Éléments pleinement commutatifs dans les groupes de Coxeter

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Séminaire Flajolet, IHP, 3 Octobre 2013

I. Coxeter groups

Coxeter group

- S a finite set; $M = (m_{st})_{s,t \in S}$ a symmetric matrix.

M must satisfy $m_{ss} = 1$ and $m_{st} \in \{2, 3, \ldots\} \cup \{\infty\}$

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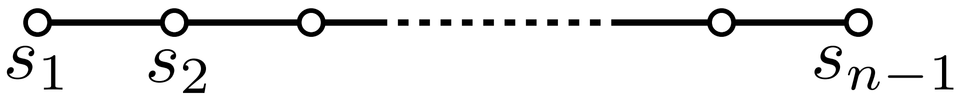
• Coxeter graph: Labeled graph encoding M, with vertices S, edge if $m_{st} \geq 3$, and label m_{st} when $m_{st} \geq 4$.

All Coxeter groups are considered irreducible $\Leftrightarrow$ Γ connected.

Coxeter group: examples

(1) A_{n-1}

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$



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Isomorphic to the symmetric group S_n via $s_i \leftrightarrow (i, i+1)$.

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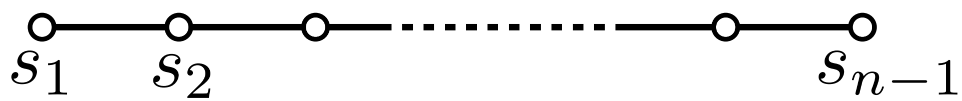

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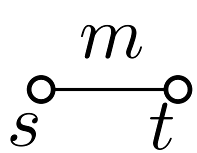

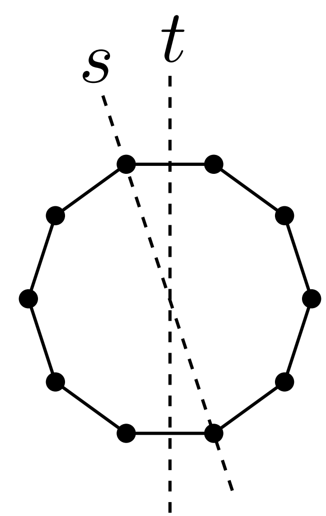

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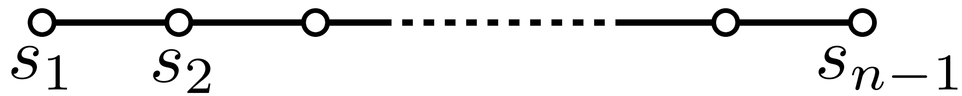

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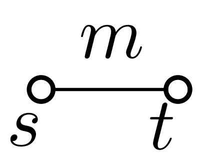

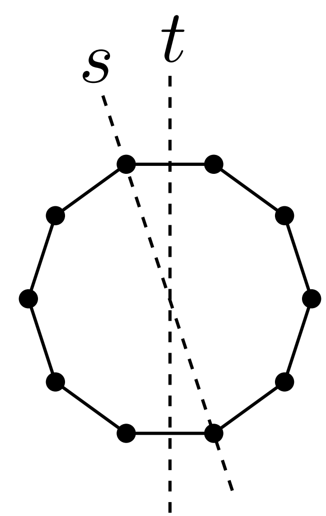

Geometry: Every Coxeter group has a geometric representation in $\mathbb{R}^n$ where $n = |S|$, where:

- Each $s \in S$ is a reflection through a hyperplane ($s^2 = 1$);
- st is a rotation of order m_{st} ($(st)^{m_{st}} = 1$).

Rough classification of Coxeter groups

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These are precisely groups of isometries of $\mathbb{R}^n$ generated by *orthogonal* reflections.

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A complete classification exists for both families, classified by their Coxeter graph.

Finite: A_{n-1}, B_n, D_n and $I_2(m), F_4, H_3, H_4, E_6, E_7, E_8$.

Affine: $\widetilde{A}_{n-1}, \widetilde{B}_n, \widetilde{C}_n, \widetilde{D}_n$ and $\widetilde{G}_2, \widetilde{F}_4, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$.

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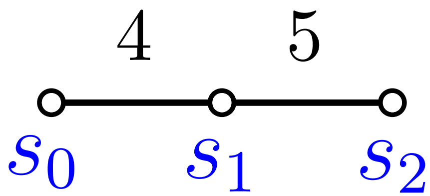
3. All the other Coxeter groups

These correspond to groups of linear transformations of $\mathbb{R}^n$ generated by reflections which are *not* orthogonal.

$\rightarrow$ Study of sub families: right-angled groups, simply laced groups, hyperbolic groups, ...

Triangle group $T(2,4,5)$



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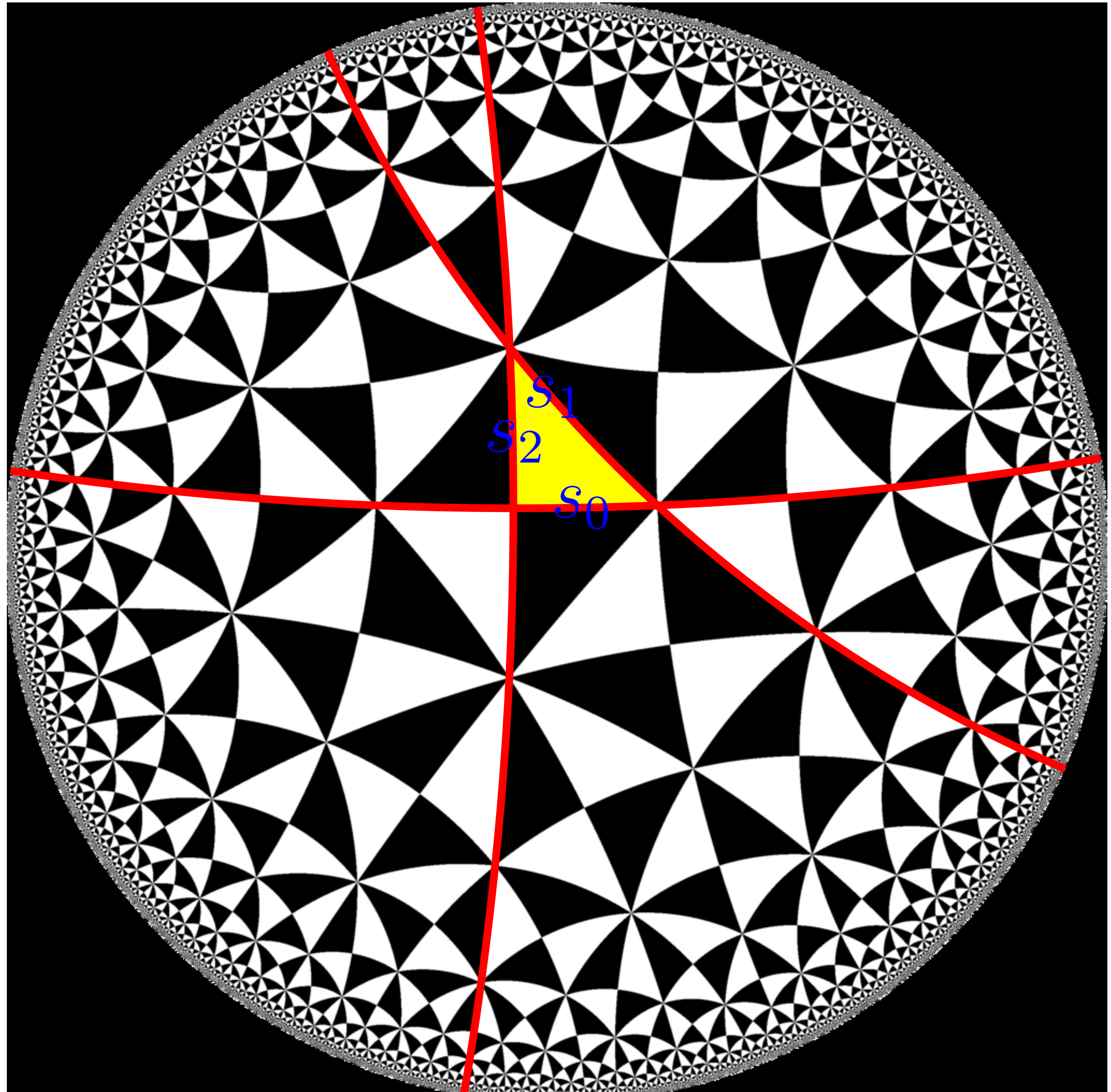

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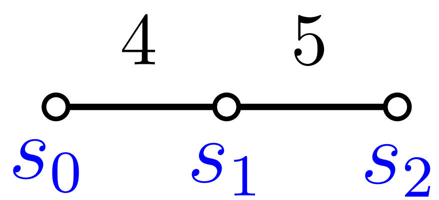

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Elements of W

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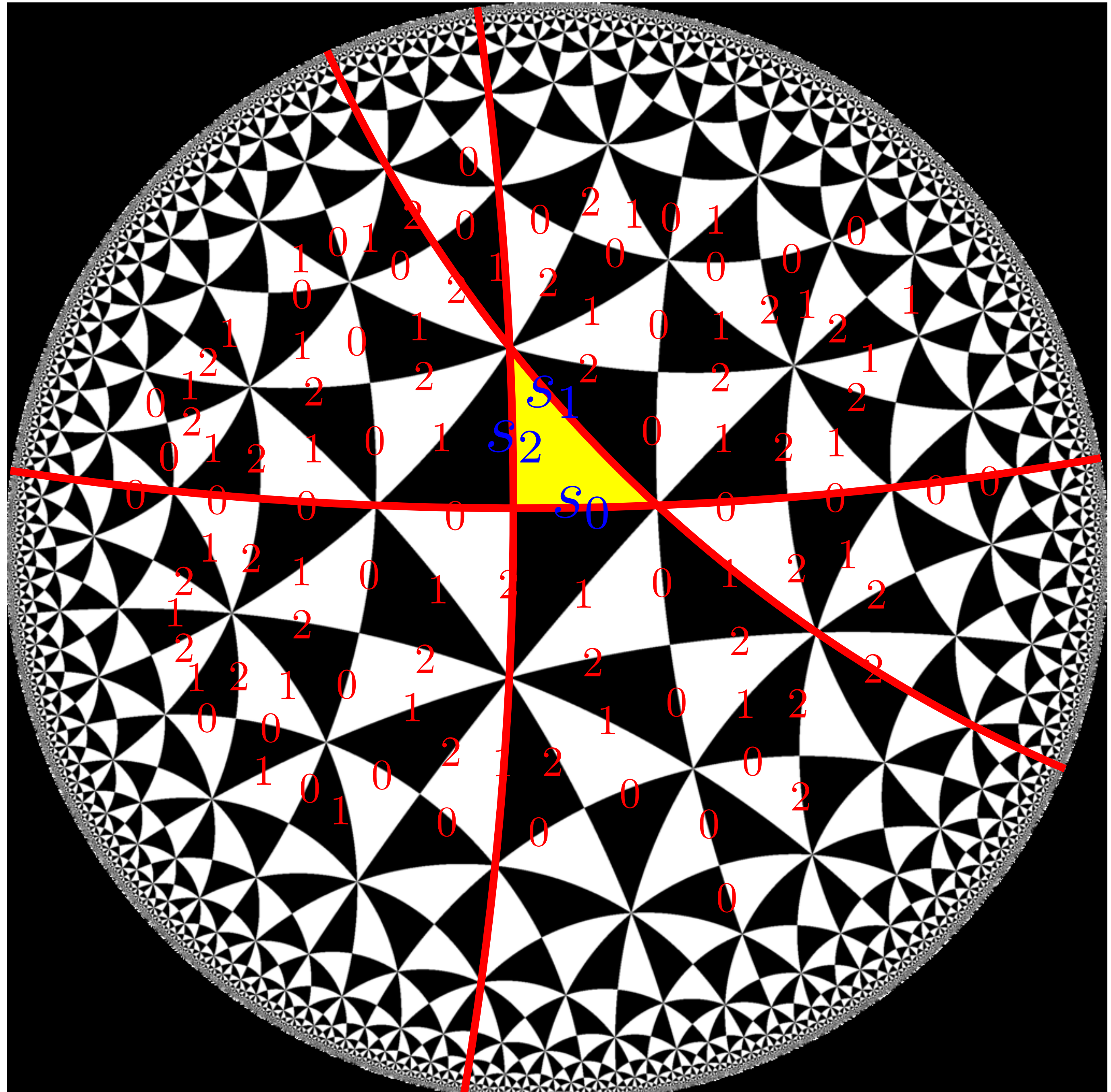
Chambers



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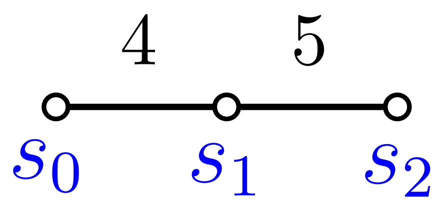

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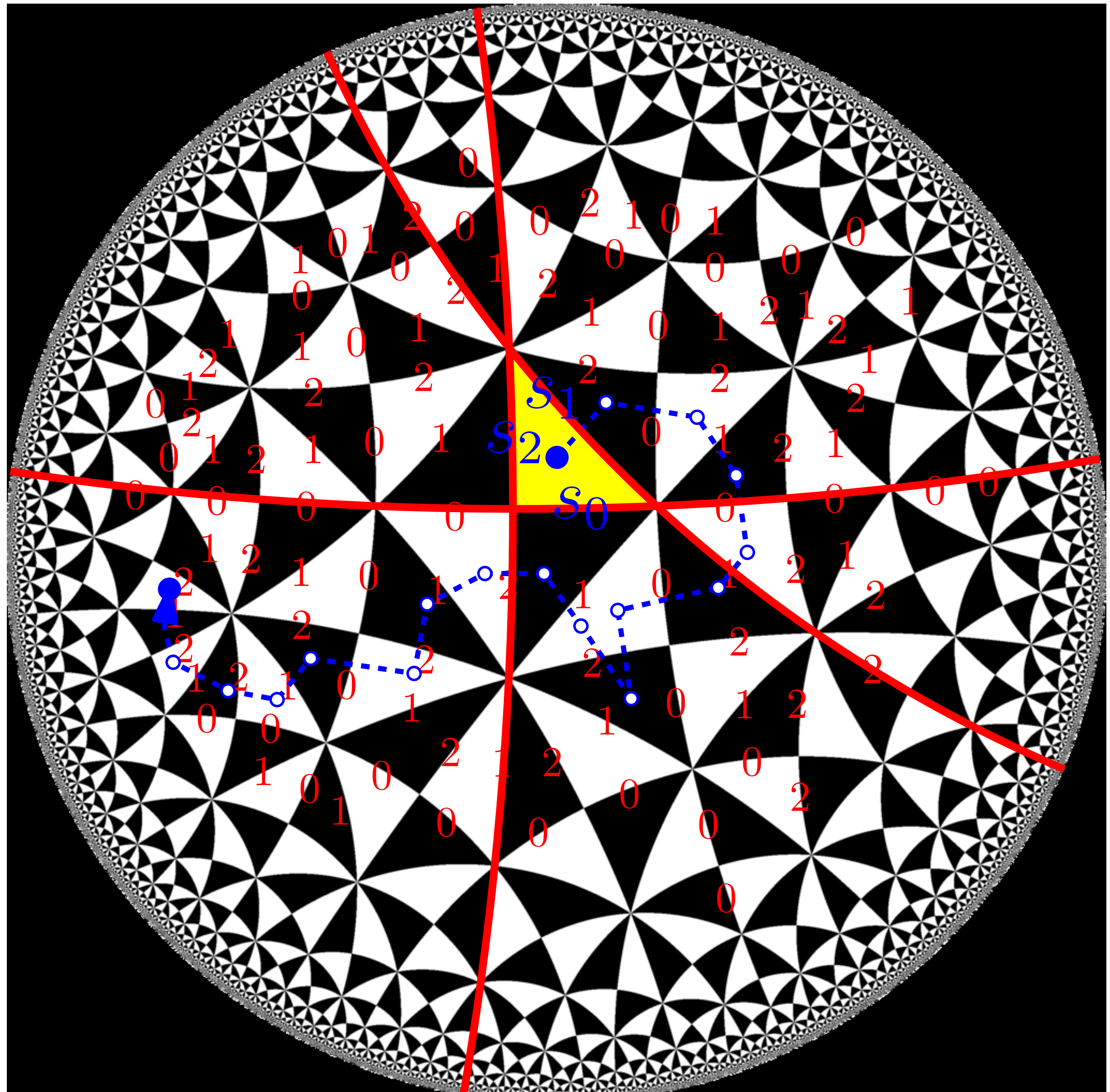

Length function

Definition Length $\ell(w)$= minimal l such that $w = s_1 s_2 \ldots s_l$.
The minimal words are the reduced decompositions of w.

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In the geometric representation, correspond to shortest paths from the fundamental chamber to the chamber of w.

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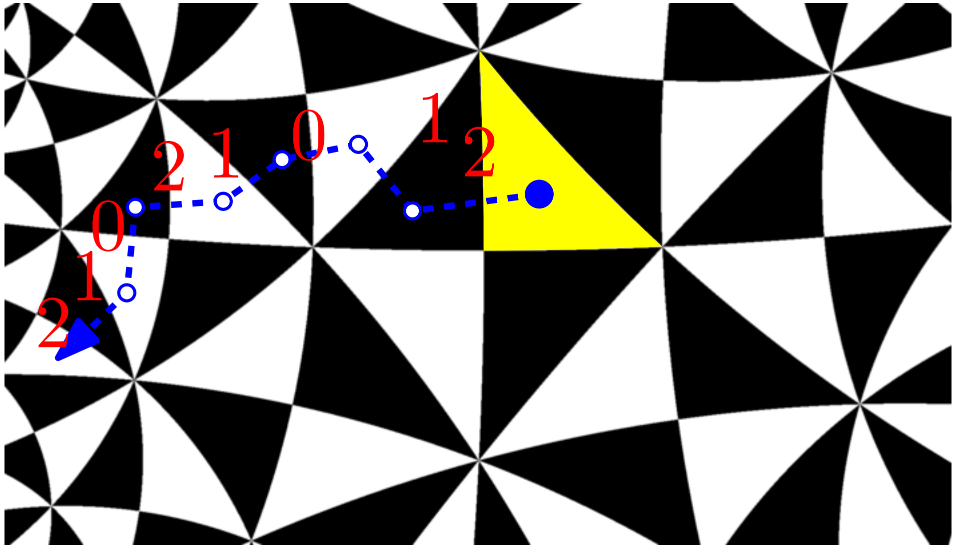

Enumeration of elements and reduced expressions.

- If W is a Coxeter group, define $W(q) := \sum_{w \in W} q^{\ell(w)}$

Theorem $W(q)$ is a rational function

(Proof by induction on $|S|$, needs a bit of Coxeter theory.)

Trivial for finite groups (polynomial), but nice product formula in that case; also nice for affine groups.

For $T(2,4,5)$ the g.f. is $\frac{\left(q^3+q^2+q+1\right)\left(q^4+q^3+q^2+q+1\right)(1+q)}{q^8-q^5-q^4-q^3+1}$

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- $Red_W(q) := \sum_{w} |Red(w)| q^{\ell(w)} = \sum_{\mathbf{w} \text{ reduced word}} q^{|\mathbf{w}|}$

Theorem [Brink, Howlett '93] $Red_W(q)$ is a rational function

They show that the language of reduced words is regular.

II. Fully commutative elements and Heaps

Fully commutative elements

Property : Given any two reduced decompositions of w, there is a sequence of braid relations which can be applied to transform one into the other.

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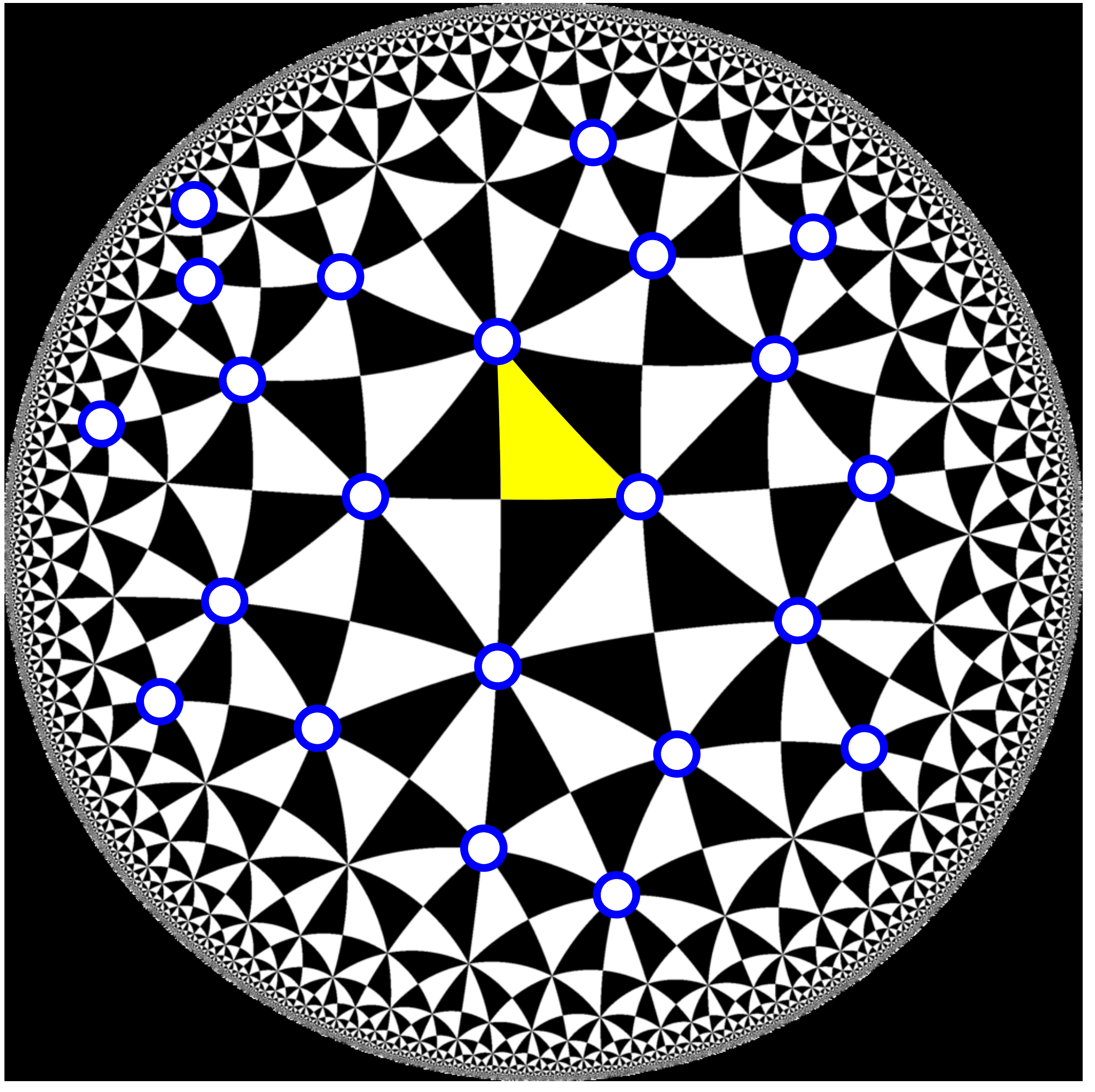
Proposition [Stembridge '96] A commutation class of reduced words corresponds to a FC element if and only no word in it contains a braid word $\underbrace{sts\cdots}_{m_{st}}$ for a $m_{st} \geq 3$.

Geometric interpretation

1. Consider all hyperplane intersections where $m_{st} \leq 3$
2. The chamber which is the furthest away is not FC.
3. Neither are the chambers behind it.



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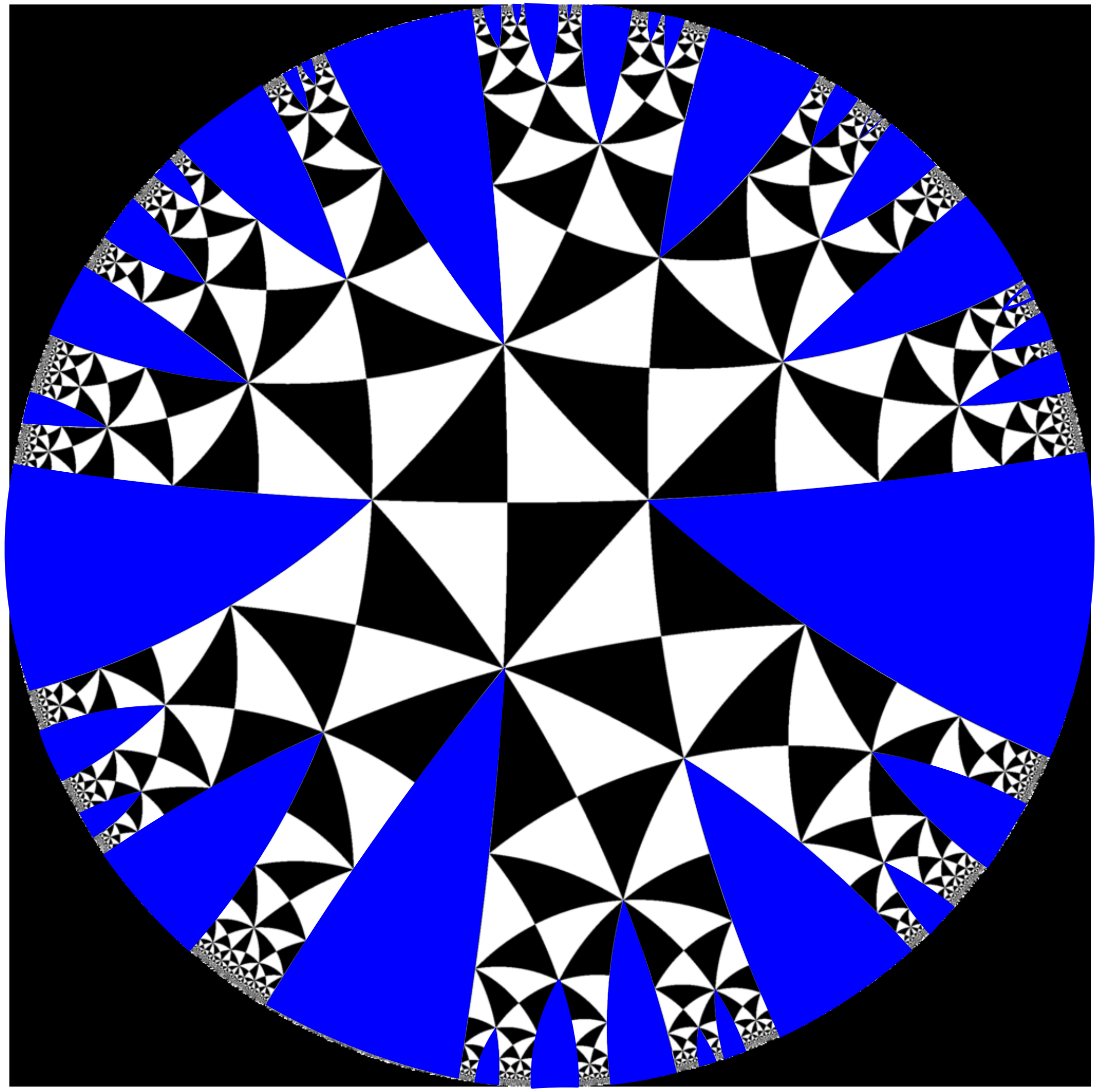

Previous work on FC elements

• The seminal combinatorics papers are [Stembridge '96,'98]:

1. First properties;
2. Classification of W with a finite number of FC elements;
3. Enumeration of these elements in each of these cases.

• [Fan '95] studies FC elements in the special case where $m_{st} \leq 3$ (*the simply laced case*).

• [Graham '95] shows that FC elements in any Coxeter group W naturally index a basis of the (generalized) Temperley-Lieb algebra of W.

• Subsequent works [Greene,Shi,Cellini,Papi] relate FC elements (and some related elements) to Kazhdan-Lusztig polynomials.

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W an irreducible affine Coxeter group.
(i) **Characterization** of FC elements.;
(ii) **Computation of** $W^{FC}(q)$;
(iii) $(W_\ell^{FC})_{\ell\geq 0}$ is **ultimately periodic.**

Affine Type	$\widetilde{A}_{n-1}$	$\widetilde{C}_n$	$\widetilde{B}_{n+1}$	$\widetilde{D}_{n+2}$	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{G}_2$	$\widetilde{F}_4, \widetilde{E}_8$
Periodicity	n	$n+1$	$(n+1)(2n+1)$	$n+1$	4	9	5	1

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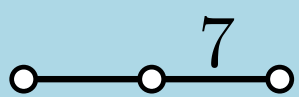
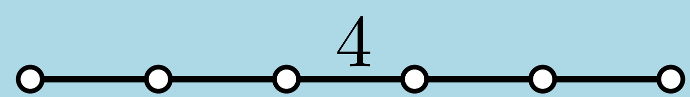
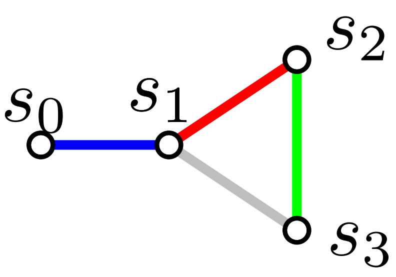
Theorem [N. '13] The sequence $(W_l^{FC})_{l\geq 0}$ is ultimately periodic if and only if W is affine, FC-finite or is one of two exceptions, namely 

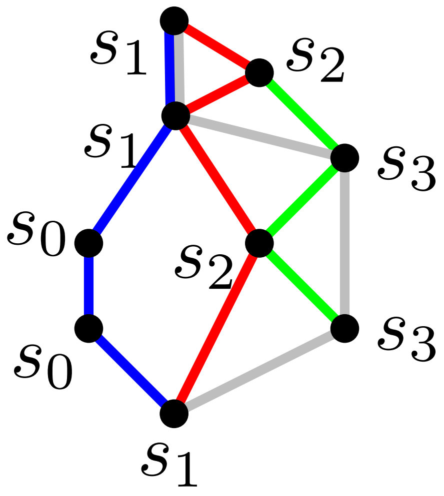
and 

Heaps

Let Γ be a finite graph.

Definition: A Γ-**heap** is a poset $(H, \leq)$ with $\epsilon : H \to S$ satisfying:
1. $\{s, t\} \in \Gamma$ an edge $\Rightarrow$ The h s.t. $\epsilon(h) \in \{s, t\}$ form a chain.
2. The poset $(H, \leq)$ is the transitive closure of these chains.





Heaps = Commutation classes

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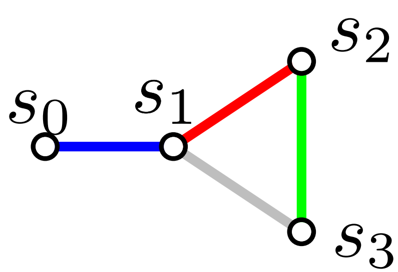

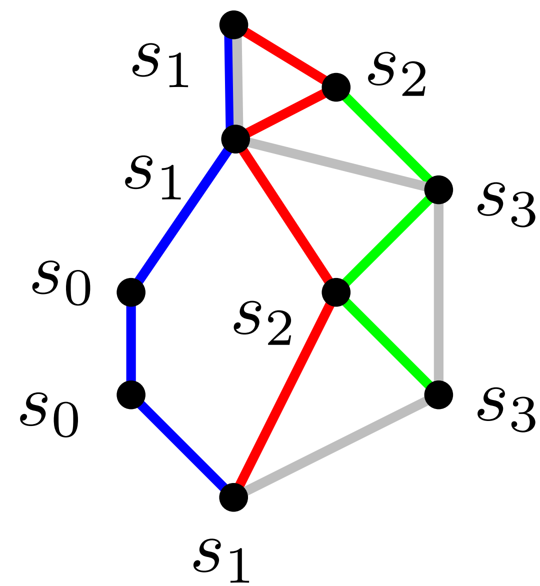

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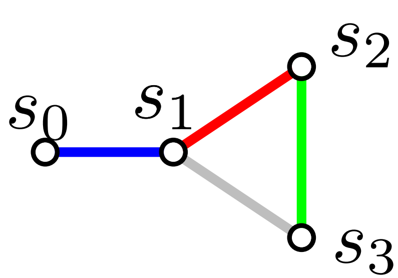

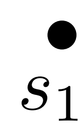

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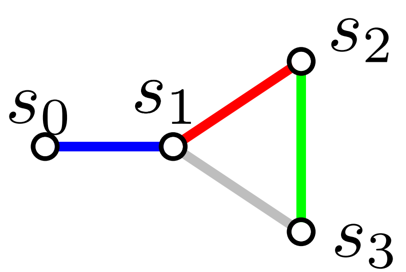

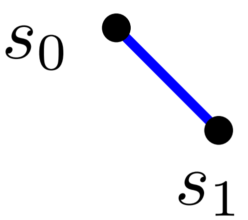

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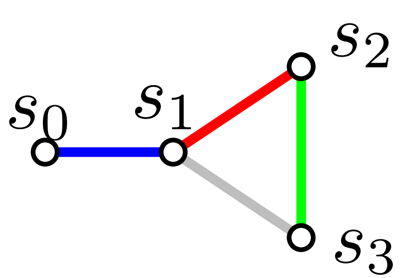

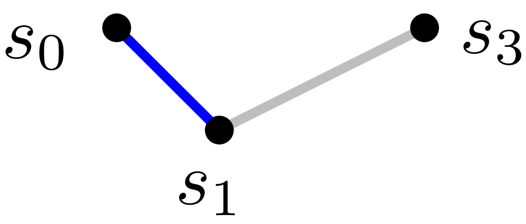

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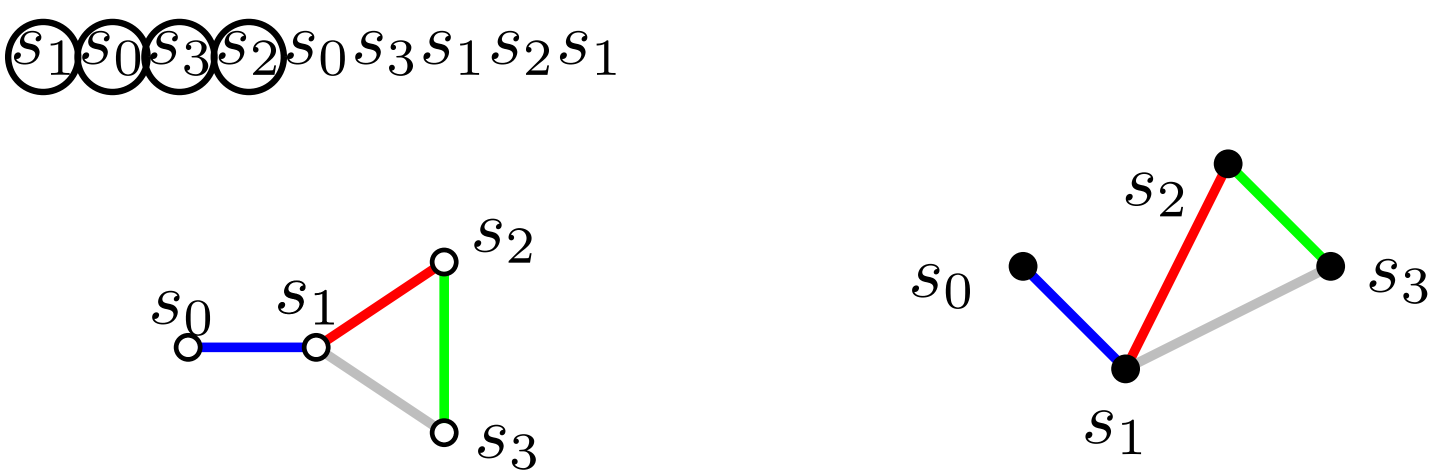

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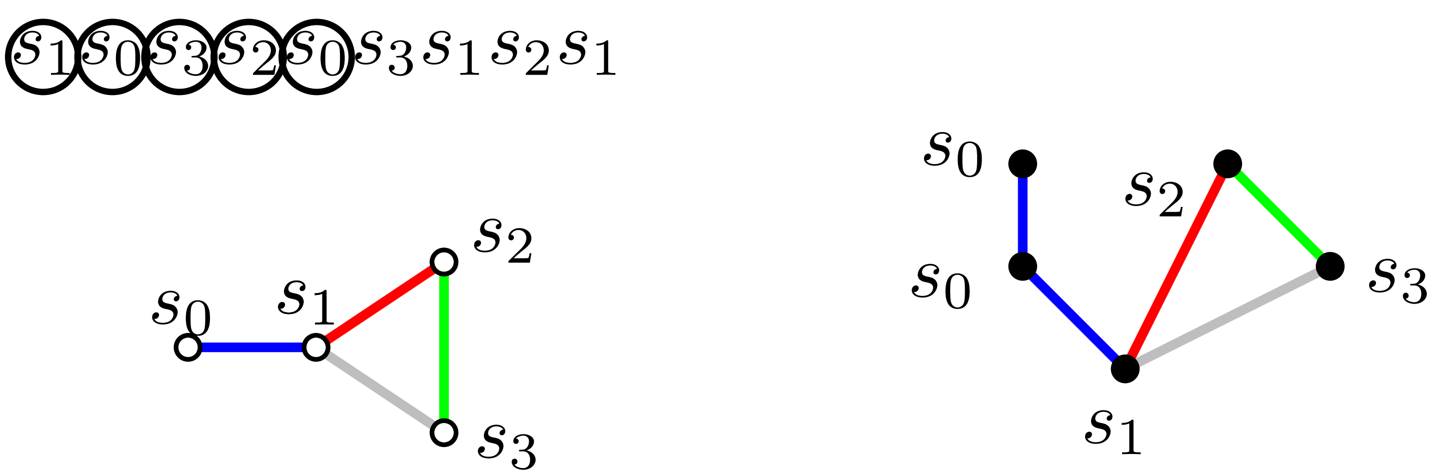

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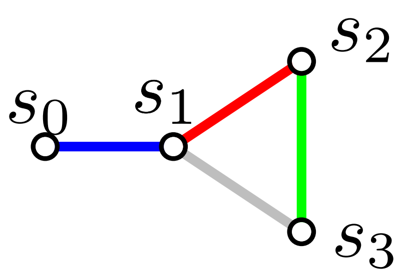

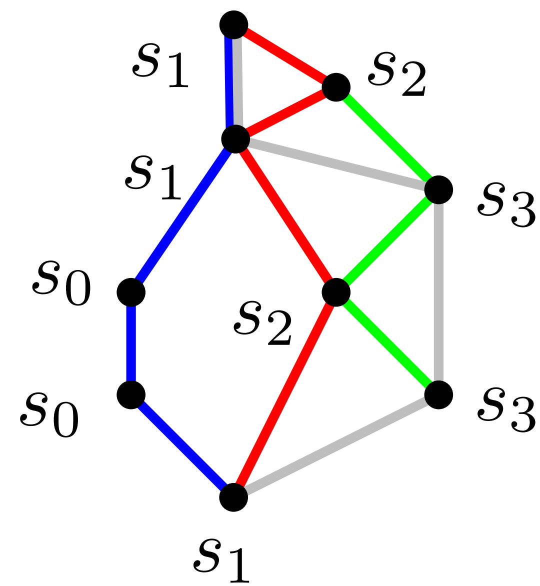

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FC heaps = Special commutation classes

Let Γ be a Coxeter graph. Recall that FC elements correspond to commutation classes of reduced words avoiding $\underbrace{sts\cdots}_{m_{st}\geq 3}$

$\rightarrow$ let us call **FC heaps** the corresponding heaps.

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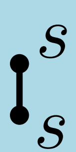
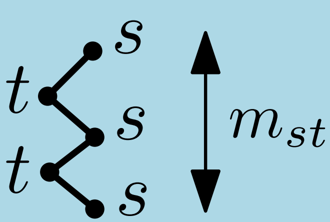
Proposition [Stembridge '95] FC heaps on Γ are characterized by the following two restrictions:

(a) No covering relation



(b) No convex chain of the form



FC heaps = Special commutation classes

Let Γ be a Coxeter graph. Recall that FC elements correspond to commutation classes of reduced words avoiding $\underbrace{sts\cdots}_{m_{st}\geq 3}$

$\rightarrow$ let us call **FC heaps** the corresponding heaps.

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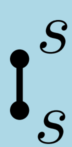

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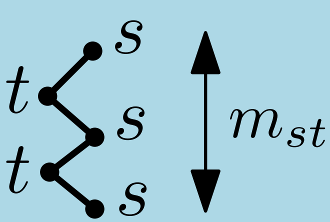

Summary

FC element w	$\longleftrightarrow$	Heap H satisfying (a) and (b)
Length $\ell(w)$	$\longleftrightarrow$	Number of elements $\lvert H\rvert$

Rationality of $Red_W^{FC}(q)$ and $W^{FC}(q)$.

Let W be a Coxeter group with Γ its graph.

• To determine if a word is a FC reduced word, construct the heap letter by letter. It turns out that only "finite information" about the heap needs to be stored.

Theorem The language Red_W^{FC} of FC reduced words can be recognized by a finite automaton.

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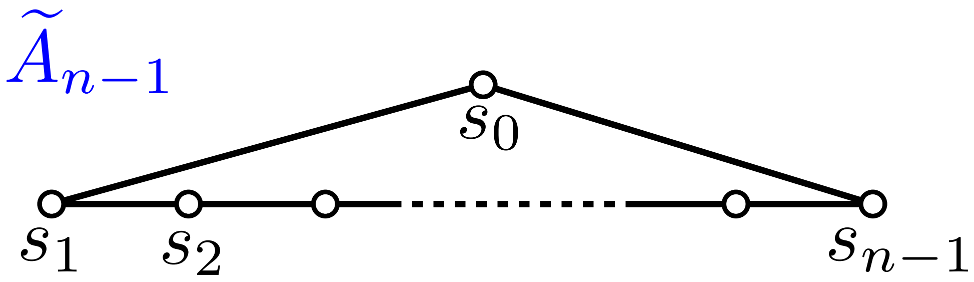
• Fix a total order of S, and associate to each Γ-commutation class its lexicographically minimal element. Now the language $Shortlex(\Gamma)$ of such words is known [Anisimov-Knuth '79] to be regular, and we get

Corollary $Shortlex(\Gamma) \cap Red_W^{FC}$ is regular.

$\Rightarrow$ its length generating function $W^{FC}(q)$ is rational.

III. FC elements in type $\widetilde{A}$

Affine permutations



$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

$$s_i s_j = s_j s_i, \quad |j - i| > 1$$

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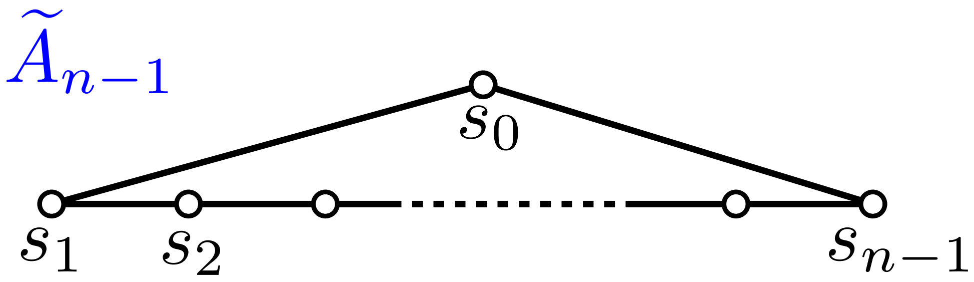

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Isomorphic to the group of permutations σ of $\mathbb{Z}$ such that:
(i) $\forall i \in \mathbb{Z}\ \sigma(i+n) = \sigma(i)+n$, and
(ii) $\sum_{i=1}^{n} \sigma(i) = \sum_{i=1}^{n} i$.

$$\ldots, 13, -12, \mid -14, -1, 17, -8, \mid \underset{\sigma(1)}{-\mathbf{10}}, \underset{\sigma(2)}{\mathbf{3}}, \underset{\sigma(3)}{\mathbf{21}}, \underset{\sigma(4)}{-\mathbf{4}}, \mid -6, 7, 25, 0, \mid -2, 11, 29, 4, \ldots$$

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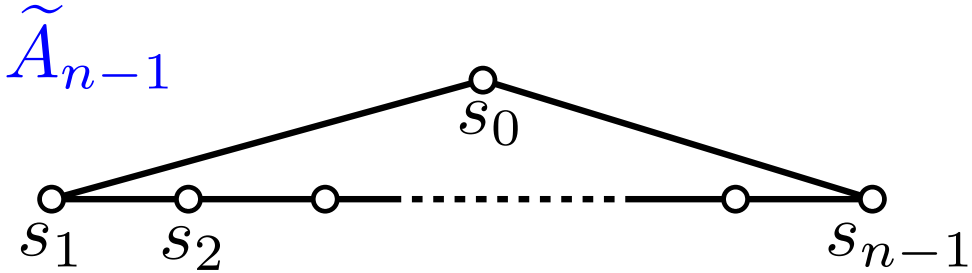

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$\sigma(1)\ \sigma(2)\ \sigma(3)\ \sigma(4)$

Theorem [Green '01] Fully commutative elements of type $\widetilde{A}_{n-1}$ correspond to 321-avoiding permutations.

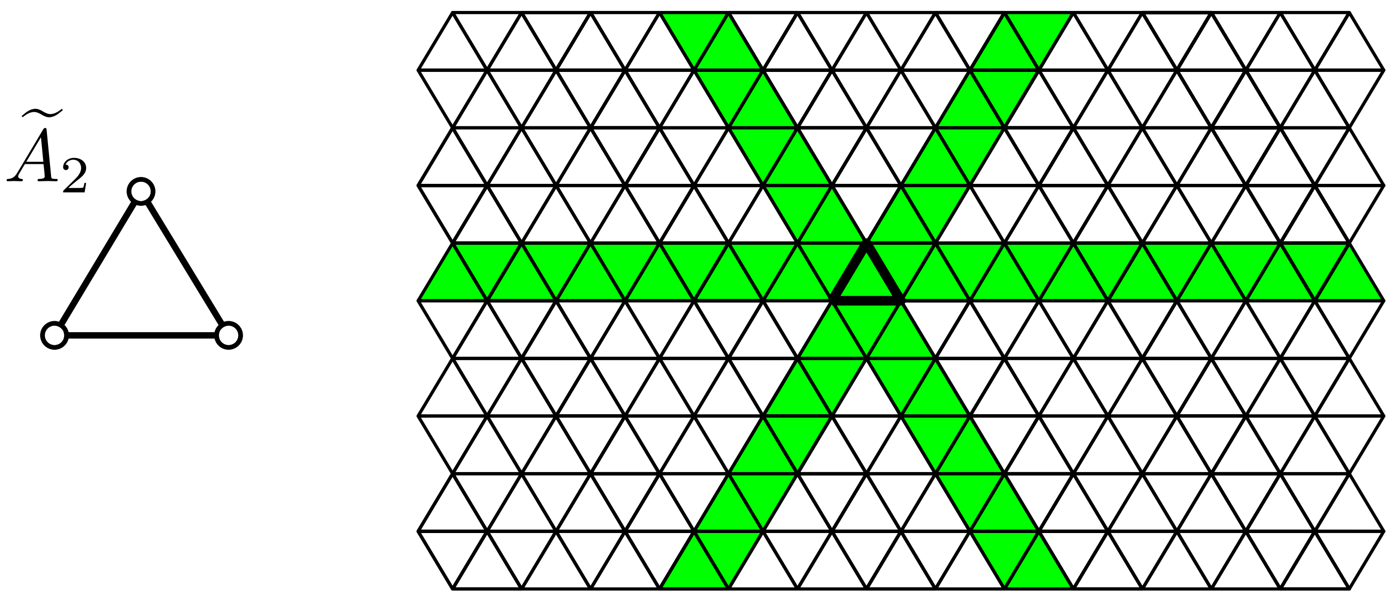
This generalizes [Billey,Jockush,Stanley '93] for type A_{n-1}, i.e. the symmetric group S_n.

Periodicity

Theorem [Hanusa-Jones '09] The sequence $(\widetilde{A}^{FC}_{n-1,l})_{l\geq 0}$ is ultimately periodic of period n.

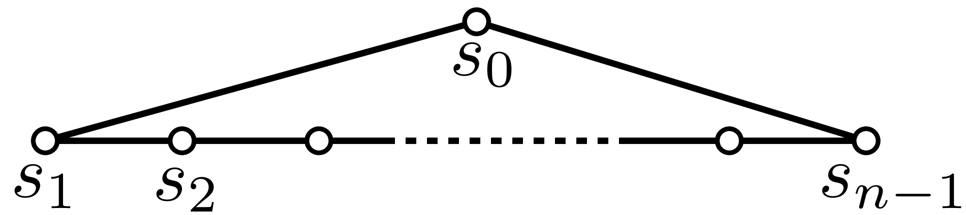
$$\widetilde{A}^{FC}_2(q) = 1 + 3q + \mathbf{6q^2} + \mathbf{6q^3} + \mathbf{6q^4} + \cdots$$

$$\widetilde{A}^{FC}_3(q) = 1 + 4q + 10q^2 + \mathbf{16q^3} + \mathbf{18q^4} + \mathbf{16q^5} + \mathbf{18q^6} + \cdots$$

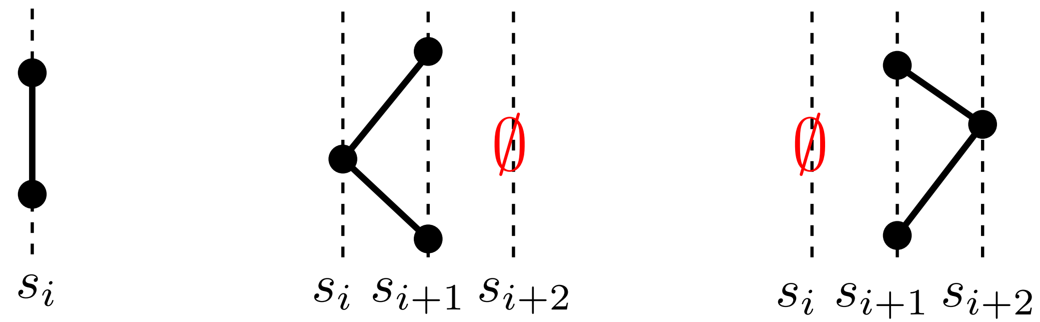


Their proof relies the representation as affine permutations.

FC heaps in type $\widetilde{A}$



$\rightarrow$ FC heaps must avoid



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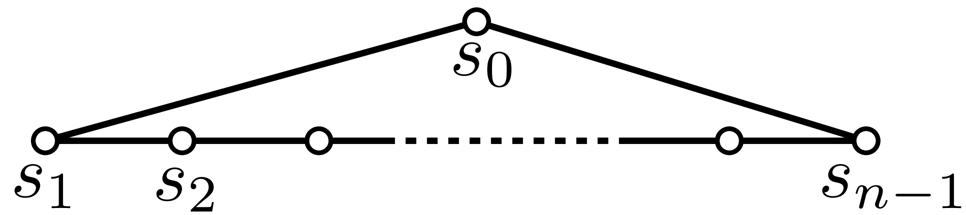

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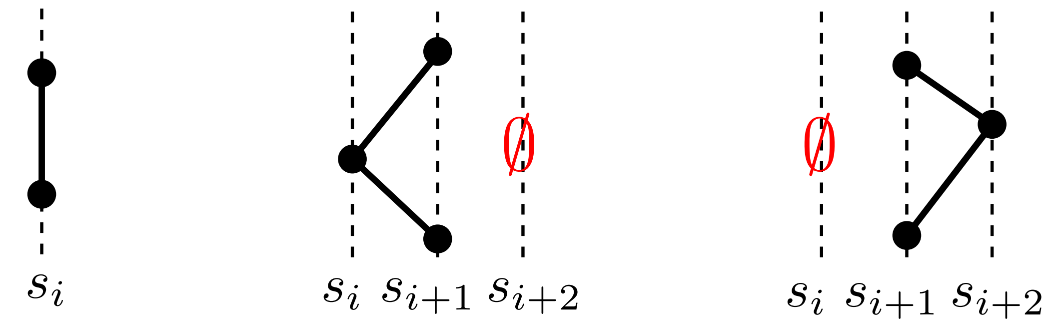
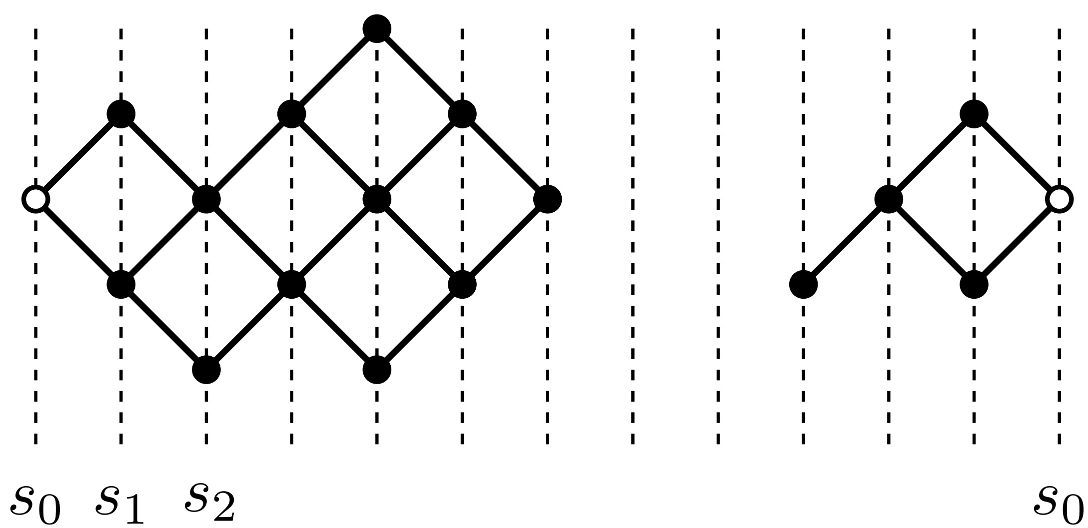
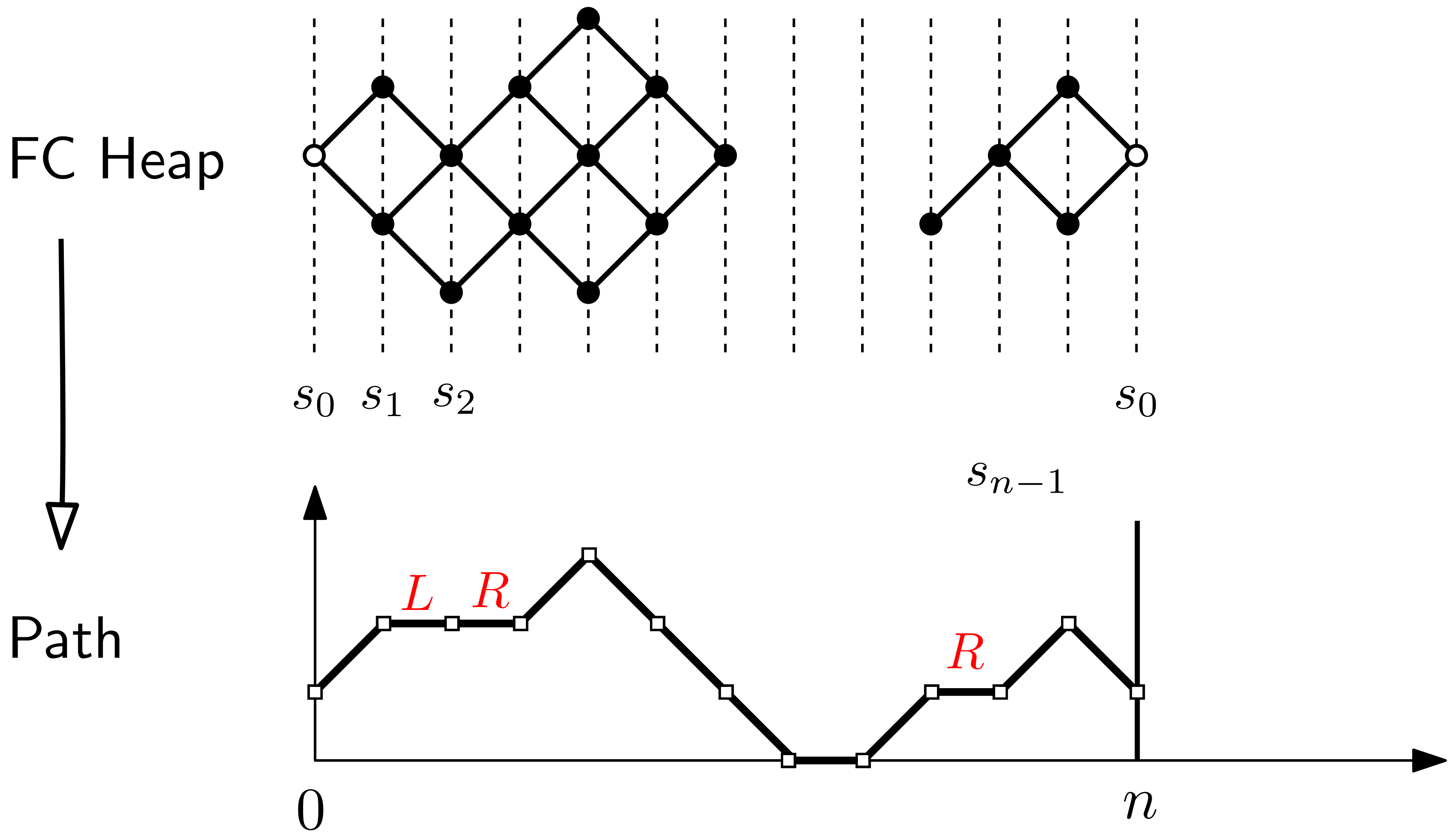


Proposition FC heaps are characterized by:
For all i, $H_{|\{s_i,s_{i+1}\}}$ is a chain with alternating labels

FC Heap



From heaps to paths



- No labels needed at height 0.
- **Size** of the heap $\rightarrow$ **Area** under the path.

From heaps to paths

$\mathcal{O}_n^* =$ Paths ≥ 0, length n:

- Starting height $=$ Ending height.
- Horizontal steps at height $h > 0$ are labeled L or R.

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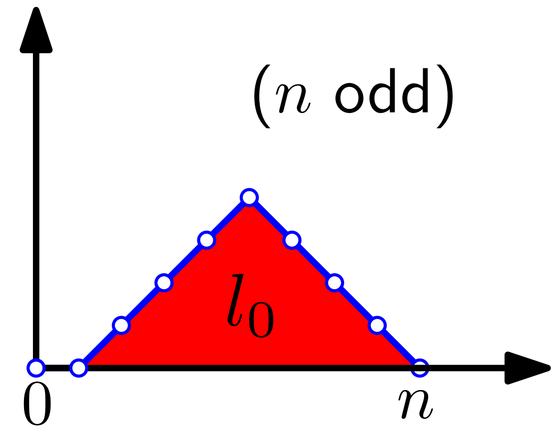
The non-trivial part of the proof is to show surjectivity.

Periodicity: for l large enough, shift the paths up by 1 unit: this is bijective, and the area under the path increases by n.
$\rightarrow$ that the length function is ultimately periodic of period n.

Enumerative results

• "Large enough length" ? Shifting is *not* bijective if the starting path P has a horizontal step at height $h = 0$
$\Rightarrow$ Area$(P) \leq l_0 = \lceil (n-1)/2 \rceil \lfloor (n+1)/2 \rfloor$.



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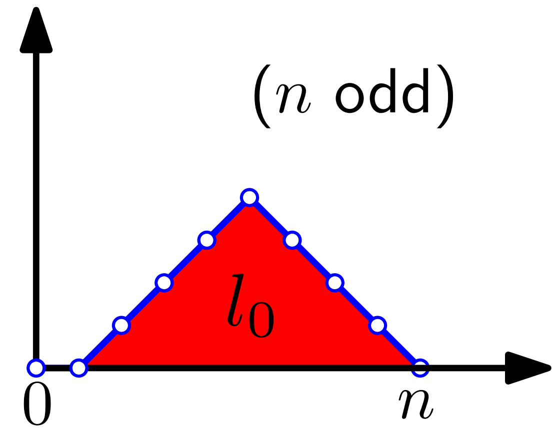


Proposition: Periodicity starts exactly at length $l_0 + 1$.

• $$\widetilde{A}^{FC}_{n-1}(q) = \frac{q^n (X_n(q) - 2)}{1 - q^n} + X_n^*(q)$$

$\sum_{n\geq 0} X_n(q) x^n = Y(x)\left(1 + qx^2 \frac{\partial(xY)}{\partial x}(xq)\right)$	$Y^*(x) = 1 + xY^*(x) + qx(Y^*(x) - 1)Y^*(qx)$
$\sum_{n\geq 0} X_n^*(q) x^n = Y^*(x)\left(1 + qx^2 \frac{\partial(xY)}{\partial x}(xq)\right)$	$Y(x) = \frac{Y^*(x)}{1 - xY^*(x)}$

Minimal period

Theorem [Jouhet, N. '13] The length function of FC elements in type $\widetilde{A}_{n-1}$ has ultimate minimal period:
$$\begin{cases} n & \text{if } n \text{ has at least two distinct prime factors} \\ p^{k-1} & \text{if } n = p^k \end{cases}$$

$$\widetilde{A}_2^{FC}(q) = 1 + 3q + \mathbf{6q^2} + \mathbf{6q^3} + \mathbf{6q^4} + \cdots$$

$$\widetilde{A}_3^{FC}(q) = 1 + 4q + 10q^2 + \mathbf{16q^3} + \mathbf{18q^4} + \mathbf{16q^5} + \mathbf{18q^6} + \cdots$$

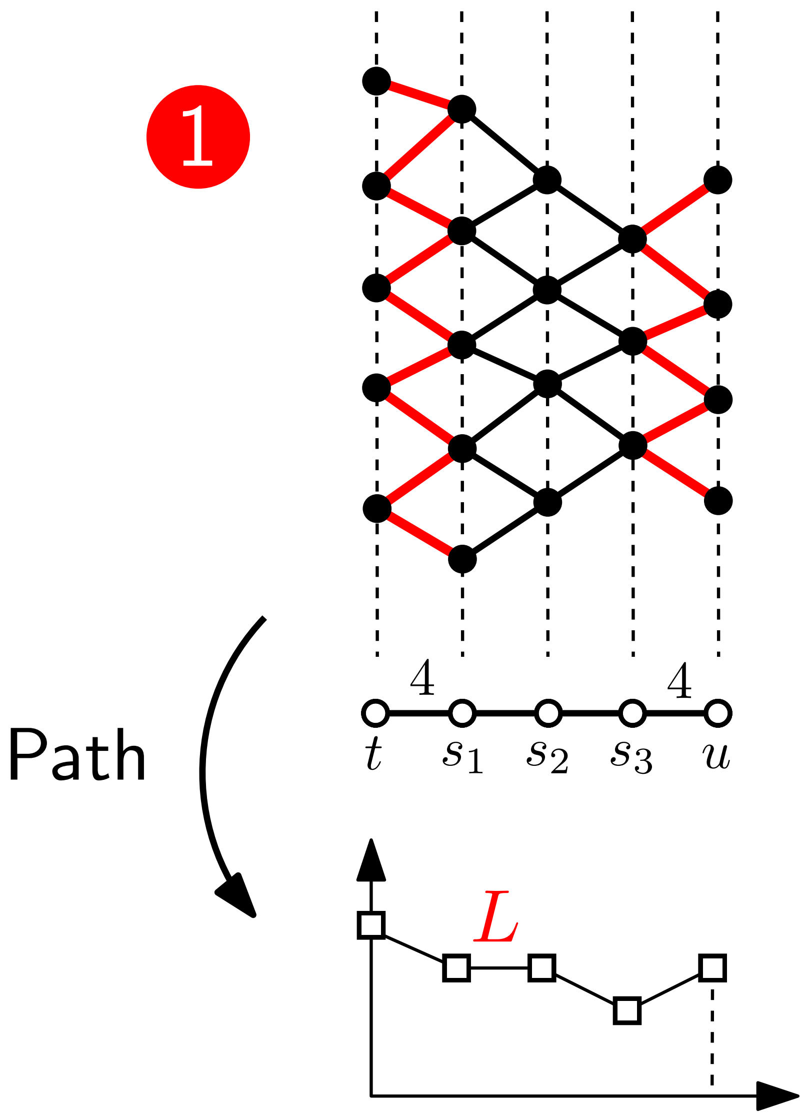
$$\widetilde{A}_4^{FC}(q) = 1 + 5q + 15q^2 + 30q^3 + 45q^4 + \mathbf{50q^5} + \mathbf{50q^6} + \mathbf{50q^7} + \mathbf{50q^8} + \mathbf{50q^9} + \cdots$$

$$\widetilde{A}_5^{FC}(q) = 1 + 6q + 21q^2 + 50q^3 + 90q^4 + 126q^5 + 146q^6 + \mathbf{150q^7} + \mathbf{156q^8} + \mathbf{152q^9} + \mathbf{156q^{10}} + \mathbf{150q^{11}} + \mathbf{158q^{12}} + \mathbf{150q^{13}} + \mathbf{156q^{14}} + \mathbf{152q^{15}} + \mathbf{156q^{16}} + \mathbf{150q^{17}} + \mathbf{158q^{18}} + \cdots$$

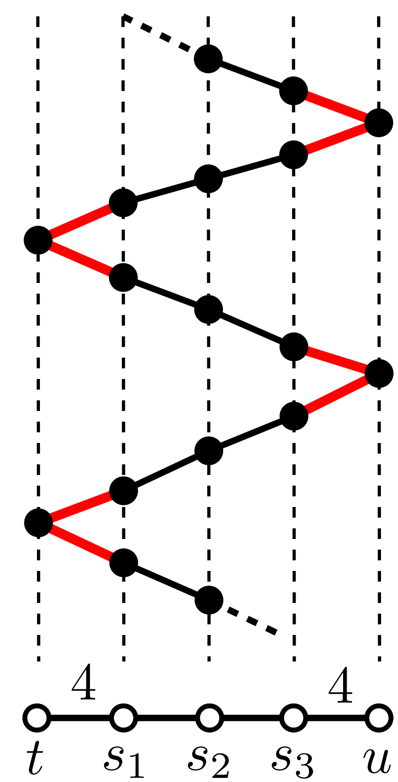
IV. FC elements in other affine types

Type $\widetilde{C}$

Two families of heaps survive **for large enough length**:



2 Finite factors of



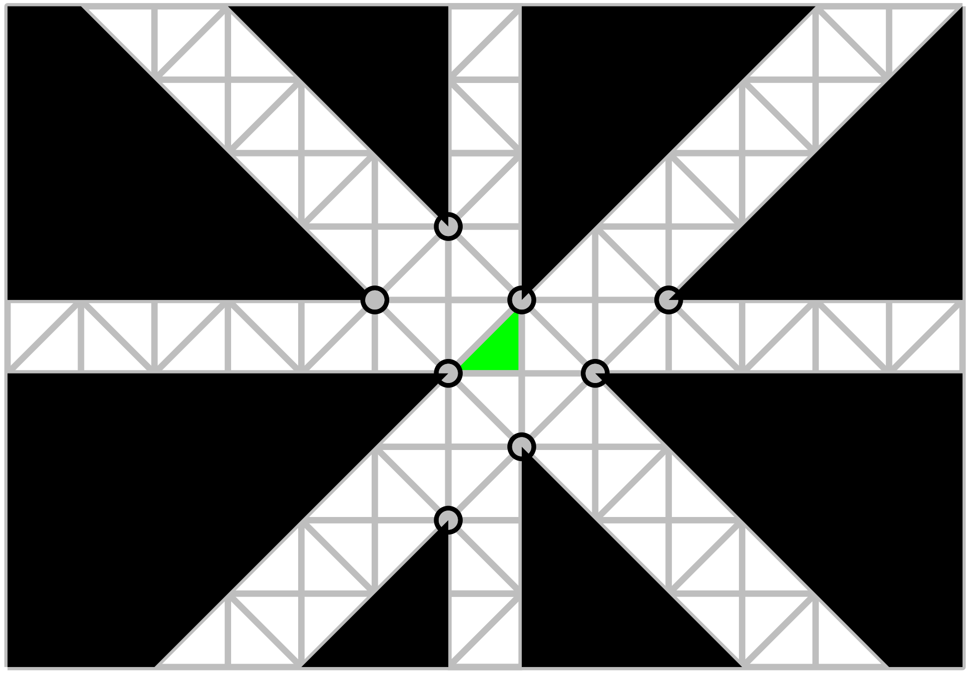
Type $\widetilde{C}$

Here a period is $n+1$. The minimal period can be determined also: it is the largest odd number dividing $n+1$ [JN '13].

The full characterization of FC elements is more complex, as is the generating function.

Types $\widetilde{B}$ and $\widetilde{D}$ very similar.

$\widetilde{C}_2$



Exceptional types