} \)

Then \( \mathcal{G}_n \leftrightarrow \mathcal{N}_n \) and more precisely \( \mathcal{G}_{i,j} \leftrightarrow \mathcal{N}_{i,j} \)
Generalized Tamari intervals

\[ \mathcal{I}_\nu := \{ \gamma, \gamma' \mid \gamma \leq \gamma' \text{ in } \operatorname{Tam}_\nu \} \]

\[ \mathcal{G}_n := \bigcup_{\nu \in \{ E, N \}^n} \mathcal{I}_\nu \]

\[ \mathcal{G}_{i,j} := \bigcup_{\nu \in \mathcal{G}(E^i N^j)} \mathcal{I}_\nu \]

Rk: \( |\mathcal{G}_{i,j}| = |\mathcal{G}_{j,i}| \) from involution

[Fang, Préville-Ratelle’17]

Let \( \mathcal{N}_n := \{ \text{rooted non-sep. maps with } n+2 \text{ edges} \} \)

Let \( \mathcal{N}_{i,j} := \{ \text{rooted non-sep. maps with } i+2 \text{ vertices and } j+2 \text{ faces} \} \)

Then \( \mathcal{G}_n \leftrightarrow \mathcal{N}_n \) and more precisely \( \mathcal{G}_{i,j} \leftrightarrow \mathcal{N}_{i,j} \)

\[ |\mathcal{G}_n| = |\mathcal{N}_n| = \frac{2(3n+3)!}{(n+2)!(2n+3)!} \]

\[ |\mathcal{G}_{i,j}| = |\mathcal{N}_{i,j}| = \frac{(2i+j+1)!(2j+i+1)!}{(i+1)!(j+1)!(2i+1)!(2j+1)!} \]

[Tutte’63]

[Brown-Tutte’64]
Generalized Tamari intervals

\[ I_\nu := \{ \gamma, \gamma' \mid \gamma \leq \gamma' \text{ in } \text{Tam}_\nu \} \]

\[ G_n := \bigcup_{\nu \in \{E,N\}^n} I_\nu \]

\[ G_{i,j} := \bigcup_{\nu \in S(E^iN^j)} I_\nu \]

\[ |G_n| = |N_n| = \frac{2(3n+3)!}{(n+2)!(2n+3)!} \]

\[ |G_{i,j}| = |N_{i,j}| = \frac{(2i+j+1)!(2j+i+1)!}{(i+1)!(j+1)!(2i+1)!(2j+1)!} \]

\[ Rk: |G_{i,j}| = |G_{j,i}| \text{ from involution} \]

[Fang, Préville-Ratelle’17] 

Let \(N_n := \{\text{rooted non-sep. maps with } n+2 \text{ edges}\}\)

Let \(N_{i,j} := \{\text{rooted non-sep. maps with } i+2 \text{ vertices and } j+2 \text{ faces}\}\)

Then \(G_n \leftrightarrow N_n\) and more precisely \(G_{i,j} \leftrightarrow N_{i,j}\)

\[ Q_{i,j} := \{\text{rooted simple quadrang. } i+2 \text{ vertices } j+2 \text{ faces}\}\]
2 Superfamilies for generalized Tamari intervals

- If $\gamma \leq \gamma'$ in $\text{Tam}_\nu$, then $\gamma$ is below $\gamma'$ (and above $\nu$)

$\Rightarrow G_{i,j} \subseteq R_{i,j}$ with $R_{i,j} := \{\text{non-crossing triples from } (0,0) \text{ to } (i,j)\}$

A triple in $R_{7,5}$
2 Superfamilies for generalized Tamari intervals

• If \( \gamma \leq \gamma' \) in \( \text{Tam}_\nu \) then \( \gamma \) is below \( \gamma' \) (and above \( \nu \))

\[
\begin{align*}
\Rightarrow \quad G_{i,j} & \subseteq R_{i,j} \\
\text{with } R_{i,j} & := \{ \text{non-crossing triples from } (0,0) \text{ to } (i,j) \}
\end{align*}
\]

• An interval \((\gamma, \gamma') \in I_n\) is called synchronized if \(\text{Can}(\gamma) = \text{Can}(\gamma')\)

Let \(S_n := \text{subfamily of synchronized intervals from } I_n\).

Then \(G_n \cong S_n \subseteq I_n\) (Rk: on the other hand \( I_n \subseteq G_{2n} \))
2 Superfamilies for generalized Tamari intervals

- If $\gamma \preceq \gamma'$ in $\text{Tam}_\nu$, then $\gamma$ is below $\gamma'$ (and above $\nu$).

$$\Rightarrow G_{i,j} \subseteq R_{i,j}$$

with $R_{i,j} := \{\text{non-crossing triples from } (0,0) \text{ to } (i,j)\}$

- An interval $(\gamma, \gamma') \in I_n$ is called **synchronized** if $\text{Can}(\gamma) = \text{Can}(\gamma')$.

Let $S_n := \text{subfamily of synchronized intervals from } I_n$.

Then $G_n \simeq S_n \subseteq I_n$ (Rk: on the other hand $I_n \subset G_{2n}$)

Let $S_{i,j} := \text{subfamily of synchronized intervals from } I_{i+j-1}$.

where common canopy word is in $G(E^i N^j)$

Then $G_{i,j} \simeq S_{i,j} \subseteq I_{i+j-1}$
A **Baxter family** is a family $\mathcal{B}_{i,j}$ indexed by two parameters $i, j$ such that

$$|\mathcal{B}_{i,j}| = 2 \frac{(i + j)! (i + j + 1)! (i + j + 2)!}{i! (i + 1)! (i + 2)! j! (j + 1)! (j + 2)!}.$$
Bijection via separating decompositions

Local conditions:

\[ \notin \{s, t\} \]

\[ s \]

\[ t \]

\[ Sep_{i,j} := \text{set of separating decompositions with } i+2 \text{ vertices, } j+2 \text{ faces} \]
**Bijection via separating decompositions**

**Local conditions:**

\[ s \notin \{s, t\} \]

\[ t \]

\[ s' \]

\[ t' \]

**Theorem:** [de Fraysseix et al. 95]

Any simple quadrangulation admits a separating decomposition.

It has a unique one that is **minimal** (no cw cycle).

**Property:** edges in each color form a tree.

**Sep}_{i,j} := set of separating decompositions with \( i+2 \) vertices, \( j+2 \) faces
Bijection via separating decompositions

close to bijection in
[F, Poulalhon, Schaeffer’09]
Bijection via separating decompositions

close to bijection in [F, Poulalhon, Schaeffer’09]

The mapping is a bijection between $\text{Sep}_{i,j}$ and $\mathcal{R}_{i,j}$
A separating decomposition is minimal iff its image is in $\mathcal{G}_{i,j}$
$\Rightarrow$ specialization into a bijection from $\mathcal{Q}_{i,j}$ to $\mathcal{G}_{i,j}$

[F, Humbert’19]
Bernardi-Bonichon bijection preserves minimality

\[ \Rightarrow \text{Bernardi-Bonichon bijection} \simeq \text{case where white vertices have blue indegree 1} \]

\[ (\text{bottom-walk} = (NE)^n) \]
Link to the Bernardi-Bonichon bijection

mapping preserves minimality

\[ \Rightarrow \text{Bernardi-Bonichon bijection} \simeq \text{case where white vertices have blue indegree 1} \]

(bottom-walk = \((NE)^n\))

More generally,

\[ (m\text{-Tamari intervals}) \quad \nu = (NE^m)^n \quad \text{minimal separating decompositions} \]

where white vertices have blue indegree \(m\)

Not yet a bijective interpretation of the formula

\[ I_n^{(m)} = \frac{m + 1}{m(nm + 1)} \left( \frac{(m + 1)^2n + m}{n - 1} \right) \]
Symmetric reformulation of the bijection

2-book embedding of a separating decomposition

[Felsner et al.'07]
Symmetric reformulation of the bijection

**Corollary:** bijection commutes with half-turn rotation

⇒ stability of $G_{i,j} \subset R_{i,j}$ under half-turn rotation
Proof of the bijection $\text{Sep}_{i,j} \leftrightarrow \mathcal{R}_{i,j}$
Proof of the bijection $\text{Sep}_{i,j} \leftrightarrow \mathcal{R}_{i,j}$
Non-minimality on arc-diagrams

non-minimal (i.e., $\exists$ clockwise cycle)

$\Downarrow$

$\exists$ clockwise 4-cycle

2-book embedding

arc-diagram
Non-minimality on arc-diagrams

non-minimal (i.e., $\exists$ clockwise cycle)

$\Downarrow$

$\exists$ clockwise 4-cycle

2-book embedding

arc-diagram

Remains to see that for $R \in \mathcal{R}_{i,j}$

$R$ is in $\mathcal{G}_{i,j}$ iff arc-diagram of $R$ has no
Bracket-vectors in the $\nu$-Tamari lattice

[Ceballos, Padrol, Sarmiento’18]

Property: $\gamma \leq \gamma'$ in $\text{Tam}_\nu$ iff $V_\nu(\gamma) \leq V_\nu(\gamma')$

$V_\nu(\gamma) = (2, 0, 2, 2, 4)$
Condition for $R \in \mathcal{R}_{i,j}$ to be in $\mathcal{G}_{i,j}$

$(\nu, \gamma, \gamma') \in \mathcal{R}_{7,5}$
Condition for $R \in \mathcal{R}_{i,j}$ to be in $\mathcal{G}_{i,j}$

$$(\nu, \gamma, \gamma') \in \mathcal{R}_{7,5}$$
Condition for \( R \in \mathcal{R}_{i,j} \) to be in \( \mathcal{G}_{i,j} \)

\[(\nu, \gamma, \gamma') \in \mathcal{R}_{7,5}\]

\[V_\nu(\gamma) \leq V_\nu(\gamma') \iff \text{no}\]
Condition for $R \in \mathcal{R}_{i,j}$ to be in $\mathcal{G}_{i,j}$

$$V_\nu(\gamma) \leq V_\nu(\gamma') \iff \text{no}$$

$$(\nu, \gamma, \gamma') \in \mathcal{R}_{7,5}$$
Other approach via the canopy

Recall that if $\gamma \leq \gamma'$ in Tam$_n$ then $\text{Can}(\gamma) \leq \text{Can}(\gamma')$ (with $N < E$)

- Let $F(x, y, z) := \text{series of Tamari intervals, with } x \#_{[E]} y \#_{[N]} z \#_{[E]}

\[ F(x, y, z) = 1 + (x + y + z) + (x^2 + y^2 + z^2 + 3xz + 3yz + 4xy) + x^3 + y^3 + z^3 + 6x^2z + 6xz^2 + 10x^2y + 10xy^2 + 6y^2z + 6yz^2 + 21xyz \]

Rk: symmetry $x \leftrightarrow y$ cf

\[ tF(t, t, t) = \sum_{n \geq 1} |\mathcal{I}_n| t^n \]

\[ F(x, y, 0) = \sum_{i, j} |\mathcal{G}_{i,j}| x^i y^j \]
3 parameters via Bernardi-Bonichon bijection

canopy-parameters via the bijection:

\[
\begin{align*}
& (E_N) & \quad \rightarrow \quad (N) \\
& (N) & \quad \rightarrow \quad (E)
\end{align*}
\]
Composition to bijection with tree-structures

[F, Poulalhon, Schaeffer’07]

3-mobile
Composition to bijection with tree-structures

[F, Poulalhon, Schaeffer’07]

3-mobile

canopy-parameters via the bijection:
Results

• Trivariate generating function expression:

\[
F = xR + yG + zRG - \frac{RG}{(1 + R)(1 + G)}
\]

where

\[
\begin{align*}
R &= (y + zR)(1 + R)(1 + G)^2 \\
G &= (x + zG)(1 + G)(1 + R)^2
\end{align*}
\]

• Simplification of the trees in the synchronized case:

known to be in bijection to quadrangulations

[Schaeffer'98, Bernardi, F'10]
New Tamari intervals and canopy symmetry

• An interval \((\gamma, \gamma') \in I_n\) is called new if (with dissection point of view) \(\gamma\) and \(\gamma'\) have no common chord.

\[\gamma \quad , \quad \gamma'\]

an interval \(\gamma \leq \gamma'\) that is not new.
New Tamari intervals and canopy symmetry

- An interval \((\gamma, \gamma') \in I_n\) is called **new** if (with dissection point of view) \(\gamma\) and \(\gamma'\) have **no common chord**

  \[ \gamma \quad \text{and} \quad \gamma' \]

- An interval \(\gamma \leq \gamma'\) that is not new

  \[ \text{an interval } \gamma \leq \gamma' \text{ that is not new} \]

- Let \(G(x, y, z) := \text{series } F(x, y, z)\) restricted to **new** Tamari intervals

  \[
  \frac{1}{z} G(x, y, z) = 1 + (x + y + z) + (x^2 + y^2 + z^2 + 3xy + 3xz + 3yz) \\
  + (x^3 + y^3 + z^3 + 6x^2y + 6xy^2 + 6x^2z + 6xz^2 + 6y^2z + 6yz^2 + 17xyz) + \cdots
  \]

  symmetry in the 3 variables!
New Tamari intervals and canopy symmetry

- An interval \((\gamma, \gamma') \in \mathcal{I}_n\) is called **new** if (with dissection point of view) \(\gamma\) and \(\gamma'\) have **no common chord**

- Let \(G(x, y, z) := \text{series } F(x, y, z)\) restricted to **new** Tamari intervals

\[
\frac{1}{z} G(x, y, z) = 1 + (x + y + z) + (x^2 + y^2 + z^2 + 3xy + 3xz + 3yz) + (x^3 + y^3 + z^3 + 6x^2y + 6xy^2 + 6x^2z + 6xz^2 + 6y^2z + 6yz^2 + 17xyz) + \cdots
\]

• **Symmetry in the 3 variables!**

- Bijective explanation via bipartite maps! [Fang’19+]