Coefficientwise total positivity (via continued fractions) for some Hankel matrices of combinatorial polynomials

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Key references:


2. Viennot, Une théorie combinatoire des polynômes orthogonaux généraux (UQAM, 1983).
Positive semidefiniteness vs. total positivity

Compare the following two properties for matrices $A \in \mathbb{R}^{m \times n}$:

- $A$ is called positive semidefinite if it is square $(m = n)$, symmetric, and all its principal minors are nonnegative (i.e. $\det A_{II} \geq 0$ for all $I \subseteq [n]$).

- $A$ is called totally positive if all its minors are nonnegative (i.e. $\det A_{IJ} \geq 0$ for all $I \subseteq [m]$ and $J \subseteq [n]$).

From the point of view of general linear algebra:

- Positive semidefiniteness is natural: it is equivalent to the positive semidefiniteness of a quadratic form on a vector space, and hence is basis-independent.

- Total positivity is unnatural: it is grossly basis-dependent.

This talk is about the “unnatural” property of total positivity.
Positive semidefiniteness vs. total positivity

Compare the following two properties for matrices $A \in \mathbb{R}^{m \times n}$:

- $A$ is called **positive semidefinite** if it is square ($m = n$), symmetric, and all its **principal** minors are nonnegative (i.e. $\det A_{II} \geq 0$ for all $I \subseteq [n]$).

- $A$ is called **totally positive** if all its minors are nonnegative (i.e. $\det A_{IJ} \geq 0$ for all $I \subseteq [m]$ and $J \subseteq [n]$).

From the point of view of general linear algebra:

- Positive semidefiniteness is **natural**: it is equivalent to the positive semidefiniteness of a quadratic form on a vector space, and hence is basis-independent.

- Total positivity is **unnatural**: it is grossly basis-dependent.

This talk is about the “unnatural” property of total positivity.

**What total positivity is really about:**
Functions $F: S \times T \to R$ where

- $S$ and $T$ are **totally ordered** sets, and

- $R$ is a **partially ordered commutative ring**
  (traditionally $R = \mathbb{R}$, but we will generalize this)
Some references on total positivity

The classics:


Two recent books:


Applications to combinatorics:


Log-concavity and log-convexity in combinatorics

A sequence \((a_i)_{i \in I}\) of nonnegative real numbers (indexed by an interval \(I \subset \mathbb{Z}\)) is called

- \textit{log-concave} if \(a_{n-1}a_{n+1} \leq a_n^2\) for all \(n\)
- \textit{log-convex} if \(a_{n-1}a_{n+1} \geq a_n^2\) for all \(n\)

Many important combinatorial sequences are log-concave (cf. Stanley 1989 review article) or log-convex.

For a triangular array \(T_{n,k}\) (\(0 \leq k \leq n\)), typically:

- “Horizontal sequences” (\(n\) fixed, \(k\) varying) are log-concave.
- “Vertical” sequence of row sums is log-convex.

\textbf{Examples:} Binomial coefficients, Stirling numbers of both kinds, Eulerian numbers, …

Proofs can be combinatorial or analytic.
Strengthenings of log-concavity and log-convexity: Toeplitz- and Hankel-total positivity

To each two-sided-infinite sequence \( \mathbf{a} = (a_k)_{k \in \mathbb{Z}} \) we associate the **Toeplitz matrix**

\[
T_{\infty}(\mathbf{a}) = (a_{j-i})_{i,j \geq 0} = \begin{pmatrix}
a_0 & a_1 & a_2 & \cdots \\
a_{-1} & a_0 & a_1 & \cdots \\
a_{-2} & a_{-1} & a_0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

If \( \mathbf{a} \) is one-sided infinite \((a_0, a_1, \ldots)\) or finite \((a_0, a_1, \ldots, a_n)\), set all “missing” entries to zero.

- We say that the sequence \( \mathbf{a} \) is **Toeplitz-totally positive** if the Toeplitz matrix \( T_{\infty}(\mathbf{a}) \) is totally positive. [Also called “Pólya frequency sequence”.

- This implies that the sequence is **log-concave**, but is much stronger.

To each one-sided-infinite sequence \( \mathbf{a} = (a_k)_{k \geq 0} \) we associate the **Hankel matrix**

\[
H_{\infty}(\mathbf{a}) = (a_{i+j})_{i,j \geq 0} = \begin{pmatrix}
a_0 & a_1 & a_2 & \cdots \\
a_1 & a_2 & a_3 & \cdots \\
a_2 & a_3 & a_4 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

- We say that the sequence \( \mathbf{a} \) is **Hankel-totally positive** if the Hankel matrix \( H_{\infty}(\mathbf{a}) \) is totally positive.

- This implies that the sequence is **log-convex**, but is much stronger.
Characterization of Toeplitz-total positivity


1. Finite sequence \((a_0, a_1, \ldots, a_n)\) is Toeplitz-TP iff the polynomial

\[ P(z) = \sum_{k=0}^{n} a_k z^k \]

has all its zeros in \((-\infty, 0]\).

2. One-sided infinite sequence \((a_0, a_1, \ldots)\) is Toeplitz-TP iff

\[
\sum_{k=0}^{\infty} a_k z^k = \frac{\prod_{i=1}^{\infty} (1 + \alpha_i z)}{\prod_{i=1}^{\infty} (1 - \beta_i z)}
\]

in some neighborhood of \(z = 0\), with \(\alpha_i, \beta_i \geq 0\) and \(\sum_i \alpha_i, \sum_i \beta_i < \infty\).

3. Similar but more complicated representation for two-sided-infinite sequences.

Proofs of \#2 and \#3 rely on Nevanlinna theory of meromorphic functions.

Open problem: Find a more elementary proof.

See Brenti for many combinatorial applications of Toeplitz-total positivity.
Characterization of Hankel-total positivity

For a sequence $a = (a_k)_{k \geq 0}$, define also the $m$-shifted Hankel matrix

$$H^{(m)}(a) = (a_{i+j+m})_{i,j \geq 0} = \begin{pmatrix}
a_m & a_{m+1} & a_{m+2} & \cdots \\
a_{m+1} & a_{m+2} & a_{m+3} & \cdots \\
a_{m+2} & a_{m+3} & a_{m+4} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

Recall that the sequence $a$ is \textit{Hankel-totally positive} in case the Hankel matrix $H^{(0)}_{\infty}(a)$ is totally positive.

**Fundamental result** (Stieltjes 1894, Gantmakher–Krein 1937, \ldots): For a sequence $a = (a_k)_{k=0}^{\infty}$ of real numbers, the following are equivalent:

(a) $H^{(0)}_{\infty}(a)$ is totally positive.

(b) Both $H^{(0)}_{\infty}(a)$ and $H^{(1)}_{\infty}(a)$ are positive-semidefinite.

(c) There exists a positive measure $\mu$ on $[0, \infty)$ such that $a_k = \int x^k \, d\mu(x)$ for all $k \geq 0$.

[That is, $(a_k)_{k \geq 0}$ is a Stieltjes moment sequence.]

(d) There exist numbers $\alpha_0, \alpha_1, \ldots \geq 0$ such that

$$\sum_{k=0}^{\infty} a_k t^k = \frac{\alpha_0}{1 - \frac{\alpha_1 t}{1 - \frac{\alpha_2 t}{1 - \cdots}}}$$

in the sense of formal power series.

[Steltjes-type continued fraction with nonnegative coefficients]
From numbers to polynomials
[or, From counting to counting-with-weights]

Some simple examples:

1. Counting subsets of \([n]\): \(a_n = 2^n\)

   Counting subsets of \([n]\) by cardinality: \(P_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k\)

2. Counting partitions of \([n]\): \(a_n = B_n\) (Bell number)

   Counting partitions of \([n]\) by number of blocks:
   \(P_n(x) = \sum_{k=0}^{n} \left\{ \frac{n}{k} \right\} x^k\) (Bell polynomial)

3. Counting non-crossing partitions of \([n]\): \(a_n = C_n\) (Catalan number)

   Counting non-crossing partitions of \([n]\) by number of blocks:
   \(P_n(x) = \sum_{k=0}^{n} N(n, k) x^k\) (Narayana polynomial)

4. Counting permutations of \([n]\): \(a_n = n!\)

   Counting permutations of \([n]\) by number of cycles:
   \(P_n(x) = \sum_{k=0}^{n} \frac{n}{k} x^k\)

   Counting permutations of \([n]\) by number of descents:
   \(P_n(x) = \sum_{k=0}^{n} \langle n \rangle_k x^k\) (Eulerian polynomial)

An industry in combinatorics: \(q\)-Narayana polynomials, \(p, q\)-Bell polynomials, \ldots
Sequences and matrices of polynomials

- Consider sequences and matrices whose entries are polynomials with real coefficients in one or more indeterminates $x$.

- $P \succeq 0$ means that $P$ has nonnegative coefficients. ("coefficientwise partial order on the ring $\mathbb{R}[x]"\)

- More generally, consider sequences and matrices with entries in a partially ordered commutative ring $R$.

We say that a sequence $(a_i)_{i \in I}$ of nonnegative elements of $R$ is

- **log-concave** if $a_{n-1}a_{n+1} - a_n^2 \leq 0$ for all $n$  

- **strongly log-concave** if $a_{k-1}a_{l+1} - a_ka_l \leq 0$ for all $k \leq l$  

- **log-convex** if $a_{n-1}a_{n+1} - a_n^2 \geq 0$ for all $n$  

- **strongly log-convex** if $a_{k-1}a_{l+1} - a_ka_l \geq 0$ for all $k \leq l$

For sequences of real numbers,

- Strongly log-concave $\iff$ log-concave with no internal zeros.  

- Strongly log-convex $\iff$ log-convex.

But on $\mathbb{R}[x]$ this is not so:

**Example:** The sequence $(a_0, a_1, a_2, a_3)$ with  

$$a_0 = a_3 = 2 + x + 3x^2$$  

$$a_1 = a_2 = 1 + 2x + 2x^2$$

is log-convex but not strongly log-convex.

We say that a matrix with entries in $R$ is **totally positive** if every minor is nonnegative (in $R$).

Toeplitz (resp. Hankel) total positivity implies the **strong** log-concavity (resp. **strong** log-convexity).
Coefficientwise Hankel-total positivity for sequences of polynomials

Many interesting sequences of polynomials \((P_n(x))_{n \geq 0}\) have been proven in recent years to be coefficientwise (strongly) log-convex:

- Binomials \(\sum_{k=0}^{n} \binom{n}{k} x^k = (1 + x)^n\) [trivial]

- Bell polynomials \(B_n(x) = \sum_{k=0}^{n} \{\binom{n}{k}\} x^k\)
  

- Narayana polynomials \(N_n(x) = \sum_{k=0}^{n} N(n, k) x^k\)
  
  (Chen–Wang–Yang 2010)

- Narayana polynomials of type B: \(W_n(x) = \sum_{k=0}^{n} (\binom{n}{k})^2 x^k\)
  

- Eulerian polynomials \(A_n(x) = \sum_{k=0}^{n} \langle \binom{n}{k} \rangle x^k\)
  

Might these sequences actually be coefficientwise Hankel-totally positive?

- In many cases I can prove that the answer is yes, by using the Flajolet–Viennot method of continued fractions.

- In several other cases I have strong empirical evidence that the answer is yes, but no proof.

- The continued-fraction approach gives a sufficient but not necessary condition for coefficientwise Hankel-total positivity.
The combinatorics of continued fractions (Flajolet 1980)

Let $\mathbf{a} = (a_n)_{n \geq 0}$ be a sequence of elements in a commutative ring $R$. We associate to $\mathbf{a}$ the formal power series

$$f(t) = \sum_{n=0}^{\infty} a_n t^n \in R[[t]]$$

We now consider two types of continued fractions:

- Continued fractions of Stieltjes type (S-type):
  
  $$f(t) = \frac{1}{1 - \frac{\alpha_1 t}{1 - \frac{\alpha_2 t}{1 - \frac{\alpha_3 t}{1 - \cdots}}}}$$

  which we denote by $S(t; \alpha)$ where $\alpha = (\alpha_n)_{n \geq 1}$.

- Continued fractions of Jacobi type (J-type):
  
  $$f(t) = \frac{1}{1 - \gamma_0 t - \frac{\beta_1 t^2}{1 - \frac{\beta_2 t^2}{1 - \frac{\beta_3 t^2}{1 - \cdots}}}}$$

  which we denote by $J(t; \beta, \gamma)$ where $\beta = (\beta_n)_{n \geq 1}$ and $\gamma = (\gamma_n)_{n \geq 0}$. 

The combinatorics of continued fractions (continued)

**Theorem** (Flajolet 1980): As an identity in $\mathbb{Z}[\alpha][[t]]$, we have

$$
\frac{1}{1 - \alpha_1 t} = \sum_{n=0}^{\infty} S_n(\alpha_1, \ldots, \alpha_n) t^n
$$

where $S_n(\alpha_1, \ldots, \alpha_n)$ is the generating polynomial for Dyck paths of length $2n$ in which each fall starting at height $i$ gets weight $\alpha_i$.

$S_n(\alpha)$ is called the **Stieltjes–Rogers polynomial** of order $n$.

**Theorem** (Flajolet 1980): As an identity in $\mathbb{Z}[\beta, \gamma][[t]]$, we have

$$
\frac{1}{1 - \gamma_0 t - \frac{\beta_1 t^2}{1 - \gamma_1 t - \frac{\beta_2 t^2}{1 - \gamma_2 t - \cdots}}} = \sum_{n=0}^{\infty} J_n(\beta, \gamma) t^n
$$

where $J_n(\beta, \gamma)$ is the generating polynomial for Motzkin paths of length $n$ in which each level step at height $i$ gets weight $\gamma_i$ and each fall starting at height $i$ gets weight $\beta_i$.

$J_n(\beta, \gamma)$ is called the **Jacobi–Rogers polynomial** of order $n$. 

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Hankel matrix of Stieltjes–Rogers polynomials

Now form the infinite Hankel matrix corresponding to the sequence \( \mathbf{S} = (S_n(\alpha))_{n \geq 0} \) of Stieltjes–Rogers polynomials:

\[
H_\infty(\mathbf{S}) = (S_{i+j}(\alpha))_{i,j \geq 0}
\]

And consider any minor of \( H_\infty(\mathbf{S}) \):

\[
\Delta_{IJ}(\mathbf{S}) = \det H_{IJ}(\mathbf{S})
\]

where \( I = \{i_1, i_2, \ldots, i_k\} \) with \( 0 \leq i_1 < i_2 < \ldots < i_k \)
and \( J = \{j_1, j_2, \ldots, j_k\} \) with \( 0 \leq j_1 < j_2 < \ldots < j_k \)

**Theorem** (Viennot 1983): The minor \( \Delta_{IJ}(\mathbf{S}) \) is the generating polynomial for families of disjoint Dyck paths \( P_1, \ldots, P_k \) where path \( P_r \) starts at \((-2i_r, 0)\) and ends at \((2j_r, 0)\), in which each fall starting at height \( i \) gets weight \( \alpha_i \).

The proof uses the Karlin–McGregor–Lindström–Gessel–Viennot lemma on families of nonintersecting paths.

**Corollary:** The sequence \( \mathbf{S} = (S_n(\alpha))_{n \geq 0} \) is a Hankel-totally positive sequence in the polynomial ring \( \mathbb{Z}[\alpha] \) equipped with the coefficientwise partial order.

Now specialize \( \alpha \) to nonnegative elements in any partially ordered commutative ring:

**Corollary:** Let \( \alpha = (\alpha_n)_{n \geq 0} \) be a sequence of nonnegative elements in a partially ordered commutative ring \( R \). Then \( (S_n(\alpha))_{n \geq 0} \) is a Hankel-totally positive sequence in \( R \).
Hankel matrix of Stieltjes–Rogers polynomials (continued)

Can also get explicit formulae for the Hankel determinants
\( \Delta_n^{(m)}(S) = \det H_n^{(m)}(S) \) for small \( m \):

**Theorem:**

\[
\Delta_n^{(0)}(S) = (\alpha_1\alpha_2)^{n-1}(\alpha_3\alpha_4)^{n-2} \cdots (\alpha_{2n-3}\alpha_{2n-2})
\]

\[
\Delta_n^{(1)}(S) = \alpha_1^n(\alpha_2\alpha_3)^{n-1}(\alpha_4\alpha_5)^{n-2} \cdots (\alpha_{2n-2}\alpha_{2n-1})
\]

These formulae are classical in the theory of continued fractions, but Viennot 1983 gives a beautiful combinatorial interpretation.

See also Ishikawa–Tagawa–Zeng 2009 for extensions to \( m = 2, 3 \).
Finding Hankel-totally positive sequences of polynomials

A general strategy:

1. Start from a sequence \((c_n)_{n \geq 0}\) of positive real numbers that is a Stieltjes moment sequence, i.e. is Hankel-totally positive.

   [This property is easy to test empirically: just expand the generating series \(\sum_{n=0}^{\infty} c_n t^n\) as an S-type continued fraction and test whether all coefficients \(\alpha_i \geq 0\).]

2. Refine this sequence somehow to a triangular array \((c_{n,k})_{0 \leq k \leq k_{\text{max}}(n)}\) satisfying \(\sum_{k=0}^{k_{\text{max}}(n)} c_{n,k} = c_n\);

   then define the polynomials \(P_n(x) = \sum_{k=0}^{k_{\text{max}}(n)} c_{n,k} x^k\).

3. By construction, the sequence \((P_n(1))_{n \geq 0}\) is Hankel-totally positive; and if we are lucky, we will find that two successively stronger properties of Hankel-total positivity also hold:

   (a) For each real number \(x \geq 0\), the sequence \((P_n(x))_{n \geq 0}\) of real numbers is Hankel-totally positive (i.e. is a Stieltjes moment sequence).

   (b) The sequence \((P_n(x))_{n \geq 0}\) of polynomials is coefficientwise Hankel-totally positive.

• Usually \((c_n)_{n \geq 0}\) will usually be a sequence of positive integers having some combinatorial interpretation, i.e. as the cardinality of some “naturally occurring” set \(S_n\).

• Then the \(c_{n,k}\) will arise from the partition of \(S_n\) into disjoint subsets \(S_{n,k}\) according to some “natural” statistic \(\kappa : S_n \to \mathbb{N}\).
Some examples of combinatorial Stieltjes moment sequences

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<th>(n)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>Continued fraction</th>
<th>(\alpha_{2k-1})</th>
<th>(\alpha_{2k})</th>
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<td>42</td>
<td>132</td>
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<td>(\alpha_1 = 2), (\alpha_2 = 1)</td>
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<td>6</td>
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<td>70</td>
<td>252</td>
<td>924</td>
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<td>(k(k + 1))</td>
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<td>((2k)(2k+1))</td>
<td>((2k)(2k+1))</td>
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</table>

So our polynomial examples will divide naturally into “families”: the Catalan family, the Bell family, the factorial family, etc.

Can also pursue this strategy in reverse:

- Find the S-type continued fraction for the generating series \(\sum_{n=0}^{\infty} c_n t^n\).
- Generalize it by inserting one or more indeterminates \(x\).
- Try to compute the corresponding polynomials \(P_n(x)\) and/or find a combinatorial interpretation for them.

**Caveat:**

- There also exist important combinatorial Stieltjes moment sequences that do not seem to have nice continued fractions.
- Some of them have polynomial refinements that are empirically Hankel-totally positive; but new methods will be needed to prove it!
Example 1: Narayana polynomials

- Narayana numbers \( N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1} \) for \( n \geq k \geq 1 \) with convention \( N(0, k) = \delta_{k0} \)
- They refine Catalan numbers: \( \sum_{k=0}^{n} N(n, k) = C_n \)
- They count numerous objects of combinatorial interest:
  - Dyck paths of length \( 2n \) with \( k \) peaks
  - Non-crossing partitions of \([n]\) with \( k \) blocks
  - Non-nesting partitions of \([n]\) with \( k \) blocks
- Define Narayana polynomials \( N_n(x) = \sum_{k=0}^{n} N(n, k) x^k \)
- Define ordinary generating function \( \mathcal{N}(t, x) = \sum_{n=0}^{\infty} t^n N_n(x) \)
- Elementary “renewal” argument on Dyck paths implies
  \[
  \mathcal{N} = \frac{1}{1 - tx - t(\mathcal{N} - 1)}
  \]
  which can be rewritten as
  \[
  \mathcal{N} = \frac{1}{1 - \frac{xt}{1 - t\mathcal{N}}}
  \]
- Leads immediately to S-type continued fraction
  \[
  \sum_{n=0}^{\infty} t^n N_n(x) = \frac{1}{1 - \frac{xt}{1 - \frac{t}{1 - \frac{xt}{t - \frac{xt}{1 - \frac{t}{1 - \cdots}}}}}}
  \]
  with coefficients \( \alpha_{2k-1} = x, \alpha_{2k} = 1. \)
Narayana polynomials (continued)

Conclusions:

1. The sequence $\mathbf{N} = (N_n(x))_{n \geq 0}$ of Narayana polynomials is coefficientwise Hankel-totally positive. The minor $\Delta_{IJ}(\mathbf{N})$ counts families of disjoint Dyck paths as specified by Viennot 1983, with weights $\alpha_{2k-1} = x$, $\alpha_{2k} = 1$.

2. The first Hankel determinants $\Delta_n^{(m)}(\mathbf{N})$ are

$$\Delta_n^{(0)}(\mathbf{N}) = x^{n(n-1)/2}$$

$$\Delta_n^{(1)}(\mathbf{N}) = x^{n(n+1)/2}$$

Remarks:

1. The strong log-convexity was known previously (Chen–Wang–Yang 2010), but with a much more difficult proof.

2. The formula for $\Delta_n^{(0)}(\mathbf{N})$ was also known (Sivasubramanian 2010), by an explicit bijective argument.
Example 2: Bell polynomials

- Stirling number $\left\{ \begin{array}{c} n \\ k \end{array} \right\} = \# \text{ of partitions of } [n] \text{ with } k \text{ blocks}$
- Convention $\left\{ \begin{array}{c} 0 \\ k \end{array} \right\} = \delta_{k0}$
- They refine Bell numbers: $\sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} = B_n$
- Define Bell polynomials $B_n(x) = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} x^k$
- Define ordinary generating function $B(t, x) = \sum_{n=0}^{\infty} t^n B_n(x)$
- Flajolet (1980) expressed $B(t, x)$ as a J-type continued fraction
- Can be transformed to an S-type continued fraction

$$\sum_{n=0}^{\infty} t^n B_n(x) = \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{1t}{1 - \cfrac{xt}{1 - \cfrac{2t}{1 - \cdots}}}}}$$

with coefficients $\alpha_{2k-1} = x$, $\alpha_{2k} = k$. 

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Bell polynomials (continued)

Conclusions:

1. The sequence $B = (B_n(x))_{n \geq 0}$ of Bell polynomials is coefficientwise Hankel-totally positive. The minor $\Delta_{I,J}(B)$ counts families of disjoint Dyck paths as specified by Viennot 1983, with weights $\alpha_{2k-1} = x, \alpha_{2k} = k$.

2. The first Hankel determinants $\Delta_n^{(m)}(B)$ are

$$\Delta_n^{(0)}(B) = x^{n(n-1)/2} \prod_{i=1}^{n-1} i!$$

$$\Delta_n^{(1)}(B) = x^{n(n+1)/2} \prod_{i=1}^{n-1} i!$$

Remarks:

1. The strong log-convexity was known previously (Chen–Wang–Yang 2011).

2. The formula for $\Delta_n^{(0)}(B)$ has also been known for a long time (Radoux 1979, Ehrenborg 2000).

3. For each real number $x \geq 0$, the sequence $(B_n(x))_{n=0}^{\infty}$ is the moment sequence for the Poisson distribution of expected value $x$:

$$B_n(x) = \sum_{k=0}^{\infty} k^n \left( e^{-x} x^k / k! \right)$$

Hence $(B_n(x))_{n=0}^{\infty}$ is a Hankel-totally positive sequence of real numbers. But the weights $e^{-x} x^k / k!$ here are not nonnegative elements of $\mathbb{R}[x]$ or $\mathbb{R}[[x]]$, so this approach cannot be used to prove the coefficientwise total positivity.
Example 3: Interpolating between Narayana and Bell

- Let \( \pi = \{B_1, B_2, \ldots, B_k\} \) be a partition of \([n]\)
- Associate to \( \pi \) a graph \( G_\pi \) with vertex set \([n]\) such that \( i, j \) are joined by an edge iff they are consecutive elements within the same block
- Always write an edge \( e \) of \( G_\pi \) as a pair \((i, j)\) with \( i < j \)
- We say that edges \( e_1 = (i_1, j_1) \) and \( e_2 = (i_2, j_2) \) of \( G_\pi \) form
  - a crossing if \( i_1 < i_2 < j_1 < j_2 \)
  - a nesting if \( i_1 < i_2 < j_2 < j_1 \)
- We define \( \text{cr}(\pi) \) [resp. \( \text{ne}(\pi) \)] to be number of crossings (resp. nestings) in \( \pi \)
- Write \( |\pi| = k \) for the number of blocks in \( \pi \)
- Now define the three-variable polynomial
  \[
  B_n(x, p, q) = \sum_{\pi \in \Pi_n} x^{|\pi|} p^{\text{cr}(\pi)} q^{\text{ne}(\pi)}
  \]
  with the convention \( B_0(x, p, q) = 1 \)
- \( B_n(x, 0, 1) = B_n(x, 1, 0) = N_n(x) \) and \( B_n(x, 1, 1) = B_n(x) \), so this polynomial generalizes the Narayana and Bell polynomials.
- Kasraoui and Zeng (2006) have constructed an involution on \( \Pi_n \) that preserves the number of blocks (as well as some other properties) and exchanges the numbers of crossings and nestings; thus \( B_n(x, p, q) = B_n(x, q, p) \).
- Define ordinary generating function \( B(t, x, p, q) = \sum_{n=0}^{\infty} t^n B_n(x, p, q) \)
Interpolating between Narayana and Bell (continued)

- Kasraoui and Zeng (2006) have expressed \( \mathcal{B}(t, x, p, q) \) as a J-type continued fraction
- Can be transformed to an S-type continued fraction

\[
\sum_{n=0}^{\infty} t^n B_n(x, p, q) = \frac{1}{1 - \frac{xt}{1 - \frac{[1]_{p,q} t}{1 - \frac{xt}{1 - \frac{[2]_{p,q} t}{1 - \cdots}}}}}
\]

with coefficients \( \alpha_{2k-1} = x, \alpha_{2k} = \left[ k \right]_{p,q} \), where

\[
\left[ k \right]_{p,q} = \frac{p^k - q^k}{p - q}
\]

Conclusions:

1. The sequence \( B = (B_n(x, p, q))_{n \geq 0} \) of three-variable polynomials is coefficientwise Hankel-totally positive. The minor \( \Delta_{IJ}(B) \) counts families of disjoint Dyck paths as specified by Viennot 1983, with weights \( \alpha_{2k-1} = x, \alpha_{2k} = \left[ k \right]_{p,q} \).

2. The first Hankel determinants \( \Delta_n^{(m)}(B) \) are

\[
\Delta_n^{(0)}(B) = x^{n(n-1)/2} \prod_{i=1}^{n-1} \left[ i \right]_{p,q}!
\]

\[
\Delta_n^{(1)}(B) = x^{n(n+1)/2} \prod_{i=1}^{n-1} \left[ i \right]_{p,q}!
\]

where

\[
\left[ n \right]_{p,q}! = \prod_{j=1}^{n} \left[ j \right]_{p,q} \quad (0.1)
\]
Example 4: Eulerian polynomials

- Eulerian number $\langle n \rangle_k = \# \text{ of permutations of } [n] \text{ with } k \text{ descents}$
- Convention $\langle 0 \rangle_k = \delta_{k0}$
- They obviously refine factorials: $\sum_{k=0}^{n} \langle n \rangle_k = n!$
- Define Eulerian polynomials $A_n(x) = \sum_{k=0}^{n} \langle n \rangle_k x^k$
- Define ordinary generating function $A(t, x) = \sum_{n=0}^{\infty} t^n A_n(x)$
- Flajolet (1980) expressed $A(t, x)$ as a J-type continued fraction
- Can be transformed to an S-type continued fraction

$$\sum_{n=0}^{\infty} t^n A_n(x) = \frac{1}{1 - \frac{t}{1 - \frac{x t}{1 - \frac{2 t}{1 - \frac{2 x t}{1 - \cdots}}}}}$$

with coefficients $\alpha_{2k-1} = k$, $\alpha_{2k} = k x$. 

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Eulerian polynomials (continued)

Conclusions:

1. The sequence $\mathbf{A} = (A_n(x))_{n \geq 0}$ of Eulerian polynomials is coefficientwise Hankel-totally positive. The minor $\Delta_{IJ}(\mathbf{A})$ counts families of disjoint Dyck paths as specified by Viennot 1983, with weights $\alpha_{2k-1} = k$, $\alpha_{2k} = kx$.

2. The first Hankel determinants $\Delta_n^{(m)}(\mathbf{A})$ are

\[
\Delta_n^{(0)}(\mathbf{B}) = x^{n(n-1)/2} \prod_{i=1}^{n-1} i!^2
\]

\[
\Delta_n^{(1)}(\mathbf{B}) = x^{n(n+1)/2} \prod_{i=1}^{n-1} i!^2
\]

Remarks:

1. The (strong) log-convexity was known previously (Liu–Wang 2007, Zhu 2013).

2. The formula for $\Delta_n^{(0)}(\mathbf{A})$ was also known (Sivasubramanian 2010), by an explicit bijective argument.

3. Shin and Zeng (2012) have a $p, q$-generalization of this S-type continued fraction $\Rightarrow$ their polynomials $A_n(x, p, q)$ form a coefficientwise (in $x, p, q$) Hankel-totally positive sequence.
Some cases I am unable (as yet) to prove . . .

There are many cases where I find empirically that a sequence \((P_n(x))_{n \geq 0}\) is coefficientwise Hankel-totally positive, but I am unable to prove it because there is no S-type continued fraction in the ring of polynomials:

- Narayana polynomials of type B
- Eğecioğlu–Redmond–Ryavec polynomials
- Inversion enumerators for trees
- Reduced binomial discriminant polynomials
  
  \[ \vdots \]
Narayana polynomials of type B

The polynomials
\[ W_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^k \]
arise as

- Coordinator polynomial of the classical root lattice \( A_n \)
- Rank generating function of the lattice of noncrossing partitions of type B on \([n]\)

I follow Chen–Tang–Wang–Yang 2010 in calling them the Narayana polynomials of type B.

- **Empirically** the sequence \( (W_n(x))_{n \geq 0} \) seems to be coefficientwise Hankel-totally positive. I have checked this through the \( 12 \times 12 \) Hankel matrix.

- There is no S-type continued fraction in the ring of polynomials: we have
\[ \alpha_1, \alpha_2, \ldots = 1+x, \frac{2x}{1+x}, \frac{1+x^2}{1+x}, \frac{x+x^2}{1+x^2}, \frac{1+x^3}{1+x^3}, \frac{x+x^3}{1+x^3}, \frac{1+x^4}{1+x^3}, \ldots \]

- However, there **is** a nice J-type continued fraction: \( \gamma_n = 1+x, \beta_1 = 2x, \beta_n = x \) for \( n \geq 2 \).

- **Maybe** I can use the J-type continued fraction to prove Hankel-total positivity. (I only discovered this 2 days ago!)
Eğecioğlu–Redmond–Ryavec polynomials

- A noncrossing graph is a graph whose vertices are points on a circle and whose edges are non-crossing line segments.

- Noy (1998) showed that the number of noncrossing trees on \( n + 2 \) vertices in which a specified vertex (say, vertex 1) has degree \( k + 1 \) is

\[
T(n, k) = \frac{k + 1}{n + 1} \binom{3n - k + 1}{n - k} = \frac{2k + 2}{3n - k + 2} \binom{3n - k + 2}{n - k}
\]

- Eğecioğlu, Redmond and Ryavec (2001) introduced the polynomials

\[
\text{ERR}_n(x) = \sum_{k=0}^{n} T(n, k) x^k
\]

- They showed that, surprisingly, the Hankel determinant \( \Delta_n^{(0)}(\text{ERR}) \) is independent of \( x \):

\[
\Delta_n^{(0)}(\text{ERR}) = \prod_{i=1}^{n} \frac{(6i - 2)}{2(4i - 1)} \frac{2i}{2i}
\]

This is the number of \((2n+1) \times (2n+1)\) alternating sign matrices that are invariant under vertical reflection.

- **Empirically** I find that the sequence \((\text{ERR}_n(x))_{n \geq 0}\) is coefficientwise Hankel-totally positive. I have checked this through the \(13 \times 13\) Hankel matrix.

- There is no S-type continued fraction in the ring of polynomials: we have

\[
\alpha_1, \alpha_2, \ldots = 2 + x, \frac{3}{2 + x}, \frac{11 + 10x}{6 + 3x}, \frac{52 + 26x}{33 + 30x}, \ldots
\]

- However, there seems to be a J-type continued fraction where \( \gamma_0 = 2 + x \) and all the other coefficients are numbers.

- **Maybe** I can use the J-type continued fraction to prove Hankel-total positivity. (I only discovered this 2 days ago too!)
Generating polynomials of connected graphs

- Let $c_{n,m} = \#$ of connected simple graphs on vertex set $[n]$ having $m$ edges

- Define the *generating polynomial of connected graphs*

  $$C_n(v) = \sum_{m=n-1}^{\binom{n}{2}} c_{n,m} v^m$$

  $$= n^{n-2}v^{n-1} + \ldots + v^{\binom{n}{2}}$$

- No useful explicit formula for the polynomials $C_n(v)$ or their coefficients is known.

- But they have the well-known exponential generating function

  $$\sum_{n=1}^{\infty} \frac{x^n}{n!} C_n(v) = \log \sum_{n=0}^{\infty} \frac{x^n}{n!} (1 + v)^{n(n-1)/2}$$

- Make change of variables $y = 1+v$ and define $\overline{C}_n(y) = C_n(y-1)$:

  $$\sum_{n=1}^{\infty} \frac{x^n}{n!} \overline{C}_n(y) = \log \sum_{n=0}^{\infty} \frac{x^n}{n!} y^{n(n-1)/2}$$

- These formulae can be considered either as identities for formal power series or as analytic statements valid when $|1 + v| \leq 1$ (resp. $|y| \leq 1$).

- In particular we have

  $$C_n(-1) = \overline{C}_n(0) = (-1)^{n-1}(n-1)!$$

- Of course we also have

  $$C_n(0) = \overline{C}_n(1) = 0 \quad \text{for } n \geq 2$$

  since $C_n(v)$ [resp. $\overline{C}_n(y)$] has an $(n-1)$-fold zero at $v = 0$ [resp. $y = 1$].
Inversion enumerator for trees

- Let $T$ be a tree with vertex set $[n]$, rooted at the vertex 1.
- An inversion of $T$ is an ordered pair $(j, k)$ of vertices such that $j > k > 1$ and the path from 1 to $k$ passes through $j$.
- Let $i_{n,\ell}$ denote the number of trees on $[n]$ having $\ell$ inversions.
- Define the inversion enumerator for trees

$$I_n(y) = \sum_{\ell=0}^{\binom{n-1}{2}} i_{n,\ell} y^\ell$$

$$= (n-1)! + \ldots + y^{\binom{n-1}{2}}$$

- The polynomial $I_n(y)$ turns out to be related to $C_n(v)$ by the beautiful formula

$$C_n(v) = v^{n-1} I_n(1 + v)$$

or equivalently

$$\overline{C}_n(y) = (y - 1)^{n-1} I_n(y)$$

- This shows in particular that $I_n(0) = (n-1)!$ and $I_n(1) = n^{n-2}$.
- It is useful to define the normalized polynomials

$$I_n^*(y) = \frac{I_n(y)}{(n-1)!}$$

which have nonnegative rational coefficients and constant term 1.
Inversion enumerator for trees (continued)

**Fact 1.** $I_n(y)$ has strictly positive coefficients.

- Nonnegativity is obvious; strict positivity takes a bit of work.

**Fact 2.** $I_n(y)$ has log-concave coefficients.

- Special case of a deep result of Huh, arXiv:1201.2915, on the log-concavity of the $h$-vector of the independent-set complex for matroids representable over a field of characteristic 0: apply it to $M^*(K_n)$.

- **Open problem:** Find an elementary direct proof.

Now form the sequence $I = (I_{n+1}(y))_{n\geq 0}$.

**Conjecture 1.** The sequence $I$ is coefficientwise Hankel-totally positive.

- I have checked this through the $8 \times 8$ Hankel matrix.
- Even the log-convexity $I_{n-1}I_{n+1} \succeq I_n^2$ seems to be an open problem!

**Conjecture 2.** The $2 \times 2$ minors $I_{m-1}I_{n+1} - I_mI_n$ ($1 \leq m \leq n$) have coefficients that are log-concave.

- I have checked this through $n = 137$.
- It is false for minors of size $3 \times 3$ and higher.
Inversion enumerator for trees (continued)

Now look at the normalized polynomials $I^* = (I_{n+1}^*(y))_{n \geq 0}$.

**Conjecture 3.** The sequence $I^*$ is coefficientwise Hankel-totally positive.

- I have checked this through the $8 \times 8$ Hankel matrix.
- The analogous result for fixed real $y \in [0, 1]$ can be proven by using a result of Laguerre on the real-rootedness of the “deformed exponential function”

$$F(x, y) = \sum_{n=0}^{\infty} \frac{x^n}{n!} y^{n(n-1)/2}$$

This is what led me to conjecture the coefficientwise Hankel-total positivity.

- I believe the result for $I^*$ implies the one for $I$, by virtue of a general fact about Hadamard products; but I need to check this more carefully!

**Conjecture 4.** All the Hankel minors of $I^*$ have coefficients that are log-concave.

- I have checked this through the $8 \times 8$ Hankel matrix.
- For the $2 \times 2$ minors, I have checked it for $1 \leq m \leq n \leq 137$. 
Binomial discriminant polynomials

- Define \( F_n(x, y) = \sum_{k=0}^{n} \binom{n}{k} x^k y^{k(k-1)/2} \)
- Can be considered as a “y-deformation” of the binomial \((1 + x)^n\).
  It is also the Jensen polynomial of the deformed exponential function.
- Now define the binomial discriminant polynomial
  \[ \overline{D}_n(y) = \text{disc}_x F_n(x, y) \]
- \( \overline{D}_n(y) \) is a polynomial with integer coefficients
- It has degree \( n(n - 1)^2/2 \) and has first and last terms
  \[ \overline{D}_n(y) = b_n^2 y^{n(n-1)(n-2)/3} + \ldots + (-1)^{n(n-1)/2} n^n y^{n(n-1)^2/2} \]
where
  \[ b_n = \prod_{k=1}^{n-1} \binom{n}{k} = \prod_{k=1}^{n} k^{2k-1-n} = \frac{n!}{\prod_{k=1}^{n} k!} \]
(does this sequence have any standard name?)
- The first few \( \overline{D}_n(y) \) are:
  \[ \overline{D}_0(y) = 1 \]
  \[ \overline{D}_1(y) = 1 \]
  \[ \overline{D}_2(y) = 4 - 4y \]
  \[ \overline{D}_3(y) = 81y^2 - 216y^3 + 162y^4 + 0y^5 - 27y^6 \]
  \[ \overline{D}_4(y) = 9216y^8 - 44032y^9 + 76032y^{10} - 46080y^{11} - 15360y^{12} \]
  \[ + 27648y^{13} - 4608y^{14} - 3072y^{15} + 0y^{16} + 0y^{17} + 256y^{18} \]
  ;
Reduced binomial discriminant polynomials

- $D_n(y)$ has a factor $y^{n(n-1)(n-2)/3}$ and also a factor $(1 - y)^{n(n-1)/2}$ [coming from the fact that the $n$ roots of $F_n(x, y)$ all coalesce as $y \to 1$].
- So define the reduced binomial discriminant polynomial

\[ J_n(y) = \frac{D_n(y)}{y^{n(n-1)(n-2)/3} (1 - y)^{n(n-1)/2}} \]

- $J_n(y)$ is a polynomial with integer coefficients
- It has degree $\binom{n}{3}$ and has first and last terms

\[ J_n(y) = b_n^2 + \ldots + n^n y^{\binom{n}{3}} \]

- $J_n(1) = \prod_{k=1}^{n} k^k$ (hyperfactorials)
- The first few $J_n(y)$ are:

\[
\begin{align*}
J_0(y) &= 1 \\
J_1(y) &= 1 \\
J_2(y) &= 4 \\
J_3(y) &= 81 + 27y \\
J_4(y) &= 9216 + 11264y + 5376y^2 + 1536y^3 + 256y^4 \\
&\vdots
\end{align*}
\]

**Conjecture 1.** The coefficients of $J_n(y)$ are nonnegative (in fact, strictly positive).

**Conjecture 2.** The coefficients of $J_n(y)$ are log-concave (in fact, strictly log-concave).

- I have checked these conjectures for $n \leq 40$.
- What are the coefficients of $J_n(y)$ counting?
- Might these coefficients be the $h$-vector for some matroid???
Reduced binomial discriminant polynomials (continued)

Now form the sequence $J = (J_n(y))_{n \geq 0}$.

**Conjecture 3.** The sequence $J$ is coefficientwise Hankel-totally positive.

- In fact, all the Hankel minors of $J$ seem to have coefficients that are *strictly positive*.
- I have checked this through the $8 \times 8$ Hankel matrix.

**Conjecture 4.** All the Hankel minors of $J$ have coefficients that are log-concave (in fact, strictly log-concave).

- I have checked this through the $8 \times 8$ Hankel matrix.
- For the $2 \times 2$ minors, I have checked it for $1 \leq m \leq n \leq 39$.

Now look at the normalized polynomials $J^* = (J^*_n(y))_{n \geq 0}$.

**Conjecture 5.** The sequence $J^*$ is coefficientwise *strongly log-convex*: that is, all the $2 \times 2$ minors $J^*_{m-1}J^*_{n+1} - J^*_mJ^*_n$ have non-negative coefficients.

- I have checked this for $1 \leq m \leq n \leq 39$.
- The $3 \times 3$ and higher minors do *not* have nonnegative coefficients.

**Conjecture 6.** All the $2 \times 2$ minors $J^*_{m-1}J^*_{n+1} - J^*_mJ^*_n$ have coefficients that are log-concave (in fact, strictly log-concave except when $m = n = 1$).

- I have checked this for $1 \leq m \leq n \leq 39$. 
(Tentative) Conclusion

- Many interesting sequences $(P_n(x))_{n \geq 0}$ of combinatorial polynomials are (or appear to be) coefficientwise Hankel-totally positive.

- In some cases this can be proven by the Flajolet–Viennot method of continued fractions.
  - Flajolet and Viennot emphasized J-type continued fractions because they are more general.
  - But S-type continued fractions, when they exist, often have simpler coefficients; and they are the most direct tool for proving Hankel-total positivity.
  - Roughly speaking:

\[
\begin{align*}
\text{J-type c.f.} & \iff \text{general orthogonal polynomials} \iff \text{Hamburger moment problem} \\
\text{S-type c.f.} & \iff \text{orthogonal polynomials on } [0, \infty) \iff \text{Stieltjes moment problem} \\
& \iff \text{Hankel-total positivity}
\end{align*}
\]

- For the other cases, new methods of proof will be needed.

- Deepest cases seem to be $I_n(y)$ and $J_n(y)$:
  - For $I_n(y)$, even the log-convexity $I_{n-1}I_{n+1} \preceq I_n^2$ is an open problem. (Bijective proof??)
  - For $J_n(y)$, even the nonnegativity $J_n \succeq 0$ is an open problem! We really need to know what $J_n(y)$ is counting!

Dédié à la mémoire de Philippe Flajolet