Trivial words in groups
Much ado about nothing

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Murray Elder Buks van Rensburg Thomas Wong

Séminaire Flajolet, June 2013

TWO PROBLEMS LINKED

Two quite different problems

- from geometric group theory — amenability of groups
- from lattice statistical mechanics — ring polymers and random knotting

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- from geometric group theory — amenability of groups
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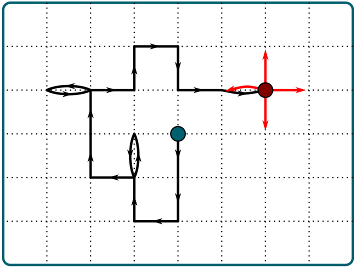
Start with simplest version of both

Random walk on $\mathbb{Z}^2$

Start at $(0, 0)$ and take steps N, S, E, W.



ACTUALLY — 1D IS EVEN SIMPLER

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Start at 0 and take steps E, W

- What is probability of ending at 0?

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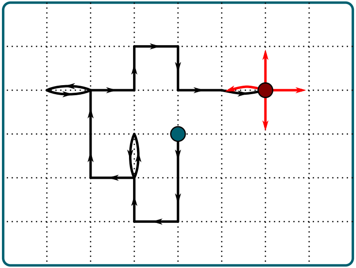
Returning to 0 — only even lengths

$$\Pr(\text{end at } 0) = \frac{\binom{2n}{n}}{2^{2n}} \sim \frac{1}{\sqrt{\pi n}}$$ polynomial decay

Back to 2D



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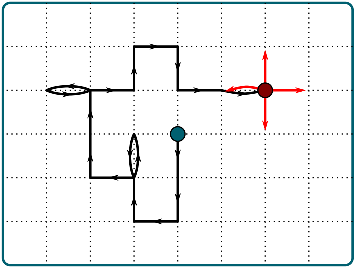

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$$\sum_n c_{n,(0,0)} \cdot z^n = 1 + 4z^2 + 36z^4 + 400z^6 + 4900z^8 + \dots$$

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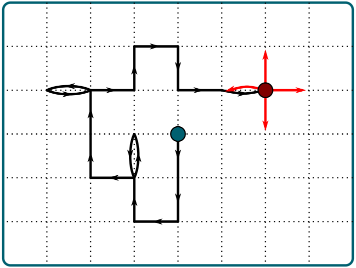

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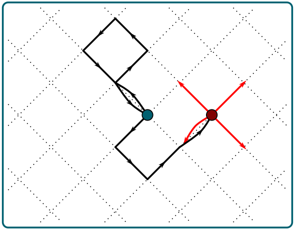
- Why are the terms $\binom{2n}{n}^2$?

ROTATE EVERYTHING



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- changes the x-ordinate by ± 1, and
- changes the y-ordinate by ± 1

So split into two independent 1d problems — each gives $\binom{2n}{n}$.

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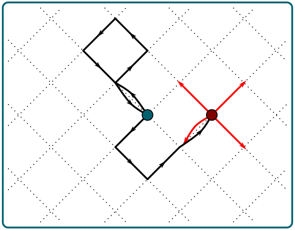

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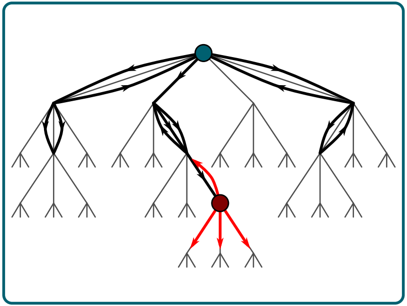
Returning to the origin — only even lengths

$$\Pr(\text{end at origin}) = \binom{2n}{n}^2 4^{-2n} \sim \frac{1}{\pi n} \qquad \text{polynomial decay}$$

Do the same thing on a tree



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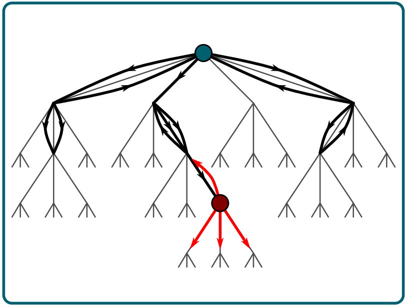

$$\sum_n t_{n,0} z^n = \frac{3}{1 + 2\sqrt{1 - 12z^2}}$$

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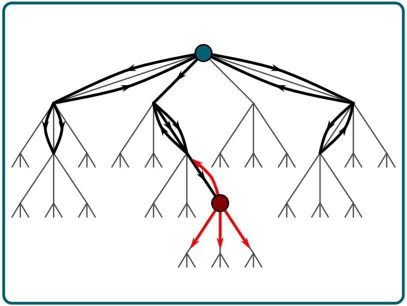

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Return to root vertex — even lengths only

$$\Pr(\text{end at root}) \sim 6\sqrt{\frac{2}{\pi n^3}} \cdot \left(\frac{\sqrt{3}}{2}\right)^n$$

exponential decay

$\mathbb{Z}^2$ AND F_2 ARE EASY CASE OF HARD PROBLEM

These random walks are special cases of bigger problem

Walks on Cayley graph of group

Let $G = \langle a, b \mid \text{ relations } \rangle$

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Amenability [Kesten, Grigorchuk, Cohen]

Let p_n be the number of words of length n in G equivalent to the identity.

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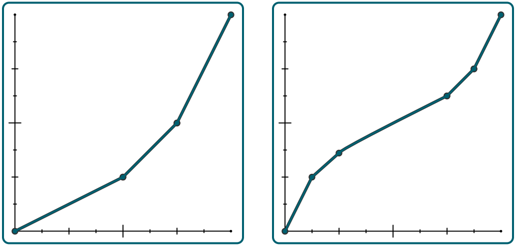
A very open problem for Thompson's group F.

Piecewise linear functions



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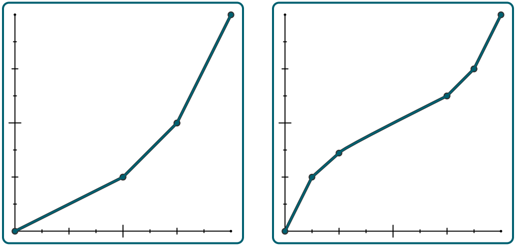

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- coordinates of breakpoints are dyadic rationals $\frac{a}{2^b}$.

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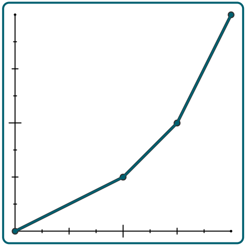

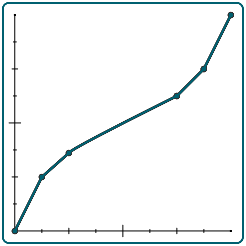

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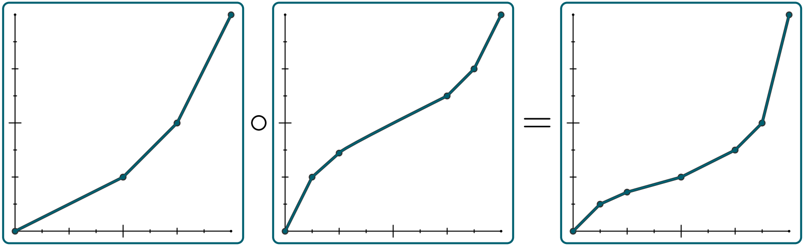

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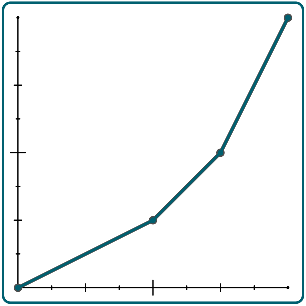
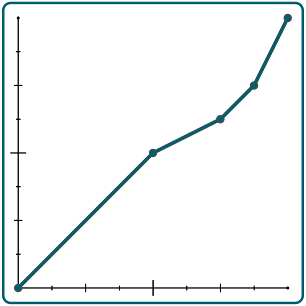
- Everything in this set can be constructed from just 2 functions





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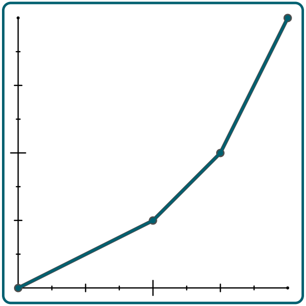

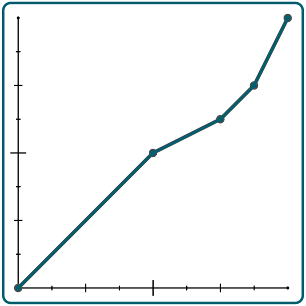

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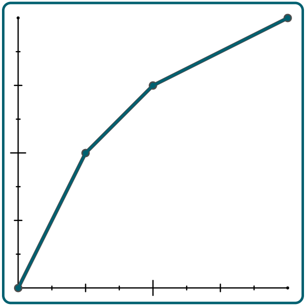

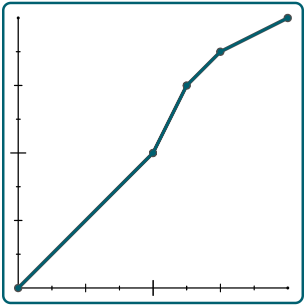

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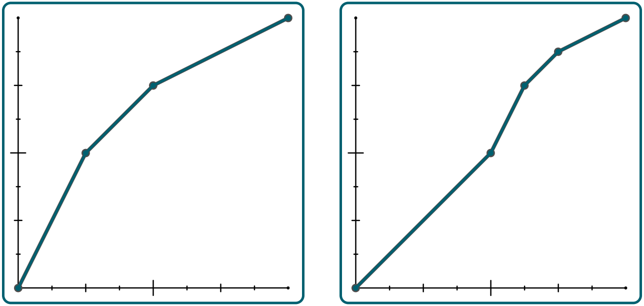

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- The generators obey 2 non-trivial relations

$$\left[x_0 x_1^{-1},\ x_0^{-1} x_1 x_0\right] = \left[x_0 x_1^{-1},\ x_0^{-2} x_1 x_0^2\right] = \text{identity}$$

Thompson's group F

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THOMPSON'S GROUP F — SOME COMBINATORIAL QUESTIONS

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Given a word in F what is the shortest equivalent word?

Growth

How many elements of F are represented by minimal words of length ℓ?

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How many words of n generators are equivalent to the identity?

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Nasty unsolved problem — why not try some stat-mech?

Some easy group theory

Murray was teaching me some group theory…

Easy problem on $\mathbb{Z}^2$

Given a sequence of steps compute distance of endpoint from origin

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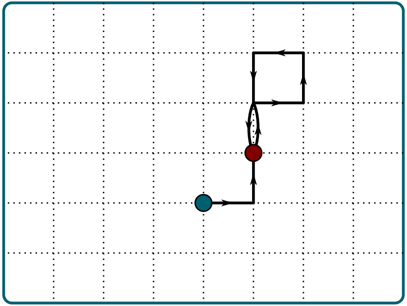
$$ab$$

- Distance is length of remainder — geodesic normal form

Again, but with pictures



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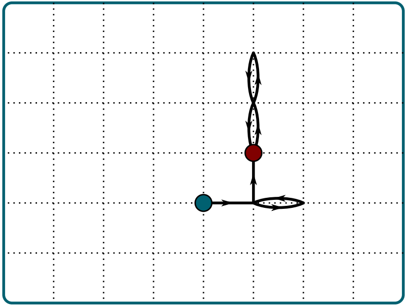

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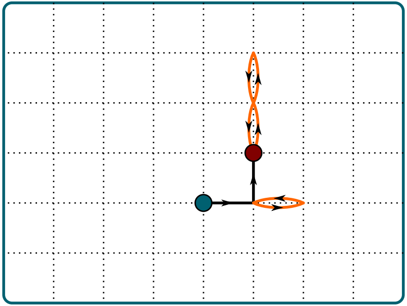
AGAIN, BUT WITH PICTURES



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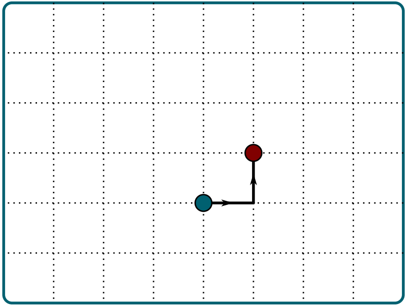

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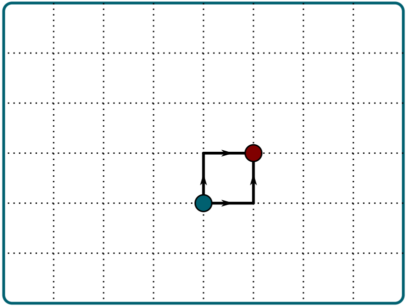

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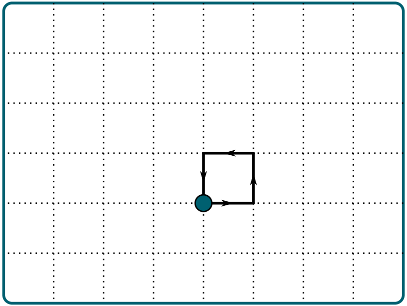
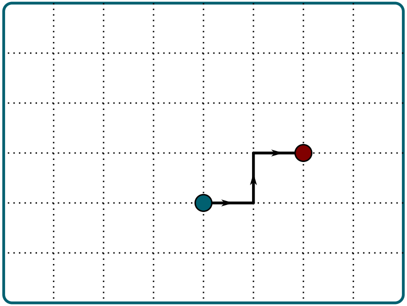


Walks on Cayley graph

$\mathbb{Z}^2$ is the group $\langle a, b \mid ab\bar{a}\bar{b} = \cdot \rangle$

- The generators are the steps
- The relation tells us we can walk around a face.

Look a bit more at commuting



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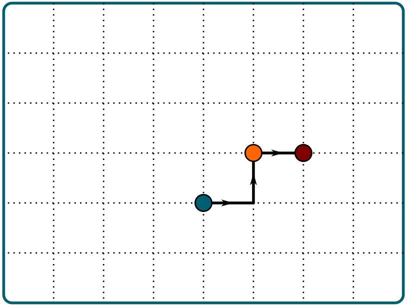

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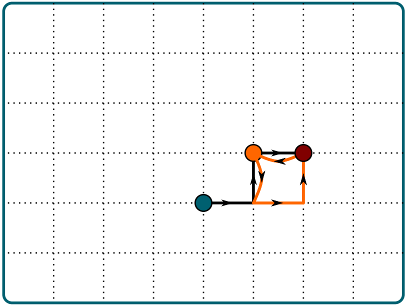

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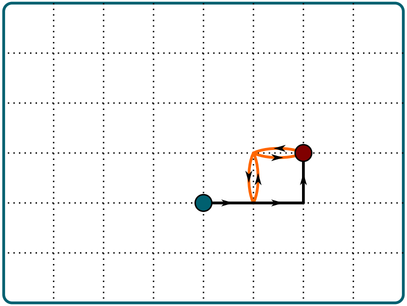

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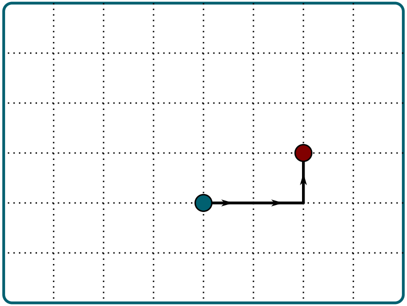

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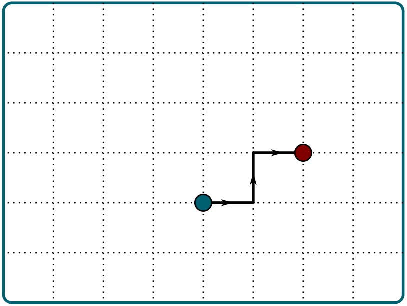

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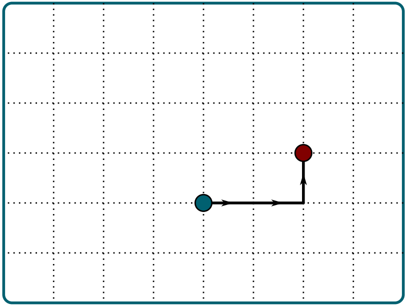

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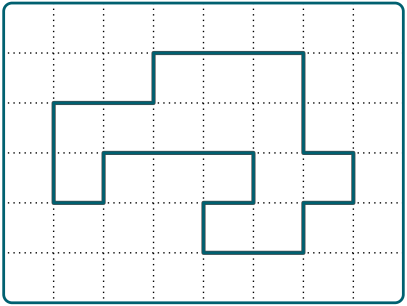

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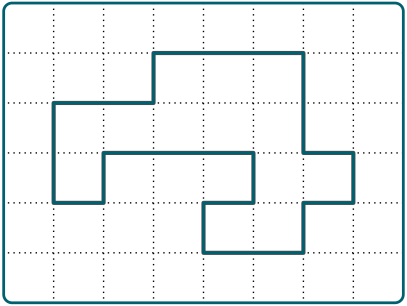

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Stubbornly unsolved, so many numerical methods developed.

Random sampling of SAPs

BFACF on $\mathbb{Z}^2$

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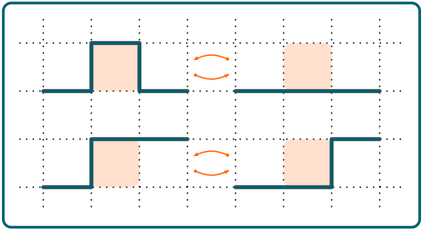

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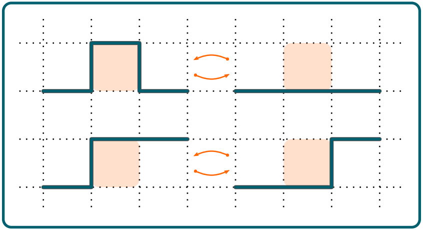

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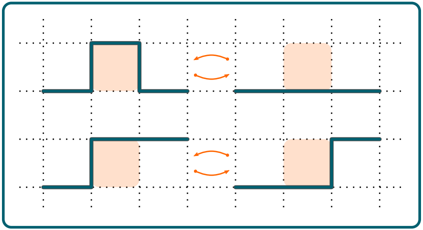
Method of choice for random knots — control over topology

BFACF $\longleftrightarrow ab = ba$

We realised that BFACF moves are just insert-relation & cancel.



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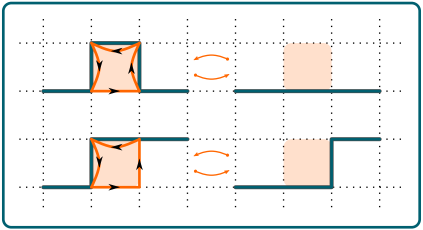

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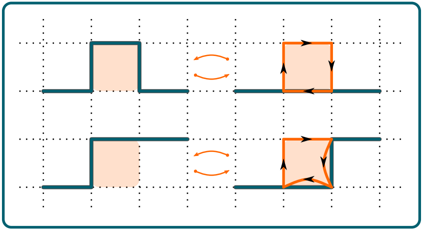

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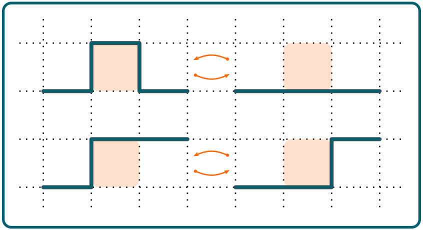

So why not do BFACF on groups?

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Start with empty word, and then do sequence of moves

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Samples freely reduced words $\equiv$ random walks with no backtracking

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- Start with $w = a^k\ \bar{r}\ \bar{a}^k$, then
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Left-insertions uniquely reversible

Insertion of r accepted only if

- cancellations occur to left of r, and
- at most $|r|$ generators canceled.

THE ALGORITHM

BFACF on finitely presented group

Start with $w = \cdot$

- Flip coin to choose left-insertion or conjugation
- Do move $w \mapsto w'$
- Accept move with probability

$$\Pr(\text{accept}) = \begin{cases} 1 & |w'| \leq |w| \\ \beta^{|w'|-|w|} & \text{otherwise} \end{cases}$$

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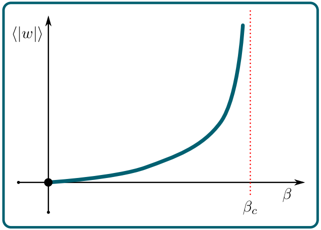

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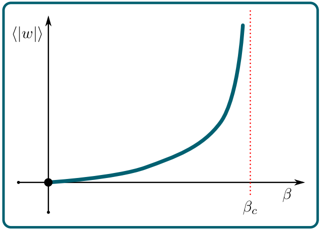

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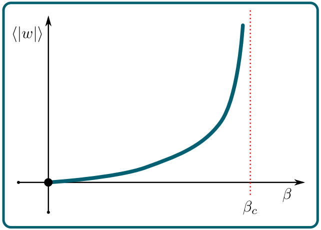

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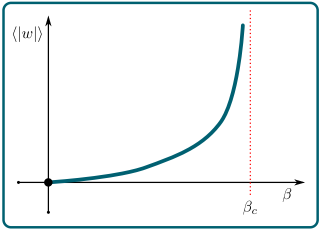

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Plot of mean length $\mapsto$ estimate of β_c $\mapsto$ decide amenability

Warm up with groups we know

Evangelise

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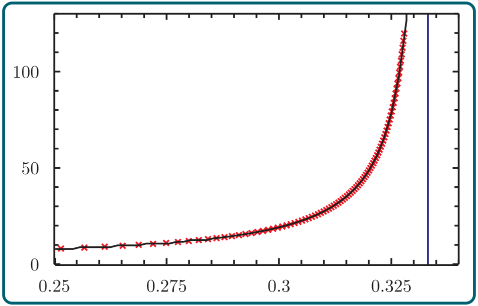

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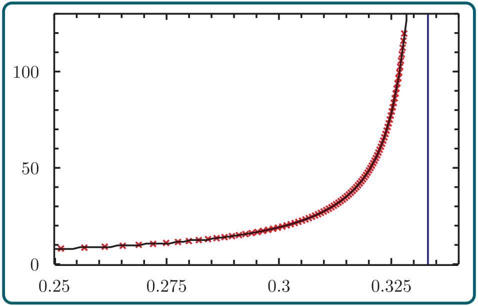

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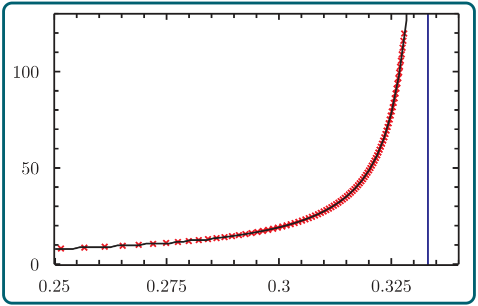

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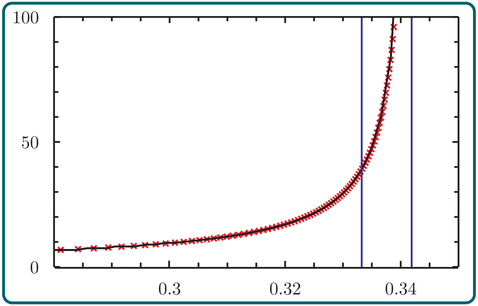

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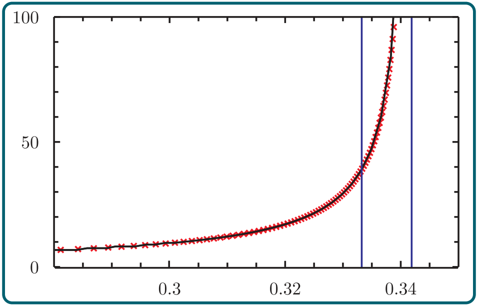

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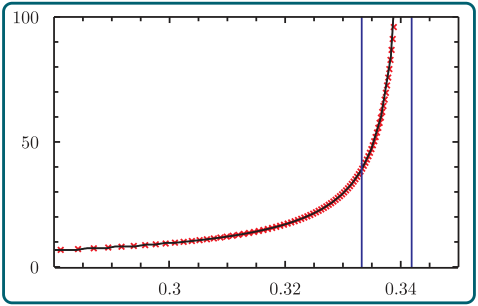

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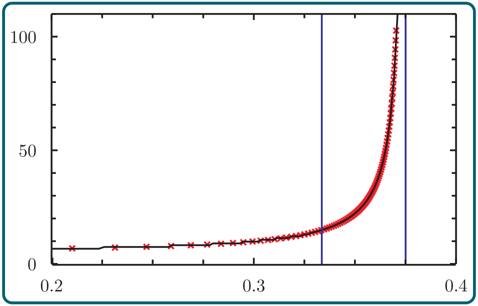

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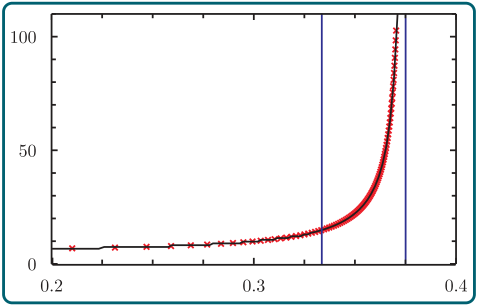

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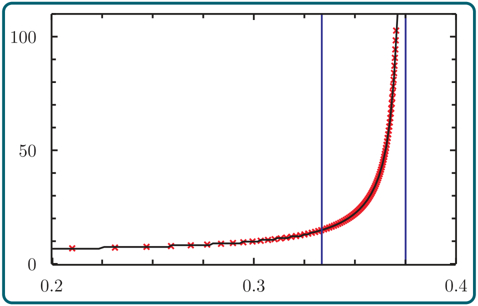

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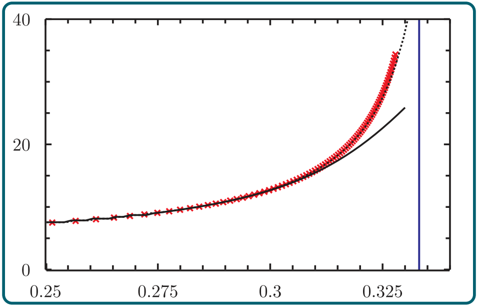

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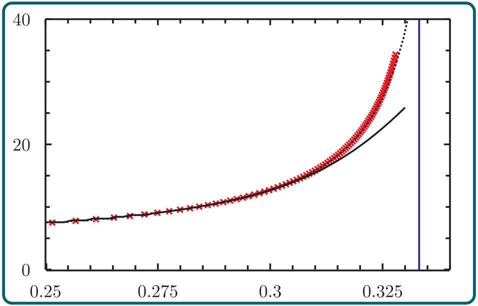

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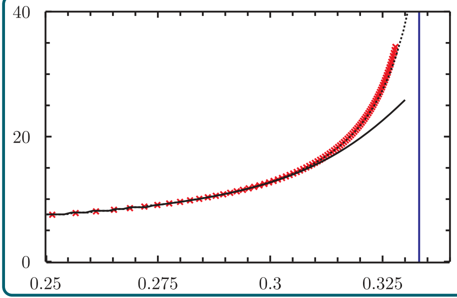

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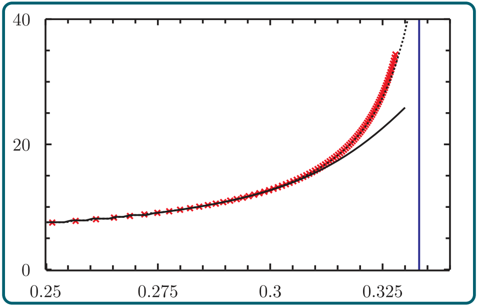

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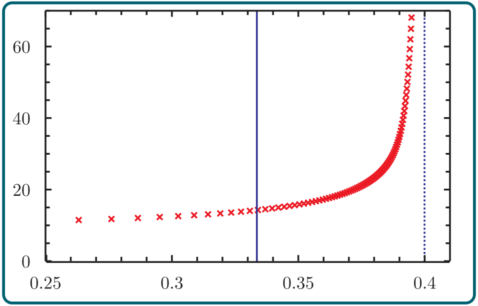

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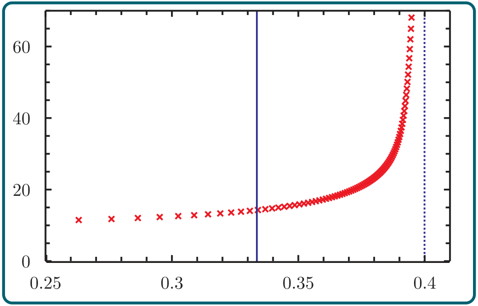

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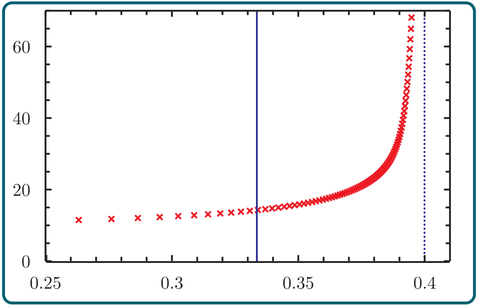

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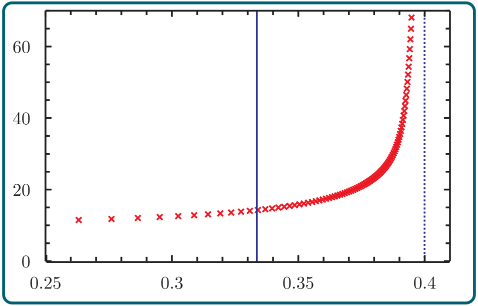

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Data says "Thompson's group is not amenable."

CHECK A DIFFERENT PRESENTATION OF F

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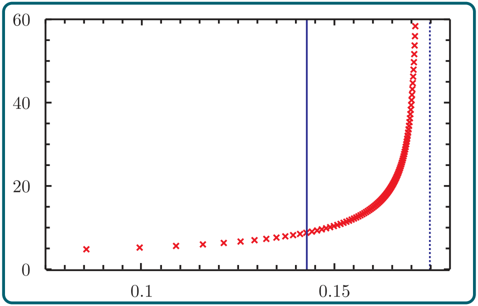

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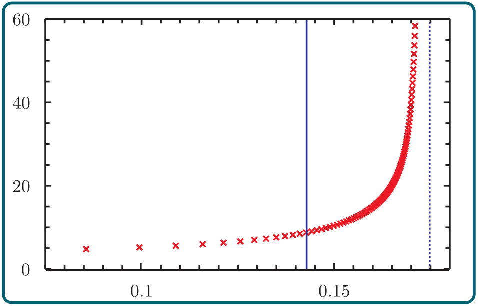

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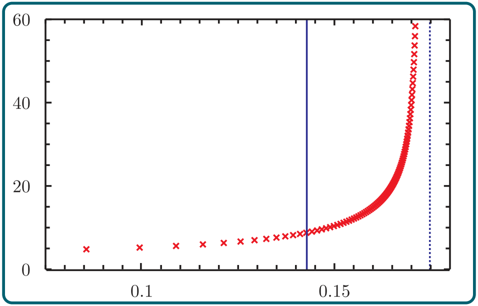

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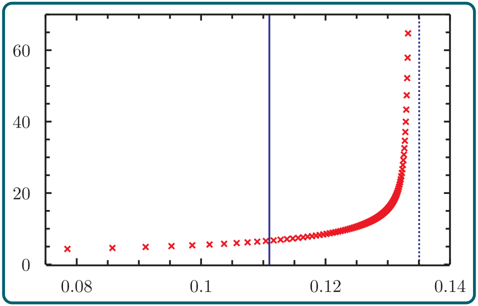

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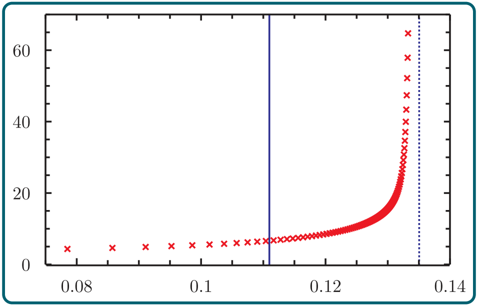

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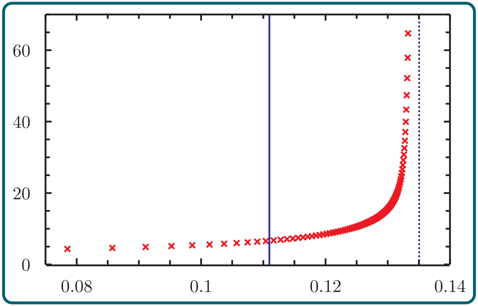

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Thanks for listening.

TRIVIAL WORDS IN BAUMSLAG-SOLITAR GROUPS

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- Key idea — cut group into cosets of $\langle a \rangle$

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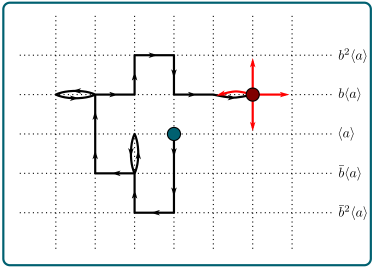

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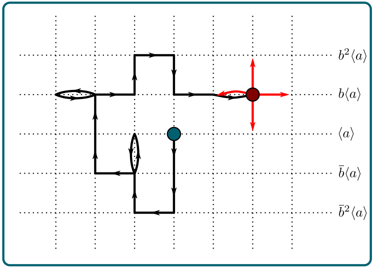

- Horizontal steps move within coset
- Vertical steps move between them.

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Count all walks ending in $\langle a \rangle$:

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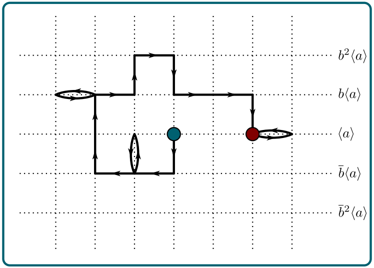

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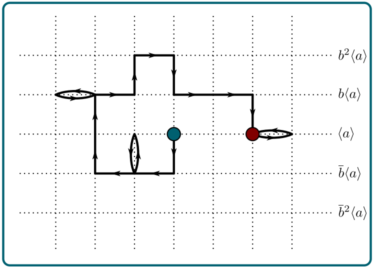

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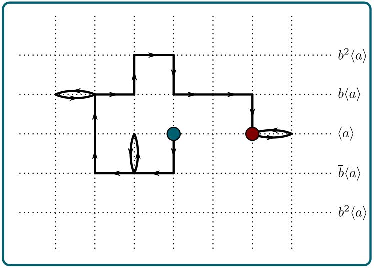

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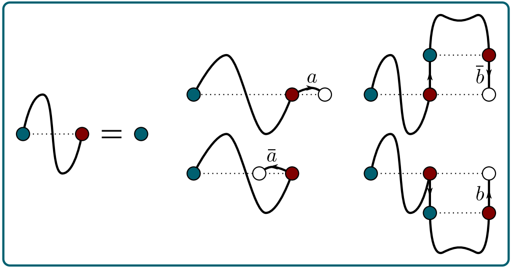
- Cut walk into pieces at each visit to $\langle a \rangle$

SCHEMATIC FACTORISATION



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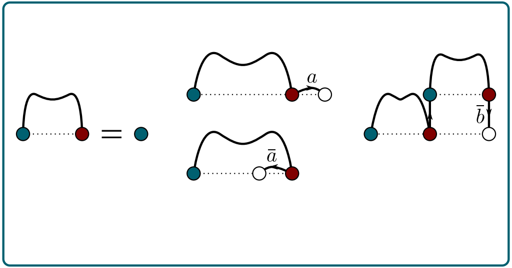

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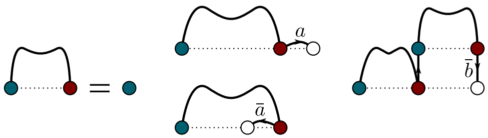

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NOW DO $BS(2,2)$

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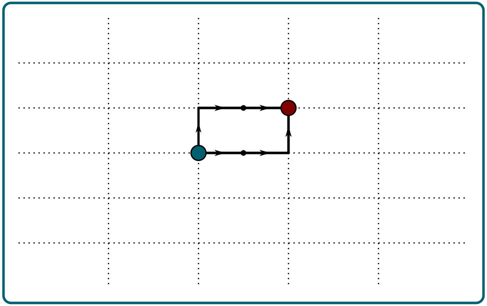

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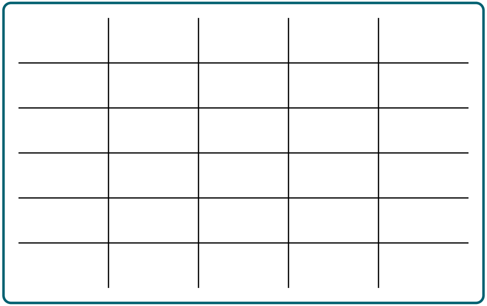

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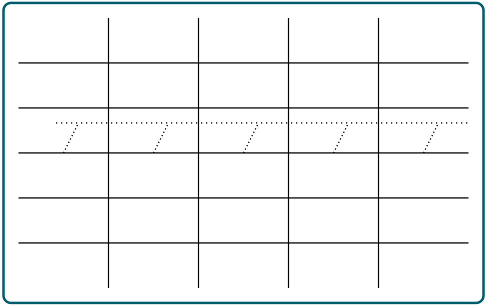

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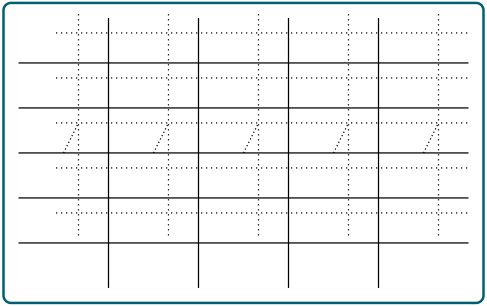

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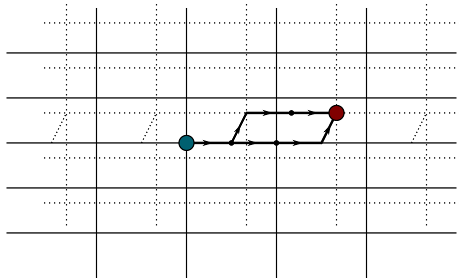

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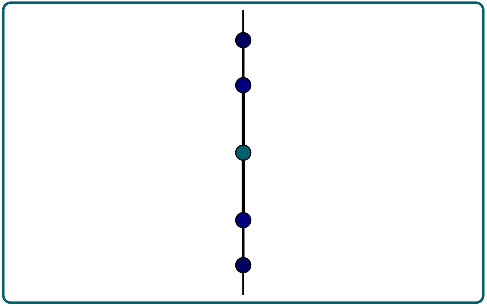

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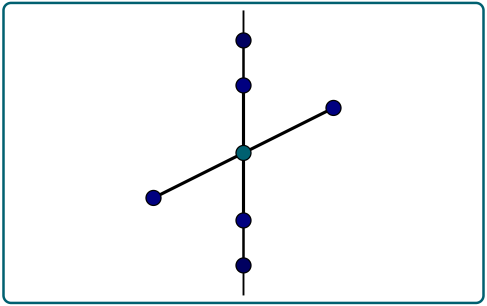

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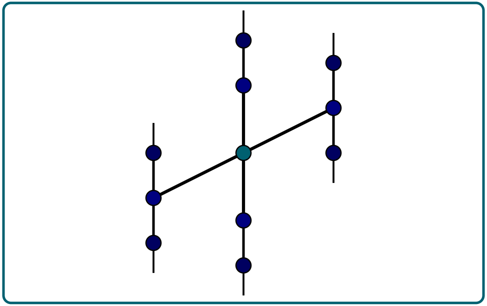

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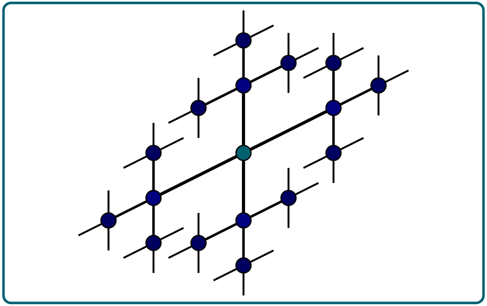

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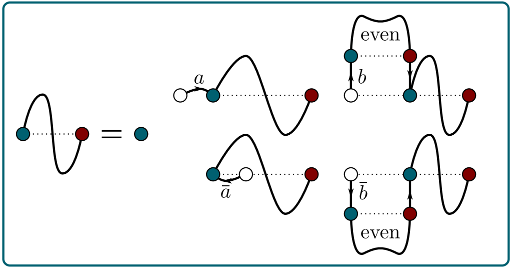

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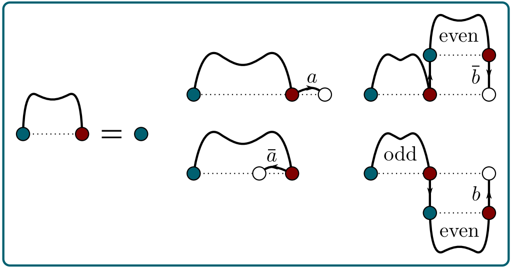

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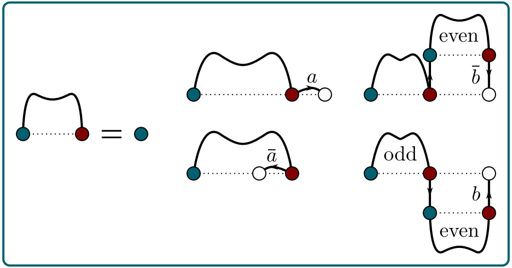

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