Asymptotics of linear divide-and-conquer recurrences

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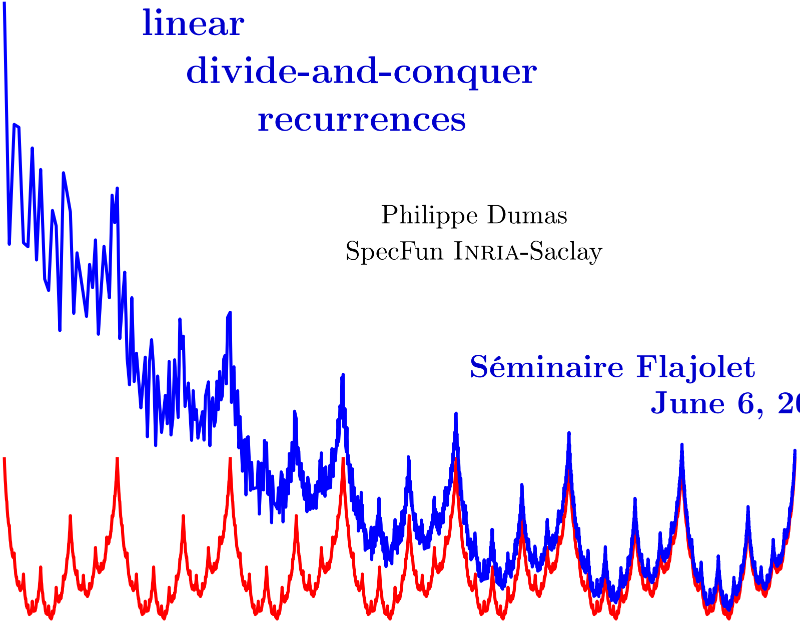
June 6, 2013

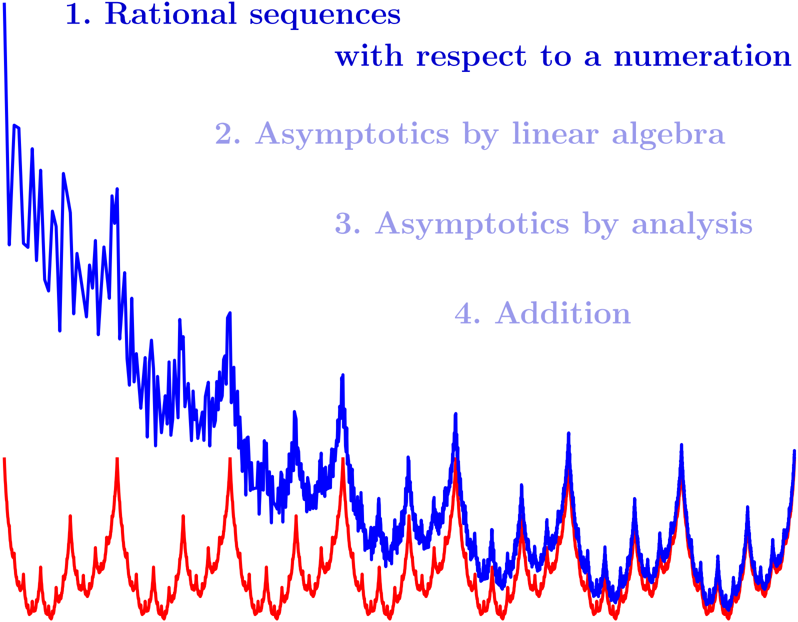


1. Rational sequences
with respect to a numeration system

2. Asymptotics by linear algebra

3. Asymptotics by analysis

4. Addition



Some divide-and-conquer recurrences: Rudin-Shapiro

3.23 We now prove some facts about D_n for large n. We prove first: for $n=2^j-1$, $D_n \geq \sqrt{\tfrac{1}{2}}\cdot\sqrt{n+1}$. We prove in fact the following: for $n=2^j-1$ there is a polynomial f(z) of degree n with coefficients ± 1 such that

(1) $|f(z)|^2 + |f(-z)|^2 = 2(n+1)$ for $|z|=1$, hence

$|f(z)| \leq \sqrt{2}\sqrt{n+1}$ so that $\left|\frac{f(z)}{\sqrt{2}\sqrt{n+1}}\right| \leq 1$, and the coefficient modulus sum is $\sqrt{\tfrac{1}{2}}\sqrt{n+1}$. We establish therefore (1), by induction. For j=1, we have (1) with $f(z) = 1+z$. Again, when k is an odd integer and

Shapiro, 1951

SOME THEOREMS ON FOURIER COEFFICIENTS

WALTER RUDIN[1]

I. **Trigonometric polynomials with coefficients** ±1. Consider the trigonometric polynomial

(1.1) $$P(e^{i\theta}) = \sum_{n=1}^{N} \epsilon_n e^{in\theta}$$

where $\epsilon_n = \pm 1$. If we set $\|P\|_\infty = \max_\theta |P(e^{i\theta})|$, the Parseval theorem shows that $\|P\|_\infty \geqq N^{1/2}$, and the following problem arises: *does there exist an absolute constant A with the property that for each N one can find* $\epsilon_1, \cdots, \epsilon_N$, *equal to* ±1, *so that*

(1.2) $$\|P\|_\infty \leqq AN^{1/2},$$

where P is given by (1.1)?

Rudin, 1959

$A = \sqrt{2}$

$P =$ Shapiro polynomial

problem:

find the lowest constant A and a polynomial P, with degree n and all coefficients equal to ± 1, that achieves the bound

$$\max_{|z|=1} |P(z)| \leq A\sqrt{n+1}$$

Some divide-and-conquer recurrences: Rudin-Shapiro

$$P_0(x) = 1 \qquad Q_1(x) = 1$$

$$P_{k+1}(x) = P_k(x) + x^{2^k} Q_k(x) \qquad Q_{k+1}(x) = P_k(x) - x^{2^k} Q_k(x)$$

$$P_1(x) = 1 + x$$

$$P_2(x) = 1 + x + x^2 - x^3$$

$$P_3(x) = 1 + x + x^2 - x^3 + x^4 + x^5 - x^6 + x^7$$

$$P_4(x) = 1 + x + x^2 - x^3 + x^4 + x^5 - x^6 + x^7 + x^8 + x^9 + x^{10} - x^{11} - x^{12} - x^{13} + x^{14} - x^{15}$$

$P =$ Shapiro polynomial

$$P_\infty(x) = 1 + x + x^2 - x^3 + x^4 + x^5 - x^6 + x^7 + x^8 + x^9 + x^{10} - x^{11} - x^{12} - x^{13} + x^{14} - x^{15} + x^{16} + x^{17} + x^{18} - x^{19} + x^{20} + x^{21} - x^{22} + x^{23} - x^{24} - x^{25} - x^{26} + x^{27} + x^{28} + x^{29} - x^{30} + x^{31} + x^{32} + \cdots$$

coefficients $\longrightarrow$ Rudin-Shapiro sequence a_n

Some divide-and-conquer recurrences: Rudin-Shapiro

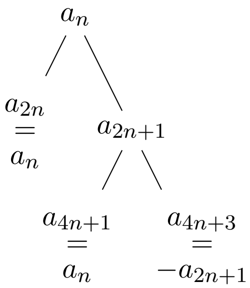
$a_0 = 1 \qquad a_1 = 1$ `(sequence A020985 in OEIS)`

$a_{2n} = a_n \qquad a_{2n+1} = (-1)^n a_n$

$$a_{2n} = a_n \qquad a_{4n+1} = a_n$$

$$a_{4n+3} = -a_{2n+1}$$

The Rudin-Shapiro sequence is rational for the radix 2.

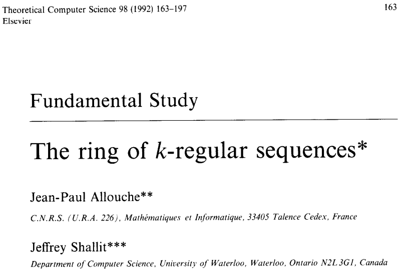


2-rational sequences: a definition

A sequence u is rational for the radix 2 if there exists a finite dimensional vector space $\mathcal{U}$, that

– contains the sequence u,

– is left stable by the operators $v_n \mapsto v_{2n}$, $v_n \mapsto v_{2n+1}$.



Theoretical Computer Science 98 (1992) 163–197 163
Elsevier

Fundamental Study

The ring of k-regular sequences*

Jean-Paul Allouche**
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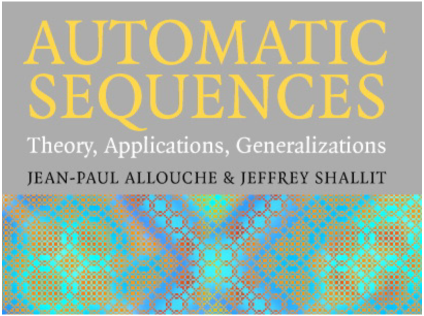
Jeffrey Shallit***
Department of Computer Science, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

Allouche, Shallit, 1992



Allouche, Shallit, 2003

Some divide-and-conquer recurrences: dichopile algo.

Laboratoire de Recherche en Informatique

Thèse

Approches combinatoires pour le test statistique à grande échelle

Présentée et soutenue publiquement le 19 novembre 2010 *par*
Johan Oudinet
pour l'obtention du
Doctorat de l'université Paris-Sud

Jury

Pr. Philippe Dague	Professeur à l'Université Paris-Sud
Pr. Alain Denise	Professeur à l'Université Paris-Sud
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Pr. Olivier Roux	Professeur à l'École Centrale de Nantes
Pr. Michèle Soria	Professeur à l'Université Paris 6

Some divide-and-conquer recurrences: dichopile algo.

problem:

draw with uniform distribution a length n word from a regular language

Combinatorics, Probability and Computing (2004) 13, 577–625.
DOI: 10.1017/S0963548304006315 Printed in the United Kingdom

Boltzmann Samplers for the Random Generation of Combinatorial Structures

PHILIPPE DUCHON,[1] PHILIPPE FLAJOLET,[2]
GUY LOUCHARD[3] and GILLES SCHAEFFER[4]

Theoretical Computer Science 132 (1994) 1–35
Elsevier

Fundamental Study

A calculus for the random generation of labelled combinatorial structures

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Some divide-and-conquer recurrences: dichopile algo.

problem:

draw with uniform distribution a length n word from a regular language

gain:

space $O(n) \longrightarrow O(\log n)$

time $O(n) \longrightarrow O(n \log n)$

that is

draw with uniform distribution a length n path ending in a final state

$\#(s, \ell) =$ number of paths starting from s ending in a final state with ℓ steps

$\longrightarrow$ storing all $\#(s, \ell)$ for $0 \leq \ell \leq n$

idea: storing only $\#(s, \ell)$ for $\ell = \frac{1}{2}n$, $\ell = \frac{3}{4}n$, $\ell = \frac{7}{8}n \ldots n$

and recompute when necessary

Some divide-and-conquer recurrences: dichopile algo.

$$f(n) = n + f(\lfloor n/2 \rfloor - 1) + g(\lceil n/2 \rceil)$$
$$g(n) = f(\lfloor n/2 \rfloor - 1) + g(\lceil n/2 \rceil)$$
$$f(1) = 1 \qquad g(1) = 0$$
$$f(0) = 0 \qquad g(0) = 0$$

System of linear divide-and-conquer equations

$$\delta_n = \nabla f_n = f_n - f_{n-1}$$

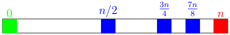
$$f(z) = \frac{z}{(1-z)^2} + z^2(1+z)f(z^2) + \left(1 + \frac{1}{z}\right) g(z^2)$$
$$g(z) = z^2(1+z)f(z^2) + \left(1 + \frac{1}{z}\right) g(z^2)$$

System of Mahler's equations

$$\delta(z) = (1-z)f(z)$$

$$\delta = 1, 1, 1, 2, 1, 2, 2, 2, 1, 3, 2, 2, 2, 3, 2, 3, 1, 3, 3, 2, 2 \ldots$$

`(not yet in OEIS :-)`



Some divide-and-conquer recurrences: dichopile algo.

$$\delta(z) = (1 - z)f(z)$$

section operators

$$S_0 \sum_{n=0}^{+\infty} u_n z^n = \sum_{n=0}^{+\infty} u_{2n} z^n \qquad S_1 \sum_{n=0}^{+\infty} u_n z^n = \sum_{n=0}^{+\infty} u_{2n+1} z^n$$

$$S_0 u(z^2) = u(z) \qquad S_1 u(z^2) = 0$$

$$S_0(uv) = S_0 u\, S_0 v + z S_1 u\, S_1 v \qquad S_1(uv) = S_1 u\, S_0 v + S_0 u\, S_1 v$$

Some divide-and-conquer recurrences: dichopile algo.

$$\delta(z) = (1-z)f(z)$$

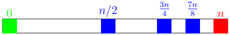
$$\delta(z) \in \mathcal{V}$$

$$\mathcal{V} = \mathbf{Q}[z]_1 \frac{1}{1-z} + \mathbf{Q}[z]_1 (1-z) f(z) + \mathbf{Q}[z]_1 \frac{1-z}{z} g(z)$$

$$S_0 \mathcal{V} \subset \mathcal{V}$$

The dichopile algorithm defines a sequence δ_n, which is a 2-rational sequence.

$$S_1 \mathcal{V} \subset \mathcal{V}$$



Some divide-and-conquer recurrences: binary partitions

XXXVII. *On a Problem in the Partition of Numbers.* By A. Cayley, *Esq.*†

It is required to find the number of partitions into a given number of parts, such that the first part is unity, and that no part is greater than twice the preceding part.

Commencing to form the partitions in question, these are

1	1	1	1	1	1	1	1	1	&c.;
	1	2	1	1	2	2	2	2	
			1	2	1	2	3	4	

[...]

And we are thus led to the series

1
1, 2
1, 2, 4, 6
1, 2, 4, 6, 10, 14, 20, 26
&c.;

where, considering 0 as the first term of each series, the first differences of any series are the terms twice repeated of the next preceding series: thus the differences of the fourth series are 1, 1, 2, 2, 4, 4, 6, 6. It is moreover clear that the first half of each series is precisely the series which immediately precedes it. We need, in fact, only consider a single infinite series, 1, 2, 4, 6, &c. It is to be remarked, moreover, that in the column of totals, the total of any line is precisely the first number in the next succeeding line.

$$b(z) = \prod_{k=0}^{+\infty} \frac{1}{1 - z^{2^k}}$$

$$(1 - z)b(z) = b(z^2)$$

$$b_n = b_{n-1} + b_{n/2}$$

Some divide-and-conquer recurrences: binary partitions

(sequence A000123 in OEIS)

And we are thus led to the series

1
1, 2
1, 2, 4, 6
1, 2, 4, 6, 10, 14, 20, 26
&c.;

where, considering 0 as the first term of each series, the first differences of any series are the terms twice repeated of the next preceding series: thus the differences of the fourth series are 1, 1, 2, 2, 4, 4, 6, 6. It is moreover clear that the first half of each series is precisely the series which immediately precedes it. We need, in fact, only consider a single infinite series, 1, 2, 4, 6, &c. It is to be remarked, moreover, that in the column of totals, the total of any line is precisely the first number in the next succeeding line.

Cayley, 1857

$$b_n = b_{n-1} + b_{n/2}$$

$$b_{2n} \underset{n\to+\infty}{\simeq} \exp\left(\frac{\ln^2 n}{2}\right)$$

Mahler, 1940

Some divide-and-conquer recurrences: binary partitions

(sequence A000123 in OEIS)

The binary partitions do not define a 2-rational sequence.

$$b_n = b_{n-1} + b_{n/2}$$

$$b_{2n+1} = b_{2n}$$

$$b_{2n} = b_{2n-1} + b_n$$

$$b_{2n} \underset{n \to +\infty}{\simeq} \exp\left(\frac{\ln^2 n}{2}\right)$$

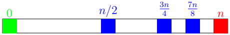
2-rational sequences: linear representation – 1

$$\mathcal{V} = \mathbf{Q}[z]_1 (1-z) f(z) + \mathbf{Q}[z]_1 \frac{1-z}{z} g(z) + \mathbf{Q}[z]_1 \frac{1}{1-z} \ni \delta(z)$$

$$\mathcal{B} = (1-z)f(z),\ z(1-z)f(z),\ \frac{1-z}{z}g(z),\ (1-z)g(z),\ \frac{1}{1-z},\ \frac{z}{1-z}$$

$S_0,\ S_1 \longrightarrow$ square matrices $A_0,\ A_1$ $\qquad$ $\delta(z) \longrightarrow$ column vector C

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



2-rational sequences: evaluation

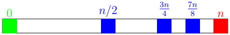
generating function	column vector
$\delta(z) = \sum_{n=0}^{+\infty} \delta_n z^n$	C
$S_1\delta(z) = \sum_{n=0}^{+\infty} \delta_{2n+1} z^n$	A_1C
$S_0S_1\delta(z) = \sum_{n=0}^{+\infty} \delta_{4n+1} z^n$	A_0A_1C
$S_1S_0S_1\delta(z) = \sum_{n=0}^{+\infty} \delta_{8n+5} z^n$	$A_1A_0A_1C$
$S_1S_1S_0S_1\delta(z) = \sum_{n=0}^{+\infty} \delta_{16n+13} z^n$	$A_1A_1A_0A_1C$

evaluation at 0

$\longrightarrow$ row vector L

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\delta_{13} = LA_1A_1A_0A_1C \qquad 13 = (1101)_2$$



2-rational sequences: linear representation – 2

A linear representation of a 2-rational sequence is a triple L, (A_0, A_1), C such that for every nonnegative integer $n = (b_K \dots b_1 b_0)_2$ the value of the sequence is

$$u_n = L A_{b_K} \cdots A_{b_1} A_{b_0} C.$$

A 2-rational sequence has an order of growth at most polynomial:

$$|u_n| \leq \|L\| \|A_{b_K}\| \cdots \|A_{b_1}\| \|A_{b_0}\| \|C\| \leq \gamma M^{\log_2 n} = \gamma n^{\alpha}.$$

1. Rational sequences with respect to a numeration system

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3. Asymptotics by analysis

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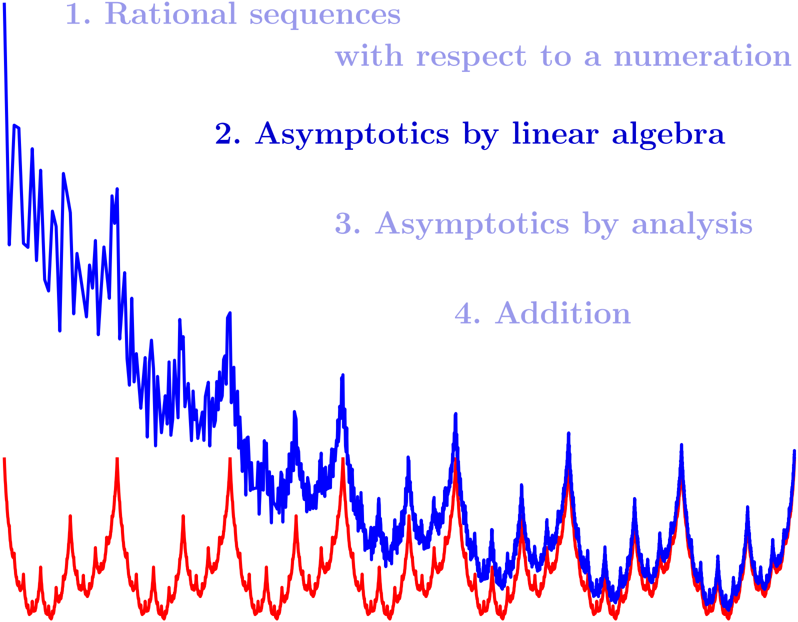

A process of computation–1

Linear Algebra and its Applications 438 (2013) 2107–2126

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Joint spectral radius, dilation equations, and asymptotic behavior of radix-rational sequences

A process of computation–1

Data: a linear representation L, A_0, A_1, C for the backward differences $u_n = \nabla s_n$ of a 2-rational sequence s_n

Result: an asymptotic expansion

$$s_N \underset{N\to+\infty}{=} \sum_{\rho,\vartheta,m} N^{\log_2 \rho} \binom{\log_2 N}{m} \times e^{i\vartheta \log_2 N} \times \Phi_{\rho,\vartheta,m}(\log_2 N) + O(N^{\log_2 r})$$

$\rho > r > 0$

ϑ real

m nonnegative integer

Φ 1-periodic function

A mere idea

notation:

$$A_w = A_{w_1} A_{w_2} \cdots A_{w_\ell}$$

for $w = w_1 w_2 \ldots w_\ell$

$$s_N = \sum_{0 \le n \le N} u_n = \sum_{\substack{0 \le n \le N \\ n=(w)_2}} L A_w C$$

$$S_K(x) = \sum_{\substack{|w|=K \\ (0.w)_2 \le x}} L A_w C \quad \text{for } 0 \le x \le 1$$

$$s_N = L(\mathrm{I}_d - A_0) \sum_{k=0}^{K} Q^k C + S_{K+1}(2^{\{t\}-1})$$

$t = \log_2 N$

$K = \lfloor t \rfloor \qquad \{t\} = t - K$

$Q = A_0 + A_1$

A mere idea

$$s_N = L(\mathrm{I}_d - A_0)\sum_{k=0}^{K} Q^k C + S_{K+1}(2^{\{t\}-1})$$

$t = \log_2 N$

$K = \lfloor t \rfloor \qquad \{t\} = t - K$

$Q = A_0 + A_1$

$$L(\mathrm{I}_d - A_0)\sum_{k=0}^{K} Q^k C \qquad \text{classical rational sequence}$$

$S_{K+1}(2^{\{t\}-1})$ to be studied

A process of computation–2

Process:

1. compute the joint spectral radius ρ_* of (A_0, A_1)
2. compute a reduced Jordan form for $Q = A_0 + A_1$
3. the eigenvalues with modulus $\leq \rho_*$ contributes to the error term
4. expand C over the Jordan basis and retain only the part for the eigenvalues $> \rho_*$
5. solve the dilation equations
6. write the asymptotic expansion for $S_K(x)$
7. translate it into an asymptotic expansion for s_N
8. gaze at the result

Joint spectral radius

KONINKL. NEDERL. AKADEMIE VAN WETENSCHAPPEN – AMSTERDAM
Reprinted from Proceedings, Series A, 63, No. 4 and Indag. Math., 22, No. 4, 1960

MATHEMATICS

A NOTE ON THE JOINT SPECTRAL RADIUS

BY

GIAN-CARLO ROTA AND W. GILBERT STRANG [1]

(Communicated by Prof. H. FREUDENTHAL at the meeting of April 30, 1960)

The notion of joint spectral radius of a set of elements of a normed algebra, introduced below, was obtained in the course of some work in matrix theory. It was later noticed that the same considerations are valid in any normed algebra, irrespective of dimension. The notion seems to be useful enough in certain contexts to warrant the following elementary discussion.

Let **B** be any bounded subset of the normed algebra $\mathfrak{A}$ with identity e. Let P_n be the set of all elements of $\mathfrak{A}$ which are the products of n elements of **B**. The *joint spectral radius* of the set **B** is defined to be the non-negative number

$$r(\mathbf{B}) = \lim_{n\to\infty} \sup_{T\in P_n} \|T\|^{1/n}.$$

That this number is well-defined follows just as in the by now classical case of the spectral radius of a single element, to which this notion reduces when the set **B** consists of a single element. Indeed, notice that $\log \sup_{T\in P_n} \|T\|$

$$\rho_* = \lim_{K\to+\infty} \max_{|w|=K} \|A_w\|^{1/K} = \inf_{K\geq 1} \max_{|w|=K} \|A_w\|^{1/K}$$

Joint spectral radius

Process:

1. compute the joint spectral radius ρ_* of (A_0, A_1)
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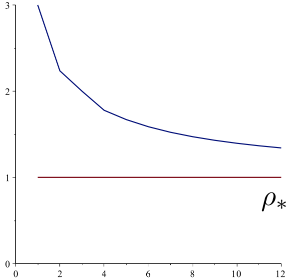
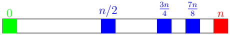
Joint spectral radius



$$\rho_* = 1 = \lim_{K \to +\infty} \max_{|w|=K} \|A_w\|_\infty^{1/K}$$



Jordan reduction

Process:

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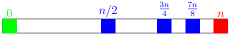
Jordan reduction

$$Q = A_0 + A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad R = P^{-1}QP$$

$$R = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad P = \frac{1}{12}\begin{bmatrix} 0 & 2 & 0 & 6 & -2 & -6 \\ 0 & 4 & 0 & -6 & 2 & 0 \\ 0 & 4 & 0 & 6 & 2 & 0 \\ 0 & 2 & 0 & -6 & -2 & 6 \\ 12 & -16 & 6 & 15 & 1 & 0 \\ 0 & 10 & -6 & -15 & -1 & 6 \end{bmatrix}$$



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Jordan reduction

$$R = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad P = \frac{1}{12}\begin{bmatrix} 0 & 2 & 0 & 6 & -2 & -6 \\ 0 & 4 & 0 & -6 & 2 & 0 \\ 0 & 4 & 0 & 6 & 2 & 0 \\ 0 & 2 & 0 & -6 & -2 & 6 \\ 12 & -16 & 6 & 15 & 1 & 0 \\ 0 & 10 & -6 & -15 & -1 & 6 \end{bmatrix}$$

$$\rho_* = 1$$

$$\text{error term}_K = O(r^K) \qquad 2 > r > \rho_* = 1$$

Jordan reduction

Process:

1. compute the joint spectral radius ρ_* of (A_0, A_1)
2. compute a reduced Jordan form for $Q = A_0 + A_1$
3. the eigenvalues with modulus $\leq \rho_*$ contributes to the error term
4. expand C over the Jordan basis and retain only the part for the eigenvalues $> \rho_*$
5. solve the dilation equations
6. write the asymptotic expansion for $S_K(x)$
7. translate it into an asymptotic expansion for s_N
8. gaze at the result

Jordan reduction

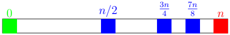
$$R = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad P = \frac{1}{12}\begin{bmatrix} 0 & 2 & 0 & 6 & -2 & -6 \\ 0 & 4 & 0 & -6 & 2 & 0 \\ 0 & 4 & 0 & 6 & 2 & 0 \\ 0 & 2 & 0 & -6 & -2 & 6 \\ 12 & -16 & 6 & 15 & 1 & 0 \\ 0 & 10 & -6 & -15 & -1 & 6 \end{bmatrix}$$

$\rho_* = 1$

$V_2^0 \quad V_2^1 \quad V_1^0 \quad V_1^1 \quad V_{-1}^0 \quad V_0^0$

Jordan basis

$$C = V_2^1 + V_2^0 + V_1^1 - 2V_{-1}^0 - V_0^0$$



Dilation equations

Process:

1. compute the joint spectral radius ρ_* of (A_0, A_1)
2. compute a reduced Jordan form for $Q = A_0 + A_1$
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5. solve the dilation equations
6. write the asymptotic expansion for $S_K(x)$
7. translate it into an asymptotic expansion for s_N
8. gaze at the result

Dilation equations

$$S_K^0(1) = 2^K L V_2^0$$

$$S_K^0(x) = 2^K L F^0(x) + O(r^K)$$

Jordan cell $J^K = \begin{bmatrix} 2^K & K2^{K-1} \\ 0 & 2^K \end{bmatrix}$

$$\begin{matrix} V_2^0 & V_2^1 \end{matrix}$$

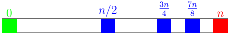
$$S_K^1(1) = K2^{K-1} L V_2^0 + 2^K L V_2^1$$

$$S_K^1(x) = K2^{K-1} L F^0(x) + 2^K L F^1(x) + O(r^K)$$

$$\underset{(6\times 2)}{F(x)}\underset{(2\times 2)}{J} = \underset{(6\times 6)}{A_0}\underset{(6\times 2)}{F(2x)} + \underset{(6\times 6)}{A_1}\underset{(6\times 2)}{F(2x-1)}$$

$$F(x) = 0 \quad \text{for } x \leq 0$$

$$F(x) = V_2 \quad \text{for } x \geq 1$$



Dilation equations

SIAM J. MATH. ANAL.
Vol. 22, No. 5, pp. 1388–1410, September 1991

TWO-SCALE DIFFERENCE EQUATIONS
I. EXISTENCE AND GLOBAL REGULARITY OF SOLUTIONS*

INGRID DAUBECHIES†‡ AND JEFFREY C. LAGARIAS†

Abstract. A *two-scale difference equation* is a functional equation of the form $f(x)=\sum_{n=0}^{N} c_n f(\alpha x-\beta_n)$, where $\alpha>1$ and $\beta_0<\beta_1<\cdots<\beta_n$ are real constants, and c_n are complex constants. Solutions of such equations arise in spline theory, in interpolation schemes for constructing curves, in constructing wavelets of compact support, in constructing fractals, and in probability theory. This paper studies the existence and uniqueness of L^1-solutions to such equations. In particular, it characterizes L^1-solutions having compact support. A time-domain method is introduced for studying the special case of such equations where $\{\alpha, \beta_0, \cdots, \beta_n\}$ are integers, which are called *lattice two-scale difference equations.* It is shown that if a lattice two-scale difference equation has a compactly supported solution in $C^m(\mathbb{R})$, then $m<(\beta_n-\beta_0)/(\alpha-1)-1$.

Key words. wavelets, subdivision algorithms, fractals

Daubechies-Lagarias, 1991

Uniform Refinement of Curves

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Dedicated to Alan J. Hoffman with friendship and esteem on the occasion of his 65th birthday.

Submitted by Hans Schneider

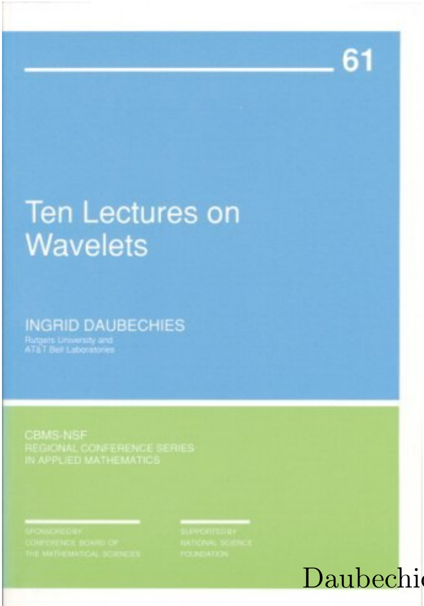
Miccheli-Prautzsch, 1989

ABSTRACT

We propose and analyze a class of algorithms for the generation of curves and surfaces. These algorithms encompass some well-known methods of subdivision for Bernstein-Bézier curves (de Casteljau's algorithm) and *B*-spline curves (Lane and Riesenfeld's algorithm). Several results concerning properties of the limiting curves as well as related questions are discussed.

$$F(x)J = A_0F(2x) + A_1F(2x-1)$$

Dilation equations



Daubechies, 1992

Subdivision Schemes in Geometric Modelling

Nira Dyn and David Levin
School of Mathematical Sciences
Tel-Aviv University
Tel-Aviv 69978, Israel

Dyn-Levin, 2002

Dilation equations

$$F_1^0(x) = \frac{1}{2}\,F_3^0(2x-1)$$

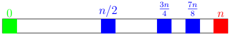
$$F_2^0(x) = \frac{1}{2}\,F_1^0(2x) + \frac{1}{2}\,F_4^0(2x) + \frac{1}{2}\,F_2^0(2x-1)$$

$$F_3^0(x) = \frac{1}{2}\,F_3^0(2x) + \frac{1}{2}\,F_1^0(2x-1) + \frac{1}{2}\,F_4^0(2x-1)$$

$$F_4^0(x) = \frac{1}{2}\,F_2^0(2x)$$

$$F_5^0(x) = \frac{1}{2}\,F_5^0(2x) + \frac{1}{2}\,F_1^0(2x-1) + \frac{1}{2}\,F_5^0(2x-1) + \frac{1}{2}\,F_6^0(2x-1)$$

$$F_6^0(x) = \frac{1}{2}\,F_1^0(2x) + \frac{1}{2}\,F_2^0(2x) + \frac{1}{2}\,F_6^0(2x) + \frac{1}{2}\,F_2^0(2x-1)$$



Dilation equations

$$F_1^1(x) = \frac{1}{2}F_3^1(2x-1) - \frac{1}{2}F_1^0(x)$$

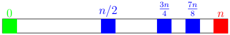
$$F_2^1(x) = \frac{1}{2}F_1^1(2x) + \frac{1}{2}F_4^1(2x) + \frac{1}{2}F_2^1(2x-1) - \frac{1}{2}F_2^0(x)$$

$$F_3^1(x) = \frac{1}{2}F_3^1(2x) + \frac{1}{2}F_1^1(2x-1) + \frac{1}{2}F_4^1(2x-1) - \frac{1}{2}F_3^0(x)$$

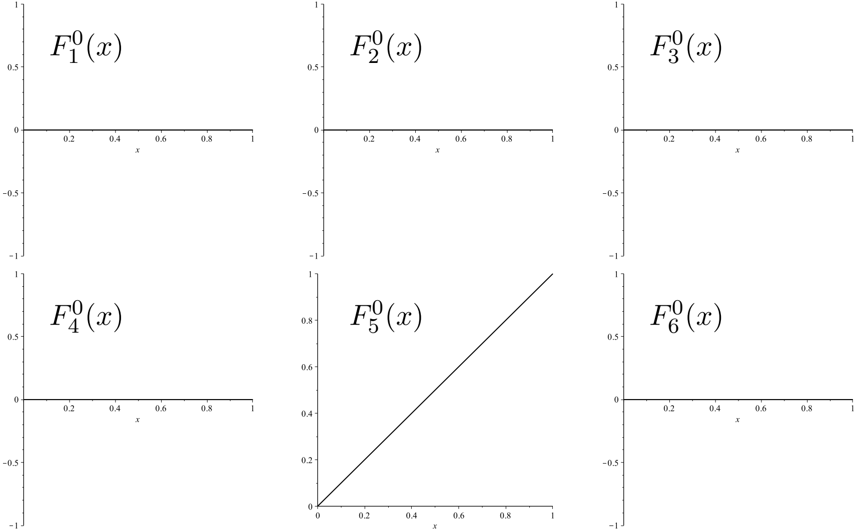
$$F_4^1(x) = \frac{1}{2}F_2^1(2x) - \frac{1}{2}F_4^0(x)$$

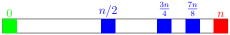
$$F_5^1(x) = \frac{1}{2}F_5^1(2x) + \frac{1}{2}F_1^1(2x-1) + \frac{1}{2}F_5^1(2x-1) + \frac{1}{2}F_6^1(2x-1) - \frac{1}{2}F_5^0(x)$$

$$F_6^1(x) = \frac{1}{2}F_1^1(2x) + \frac{1}{2}F_2^1(2x) + \frac{1}{2}F_6^1(2x) + \frac{1}{2}F_2^1(2x-1) - \frac{1}{2}F_6^0(x)$$



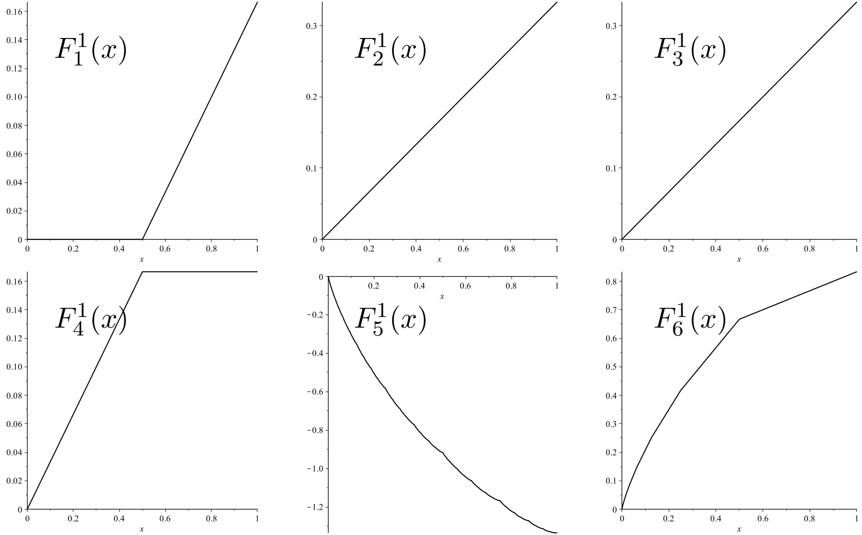
Dilation equations

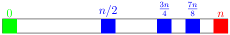




cascade algorithm

Dilation equations





Asymptotic expansion for $S_K(x)$

Process:

1. compute the joint spectral radius ρ_* of (A_0, A_1)
2. compute a reduced Jordan form for $Q = A_0 + A_1$
3. the eigenvalues with modulus $\leq \rho_*$ contributes to the error term
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7. translate it into an asymptotic expansion for s_N
8. gaze at the result

Asymptotic expansion for $S_K(x)$

$$S_K^0(x) = 2^K F_5^0(x) + O(r^K)$$

$$S_K^1(x) = K 2^{K-1} F_5^0(x) + 2^K F_5^1(x) + O(r^K) \qquad 1 < r < 2$$

$$S_K(x) \underset{K \to +\infty}{=} 2^{K-1}(K+2)x + 2^K F_5^1(x) + O(r^K)$$

Asymptotic expansion for s_N

Process:

1. compute the joint spectral radius ρ_* of (A_0, A_1)
2. compute a reduced Jordan form for $Q = A_0 + A_1$
3. the eigenvalues with modulus $\leq \rho_*$ contributes to the error term
4. expand C over the Jordan basis and retain only the part for the eigenvalues $> \rho_*$
5. solve the dilation equations
6. write the asymptotic expansion for $S_K(x)$
7. translate it into an asymptotic expansion for s_N
8. gaze at the result

Asymptotic expansion for s_N

$$s_N = L(\mathrm{I}_d - A_0) \sum_{k=0}^{K} Q^k C + S_{K+1}(2^{\{t\}-1})$$

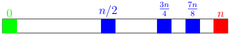
$$S_K(x) \underset{K\to+\infty}{=} 2^{K-1}(K+2)x + 2^K F_5^1(x) + O(r^K) \qquad 1 < r < 2$$

$$f_N \underset{N\to+\infty}{=} \frac{N}{2}\log_2 N + N\Phi(\log_2 N) + O(N^{\varepsilon}) \qquad 0 < \varepsilon < 1$$

$$\Phi(t) = \frac{3-\{t\}}{2} + 2^{1-\{t\}} F_5^1(2^{\{t\}-1})$$

$\Phi(t)$ 1-periodic

$\Phi(t)$ Hölder with exponent $\log_2(2/r) = 1 - \varepsilon$



Asymptotic expansion for s_N

Process:

1. compute the joint spectral radius ρ_* of (A_0, A_1)
2. compute a reduced Jordan form for $Q = A_0 + A_1$
3. the eigenvalues with modulus $\leq \rho_*$ contributes to the error term
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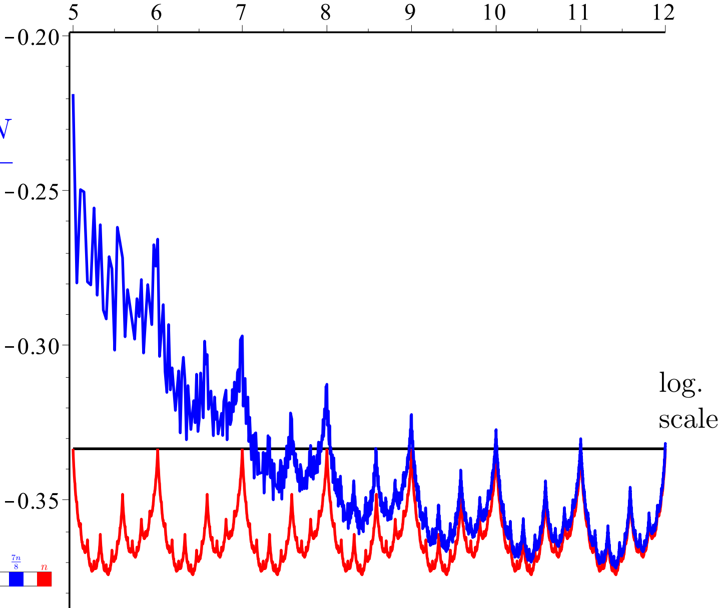
6. write the asymptotic expansion for $S_K(x)$
7. translate it into an asymptotic expansion for s_N
8. gaze at the result

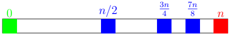
Picture

$$\frac{f_N - \frac{N}{2}\log_2 N}{N}$$

against

$\Phi(\log_2 N)$



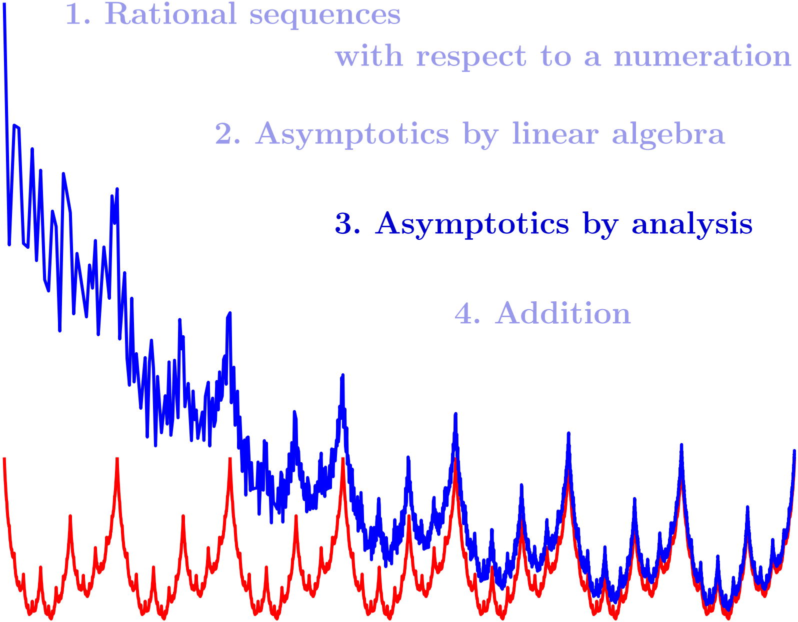


1. Rational sequences
with respect to a numeration system

2. Asymptotics by linear algebra

3. Asymptotics by analysis

4. Addition



D&C from standpoint of analytic number theory

Theoretical Computer Science 123 (1994) 291–314 291
Elsevier

Mellin transforms and asymptotics: digital sums

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Flajolet *et alii*, 1994

PACIFIC JOURNAL OF MATHEMATICS
Vol. 107, No. 1, 1983

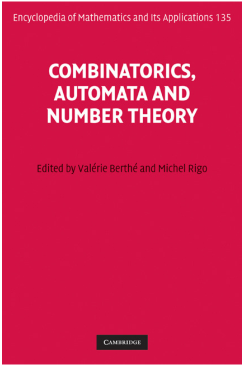
ON SUMS OF RUDIN-SHAPIRO COEFFICIENTS II

JOHN BRILLHART, PAUL ERDÖS AND PATRICK MORTON

Let $\{a(n)\}$ be the Rudin-Shapiro sequence, and let $s(n) = \sum_{k=0}^{n} a(k)$ and $t(n) = \sum_{k=0}^{n}(-1)^k a(k)$. In this paper we show that the sequences $\{s(n)/\sqrt{n}\}$ and $\{t(n)/\sqrt{n}\}$ do not have cumulative distribution functions, but do have logarithmic distribution functions (given by a specific Lebesgue integral) at each point of the respective intervals $[\sqrt{3/5}, \sqrt{6}]$ and $[0, \sqrt{3}]$. The functions $a(x)$ and $s(x)$ are also defined for real $x \geq 0$, and the function $[s(x) - a(x)]/\sqrt{x}$ is shown to have a Fourier expansion whose coefficients are related to the poles of the Dirichlet series $\sum_{n=1}^{\infty} a(n)/n^{\tau}$, where $\operatorname{Re} \tau > \frac{1}{2}$.

Brillhart, Erdős, Morton, 1983

D&C from standpoint of analytic number theory



8

Analysis of digital functions and applications

8.1 Introduction: digital functions

Digital functions in a rather informal and general sense are functions defined in a way depending on the digits in some digital representation of the integers. In the simplest case the digital representation is the q-adic representation and the dependence of the function on the digits is additive as for the sum-of-digits function given by

$$s_q\left(\sum_{k=0}^{K} \varepsilon_k q^k\right) = \sum_{k=0}^{K} \varepsilon_k,$$

which also serves as the most prominent example for such functions. As a very general reference for results on digital functions, we refer to (Allouche and Shallit 2003). We remark that depending on the point of view such maps $f : \mathbb{N} \to A$ can be seen as (arithmetic) functions or sequences. The aim of this chapter is to study various asymptotic and limiting properties of such functions.

For the convenience of the reader we collect the basic definitions as given in (Allouche and Shallit 2003).

Drmota, Grabner, 2010

A process of computation, anew

The analytic approach

- has a wider scope of application than the linear algebra approach,
- is trickier to apply.

A process of computation, anew

Process:

1. define the Dirichlet series associated to the backward differences
2. compute its absolute convergence abscissa σ_a
3. extend it to the left
4. apply the Mellin-Perron formula
5. shift the vertical line of integration to the left and collect the residues
6. write the asymptotic expansion for s_N
7. gaze at the result

Dirichlet series

Process:

1. define the Dirichlet series associated to the backward differences
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6. write the asymptotic expansion for s_N

Dirichlet series

Dirichlet series: $U(s) = \sum_{n \geq 1} \frac{U_n}{n^s}$

Abscissa of absolute convergence

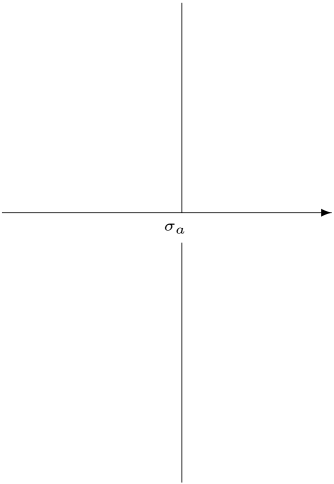
Process:

1. define the Dirichlet series associated to the backward differences
2. compute its absolute convergence abscissa σ_a
3. extend it to the left
4. apply the Mellin-Perron formula
5. shift the vertical line of integration to the left and collect the residues
6. write the asymptotic expansion for s_N

Abscissa of absolute convergence

Dirichlet series: $U(s) = \sum_{n \geq 1} \frac{U_n}{n^s}$

Abscissa: $\sigma_a = 1 + \log_2 \rho_*$



Usually difficult to compute.

Extension as a meromorphic function

Process:

1. define the Dirichlet series associated to the backward differences
2. compute its absolute convergence abscissa σ_a
3. extend it to the left
4. apply the Mellin-Perron formula
5. shift the vertical line of integration to the left and collect the residues
6. write the asymptotic expansion for s_N

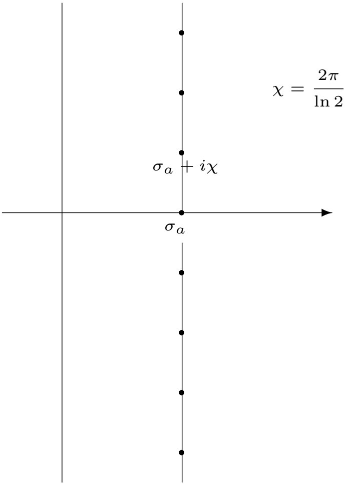
Extension as a meromorphic function

Dirichlet series: $U(s) = \sum_{n \geq 1} \frac{U_n}{n^s}$

Abscissa: $\sigma_a = 1 + \log_2 \rho_*$

Extension: $U(s)(\mathrm{I}_d - 2^{-s} Q) = \nabla U(s)$

$$\nabla U(s) = U_1 + \sum_{n=1}^{+\infty} \left(\frac{1}{(2n+1)^s} - \frac{1}{(2n)^s} \right) U_n A_1$$



Perhaps no pole.

Mellin-Perron formula

Process:

1. define the Dirichlet series associated to the backward differences
2. compute its absolute convergence abscissa σ_a
3. extend it to the left
4. apply the Mellin-Perron formula
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Mellin-Perron formula

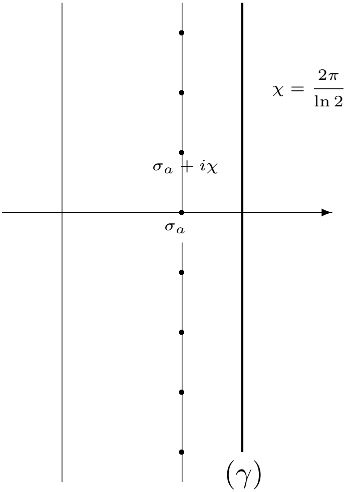
Dirichlet series: $U(s) = \sum_{n \geq 1} \frac{U_n}{n^s}$

Abscissa: $\sigma_a = 1 + \log_2 \rho_*$

Extension: $U(s)(\mathrm{I}_d - 2^{-s}Q) = \nabla U(s)$

Mellin-Perron formula:

$$\sum_{1 \leq k < N} U_k + \frac{1}{2} U_N = \frac{1}{2\pi i} \int_{(\gamma)} U(s)\, N^s \frac{ds}{s}$$



Cauchy's residues theorem

Process:

1. define the Dirichlet series associated to the backward differences
2. compute its absolute convergence abscissa σ_a
3. extend it to the left
4. apply the Mellin-Perron formula
5. shift the vertical line of integration to the left and collect the residues
6. write the asymptotic expansion for s_N

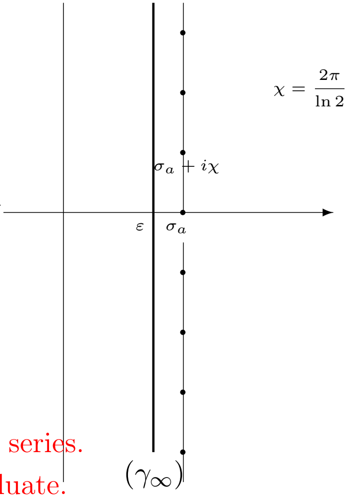
Cauchy's residue theorem

$$\sum_{1\le k<N} U_k + \frac{1}{2}U_N$$

$$= N^{\sigma_a} \sum_{k=-\infty}^{+\infty} c_k \exp\left(\frac{2ik\pi \ln N}{\ln 2}\right)$$

$$+\frac{1}{2\pi i}\int_{(\gamma_\infty)} u(s)\, N^s \frac{ds}{s}$$

$$= N^{\sigma_a}\Phi(\log_2 N) + O(N^{\varepsilon})$$



Perhaps divergence of the trigonometric series.

Order of growth at $\pm i\infty$ difficult to evaluate.

Mellin-Perron formula may not apply directly.

Virtuosity

Constr. Approx. (2005) 21: 149–179
DOI: 10.1007/s00365-004-0561-x

CONSTRUCTIVE APPROXIMATION

$$\text{(3.13)}\quad \sum_{1\le k<n}\Phi(k) = \sum_{0\le k<n}(n-1-k)2^{\nu(k)} = \frac{n^2}{2}-\frac{\Phi(n)}{2}+\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{n^{s+2}}{s(s+1)(s+2)}\sum_{j\ge1}\nabla\Delta\Phi(j)j^{-s}\,ds.$$

By (3.3),

$$\sum_{j\ge1}\nabla\Delta\Phi(j)j^{-s} = \frac{A_2(s)}{1-3\cdot2^{-s-1}} = \frac{1-2^{-s-1}-B_2(s)/2}{1-3\cdot2^{-s-1}}.$$

$$\begin{aligned}
V_1''(0) &= -0.40632\,91671\,14929\,22563\,37014\,58481\,78635\,30386\,92416\,64842\ldots,\\
V_1'''(0) &= 1.12746\,03441\,76855\,00723\,94784\,63671\,80426\,48344\,45077\,21808\ldots,\\
V_2'(0) &= 0.31047\,16129\,81928\,91222\,32068\,52261\,52855\,96918\,44215\,57523\ldots,\\
V_2''(0) &= -1.20785\,26305\,05474\,15248\,60897\,62038\,67711\,07449\,26970\,51090\ldots,\\
V_3'(0) &= 0.79612\,43185\,47763\,30582\,71007\,27435\,50514\,41134\,19022\,61579\ldots.
\end{aligned}$$

Digital Sums and Divide-and-Conquer Recurrences: Fourier Expansions and Absolute Convergence

Peter J. Grabner and Hsien-Kuei Hwang

Proposition 4. *Let α and β be two positive constants. Consider the recurrence*

$$\text{(3.1)}\qquad f_n = \alpha f_{\lfloor n/2\rfloor} + \beta f_{\lceil n/2\rceil} + g_n \qquad (n\ge 2),$$

with f_1 and the sequence $\{g_n\}_{n\ge2}$ given. Let the abscissa of convergence of the Dirichlet series $W(s) := \sum_{n\ge1}\nabla\Delta f_n\, n^{-s}$ be σ_f. Suppose that $c > \max\{0, \sigma_f, \log_2(\alpha+\beta)-1\}$. Then the solution of (3.1) *satisfies*

$$\text{(3.2)}\qquad \frac{f_n}{n} = f_1 + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{n^s}{s(s+1)}\frac{W(s)}{1-(\alpha+\beta)2^{-s-1}}\,ds,$$

where $(\Delta f(x) := \Delta f(\lfloor x\rfloor),\ f_0 = g_0 = g_1 = 0)$,

$$\text{(3.3)}\quad W(s) = (\alpha+\beta-2)f_1(1-2^{-s-1}) + \sum_{n\ge1}\frac{\nabla\Delta g_n}{n^s} + \frac{(\alpha-\beta)s}{2^s}\int_1^\infty\frac{\Delta f(x)}{x^{s+1}}\,\xi(x)\,dx.$$

Grabner, Hwang, 2004

$$\begin{aligned}
\psi_k(s) &= \sum_{n\ge1}\frac{\nu(2n)^k}{(2n+\frac12)^s} + \sum_{n\ge0}\frac{\nu(2n+1)^k}{(2n+\frac32)^s}\\
&= 2^{-s}\zeta(s,\tfrac34) + 2^{1-s}\psi_k(s) + 2^{1-s}\sum_{m\ge1}\binom{s+2m-1}{2m}\frac{\psi_k(s+2m)}{16^m}\\
&\quad + 2^{-s}\sum_{1\le j<k}\binom{kj}{\sum_{m\ge0}}\binom{s+m-1}{m}\frac{(-1)^m}{4^m}\psi_j(s+m).
\end{aligned}$$

Solving for $\psi_k(s)$, we then obtain

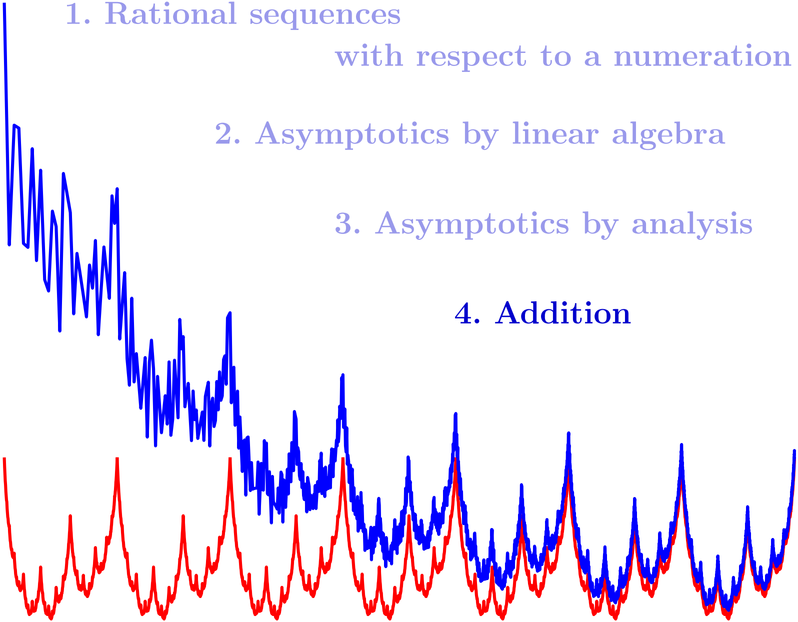
$$\text{(4.5)}\quad \psi_k(s) = \frac{1}{2^s-2}\zeta(s,\tfrac34) + \frac{2}{2^s-2}\sum_{m\ge1}\binom{s+2m-1}{m}\frac{\psi_k(s+2m)}{16^m} + \frac{1}{2^s-2}\sum_{1\le j<k}\binom{kj}{\sum_{m\ge0}}\binom{s+m-1}{m}\frac{(-1)^m}{4^m}\psi_j(s+m).$$

1. Rational sequences
with respect to a numeration system

2. Asymptotics by linear algebra

3. Asymptotics by analysis

4. Addition



A huge amount of publications

INTEGERS: ELECTRONIC JOURNAL OF COMBINATORIAL NUMBER THEORY 5(3) (2005), #A09

COUNTING OPTIMAL JOINT DIGIT EXPANSIONS

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Grabner, Heuberger, Prodinger, 2005

linear representation

probability, Fourier transform

nonnegative linear representation

A Master Theorem for Discrete Divide and Conquer Recurrences*

Michael Drmota† Wojciech Szpankowski‡

Dedicated to Philippe Flajolet 1948-2011

Abstract

Divide-and-conquer recurrences are one of the most studied equations in computer science. Yet, *discrete* versions of these recurrences, namely

$$T(n) = a_n + \sum_{j=1}^{m} b_j T\left(\lfloor p_j n + \delta_j \rfloor\right) + \sum_{j=1}^{m} b'_j T\left(\lceil p_j n + \delta'_j \rceil\right)$$

Drmota, Szpankowski, 2011

very general, mixing of radices

analytic approach

positive coefficients

Lazy approach

Lazy process:

1. evaluate the joint spectral radius
2. take into account only the dominant eigenvalue
3. solve only the first dilation equation
4. find an equivalent for $S_K(x)$
5. translate it for s_N

$$\max_{|w|=2} \|A_w\|^{1/2} \simeq 1.3 < 2$$

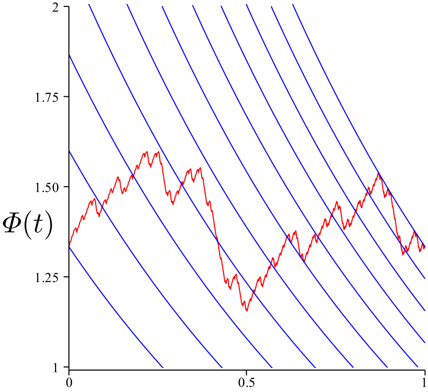
Q's dominant eigenvalue $\lambda = 2$

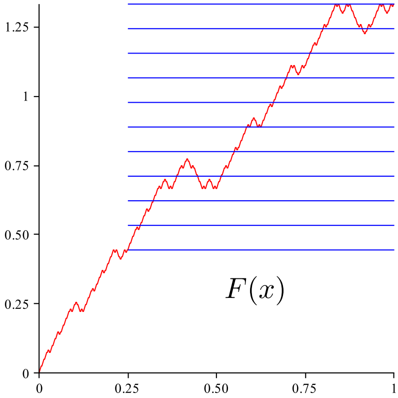
$$F^0(x) = (0,0,0,0,x,0)$$

$$s_K(x) \underset{K\to+\infty}{\sim} 2^{K-1}Kx \qquad f_N \underset{N\to+\infty}{\sim} \frac{1}{2} N \log_2 N$$

Autosimilarity loss: Newman-Coquet

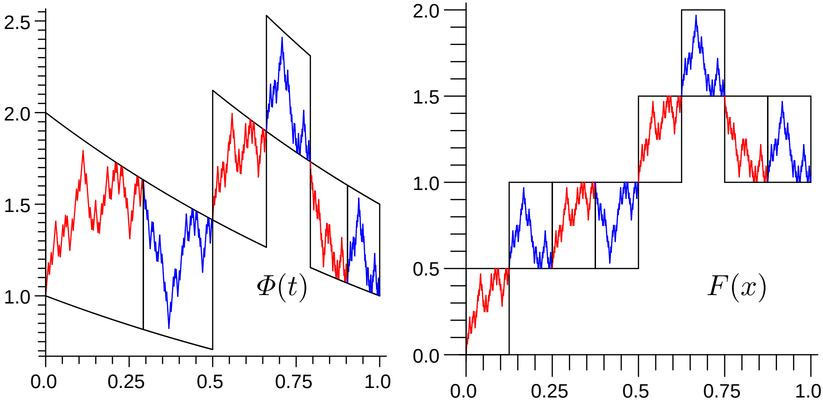
$$\sum_{n \le N} (-1)^{s_2(3n)} \underset{N \to +\infty}{=} N^{\log_4 3} \Phi(\log_4 N) + O(1) \qquad \Phi(t) = 3^{1-\{t\}} F(4^{\{t\}-1})$$





Symmetry loss: Rudin-Shapiro

$$\sum_{n\leq N} a_n \underset{N\to+\infty}{=} \sqrt{N}\Phi(\log_4 N) + O(1) \qquad \Phi(t) = 2^{1-\{t\}}F(4^{\{t\}-1})$$



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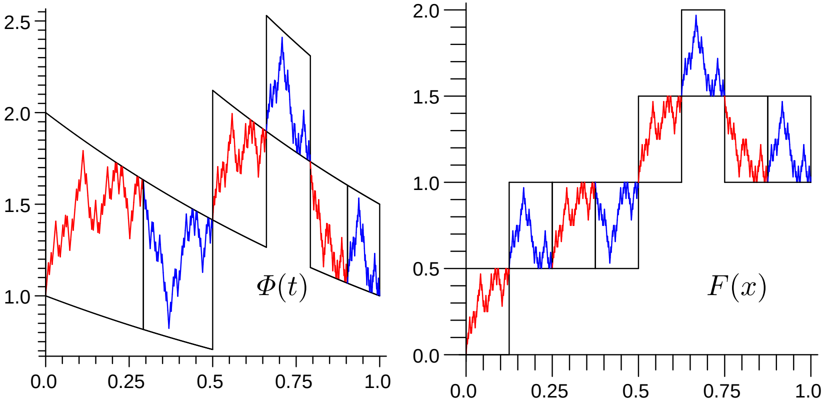


$$2^s\Phi(s) + 2^t\Phi(t) = 4 \text{ for } s \in [0, 1/2] \text{ and } t \in [\log_4 3, 1] \text{ with } 4^t - 4^s = 2$$

$$\log_4 3 \simeq 0.79$$

Periodicity versus pseudo-periodicity

Result: an asymptotic expansion for s_N

$$s_N \underset{N\to+\infty}{=} \sum_{\rho,\vartheta,m} N^{\log_2 \rho} \binom{\log_2 N}{m} \times e^{i\vartheta \log_2 N} \times \Phi_{\rho,\vartheta,m}(\log_2 N) + O(N^{\log_2 r})$$

$\lambda = \rho e^{i\vartheta}$ eigenvalue of Q

$t \longmapsto e^{i\vartheta t}$ T-periodic with $T \in \mathbb{N}^*$ iff $\vartheta \in \pi\mathbb{Q}$

$t \longmapsto \Phi(t)$ 1-periodic

hence

$t \longmapsto e^{i\vartheta t}\Phi(t)$ T-periodic with $T \in \mathbb{N}^*$ iff $\vartheta \in \pi\mathbb{Q}$

Periodicity versus pseudo-periodicity: rosette

$$L = [\ *\quad *\ ],$$

$$A_0 = \begin{bmatrix} \cos\vartheta & 0 \\ 0 & \cos\vartheta \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & -\sin\vartheta \\ \sin\vartheta & 0 \end{bmatrix},$$

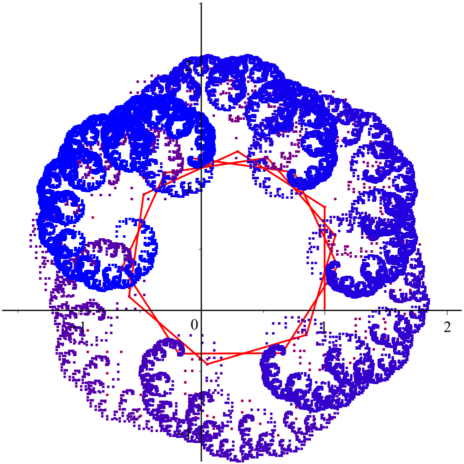
$$C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\rho_* = \max(|\cos\vartheta|, |\sin\vartheta|) < 1 \text{ for } \vartheta \notin \frac{\pi}{2}\mathbb{Z}$$

$$Q = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} \qquad \lambda = e^{\pm i\vartheta} \qquad |\lambda| = 1 > \rho_*$$

Periodicity versus pseudo-periodicity: rosette

$\vartheta = 1 \notin \pi\mathbb{Q}$



Conclusion

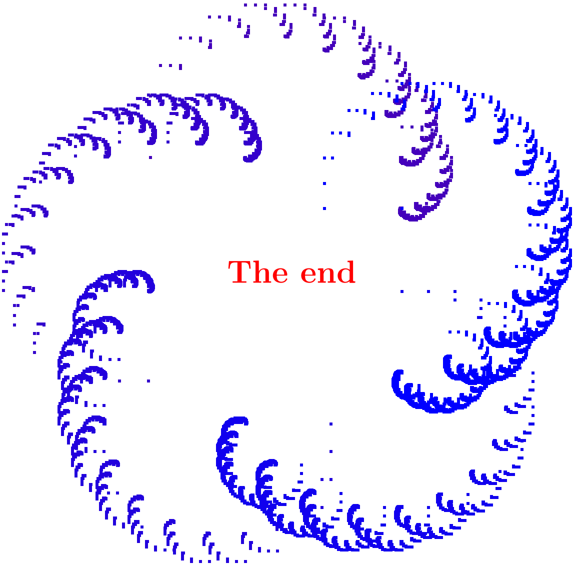
Rational sequences with respect to a numeration system

- are a direct generalization of classical rational sequences,
- provide the most basic case of linear divide-and-conquer recurrences (constant coefficients),
- have an asymptotic behaviour that can be made known both by algebra and analysis.

The linear approach provides us with a

- not too sophisticated,
- not too difficult method

to deal with the asymptotic behaviour of 2-rational sequence.



The end