Triangulations with spins: algebraicity and local limit

Laurent Ménard (Paris Nanterre)
joint work with Marie Albenque and Gilles Schaeffer (CNRS and LIX)

Séminaire Philippe Flajolet, septembre 2017
1. Motivation (is Watabiki right?)
2. Local weak topology
3. Combinatorics of triangulations with spins
4. Local limit of triangulations with spins
Planar Maps as discrete planar metric spaces

Definition:
A planar map is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).
Planar Maps as discrete planar metric spaces

**Definition:**
A **planar map** is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).
**Planar Maps as discrete planar metric spaces**

**Definition:**
A **planar map** is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).

- **faces:** connected components of the complement of edges
- **p-angulation:** each face is bounded by \( p \) edges
Planar Maps as discrete planar metric spaces

**Definition:**
A **planar map** is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).

**faces:** connected components of the complement of edges

**$p$-angulation:** each face is bounded by $p$ edges
Planar Maps as discrete planar metric spaces

**Definition:**
A **planar map** is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).

**faces:** connected components of the complement of edges

**p-angulation:** each face is bounded by $p$ edges
Definition:
A planar map is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).

**faces:** connected components of the complement of edges

**p-angulation:** each face is bounded by $p$ edges

This is a triangulation
**Planar Maps as discrete planar metric spaces**

**Definition:**
A **planar map** is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).

**M Planar Map:**
- \(V(M) := \text{set of vertices of } M\)
- \(d_{gr} := \text{graph distance on } V(M)\)
- \((V(M), d_{gr}) \text{ is a (finite) metric space}\)
Planar Maps as discrete planar metric spaces

Definition:
A **planar map** is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).

- **M** Planar Map:
  - \( V(M) := \) set of vertices of \( M \)
  - \( d_{gr} := \) graph distance on \( V(M) \)
  - \((V(M), d_{gr})\) is a (finite) metric space

**Rooted** map: mark an oriented edge of the map
"Classical" large random triangulations

Take a triangulation with $n$ edges uniformly at random. What does it look like if $n$ is large?

Two points of view: global/local, continuous/discrete
"Classical" large random triangulations

Take a triangulation with $n$ edges uniformly at random. What does it look like if $n$ is large?

Two points of view: global/local, continuous/discrete

Global:
Rescale distances to keep diameter bounded

[Le Gall 13, Miermont 13]: converges to the Brownian map.

- Gromov-Hausdorff topology
- Continuous metric space
- Homeomorphic to the sphere
- Hausdorff dimension 4
- Universality
"Classical" large random triangulations

Take a triangulation with \( n \) edges uniformly at random. What does it look like if \( n \) is large?

Two points of view: global/local, continuous/discrete

**Local:**
Don’t rescale distances and look at neighborhoods of the root
"Classical" large random triangulations

Take a triangulation with $n$ edges uniformly at random. What does it look like if $n$ is large?

Two points of view: global/local, continuous/discrete

Local:
Don’t rescale distances and look at neighborhoods of the root

[Angel – Schramm 03, Krikun 05]: Converges to the Uniform Infinite Planar Triangulation

- Local topology
- Metric balls of radius $R$ grow like $R^4$
- "Universality" of the exponent 4.
Adding matter: Ising model on triangulations

How does Ising model influence the underlying map?
Adding matter: Ising model on triangulations

How does Ising model influence the underlying map?

First, Ising model on a finite deterministic graph:

\[ G = (V, E) \text{ finite graph} \]

Spin configuration on \( G \):
\[ \sigma : V \rightarrow \{-1, +1\} \]
Adding matter: Ising model on triangulations

How does Ising model influence the underlying map?

First, Ising model on a finite deterministic graph:

$G = (V, E)$ finite graph

Spin configuration on $G$:

$\sigma : V \rightarrow \{-1, +1\}$.  

Ising model on $G$: take a random spin configuration with probability

$$P(\sigma) \propto e^{-\frac{\beta}{2} \sum_{v \sim v'} 1_{\{\sigma(v) \neq \sigma(v')\}}}$$

$\beta > 0$: inverse temperature.
Adding matter: Ising model on triangulations

How does Ising model influence the underlying map?

First, Ising model on a finite deterministic graph:

\[ G = (V, E) \text{ finite graph} \]

Spin configuration on \( G \):

\[ \sigma : V \rightarrow \{-1, +1\}. \]

Ising model on \( G \): take a random spin configuration with probability

\[ P(\sigma) \propto e^{-\frac{\beta}{2} \sum_{v \sim v'} 1_{\{\sigma(v) \neq \sigma(v')\}}} \]

\( \beta > 0 \): inverse temperature.

Combinatorial formulation: \( P(\sigma) \propto \nu^{m(\sigma)} \)

with \( m(\sigma) = \) number of monochromatic edges and \( \nu = e^{\beta} \).
Adding matter: Ising model on triangulations

\[ \mathcal{T}_n = \{ \text{rooted planar triangulations with } 3n \text{ edges} \}. \]

Random triangulation in \( \mathcal{T}_n \) with probability \( \propto \nu^m(T,\sigma) \)?
Adding matter: Ising model on triangulations

\( \mathcal{T}_n = \{ \text{rooted planar triangulations with } 3n \text{ edges} \} \).

Random triangulation in \( \mathcal{T}_n \) with probability \( \propto \nu^m(T,\sigma) \)?

Generating series of **Ising-weighted triangulations**:

\[
Q(\nu, t) = \sum_{T \in \mathcal{T}_f} \sum_{\sigma : V(T) \to \{-1, +1\}} \nu^m(T,\sigma) t^e(T).
\]
Adding matter: Ising model on triangulations

\( \mathcal{T}_n = \{\text{rooted planar triangulations with } 3n \text{ edges}\} \).

Random triangulation in \( \mathcal{T}_n \) with probability \( \propto \nu^m(T,\sigma) \)?

Generating series of Ising-weighted triangulations:

\[
Q(\nu, t) = \sum_{T \in \mathcal{T}_f} \sum_{\sigma: V(T) \to \{-1, +1\}} \nu^m(T,\sigma) t^e(T).
\]

**Theorem** [Bernardi – Bousquet-Mélou 11]

For every \( \nu \) the series \( Q(\nu, t) \) is algebraic, has \( \rho_\nu > 0 \) as unique dominant singularity and satisfies

\[
[t^{3n}] Q(\nu, t) \sim_{n \to \infty} \begin{cases} 
\kappa \rho_{\nu c} n^{-7/3} & \text{if } \nu = \nu_c = 1 + \frac{1}{\sqrt{7}}, \\
\kappa \rho_\nu n^{-5/2} & \text{if } \nu \neq \nu_c.
\end{cases}
\]

This suggests an unusual behavior of the underlying maps for \( \nu = \nu_c \).

See also [Boulatov – Kazakov 1987], [Bousquet-Mélou – Schaeffer 03] and [Bouttier – Di Francesco – Guitter 04].
Adding matter: Watabiki’s (controversial?) predictions

Counting exponent:
\[ \text{coeff } [t^n] \text{ of generating series of (decorated) maps } \sim \kappa \rho^{-n} n^{-\alpha} \]

Central charge \( c \):
\[
\alpha = \frac{25 - c + \sqrt{(1 - c)(25 - c)}}{12}
\]

Hausdorff dimension: [Watabiki 93]
\[
D_H = 2 \frac{\sqrt{25 - c} + \sqrt{49 - c}}{\sqrt{25 - c} + \sqrt{1 - c}}
\]
Adding matter: Watabiki’s (controversial?) predictions

Counting exponent:
coeff \([t^n]\) of generating series of (decorated) maps \(\sim \kappa \rho^{-n} n^{-\alpha}\)

Central charge \(c\):

\[
\alpha = \frac{25 - c + \sqrt{(1 - c)(25 - c)}}{12}
\]

Hausdorff dimension: [Watabiki 93]

\[
D_H = 2 \frac{\sqrt{25 - c} + \sqrt{49 - c}}{\sqrt{25 - c} + \sqrt{1 - c}}
\]

- \(\alpha = 5/2\) gives \(D_H = 4\)
- \(\alpha = 7/3\) gives \(D_H = \frac{7 + \sqrt{97}}{4} \approx 4.21\)
Local convergence of triangulations with spins

Probability measure on triangulations of $\mathcal{T}_n$ with a spin configuration:

$$\mathbb{P}_n^{\nu}\left(\{(T,\sigma)\}\right) = \frac{\nu^m(T,\sigma)}{[t^{3n}]Q(\nu,t)}.$$
Local convergence of triangulations with spins

Probability measure on triangulations of $\mathcal{T}_n$ with a spin configuration:

$$\mathbb{P}_n^\nu \left( \{ (T, \sigma) \} \right) = \frac{\nu^m(T, \sigma)}{[t^{3n}]Q(\nu, t)}.$$ 

**Theorem** [Albenque – M. – Schaeffer]
As $n \to \infty$, the sequence $\mathbb{P}_n^\nu$ converges **weakly** to a probability measure $\mathbb{P}^\nu$ for the **local topology**. The measure $\mathbb{P}^\nu$ is supported on infinite triangulations with **one end**.
Local topology

\[ T_f := \{ \text{finite rooted planar triangulations with spins} \}. \]

**Definition:**

The **local topology** on \( T_f \) is induced by the distance:

\[
d_{loc}(T, T') := \left( 1 + \max\{ r \geq 0 : B_r(T) = B_r(T') \} \right)^{-1}
\]

where \( B_r(T) \) is the submap (with spins) of \( T \) composed by the faces of \( T \) with a vertex at distance \(< r\) from the root.
Definition:
The local topology on $\mathcal{T}_f$ is induced by the distance:

$$d_{loc}(T, T') := (1 + \max\{r \geq 0 : B_r(T) = B_r(T')\})^{-1}$$

where $B_r(T)$ is the submap (with spins) of $T$ composed by the faces of $T$ with a vertex at distance $< r$ from the root.

- $(\mathcal{T}, d_{loc})$: closure of $(\mathcal{T}_f, d_{loc})$. It is a Polish space.
- $\mathcal{T}_\infty := \mathcal{T} \setminus \mathcal{T}_f$ set of infinite planar triangulations with spins.
Local topology: Hulls

Balls $B_r(T)$ not practical (multiple holes). Take **hulls** instead:

$$\overline{B}_r(T) := \text{everything not in the largest connected component of } T \setminus B_r(T)$$
Local topology: Hulls

Balls $B_r(T)$ not practical (multiple holes). Take hulls instead:

$$\overline{B}_r(T) := \text{everything not in the largest connected component of } T \setminus B_r(T)$$

Problem: Hulls are not nested!
Local topology: Pointed hulls

For \((T, v) \in \mathcal{T}_f^\bullet := \{ \text{finite rooted triangulations with pointed vertex} \}\)

\[
B_r^\bullet (T, v) = \begin{cases} 
(T, v) & \text{if } v \in B_r(T); \\
B_r(T) \text{ and the connected components of } T \setminus B_r(T) \text{ that do not contain } v & \text{if } v \notin B_r(T).
\end{cases}
\]
Local topology: Pointed hulls

For \((T, v) \in T_f^\bullet := \{ \text{finite rooted triangulations with pointed vertex} \}\)

\[
B^\bullet_r(T, v) = \begin{cases} 
(T, v) & \text{if } v \in B_r(T); \\
B_r(T) \text{ and the connected components of } T \setminus B_r(T) \text{ that do not contain } v & \text{if } v \not\in B_r(T). 
\end{cases}
\]

Convergence for \(d^\bullet_{loc} \Rightarrow \) convergence for \(d_{loc}\) with the same limit.
Weak convergence for the local topology

**Portemanteau theorem + Levy – Prokhorov metric:**
The measures $\mathbb{P}^n$ converge weakly to $\mathbb{P}^\nu$ if

1. For every $r > 0$ and every possible hull $\Delta$

\[
\mathbb{P}_n^\bullet \left( \{(T, v) \in \mathcal{T}_n : B_r^\bullet(T, v) = \Delta \} \right) \xrightarrow{n \to \infty} \mathbb{P}^\nu \left( \{T \in \mathcal{T}_\infty : B_r^\bullet(T) = \Delta \} \right).
\]
Weak convergence for the local topology

**Portemanteau theorem + Levy – Prokhorov metric:**
The measures $\mathbb{P}_n^\bullet$ converge weakly to $\mathbb{P}^\nu$ if

1. For every $r > 0$ and every possible hull $\Delta$

\[
\mathbb{P}_n^\bullet \left( \left\{ (T, v) \in \mathcal{T}_n : B_r^\bullet(T, v) = \Delta \right\} \right) \xrightarrow{n \to \infty} \mathbb{P}^\nu \left( \left\{ T \in \mathcal{T}_\infty : B_r^\bullet(T) = \Delta \right\} \right).
\]

**Problem:** not sufficient since the spaces $(\mathcal{T}, d_{loc})$ or $(\mathcal{T}, d_{loc}^\bullet)$ are not compact!

Ex: degree $n$
Weak convergence for the local topology

**Portemanteau theorem + Levy – Prokhorov metric:**
The measures $P^\bullet_n$ converge weakly to $P^\nu$ if

1. For every $r > 0$ and every possible hull $\Delta$

$$P^\bullet_n\left(\left\{(T,v) \in \mathcal{T}_n : B^\bullet_r(T,v) = \Delta\right\}\right) \xrightarrow{n \to \infty} P^\nu\left(\left\{T \in \mathcal{T}_\infty : B^\bullet_r(T) = \Delta\right\}\right).$$

**Problem:** not sufficient since the spaces $(\mathcal{T}, d_{loc})$ or $(\mathcal{T}, d^\bullet_{loc})$ are not compact!

2. No loss of mass at the limit:
   The measure $P^\nu$ defined by the limits in 1. is a probability measure.
Weak convergence for the local topology

**Portemanteau theorem + Levy – Prokhorov metric:**
The measures $\mathbb{P}_n$ converge weakly to $\mathbb{P}^\nu$ if

1. For every $r > 0$ and every possible hull $\Delta$

$$
\mathbb{P}_n\left(\{(T, v) \in \mathcal{T}_n : B_r^\bullet(T, v) = \Delta\}\right) \xrightarrow{n \to \infty} \mathbb{P}^\nu\left(\{T \in \mathcal{T}_\infty : B_r^\bullet(T) = \Delta\}\right).
$$

**Problem:** not sufficient since the spaces $(\mathcal{T}, d_{loc})$ or $(\mathcal{T}, d_{loc}^\bullet)$ are not compact!

2. No loss of mass at the limit:
The measure $\mathbb{P}^\nu$ defined by the limits in 1. is a probability measure.

True if $\forall r \geq 0, \sum_{r - \text{hulls } \Delta} \mathbb{P}^\nu\left(\{T \in \mathcal{T}_\infty : B_r^\bullet(T) = \Delta\}\right) = 1.$
Local convergence and generating series

Need to evaluate, for every possible hull $\Delta$

$P^\bullet_n(\Delta ???)$
Local convergence and generating series

Need to evaluate, for every possible hull $\Delta$

Simple (rooted) cycle, spins given by a word $\omega$
Local convergence and generating series

Need to evaluate, for every possible hull $\Delta$:

$$\mathbb{P}_n^\bullet(\Delta) = \frac{\nu^m(\Delta) - m(\omega) [t^{3n} - e(\Delta) + |\omega|] Z_{\omega}(\nu, t)}{[t^{3n}] Q^\bullet(\nu, t)}$$

Simple (rooted) cycle, spins given by a word $\omega$

$Z_{\omega}(\nu, t) :=$ generating series of triangulations with simple boundary $\omega$
Local convergence and generating series

Need to evaluate, for every possible hull $\Delta$

$$P_n^\bullet \left( \begin{array}{c} \Delta \\ ??? \end{array} \right) = \frac{\nu^m(\Delta) - m(\omega) \left[ t^{3n - e(\Delta)} + |\omega| \right] Z_\omega^\bullet(\nu, t)}{[t^{3n}] Q^\bullet(\nu, t)}$$

Simple (rooted) cycle, spins given by a word $\omega$

$Z_\omega(\nu, t) :=$ generating series of triangulations with simple boundary $\omega$

**Theorem** [Albenque – M. – Schaeffer]

For every $\omega$, the series $t|\omega| Z_\omega(\nu, t)$ is algebraic, has $\rho_c$ as unique dominant singularity and satisfies

$$[t^{3n}] t|\omega| Z_\omega(\nu, t) \sim_{n \to \infty} \begin{cases} \kappa_\omega(\nu_c) \rho_{\nu_c}^{-n} n^{-7/3} & \text{if } \nu = \nu_c = 1 + \frac{1}{\sqrt{7}}, \\ \kappa_\omega(\nu) \rho_{\nu}^{-n} n^{-5/2} & \text{if } \nu \neq \nu_c. \end{cases}$$
Triangulations with simple boundary

Fix a word $\omega$, with injections from and into triangulations of the sphere:

$$[t^{3n}] t^{\omega} Z_\omega = \Theta \left( \rho^{-n} n^{-\alpha} \right), \text{ with } \alpha = 5/2 \text{ or } 7/3 \text{ depending on } \nu.$$ 

To get exact asymptotics we need, as series in $t^3$,

1. algebraicity,
2. no other dominant singularity than $\rho_\nu$. 
Triangulations with simple boundary

Fix a word $\omega$, with injections from and into triangulations of the sphere:

$$[t^{3n}]t^{|\omega|} Z_\omega = \Theta \left( \rho_\nu^{-n} n^{-\alpha} \right), \text{ with } \alpha = \frac{5}{2} \text{ of } \frac{7}{3} \text{ depending on } \nu.$$  

To get exact asymptotics we need, as series in $t^3$,

1. algebraicity,
2. no other dominant singularity than $\rho_\nu$.

Tutte’s equation (or peeling equation, or loop equation...):

$$Z_\omega = \left( Z_{\ominus \omega} + Z_{\ominus \omega} + \sum_{\omega=\omega_1 \omega_2} Z_{\omega_1} \cdot Z_{\omega_2} \right) \times \nu^{1_{\overline{\omega} = \overline{\omega}}} t$$
Triangulations with simple boundary

Fix a word $\omega$, with injections from and into triangulations of the sphere:

$$[t^{3n}] t^{\mid \omega \mid} Z_\omega = \Theta \left( \rho^{-n} n^{-\alpha} \right), \text{ with } \alpha = \frac{5}{2} \ text{of } \frac{7}{3} \text{depending on } \nu. \text{To get exact asymptotics we need, as series in } t^3,$$

1. algebraicity,
2. no other dominant singularity than $\rho_\nu$.

Tutte’s equation (or peeling equation, or loop equation... ):

$$Z_\omega = \left( Z_{\oplus \omega} + Z_{\ominus \omega} + \sum_{\omega = \omega_1 \ominus \omega_2} Z_{\omega_1} \cdot Z_{\omega_2} \right) \times \nu^{1_{\overline{\omega} = \overline{\omega}}} t$$

Double recursion on $|\omega|$ and number of $\ominus$’s:
enough to prove 1. and 2. for the $t^p Z_{\oplus p}$’s
Positive boundary conditions: two catalytic variables

\[ A(x) := \sum_{p \geq 1} Z_{\oplus p} x^p = \nu t x^2 + \frac{\nu t}{x} (A(x))^2 \]
Positive boundary conditions: two catalytic variables

\[ A(x) := \sum_{p \geq 1} Z_{\oplus p} x^p = \nu t x^2 + + \frac{\nu t}{x} (A(x))^2 \]

Peeling equation at interface \( \ominus - \oplus \):

\[ S(x, y) := \sum_{p, q \geq 1} Z_{\ominus p \ominus q} x^p y^q \]
Positive boundary conditions: two catalytic variables

\[ A(x) := \sum_{p \geq 1} Z_{\oplus p} x^p = \nu t x^2 + \frac{\nu t}{x} \left( A(x) - x Z_{\oplus} \right) + \nu t [y] S(x, y) + \frac{\nu t}{x} (A(x))^2 \]

Peeling equation at interface \( \ominus - \ominus \):

\[ S(x, y) := \sum_{p, q \geq 1} Z_{\oplus p \ominus q} x^p y^q \]
Positive boundary conditions: two catalytic variables

\[ A(x) := \sum_{p \geq 1} Z_{\oplus p} x^p = \nu t x^2 + \frac{\nu t}{x} \left( A(x) - x Z_{\oplus} \right) + \nu t [y] S(x, y) + \frac{\nu t}{x} \left( A(x) \right)^2 \]

Peeling equation at interface \( \ominus \rightarrow \ominus \):

\[ S(x, y) := \sum_{p, q \geq 1} Z_{\ominus p \ominus q} x^p y^q \]
\[ = t x y + \frac{t}{x} \left( S(x, y) - x [x] S(x, y) \right) + \frac{t}{y} \left( S(x, y) - y [y] S(x, y) \right) \]
\[ + \frac{t}{x} S(x, y) A(x) + \frac{t}{y} S(x, y) A(y) \]
From two catalytic variables to one: Tutte’s invariants

**Kernel method:** equation for $S$ reads

$$K(x, y) \cdot S(x, y) = R(x, y)$$

where

$$K(x, y) = 1 - \frac{t}{x} - \frac{t}{y} - \frac{t}{x}A(x) - \frac{t}{y}A(y).$$
Kernel method: equation for $S$ reads

$$K(x, y) \cdot S(x, y) = R(x, y)$$

where

$$K(x, y) = 1 - \frac{t}{x} - \frac{t}{y} - \frac{t}{x}A(x) - \frac{t}{y}A(y).$$

1. Find two series $Y_1$ and $Y_2$ in $\mathbb{Q}(x)[[t]]$ such that $K(x, Y_i/t) = 0$. 
From two catalytic variables to one: Tutte’s invariants

**Kernel method:** equation for $S$ reads

$$K(x, y) \cdot S(x, y) = R(x, y)$$

where

$$K(x, y) = 1 - \frac{t}{x} - \frac{t}{y} - \frac{t}{x} A(x) - \frac{t}{y} A(y).$$

1. Find two series $Y_1$ and $Y_2$ in $\mathbb{Q}(x)[[t]]$ such that $K(x, Y_i/t) = 0$. It gives

$$\frac{1}{Y_1} (A(Y_1/t) + 1) = \frac{1}{Y_2} (A(Y_2/t) + 1).$$
From two catalytic variables to one: Tutte’s invariants

**Kernel method:** equation for $S$ reads

$$K(x, y) \cdot S(x, y) = R(x, y)$$

where

$$K(x, y) = 1 - \frac{t}{x} - \frac{t}{y} - \frac{t}{x} A(x) - \frac{t}{y} A(y).$$

1. Find **two** series $Y_1$ and $Y_2$ in $\mathbb{Q}(x)[[t]]$ such that $K(x, Y_i/t) = 0$.

It gives

$$\frac{1}{Y_1} (A(Y_1/t) + 1) = \frac{1}{Y_2} (A(Y_2/t) + 1).$$

$I(y) := \frac{1}{y} (A(y/t) + 1)$ is called an **invariant**.
From two catalytic variables to one: Tutte’s invariants

Kernel method: equation for $S$ reads

$$K(x, y) \cdot S(x, y) = R(x, y)$$

where

$$K(x, y) = 1 - \frac{t}{x} - \frac{t}{y} - \frac{t}{x}A(x) - \frac{t}{y}A(y).$$

1. Find two series $Y_1$ and $Y_2$ in $\mathbb{Q}(x)[[t]]$ such that $K(x, Y_i/t) = 0$.

   It gives $\frac{1}{Y_1} (A(Y_1/t) + 1) = \frac{1}{Y_2} (A(Y_2/t) + 1)$.

   $I(y) := \frac{1}{y} (A(y/t) + 1)$ is called an invariant.

2. Work a bit with the help of $R(x, Y_i/t) = 0$ to get a second invariant $J(y)$ depending only on $t, Z \oplus (t), y$ and $A(y/t)$. 
From two catalytic variables to one: Tutte’s invariants

Kernel method: equation for $S$ reads

$$K(x, y) \cdot S(x, y) = R(x, y)$$

where

$$K(x, y) = 1 - \frac{t}{x} - \frac{t}{y} - \frac{t}{x} A(x) - \frac{t}{y} A(y).$$

1. Find two series $Y_1$ and $Y_2$ in $\mathbb{Q}(x)[[t]]$ such that $K(x, Y_i/t) = 0$.

It gives

$$\frac{1}{Y_1}(A(Y_1/t) + 1) = \frac{1}{Y_2}(A(Y_2/t) + 1).$$

$I(y) := \frac{1}{y} (A(y/t) + 1)$ is called an invariant.

2. Work a bit with the help of $R(x, Y_i/t) = 0$ to get a second invariant $J(y)$ depending only on $t, Z\oplus(t), y$ and $A(y/t)$.

3. Prove that $J(y) = C_0(t) + C_1(t)I(y) + C_2(t)I^2(y)$ with $C_i$’s explicit polynomials in $t, Z\oplus(t)$ and $Z\oplus^2(t)$.

Equation with one catalytic variable for $A(y)$ with $Z\oplus$ and $Z\oplus^2$ !
Explicit solution for positive boundary conditions

Equation with one catalytic variable reads:

\[ 2t^2 \nu (1 - \nu) \left( \frac{A(y)}{y} - Z_\oplus \right) = y \cdot \text{Pol} \left( \nu, \frac{A(y)}{y}, Z_\oplus, Z_\oplus^2, t, y \right) \]

[Bousquet-Mélou – Jehanne 06] gives algebraicity and strategy to solve this equation.
Explicit solution for positive boundary conditions

Equation with one catalytic variable reads:

\[ 2t^2 \nu (1 - \nu) \left( \frac{A(y)}{y} - Z_\oplus \right) = y \cdot \text{Pol} \left( \nu, \frac{A(y)}{y}, Z_\oplus, Z_{\oplus^2}, t, y \right) \]

[Bousquet-Mélou – Jehanne 06] gives algebraicity and strategy to solve this equation.

Much easier: [Bernardi – Bousquet Mélou 11] gives us \( Z_\oplus \) and \( Z_{\oplus^2} \)!
Explicit solution for positive boundary conditions

Equation with one catalytic variable reads:

\[ 2t^2 \nu (1 - \nu) \left( \frac{A(y)}{y} - Z_\oplus \right) = y \cdot \text{Pol} \left( \nu, \frac{A(y)}{y}, Z_\oplus, Z_\oplus^2, t, y \right) \]

[Bousquet-Mélou – Jehanne 06] gives \textit{algebraicity} and strategy to solve this equation.

Much easier: [Bernardi – Bousquet Mélou 11] gives us \( Z_\oplus \) and \( Z_\oplus^2 \)!

Maple: \textit{rational parametrization} !

\[ t^3 = U \frac{P_1(\mu, U)}{4(1 - 2U)^2(1 + \mu)^3} \]

\[ ty = V \frac{P_2(\mu, U, V)}{(1 - 2U)(1 + \mu)^2(1 - V)^2} \]

\[ t^3 A(t, ty) = \frac{VP_3(\mu, U, V)}{4(1 - 2U)^2(1 + \mu)^3(1 - V)^3} \]

with \( \nu = \frac{1+\mu}{1-\mu} \) and \( P_i \)'s explicit polynomials.
1. Fix \( r \geq 0 \) and take \( \Delta \) a \( r \)-hull with boundary spins \( \partial \Delta \):

\[
\mathbb{P}_{n}^{\bullet} (B_{r}^{\bullet}(T, v) = \Delta) = \frac{\nu^{m(\Delta) - m(\partial \Delta)} [t^{3n - e(\Delta)} + |\partial \Delta|] Z_{\partial \Delta}^{\bullet}(\nu, t)}{[t^{3n}] Q^{\bullet}(\nu, t)} \rightarrow_{n \to \infty} \frac{\kappa_{\partial \Delta}}{\kappa} \nu^{m(\Delta) - m(\partial \Delta)} \rho(|\Delta| - 2|\partial \Delta|)/3.
\]
Going back to local convergence

1. Fix \( r \geq 0 \) and take \( \Delta \) a \( r \)-hull with boundary spins \( \partial \Delta \):

\[
\mathbb{P}_n^\bullet (B_r^\bullet (T, v) = \Delta) = \frac{\nu^m(\Delta) - m(\partial \Delta)}{[t^{3n}] Q^\bullet (\nu, t)} \frac{[t^{3n}] Z_{\partial \Delta} (\nu, t)}{t^{3n}} \to_{n \to \infty} \frac{\kappa_{\partial \Delta}}{\kappa} \frac{\nu^m(\Delta) - m(\partial \Delta)}{\kappa} \frac{\rho (|\Delta| - 2|\partial \Delta|)/3}{\kappa}.}

2. Remains to prove, for every \( r \):

\[
\sum_{r-\text{hulls } \Delta} \frac{\kappa_{\partial \Delta}}{\kappa} \frac{\nu^m(\Delta) - m(\partial \Delta)}{\kappa} \frac{\rho (|\Delta| - 2|\partial \Delta|)/3}{\kappa} = 1.
\]
No loss of mass at the limit

Decompose triangulations by hulls:

\[ Q^\bullet(\nu, t) = Q^{\leq r}(\nu, t) + \sum_{r-\text{hulls } \Delta} \sum_{(T,v) : B^\bullet_r(T,v) = \Delta} \nu^m(\Delta) + m(T\setminus \Delta)_t |\Delta| + |T\setminus \Delta| \]

pointed at dist. \( \leq r \) from the root

\[ = Q^{\leq r}(\nu, t) + \sum_{r-\text{hulls } \Delta} \nu^m(\Delta) - m(\partial \Delta)_t |\Delta| - |\partial \Delta| Z^\bullet_{\partial \Delta}(\nu, t) \]
No loss of mass at the limit

Decompose triangulations by hulls:

\[ Q^\bullet(\nu, t) = Q^{\leq r}(\nu, t) + \sum_{r-\text{hulls } \Delta} \sum_{(T,v): B_r^\bullet(T,v) = \Delta} \nu^m(\Delta) + m(T \setminus \Delta) t |\Delta| + |T \setminus \Delta| \]

pointed at dist. \( \leq r \) from the root

\[ = Q^{\leq r}(\nu, t) + \sum_{r-\text{hulls } \Delta} \nu^m(\Delta) - m(\partial \Delta) t |\Delta| - |\partial \Delta| Z_{\partial \Delta}^\bullet(\nu, t) \]

Since \([t^{3n}] Q^\bullet(\nu, t) \gg [t^{3n}] Q^{\leq r}(\nu, t)\), extracting \([t^{3n}]\) gives

\[ [t^{3n}] Q^\bullet(\nu, t) \sim \sum_{r-\text{hulls } \Delta} \nu^m(\Delta) - m(\partial \Delta) [t^{3n} - |\Delta| + |\partial \Delta|] Z_{\partial \Delta}^\bullet(\nu, t) \]

\[ \kappa \rho^{-n} n^{-\alpha + 1} \sim \sum_{r-\text{hulls } \Delta} \nu^m(\Delta) - m(\partial \Delta) \kappa_{\partial \Delta} \rho^{-n + (|\Delta| - 2|\partial \Delta|)/3} n^{-\alpha + 1} \]
The story so far

What we know:

- Convergence in law for the local topology.
- The limiting random triangulation has one end a.s.
The story so far

What we know:

- Convergence in law for the local topology.
- The limiting random triangulation has one end a.s.
- A spatial Markov property.
- Some links with Boltzmann triangulations.
The story so far

What we know:

- Convergence in law for the local topology.
- The limiting random triangulation has one end a.s.
- A spatial Markov property.
- Some links with Boltzmann triangulations.

What we would like to know:

- Singularity with respect to the UIPT?
- Volume growth?
The story so far

What we know:

- Convergence in law for the local topology.
- The limiting random triangulation has one end a.s.
- A spatial Markov property.
- Some links with Boltzmann triangulations.

What we would like to know:

- Singularity with respect to the UIPT?
- Volume growth?
- At least volume growth $\neq 4$ at $\nu_c$?
Conference Dynamics on random graphs and random planar maps

October 23 to 27, 2017 in Marseille France

Org. LM, Pierre Nolin, Bruno Schapira and Arvind Singh

Thank you for your attention!