

[Combinatorics of] T -systems
And related Discrete integrable systems

Seminaire Flajolet March 30, 2017, IHP PARIS

[Joint work with Philippe Di Francesco]

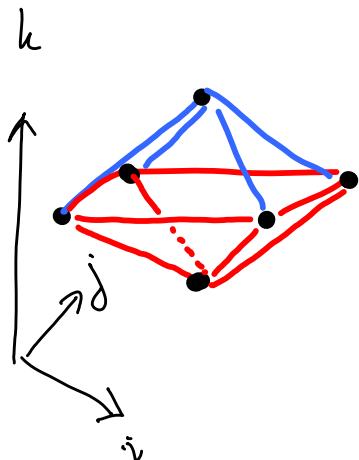
Outline:

- 1) The octahedron relation
- 2) Boundary conditions on the octahedron relation
 - T-systems
 - Frieze patterns
 - Zamolodchikov periodicity
 - Q-systems
- 3) Solution of system using networks
- 4) Quantization

1) Octahedron equation:

a map $\mathbb{Z}^3 \rightarrow A = \bullet$ a commutative algebra;

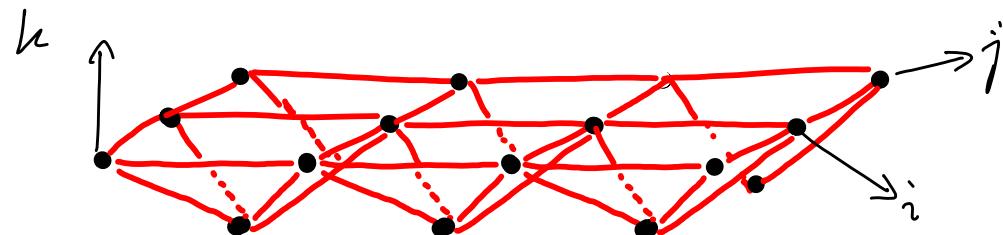
- generators $\{x_{i,j,k} \mid i,j,k \in \mathbb{Z}, i+j+k \equiv 0(2)\}$
- relations: ∇ octahedron in \mathbb{Z}^3 ,



$$x_{i,j,k+1} = \frac{x_{i,j+1,k}x_{i,j-1,k} + x_{i+1,j,k}x_{i-1,j,k}}{x_{i,j,k-1}}$$

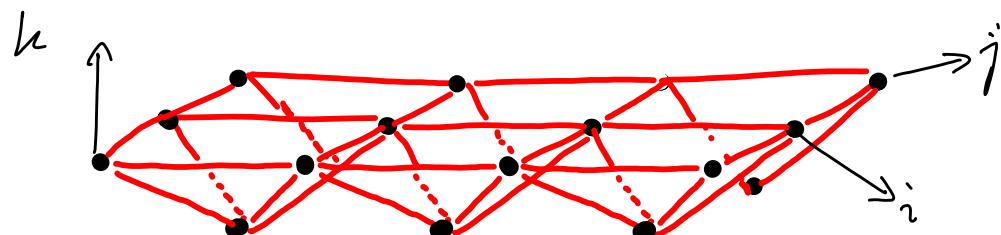
for all $(i,j,k) \in \mathbb{Z}^3$ with fixed parity $i+j+k \equiv 0(2)$

From the stepped (eggbox) surface in \mathbb{Z}^3 :



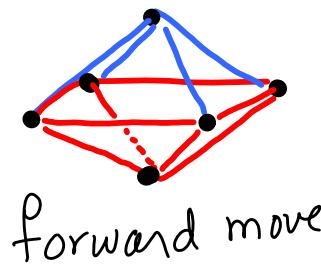
$$S_0 = \left\{ (i, j, i+j \pmod{2}) \right\}_{i, j \in \mathbb{Z}}$$

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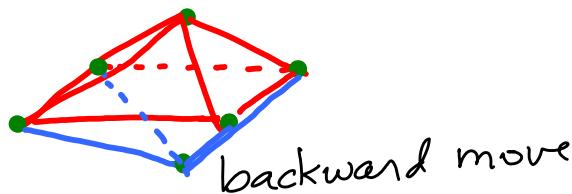


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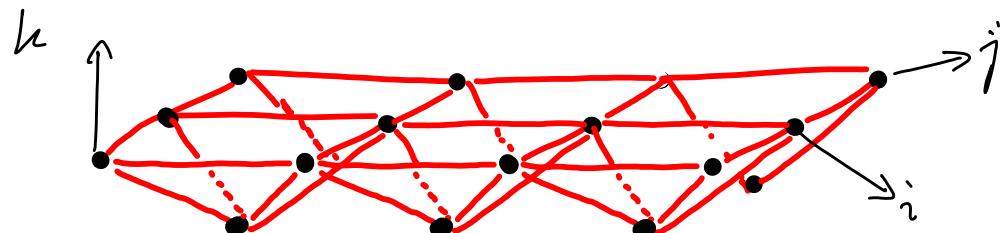
- all other variables on the even \mathbb{Z}^3 -lattice are determined via the octahedron relations.



or

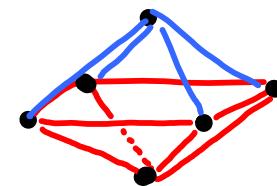


From the stepped (eggbox) surface in \mathbb{Z}^3 :



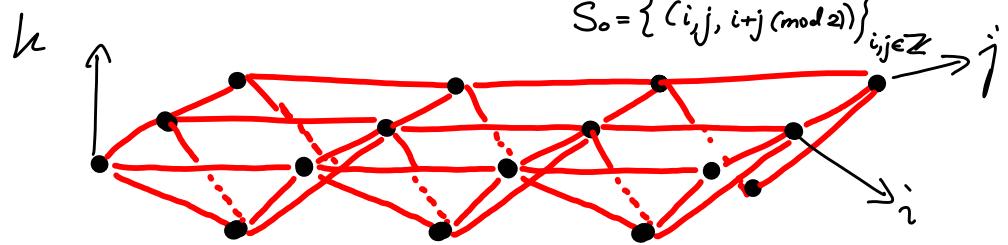
$$S_0 = \left\{ (i, j, i+j \pmod{2}) \right\}_{i, j \in \mathbb{Z}}$$

- all other variables on the even \mathbb{Z}^3 -lattice are determined via the octahedron relations.
- We view the relation as a discrete dynamical system in the time variable k , with **valid initial data** given by the variables on any stepped surface S

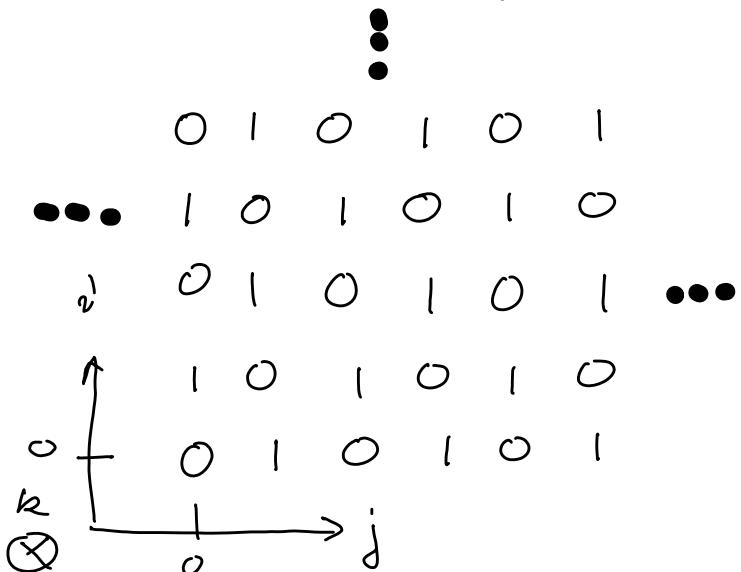
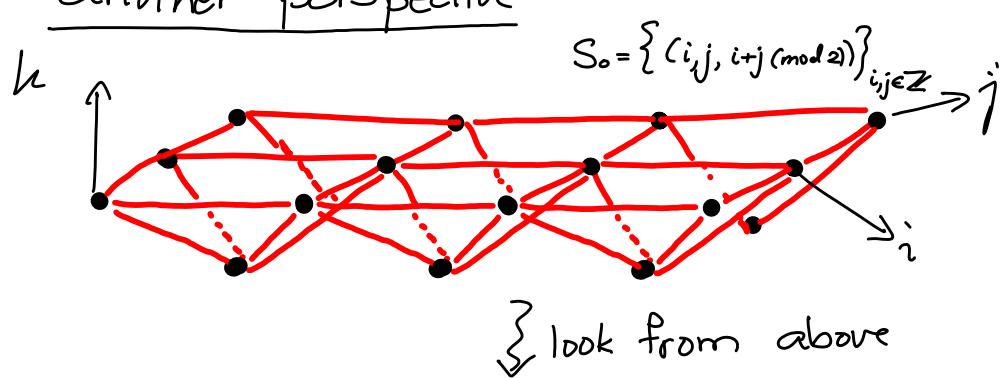


$$S = \left\{ (i, j, k_0(i, j)) : |k_0(i+1, j) - k_0(i, j)| = 1 = |k_0(i, j+1) - k_0(i, j)| \quad \forall i, j. \right\}$$

Another perspective



Another perspective

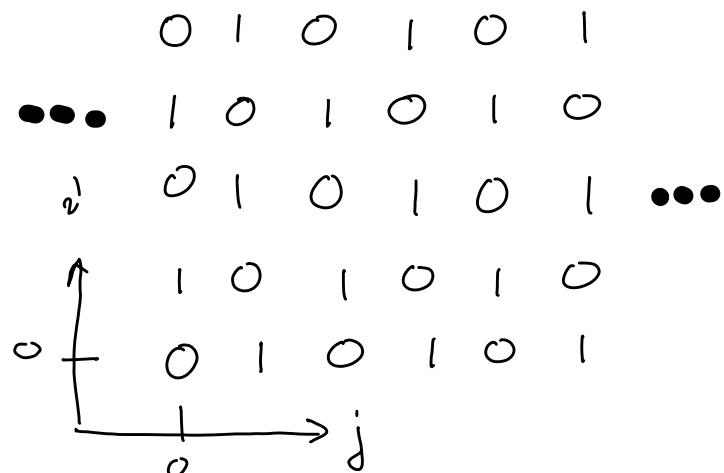
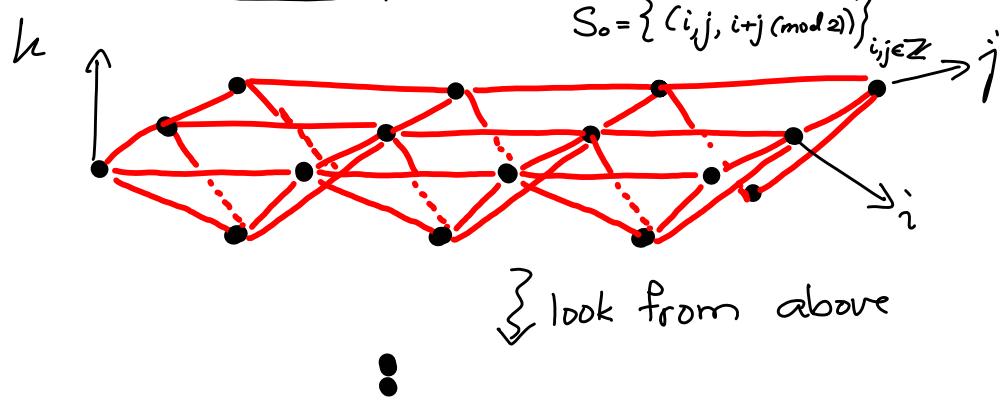


the integers at the lattice sites
denote the value of k :

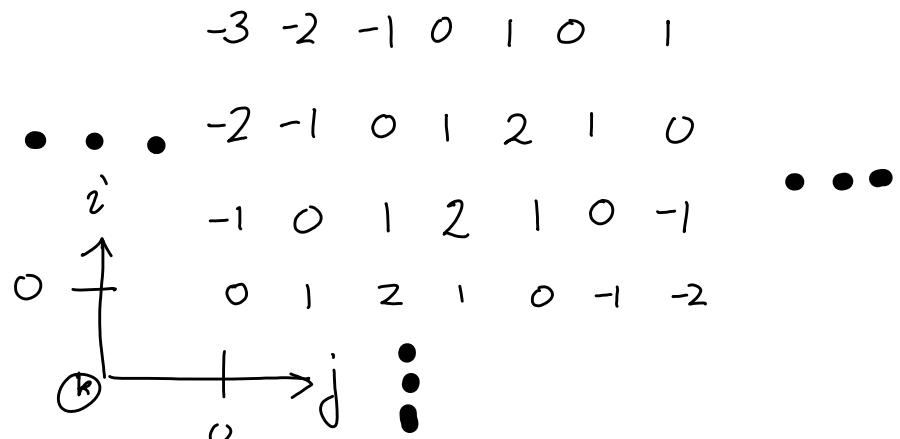
$$i - 1 \leftrightarrow x_{i,j,k}$$

↓
j

Another perspective



arbitrary stepped surface:



Cluster algebra structure

$$x_{i,j,k+1} = \frac{x_{i,j+k} x_{i,j-k} + x_{i+j,k} x_{i-j,k}}{x_{i,j,k-1}}$$

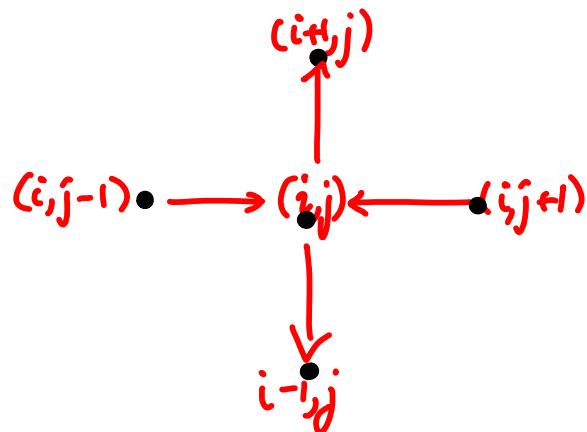
looks like \rightsquigarrow $x'_{ij} = \frac{x_{ij+1} x_{ij-1} + x_{i+1,j} x_{i-1,j}}{x_{ij}}$



Cluster algebra structure

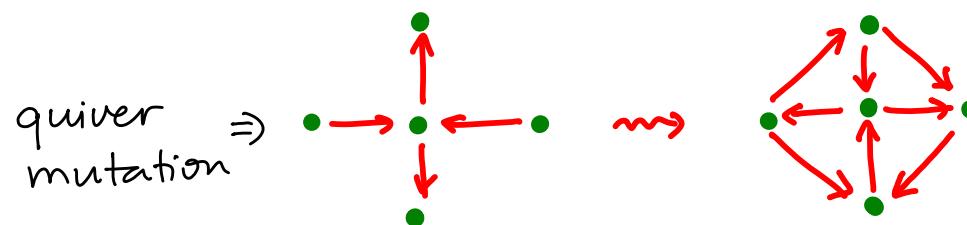
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looks like \rightsquigarrow $x'_{ij} = \frac{x_{ij+1} x_{ij-1} + x_{i+1,j} x_{i-1,j}}{x_{ij}}$



$$x'_{ij} = \frac{\pi^+ x_{ij'} + \pi^- x_{ij'}}{x_{ij}}$$

an exchange relation
in a cluster algebra

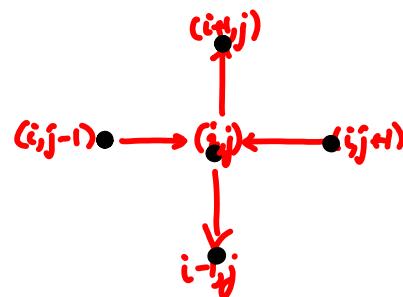


Cluster algebra structure

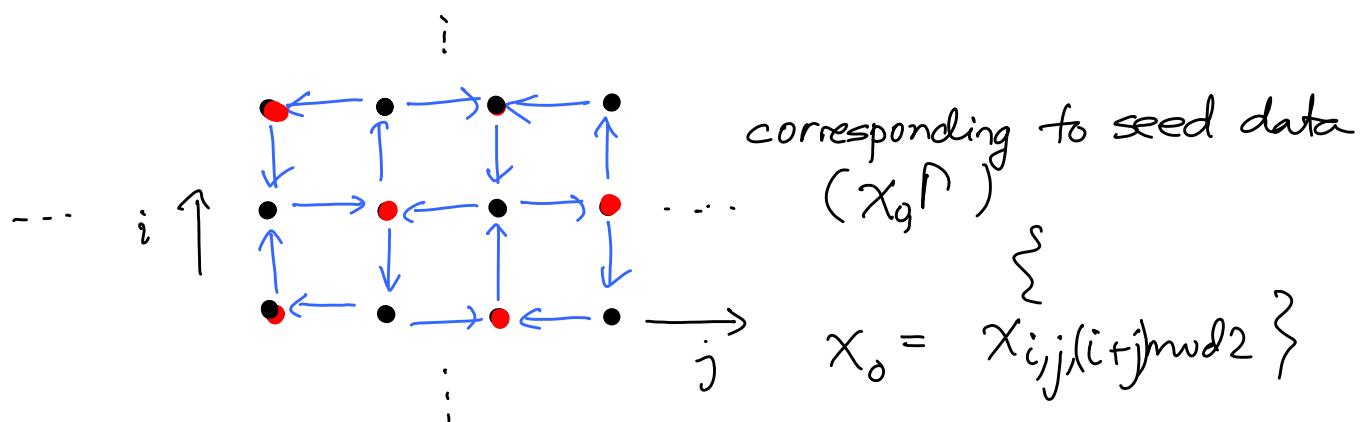
$$x_{i,j,k+1} = \frac{x_{i,j+k} x_{i,j-k} + x_{i+j,k} x_{i-j,k}}{x_{i,j,k-1}}$$

looks like \sim

$$x'_{ij} = \frac{x_{ij+1} x_{ij-1} + x_{i+1,j} x_{i-1,j}}{x_{ij}}$$



Theorem: All T-system relations are mutations in the (infinite-rank) cluster algebra with quiver Γ



About this cluster algebra:

- There are many exchange relations outside the T-system evolution.
- Therefore there are clusters with variables which are not x_{ijk}
- In a cluster algebra, any cluster variable is a (positive) Laurent polynomial in any initial data.
- To get an integer pattern, set any seed data to 1's:
frieze patterns

2) Boundary conditions on A

BC1: $x_{0,j,k} = 1$

↷ The half-space with $i \geq 0$

- The octahedron relation becomes the Desnanot-Jacobi relation for determinants

$$x_{i+1,j,k} x_{i,j,k+1} = x_{i,j,k-1} x_{i,j,k+1} - x_{i,j+1,k} x_{i,j-1,k}$$

$$\left| \begin{pmatrix} i+1 \\ i+1 \end{pmatrix} \right| \left| \begin{pmatrix} \cancel{i-1} \\ \cancel{i-1} \end{pmatrix} \right| = \left| \begin{pmatrix} i \\ i \end{pmatrix} \right| \left| \begin{pmatrix} \cancel{i} \\ \cancel{i} \end{pmatrix} \right| - \left| \begin{pmatrix} \cancel{i+1} \\ \cancel{i+1} \end{pmatrix} \right| \left| \begin{pmatrix} i \\ i \end{pmatrix} \right|$$

$$\det M \times \det M_{i,i+1}^{1,i+1} = \det M_i^1 \det M_{i+1}^{i+1} - \det M_i^{i+1} \det M_{i+1}^1$$

for M any $(i+1) \times (i+1)$ matrix.

2) Boundary conditions on A

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$$x_{i+i,j,k} x_{i,j,j+k} = x_{i,j,k-1} x_{i,j,k+1} - x_{i,j+1,k} x_{i,j-1,k}$$

$$\left| \begin{pmatrix} i+1 \\ i \end{pmatrix} \right| \left| \begin{pmatrix} i-1 \\ i \end{pmatrix} \right| = \left| \begin{pmatrix} i \\ i \end{pmatrix} \right| \left| \begin{pmatrix} i \\ i \end{pmatrix} \right| - \left| \begin{pmatrix} i \\ i \end{pmatrix} \right| \left| \begin{pmatrix} i \\ i \end{pmatrix} \right| \quad \text{with initial data } x_{0,j,k} = 1, x_{i,j,k} = \det(x_{j:k})$$

$$x_{i,j,k} = \det(W_{j,k}^{(i)})$$

$$W_{j,k}^{(i)} = \begin{pmatrix} x_{j,k-i+1} & x_{j+1,k-i+2} & \cdots & x_{j+i,k} \\ x_{j-1,k-i+2} & x_{j,k-i+3} & \ddots & \vdots \\ \vdots & & \ddots & \ddots \\ x_{j-i+1,k} & & & x_{j,k+i-1} \end{pmatrix}_{i \times i} \Rightarrow \bullet x_{i,j,k} \text{ is a polynomial in } \{x_{1,j,k}\} \text{ for } i \geq 1$$

$$x_{1,j,k} = x_{j,k}$$

Examples:

• $i=0$:

$$\begin{aligned} x_{1,j,k} x_{-1,j,k} &= x_{0,j,k+1} x_{0,j,k-1} - x_{0,j+1,k} x_{0,j-1,k} \\ &= 1 - 1 = 0 \\ \Rightarrow x_{-1,j,k} &= 0 \end{aligned}$$

• $i=1$:

$$\begin{aligned} x_{2,j,k} x_{0,j,k} &= x_{1,j,k+1} x_{1,j,k-1} - x_{1,j+1,k} x_{1,j-1,k} \\ \Rightarrow x_{2,j,k} &= \begin{vmatrix} x_{j,k-1} & x_{j+1,k} \\ x_{j-1,k} & x_{j,k+1} \end{vmatrix} \end{aligned}$$

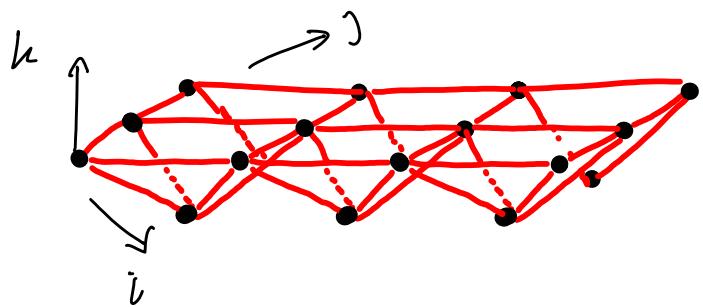
• $i=2$:

$$x_{3,j,k} = \begin{vmatrix} x_{j,k-2} & x_{j+1,k-1} & x_{j+2,k} \\ x_{j-1,k-1} & x_{j,k} & x_{j+1,k+1} \\ x_{j-2,k} & x_{j-1,k+1} & x_{j,k+2} \end{vmatrix}$$

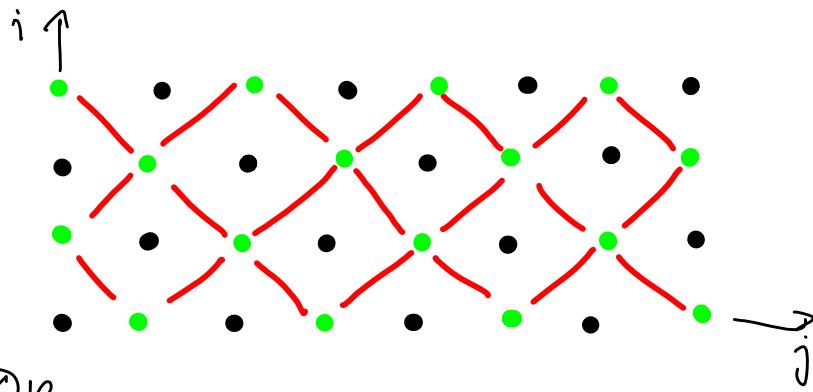
Boundary condition 2: $x_{N,j,k} = 1 \quad \forall j, k$: A_{N-1} T-system



The octahedron relation on a strip



look
from
above
~~~~~

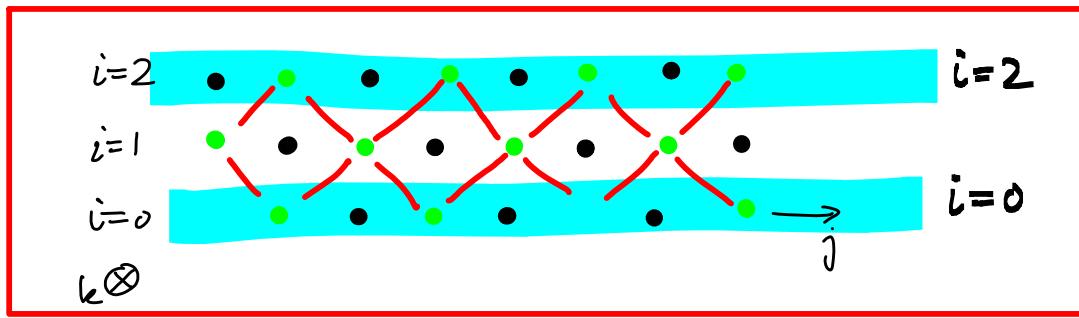


$$x_{i,j,k+1} x_{i,j,k-1} = x_{i,j+1,k} x_{i,j-1,k} + x_{i+1,j,k} x_{i-1,j,k}, \quad x_{0,j,k} = x_{N,j,k} = 1$$

Example:  $N=2$

$A_1$  T-system

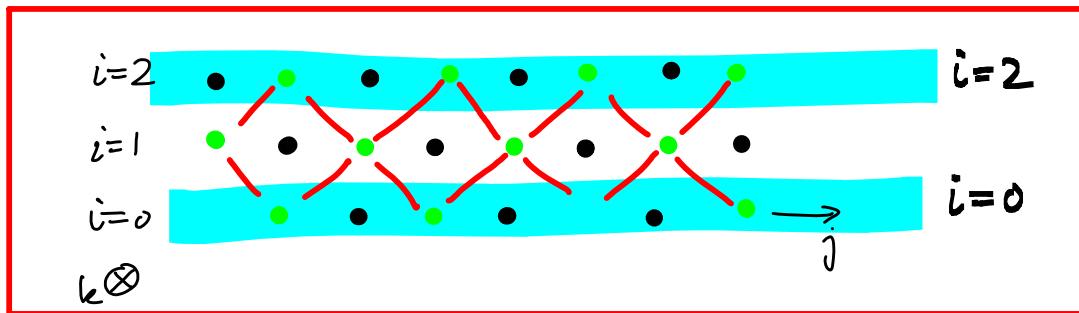
$$X_{i,j,k+1} X_{i,j,k-1} = X_{i,j+1,k} X_{i,j-1,k+1}$$



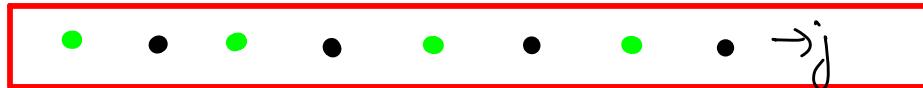
Example :  $N = 2$

## $A_1$ T-system

$$x_{i,j,k+1} x_{i,j,k-1} = x_{i,j+1,k} x_{i,j-1,k} + 1$$



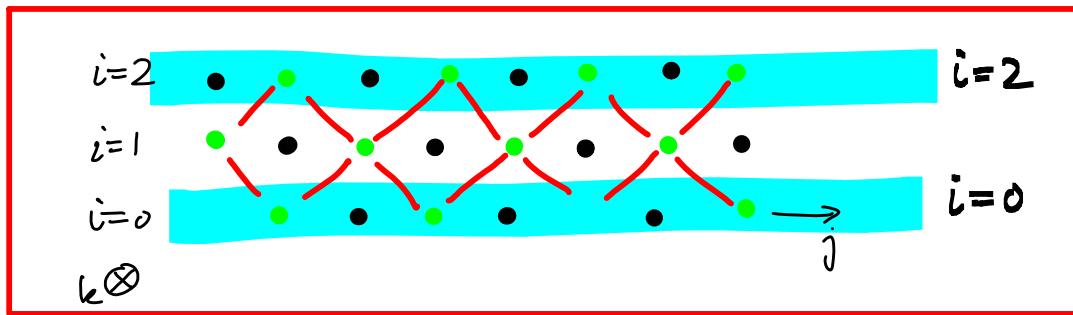
↓ Restrict to the strip



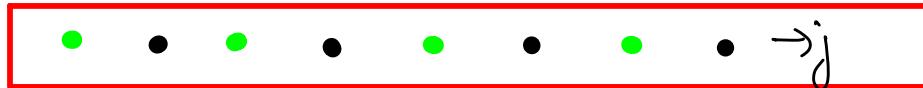
Example:  $N=2$

$A_1$  T-system

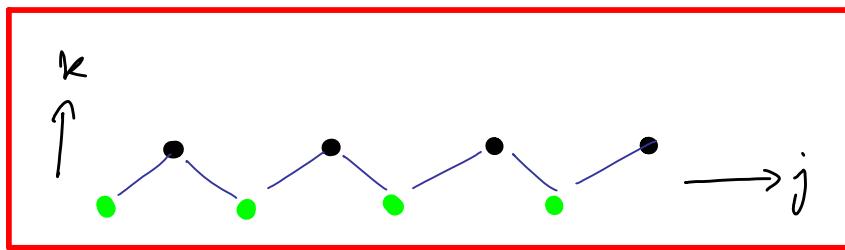
$$X_{i,j,k+1} X_{i,j,k-1} = X_{i,j+1,k} X_{i,j-1,k+1}$$



↓ Restrict to the strip

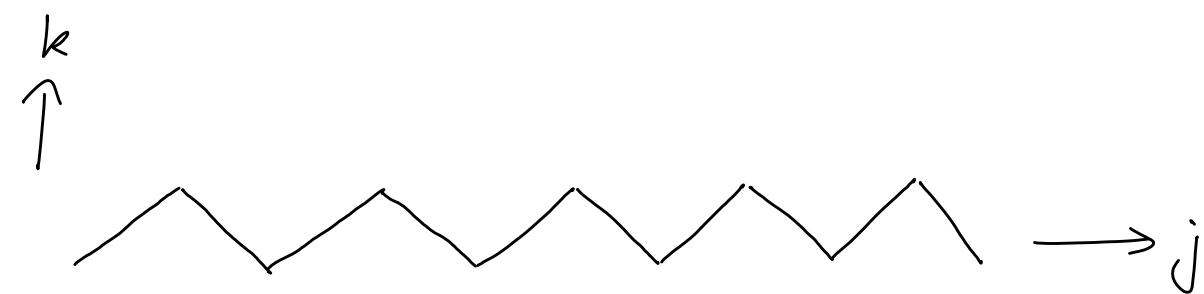


↓ rotate to  $j-k$  plane



The  $A_1$ -evolution takes place  
on the odd sublattice of  $\mathbb{Z}^2$

$A_1$  evolution:



## About T-systems:

- There is a T-system, satisfied by the q-characters of quantum affine algebras or transfer matrices of generalized Heisenberg spin chains, for any simple Lie algebra  $\mathfrak{g}$  (and beyond) ...
- History:  
80's: Kirillov-Reshetikhin, 90's: Kuniba, Nakanishi, Suzuki, 2000's: Nakajima, Hernandez
- In this talk: combinatorial games with type A only.

## Linear recursion relation & conserved quantities

- BC2 in light of Desnauot-Jacobi, so:  $x_{N;j,k} = |W_{j,k}^{(N)}| = 1 \quad \forall j, k$

$$\Rightarrow \bullet |W_{i,j}^{(N+1)}| = 0 \quad \forall i, j$$

$$W_{j,k}^{(N+1)} = \begin{pmatrix} x_{j,k-N} & x_{j+1,k-N+1} & \cdots & x_{j+N,k} \\ x_{j-1,k-N+1} & x_{j,k-N+2} & \ddots & \\ \vdots & & & \\ x_{j-N,k} & & & x_{j,k+N} \end{pmatrix}_{(N+1) \times (N+1)}$$

## Linear recursion relation & conserved quantities

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- expanding the determinant along a row/column:

$$\boxed{\sum_{\ell=0}^N (-1)^{\ell} x_{j+\ell, k+\ell} C_{kj}^{(\ell)} = 0 = \sum_{\ell=0}^{N+1} (-1)^{\ell} x_{j-\ell, k+\ell} D_{kj}^{(\ell)}}$$

linear recursion  
relations

## Linear recursion relation & conserved quantities

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$$W_{j,k}^{(N+1)} = \begin{pmatrix} x_{j,k-N} & x_{j+1,k-N+1} & \cdots & x_{j+N,k} \\ x_{j-1,k-N+1} & x_{j,k-N+2} & \ddots & \\ \vdots & & \ddots & \\ x_{j-N,k} & & & x_{j,k+N} \end{pmatrix}_{(N+1) \times (N+1)}$$

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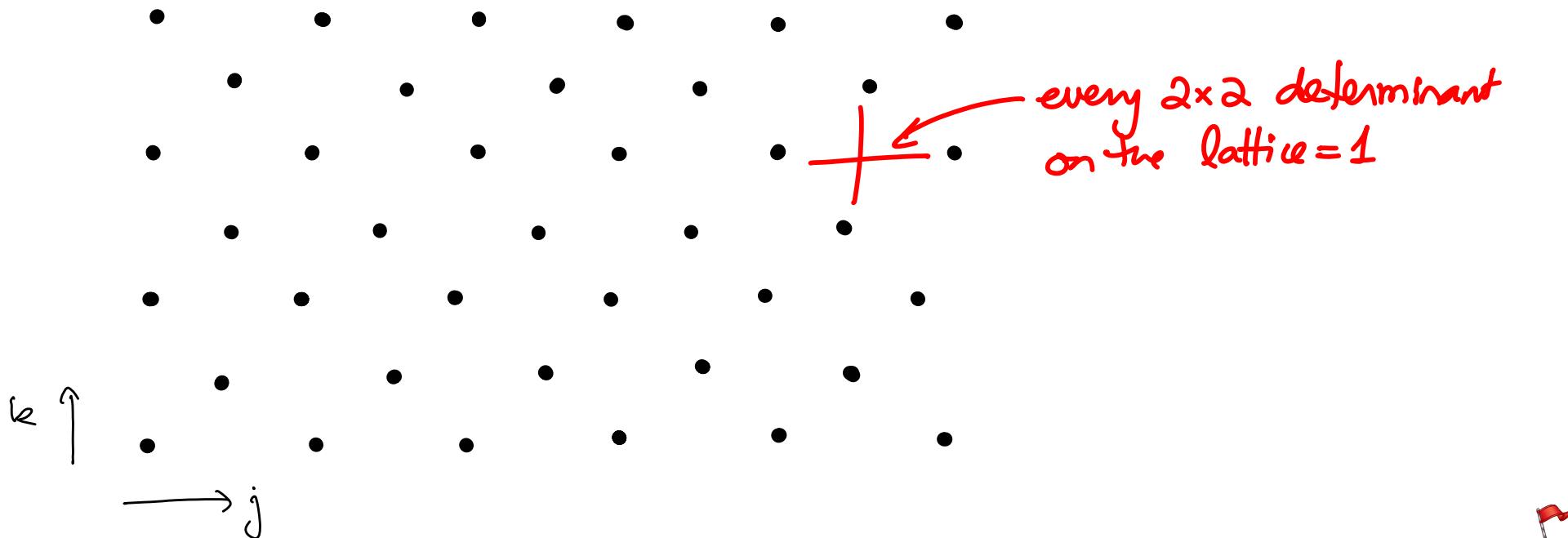
$$\sum_{l=0}^N (-1)^l x_{j+l,k+l} C_{kj}^{(l)} = 0 = \sum_{l=0}^{N+1} (-1)^l x_{j-l,k+l} D_{kj}^{(l)}$$

linear recursion relations

- Theorem:  $C_{kj}^{(l)} = C_{k+j}^{(l)}$  and  $D_{kj}^{(l)} = D_{k-j}^{(l)} \rightsquigarrow$  Invariants

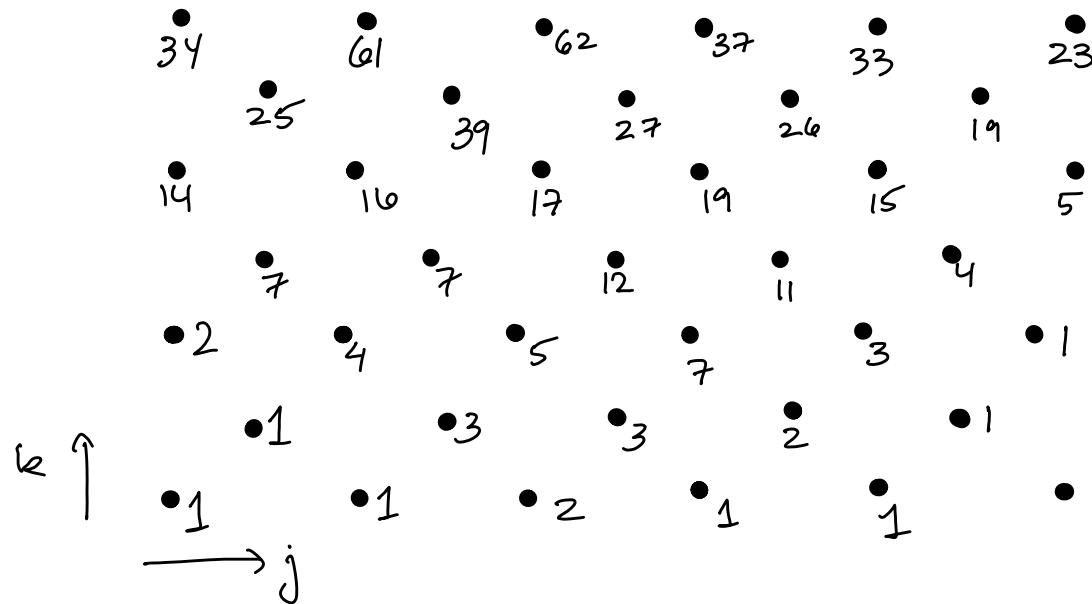
Example:  $N=2$

$$\begin{vmatrix} x_{j,k-1} & x_{j+1,k} \\ x_{j+1,k} & x_{j,k+1} \end{vmatrix} = 1 \leftrightarrow \boxed{x_{i,j,k+1} x_{i,j+1,k} - x_{i,j+1,k} x_{i,j+1,k+1}} \\ A_1 \text{- system}$$



Example:  $N=2$

$$\begin{vmatrix} x_{j,k-1} & x_{j+1,k} \\ x_{j+1,k} & x_{j,k+1} \end{vmatrix} = 1 \iff x_{i,j,k+1} x_{i,j+1,k} = x_{i,j+1,k} x_{i,j+1,k+1}$$



If we choose  $x_{i,k} \in \mathbb{N}$   
this is called a Conway  
Coxeter Freeze pattern

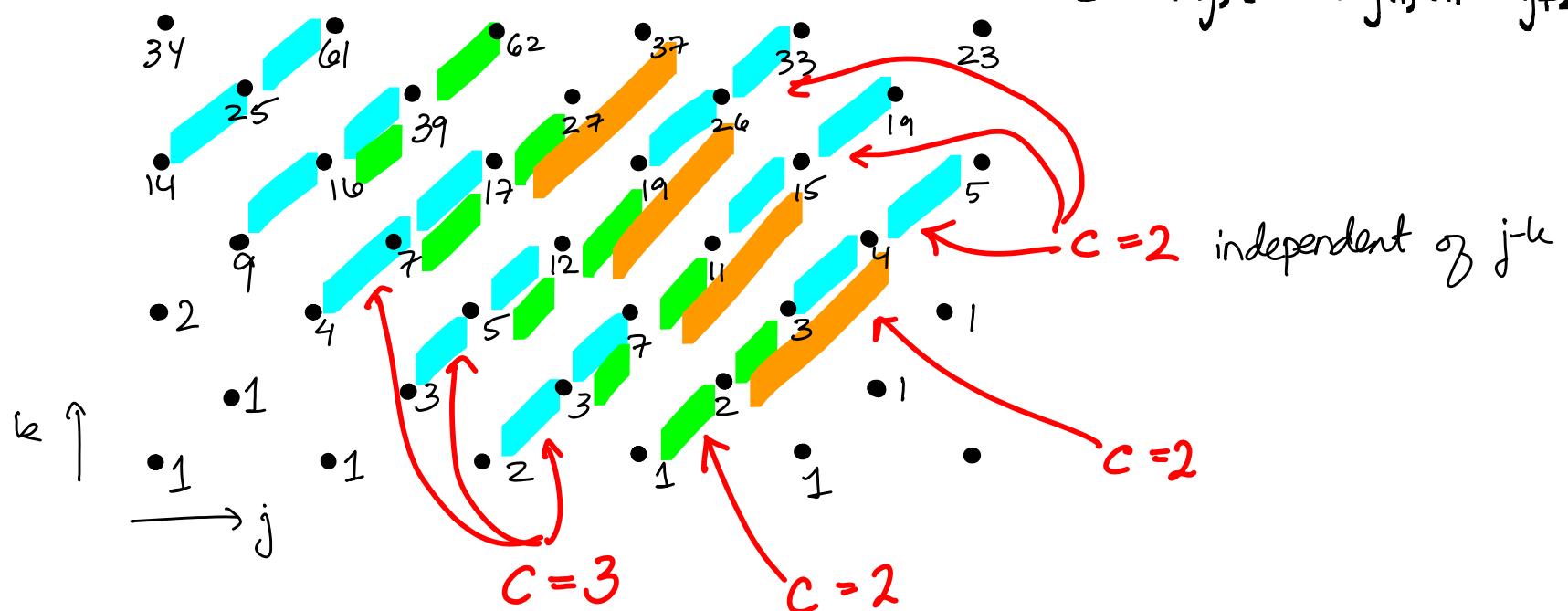
(choice of initial data)

Example:  $N=2$

$$\begin{vmatrix} x_{j,k-1} & x_{j+1,k} \\ x_{j+1,k} & x_{j,k+1} \end{vmatrix} = 1 \leftrightarrow$$

$$x_{i,j,k+1} x_{i,j+1,k} - x_{i,j+1,k} x_{i,j+1,k+1} = 1$$

$$O = x_{j,k} - C^1 x_{j+1,k+1} + x_{j+2,k+2}$$

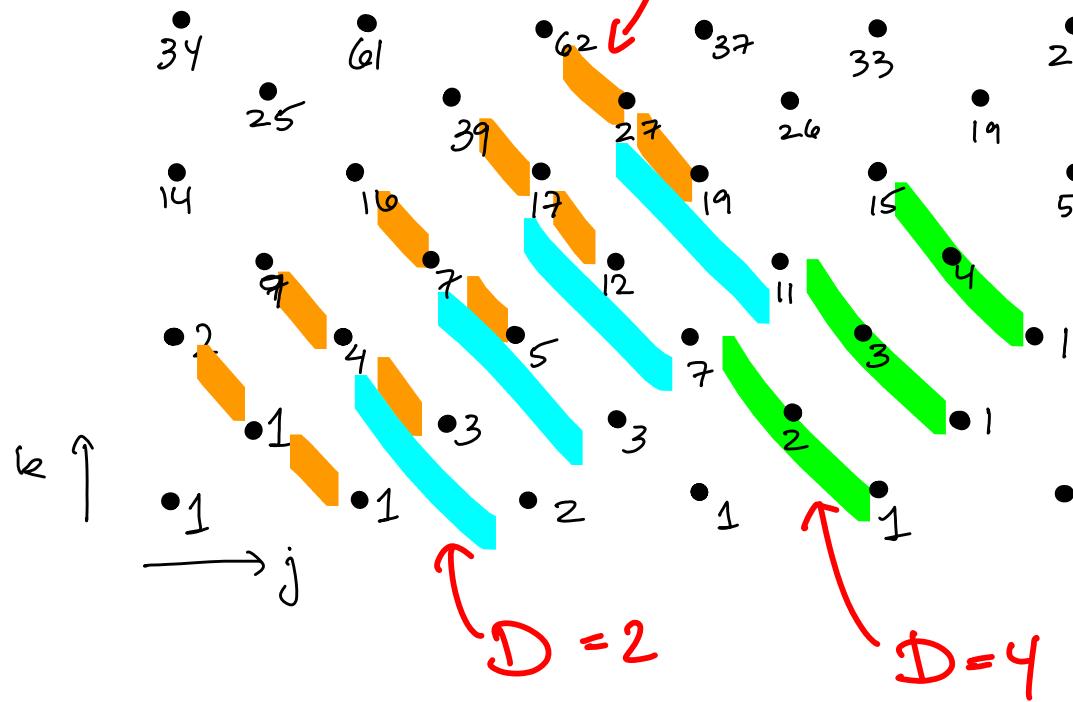


Example:  $N=2$

$$\begin{vmatrix} x_{j,k-1} & x_{j+1,k} \\ x_{j+1,k} & x_{j,k+1} \end{vmatrix} = 1 \iff$$

$$x_{i,j,k+1} x_{i,j+1,k-1} = x_{i,j+1,k} x_{i,j-1,k} + 1$$

$$D=3$$



$$0 = x_{j,k} - D^{(1)} x_{j-1,k+1} + x_{j-2,k+2}$$

$D$  independent of  $j+k$ .

## Integrability : The linear recursions

Theorem:

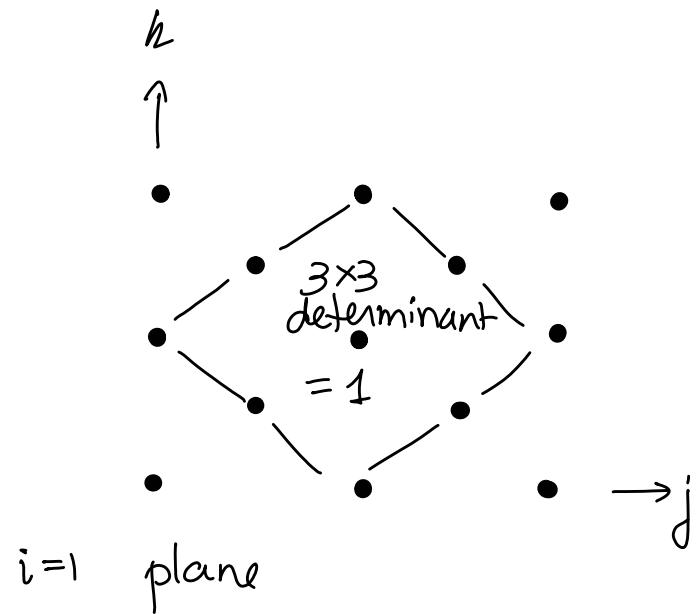
$$\sum_{\ell=0}^N (-1)^\ell C_{j+h}^{(\ell)} x_{j+\ell, h+\ell} = 0, \quad C_{j+h}^{(\ell)} \text{ independent of } j-h$$

$$\sum_{\ell=0}^N (-1)^\ell D_{j-h}^{(\ell)} x_{j-\ell, h+\ell} = 0, \quad D_{j-h}^{(\ell)} \text{ independent of } j+h$$

C's and D's are conserved quantities along the directions  $j-h$  &  $j+h$ .

$C^{(0)} = C^{(N)} = D^{(0)} = D^{(N)} = 1$  so there are  $(N-1)$  non-trivial coefficients in each linear equation.

$SL_3$       Freeze pattern



$$H_{j,k}, W_{j,k}^{(3)} = 1 = \begin{vmatrix} x_{j,k-2} & x_{j+1,k-1} & x_{j+2,k} \\ x_{j,k-1} & x_{j,k} & x_{j+1,k+1} \\ x_{j-2,k} & x_{j-1,k+1} & x_{j,k+2} \end{vmatrix}$$

$$x_{j,k} \equiv x_{i,j,k}$$

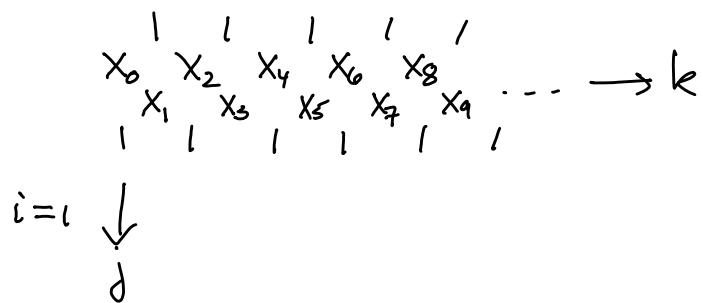
Any cluster variable = positive Laurent  
in initial data  $\Rightarrow$  choose initial  
data = 1's  $\Rightarrow$  integer pattern.

Boundary condition 3:



Zamolodchikov periodicity for T-systems  $x_{i,0,k} = x_{i,l+1,k} = 1$

Example:  $N=2, l=2$



$$x_0$$

$$x_1$$

$$x_2 = \frac{1+x_1}{x_0}$$

$$x_3 = \frac{1+x_0+x_1}{x_1}$$

$$x_{i+5} = x_i$$

$$x_4 = \frac{1+x_0}{x_1}$$

$$x_5 = x_6$$

$$x_6 = x_1$$

## Zamolodchikov periodicity for T-systems $x_{i,0,k} = x_{i,l+k} = 1$

Example:  $N=2, l=3$

$$\begin{matrix} & | & | & | & | & | & | \\ \downarrow & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ j & | & | & | & | & | & | & | & | & | & | \end{matrix} \rightarrow k$$

$$\left. \begin{matrix} x_0 \\ x_1 \end{matrix} \quad x'_1 \end{matrix} \right\} \text{initial data}$$

$$x_2 = (x_1 x'_1 + 1) x_0^{-1}$$

$$x_3 = (1 + x_0 + x_1 x'_1) (x_1 x_0)^{-1}$$

$$x'_3 = (1 + x_0 + x_1 x'_1) (x_1 x'_1)^{-1}$$

$$x_4 = (1 + 2x_0 + x_0^2 + x_1 x'_1) (x_0 x_1 x'_1)^{-1}$$

$$x_5 = (1 + x_0) (x'_1)^{-1}$$

$$x'_5 = (1 + x_0) (x_1)^{-1}$$

$$x_6 = x_0$$

$$x_7 = x'_1 \quad x'_7 = x_1$$

$$\Rightarrow x_{l+2} = x_a$$

$$x'_{l+2} = x'_a$$

Evolution takes place in a tube in  $\mathbb{Z}^3$  of width  $l \times (N-1)$ .

Zamolodchikov periodicity for T-systems  $x_{i,0,k} = x_{i,l+1,k} = 1$

Theorem: For the  $A_{N-1}$  T-system with boundary condition

$$x_{i,0,k} = 1 = x_{i,l+1,k}$$

There is a (quasi-) periodicity

$$x_{i,j,k+l+N+1} = x_{N-i,l+1-j,k}.$$

full period  $\Leftrightarrow p = 2(l+N+1)$ .

## BC3': Q-systems

$$x_{i,j+2,k} = x_{i,j,k} =: Q_{i,k}$$



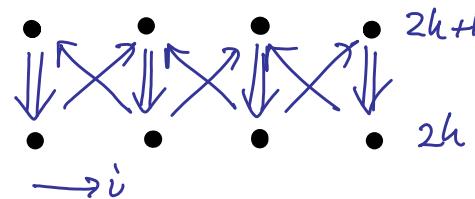
$$x_{i,j,k+1} x_{i,j,k-1} = x_{i,j+1,k} x_{i,j-1,k} + x_{i+1,j,k} x_{i-1,j,k}$$



$$Q_{i,k+1} Q_{i,k-1} = Q_{i,k}^2 + Q_{i+1,k} Q_{i-1,k}$$

- If  $x_{0,k} = x_{N,k} = 1 \Rightarrow "A_{N-1} \text{ Q-system}"$

- As a cluster algebra, quiver =



finite rank  
cluster algebra

## About Q-systems

$$\bullet Q_{i,k+1}Q_{i,k-1} = Q_{i,k}^2 - Q_{i+1,k}Q_{i-1,k}$$

$$\bullet Q_{0,k} = Q_{N+1,k} = 1$$

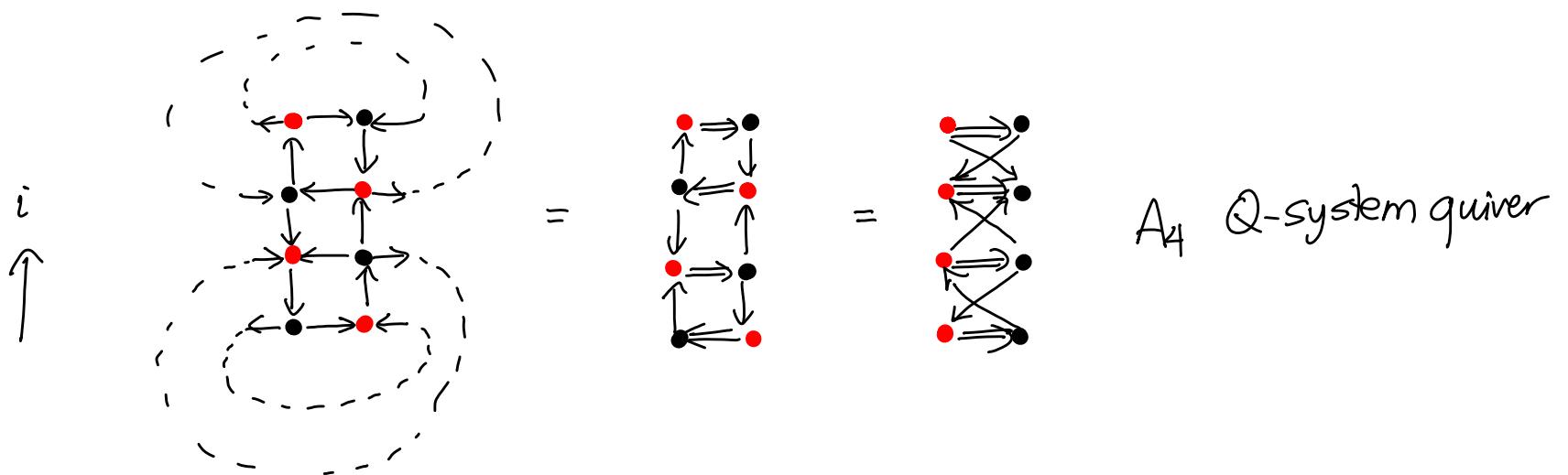
- There is a Q-system for each  $\mathfrak{g}$  (simple and beyond).

- For type A, with the initial data  $Q_{i,0}=1$ ,  $Q_{i,1}=e_i(\underline{\lambda})$  elementary symmetric functions  $\Rightarrow Q_{i,k} = \sum_{\substack{\text{Young diagram } \lambda \\ \text{with } k \text{ boxes}}} \{_{\lambda}\}_i$  Schur function.

$$\sum_{\substack{\text{Young diagram } \lambda \\ \text{with } k \text{ boxes}}} \{_{\lambda}\}_i$$

- For  $\mathfrak{g}$  simple, with initial data  $Q_{i,0}=1$ ,  $Q_{i,1} = \text{ch KR}(w_i)$   
 $\Rightarrow Q_{i,k} = \text{ch KR}(kw_i)$  Kirillov-Reshetikhin modules

T-system wrapped around the cylinder with period 2:



## A<sub>N-1</sub> Q-systems

- Wronskian matrix

$$W_{j,k}^{(i)} \rightsquigarrow W_k^{(i)}$$

$$W_k^{(i)} = \begin{pmatrix} x_{k-i+1} & x_{k-i+2} & \cdots & x_n \\ x_{k-i+2} & x_{k-i+3} & & \\ \vdots & & & x_{k+i-1} \\ x_n & & & \end{pmatrix}_{i \times i}$$

$$x_k = Q_{1,k}$$

Then  $Q_{1,k} = |W_k^{(i)}|$

- Linear recursion

$$W_k^{(N+1)} = 0 \quad \forall k \Rightarrow$$

$$\sum_{\ell=0}^N C^{(\ell)} Q_{1,k-\ell} = 0$$

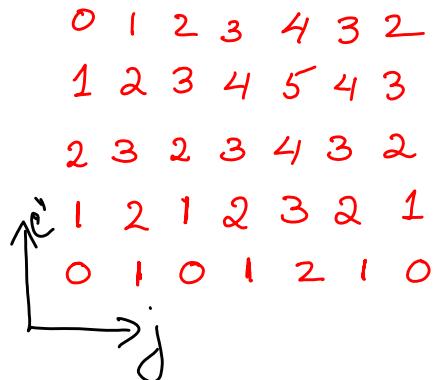
Theorem:  $C^{(\ell)}$  independent of  $k$  (conserved quantities)

$C^{(0)} = C^{(n)} = 1$  and  $C^{(\ell)}$  are determinants of minors of  $W_k^{(N+1)}$   
 (DFK + H. Williams, P. Vichitkunakorn + many others...)

### 3) Solutions of the T-system in terms of Networks



① Start with stepped surface  $S = (i, j, k(i, j))$



### 3) Solutions of the T-system in terms of Networks

(a) Start with stepped surface  $S = (i, j, k(i, j))$

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 3 | 2 |
| 1 | 2 | 3 | 4 | 5 | 4 | 3 |
| 2 | 3 | 2 | 3 | 4 | 3 | 2 |
| 1 | 2 | 1 | 2 | 3 | 2 | 1 |
| 0 | 1 | 0 | 1 | 2 | 1 | 0 |

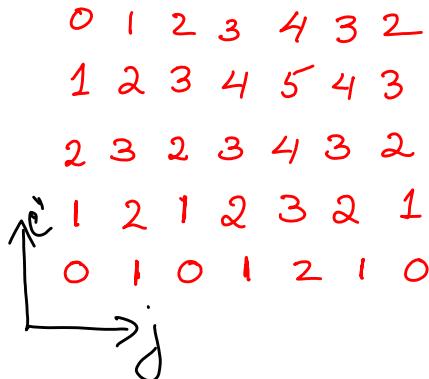
(b) Draw arrows on the horizontal:  $k \rightarrow k+1$



$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 3 \rightarrow 2$   
 $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 4 \rightarrow 3$   
 $2 \leftarrow 3 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 3 \leftarrow 2$   
 $1 \rightarrow 2 \leftarrow 1 \rightarrow 2 \rightarrow 3 \leftarrow 2 \rightarrow 1$   
 $0 \rightarrow 1 \leftarrow 0 \rightarrow 1 \rightarrow 2 \leftarrow 1 \rightarrow 0$

### 3) Solutions of the T-system in terms of Networks

(a) Start with stepped surface  $S = (i, j, k(i, j))$

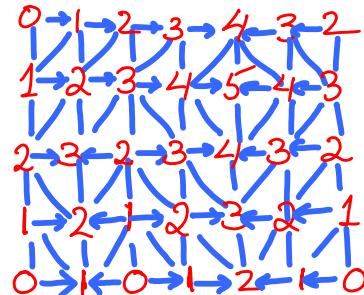


(b) Draw arrows on the horizontal:  $k \rightarrow k+1$

$$\begin{array}{c} 0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 3 \rightarrow 2 \\ 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 4 \rightarrow 3 \\ 2 \leftarrow 3 \leftarrow 2 \rightarrow 3 \rightarrow 4 \leftarrow 3 \leftarrow 2 \\ 1 \leftarrow 2 \leftarrow 1 \rightarrow 2 \rightarrow 3 \leftarrow 2 \leftarrow 1 \\ 0 \rightarrow 1 \leftarrow 0 \rightarrow 1 \rightarrow 2 \leftarrow 1 \end{array}$$

(c) add vertical edges  
add diagonal edges

$$k \nearrow^k \text{ or } \swarrow^k$$

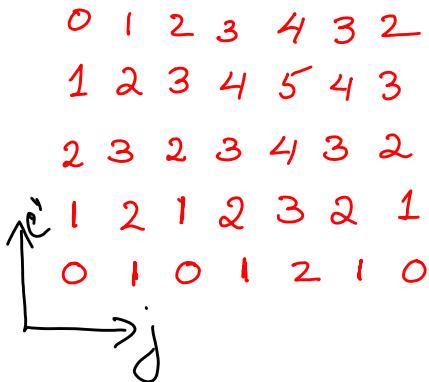


Note: If there is a choice...

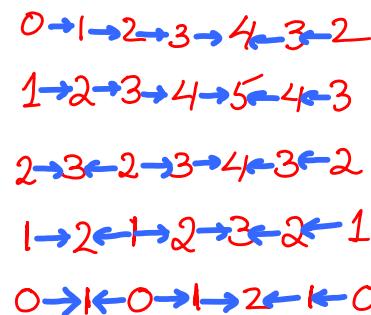
$$\begin{matrix} & & k & & k-1 \\ & & \backslash & & / \\ k-1 & & k & & k-1 \end{matrix} \quad \text{or} \quad \begin{matrix} & & k & & k-1 \\ & & / & & \backslash \\ k-1 & & k & & k \end{matrix}$$

### 3) Solutions of the T-system in terms of Networks

(a) Start with stepped surface  $S = (i, j, k(i, j))$

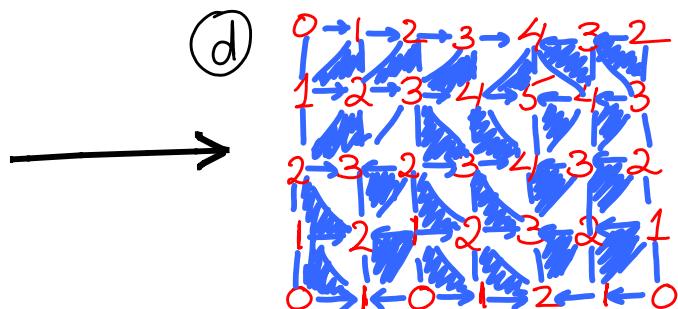
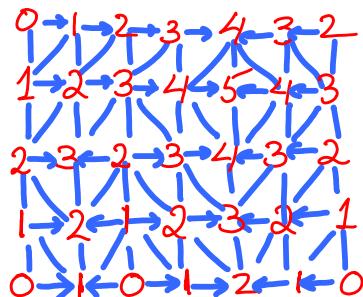


(b) Draw arrows on the horizontal:  $k \rightarrow k+1$

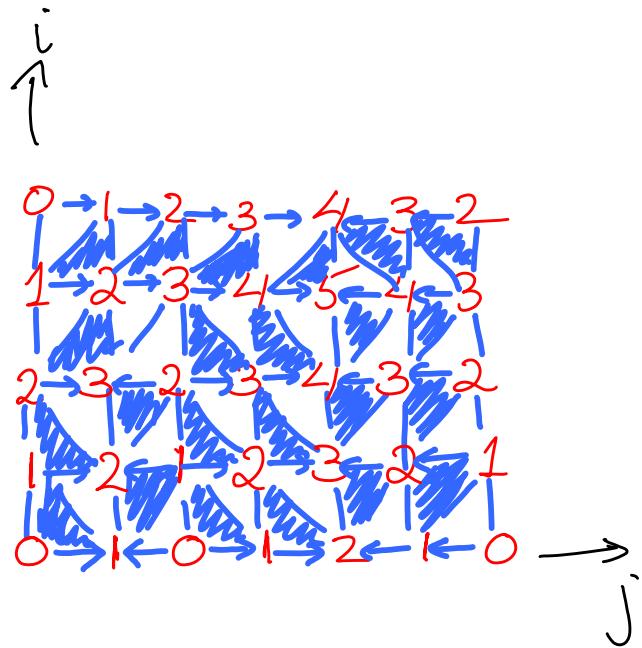


(c) add vertical edges  
add diagonal edges

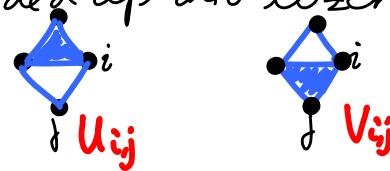
$k \nearrow^k$  or  $\searrow^k$



Color to the left of each arrow.



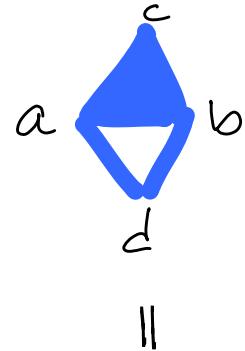
is a region divided up into lozenges



Example: A<sub>2</sub>

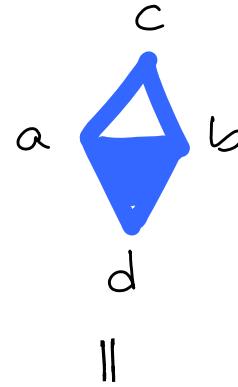
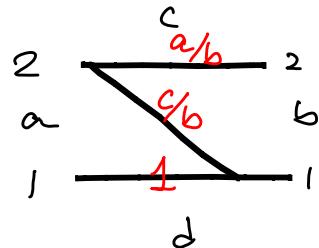
0 1 0 1 2 1 2 3 2 1 0

## Network matrix



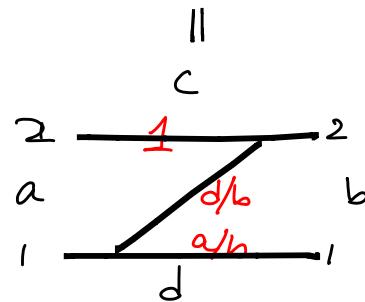
$$U = \begin{pmatrix} 1 & 0 \\ \frac{c}{b} & \frac{a}{b} \end{pmatrix}$$

||



$$V = \begin{pmatrix} \frac{a}{b} & \frac{d}{b} \\ 0 & 1 \end{pmatrix}$$

||

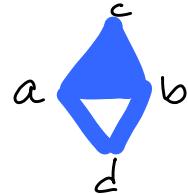


*lattice chip*

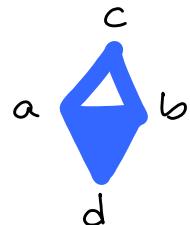
*matrix*

*network*

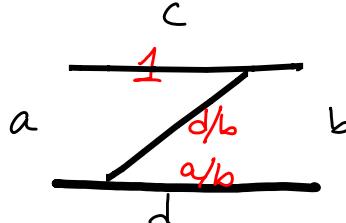
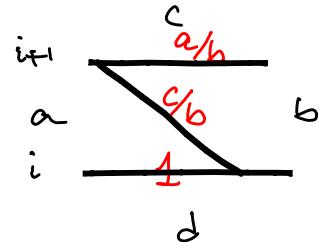
## Relations for network matrices:



$$U = \begin{pmatrix} 1 & 0 \\ \frac{c}{b} & \frac{a}{b} \end{pmatrix}$$



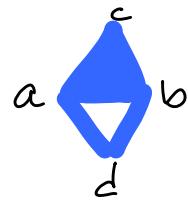
$$V = \begin{pmatrix} \frac{a}{b} & \frac{a}{b} \\ 0 & 1 \end{pmatrix}$$



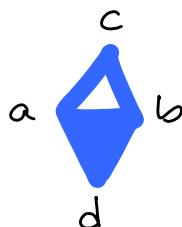
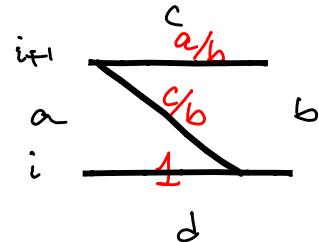
$$U_i = i \begin{pmatrix} 1 & & & \\ & \boxed{u} & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}, V_i = i \begin{pmatrix} 1 & & & \\ & \checkmark & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}$$

embed network into  $GL(N)$

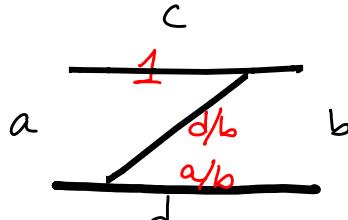
## Relations for network matrices:



$$U = \begin{pmatrix} 1 & 0 \\ \frac{c}{b} & \frac{a}{b} \end{pmatrix}$$



$$V = \begin{pmatrix} \frac{a}{b} & \frac{a}{b} \\ 0 & 1 \end{pmatrix}$$



$$U_i = \begin{pmatrix} 1 & & \\ & \boxed{u} & \\ & 1 & \dots \\ & \dots & 1 \end{pmatrix}, V_i = \begin{pmatrix} 1 & & \\ & \checkmark & \\ & \dots & \dots \\ & \dots & 1 \end{pmatrix}$$

## Relations

①

and

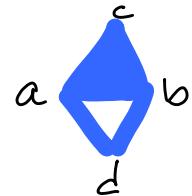
$$U_2 V_1 = V_1' U_2'$$

$$V_2 U_1 = U_1' V_2'$$

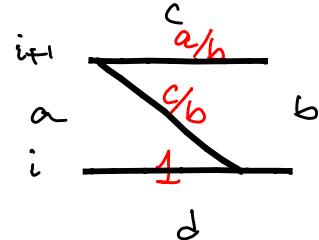
(How we draw the diagonal )  
in  $k - k-1$  is not important



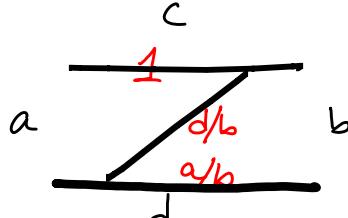
## Relations for network matrices:



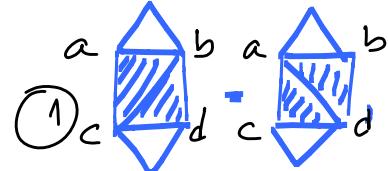
$$U = \begin{pmatrix} 1 & 0 \\ \frac{c}{b} & \frac{a}{b} \end{pmatrix}$$



$$V = \begin{pmatrix} \frac{a}{b} & \frac{a}{b} \\ 0 & 1 \end{pmatrix}$$



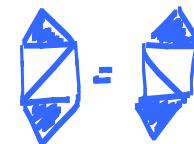
$$U_i = \begin{pmatrix} 1 & & & \\ & \boxed{u} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, V_i = \begin{pmatrix} 1 & & & \\ & \checkmark & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$



$$U_2 V_1 = V_1' U_2'$$

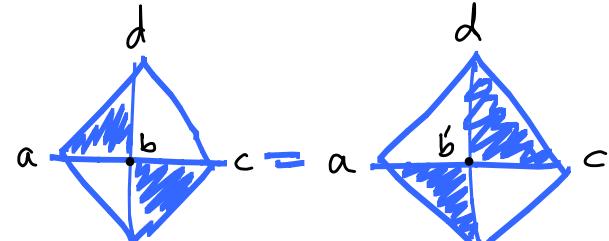
(choice of how to draw diagonal  
 $\begin{matrix} k & k+r \\ k+r & k \end{matrix}$  or  $\begin{matrix} k & k-r \\ k-r & k \end{matrix}$ )

and



$$V_2 U_1 = U_1' V_2$$

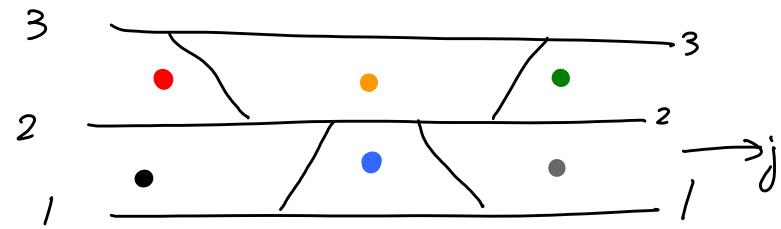
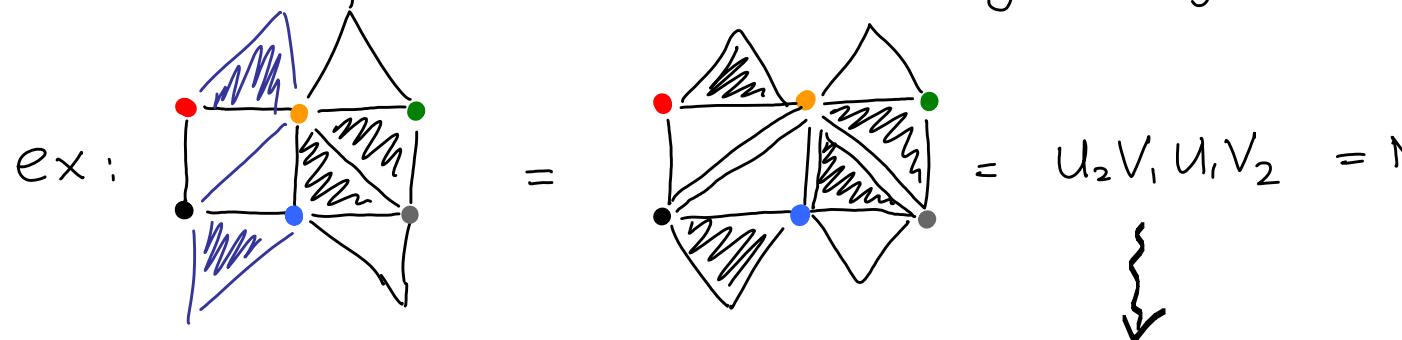
②



$$U_{i,j} V_{i,j+1} = V_{i,j}' U_{i,j+1}$$

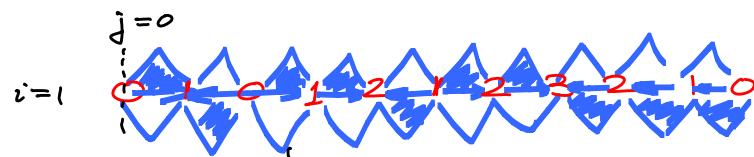
$$\text{where } bb' = ac + de$$

Definition The Network matrix of a tiled surface is the ordered product of the elementary lozenge matrices

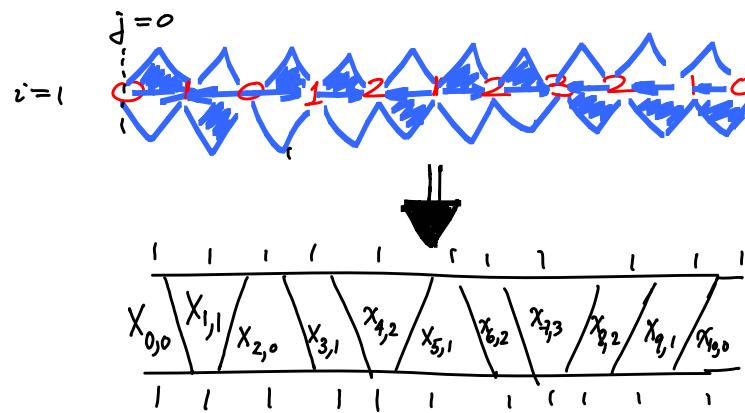


Network matrix  $N_{ab}$  = partition function of paths from  $a$  to  $b$

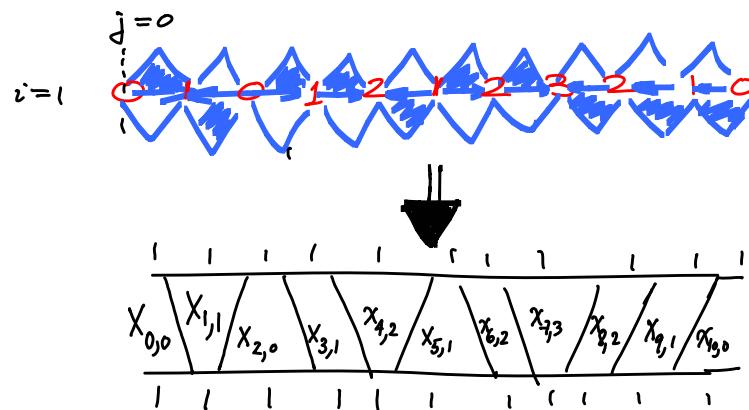
Example:  $A_2$



Example:  $A_2$

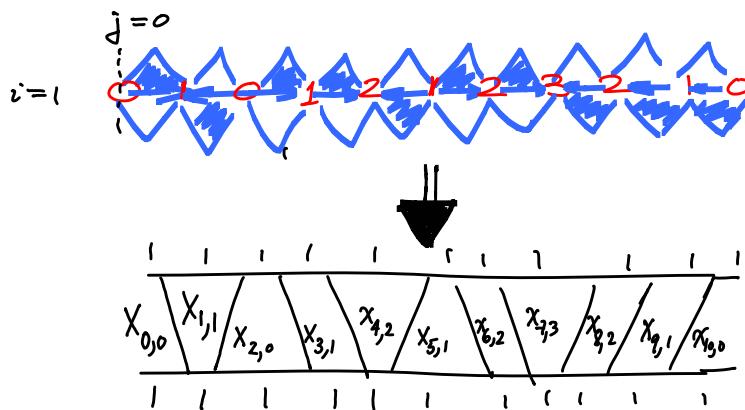


Example:  $A_2$



Network Matrix  $N := U(x_{0,0}, x_{1,1}) \cup (x_{1,1}, x_{2,0}) \cup (x_{2,0}, x_{3,1}) \dots \cup (x_{9,1}, x_{10,0})$   $2 \times 2$  matrix

Example:  $A_2$

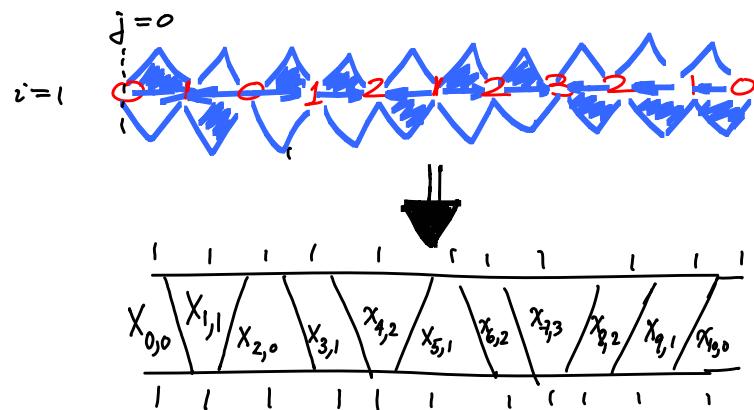


Network Matrix  $N := U(x_{0,0}, x_{1,1}) V(x_{1,1}, x_{2,0}) U(x_{2,0}, x_{3,1}) \dots V(x_{9,1}, x_{10,0})$   $2 \times 2$  matrix

$$\text{Relation } U(x_{j,k}, x_{j+1,k+1}) V(x_{i+1,k+1}, x_{i+2,k}) = V(x_{j,k}, x_{j+1,k+1}) U(x_{j+1,k+1}, x_{j+2,k})$$

$$\begin{array}{c} x_{j,k} \quad x_{j+1,k+1} \\ \diagup \quad \diagdown \\ \text{---} \end{array} = x_{j,k} \begin{array}{c} x_{j+1,k+1} \quad x_{j+2,k} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \Leftrightarrow x_{j+1,k+1} x_{j+1,k+1} = x_{j,k} x_{j+1,k} + l$$

Example:  $A_2$



Network Matrix  $N := U(x_{0,0}, x_{1,0}) V(x_{1,1}, x_{2,0}) U(x_{2,0}, x_{3,1}) \dots V(x_{9,1}, x_{10,0})$   $2 \times 2$  matrix

$$\text{Relation } U(x_{j,k}, x_{j+1,k+1}) V(x_{i+1,k+1}, x_{i+2,k}) = V(x_{j,k}, x_{j+1,k+1}) U(x_{j+1,k+1}, x_{j+2,k})$$

$$\begin{array}{c} x_{j,k} \\ \diagup \quad \diagdown \\ x_{j+1,k+1} \end{array} = x_{j,k} \begin{array}{c} \diagup \quad \diagdown \\ x_{j+1,k+1} \quad x_{j+2,k} \\ | \end{array} \Leftrightarrow x_{j+1,k+1} x_{j+1,k+1} = x_{j,k} x_{j+1,k} + 1$$

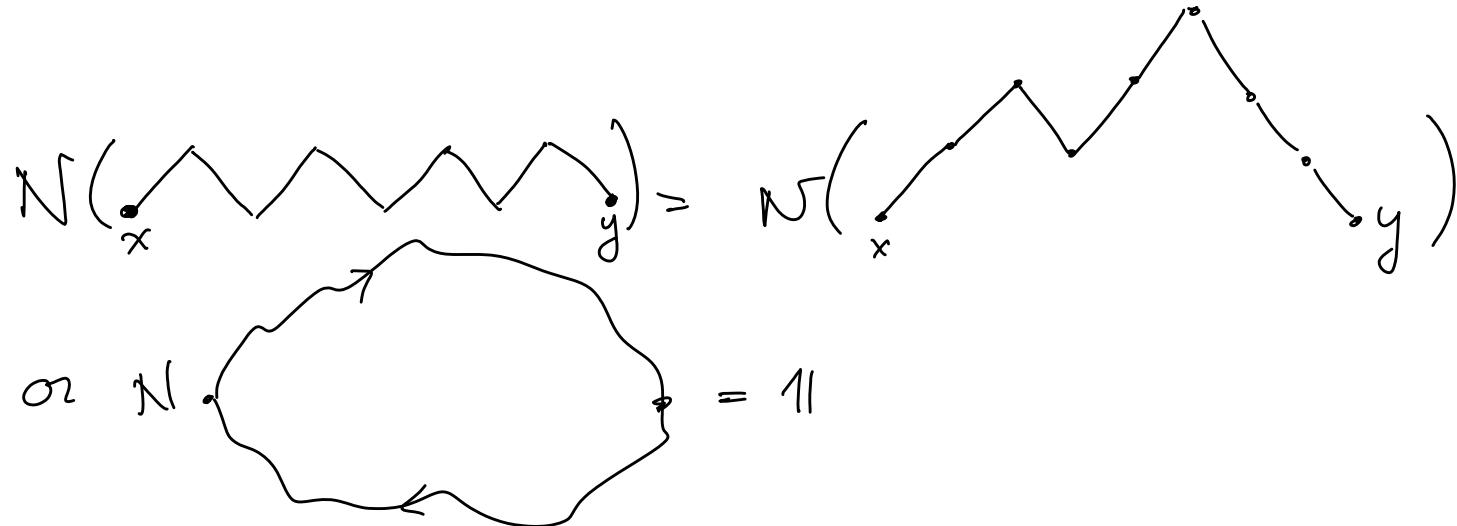
$\Rightarrow$  Network matrices are invariant under exchange relation!

Zero Curvature condition: Network matrices with the same b.c. are equal.

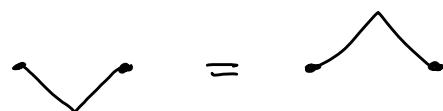
Zero curvature:

$$N(x \text{---} y) = N(x \text{---} y)$$

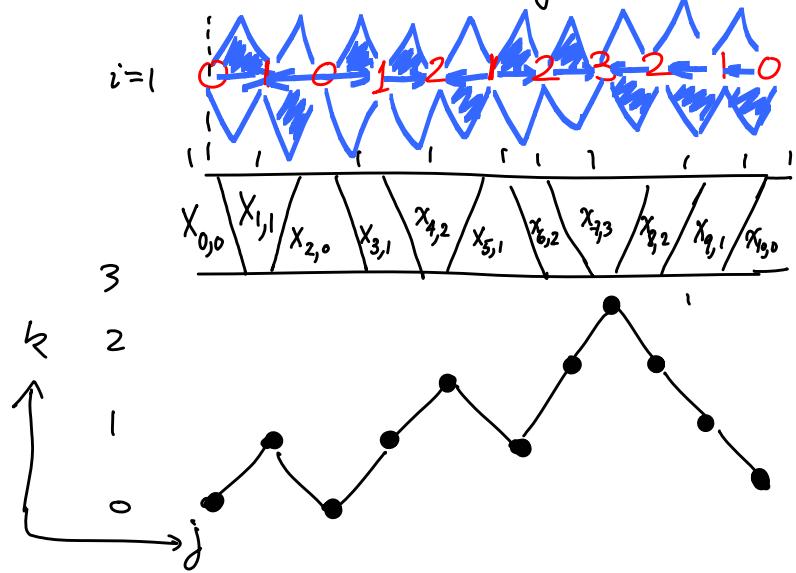
or  $N(\text{---}) = 11$



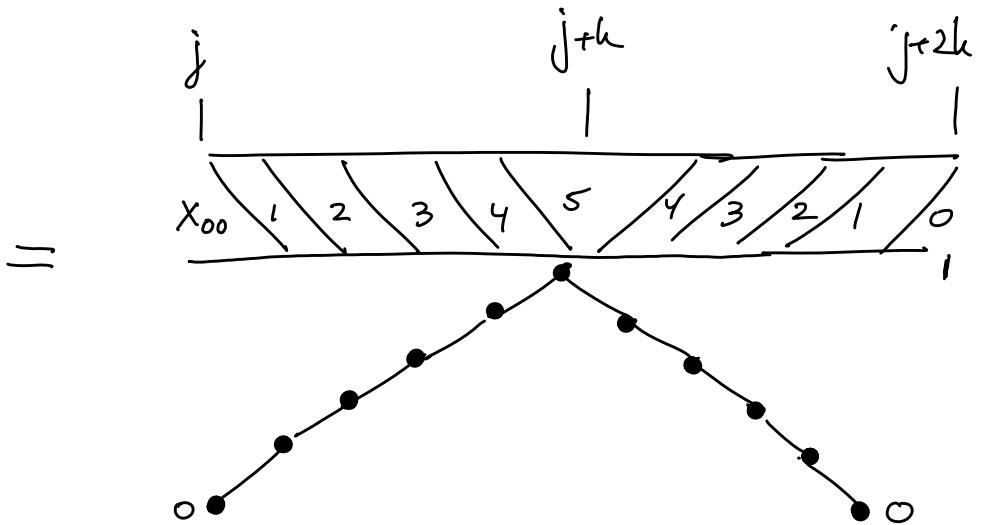
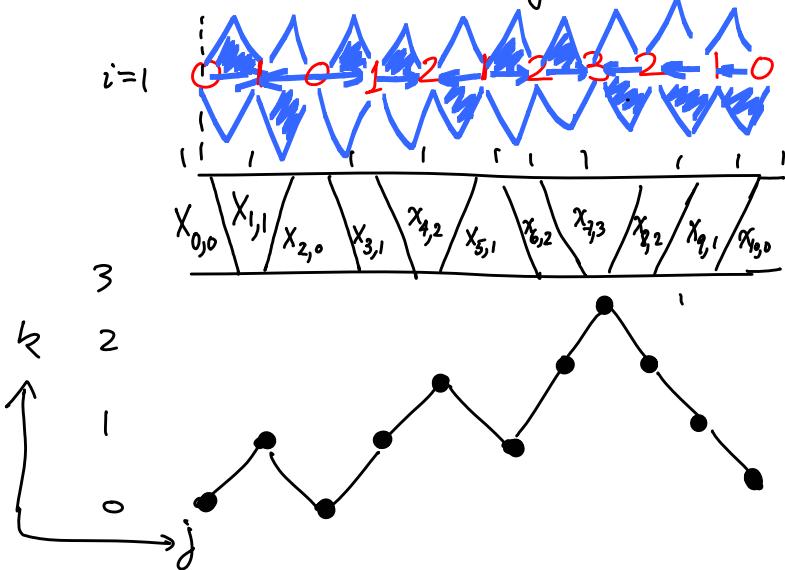
due to local exchange relation

$$\text{---} = \text{---}$$


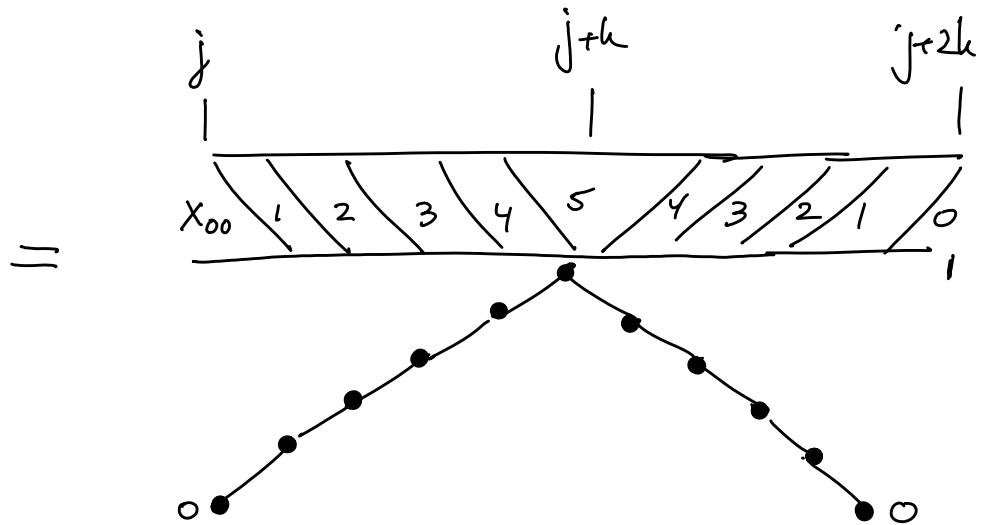
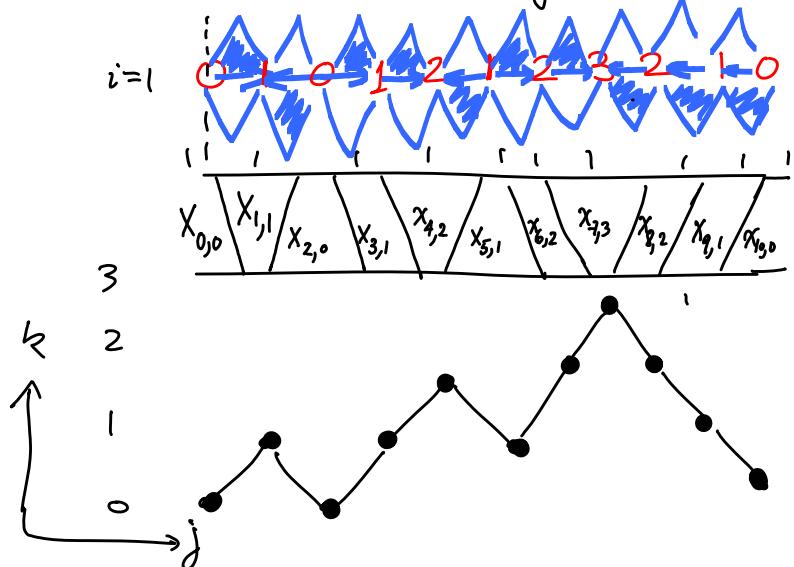
How to compute  $x_{j,k}$  in terms of the network matrices?



How to compute  $x_{j,k}$  in terms of the network matrices?



How to compute  $x_{j,k}$  in terms of the network matrices?



Compute the right network matrix:

$$U(x_0, x_1)U(x_2, x_3)\dots U(x_{k-1}, x_k) = \begin{pmatrix} 1 & 0 \\ * & \frac{x_0}{x_k} \end{pmatrix} \quad \left. \begin{array}{l} \\ \end{array} \right\}$$

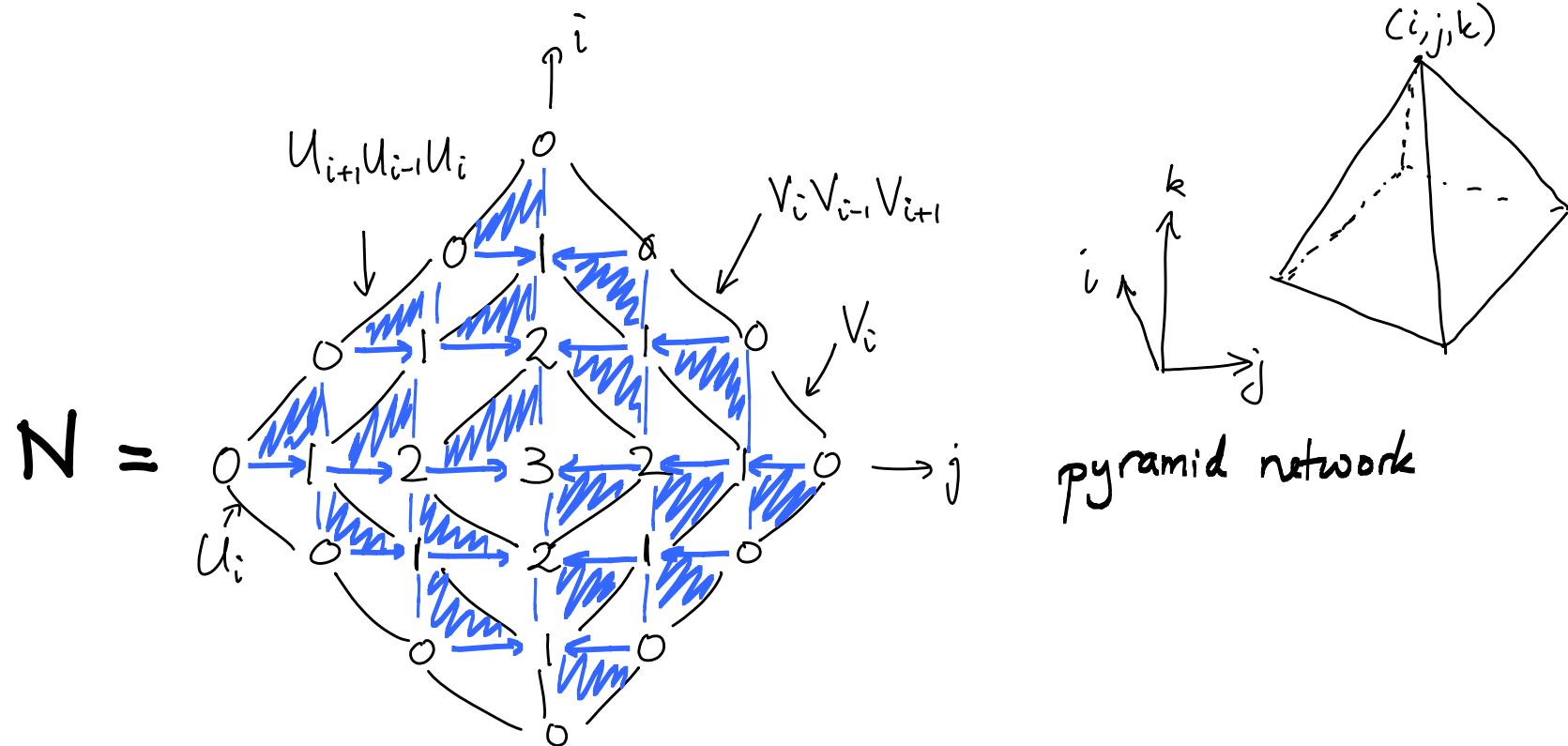
$$V(x_k, x_{k-1})V(x_{k-2}, x_{k-3})\dots V(x_1, x_0) = \begin{pmatrix} x_k & 0 \\ 0 & 1 \end{pmatrix}$$

$$\pi_U \times \pi_V = \begin{pmatrix} x_k & x_0^{-1} & * \\ * & * & * \end{pmatrix} = N$$

$$\Rightarrow N_{11} x_{j+2k, 0} = x_{j+k, k}.$$

How to compute  $x_{ijk}$  in general?

- Compute the network matrix under the pyramid whose top is  $x_{ijk}$



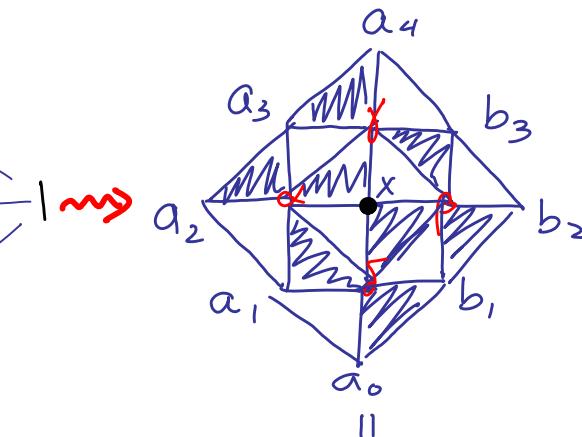
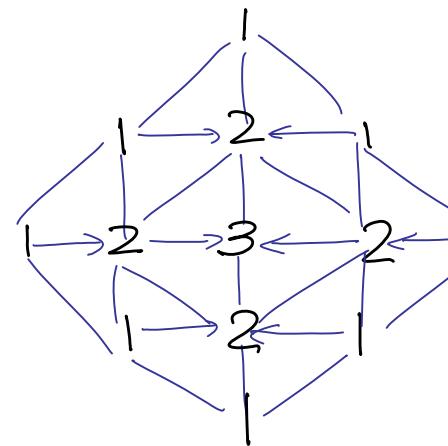
Zero curvature: Network matrices related by mutations are equal

All network matrices with the same boundary are equal.

$$\begin{matrix} & & 1 \\ & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{matrix} = \begin{matrix} & & 1 \\ & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 & 1 \end{matrix} \quad i=0$$
$$\begin{matrix} & & 1 \\ & 1 & 0 & 1 \\ 1 & & & \\ & & 1 \end{matrix} \quad \begin{matrix} & & 1 \\ & 1 & 2 & 1 \\ 1 & & & \\ & & 1 \\ j=0 & & & \end{matrix}$$

Example:

|   |   |   |   |     |
|---|---|---|---|-----|
|   |   |   |   | 1   |
| 1 | 2 | 1 |   |     |
| 1 | 2 | 3 | 2 | 1 ↗ |
| 1 | 2 | 1 |   |     |
|   |   |   |   | 1   |



$x \sim$  partition function of  
non-intersecting paths

$$\begin{matrix} 1 \rightarrow 1' \\ 2 \rightarrow 2' \end{matrix}$$

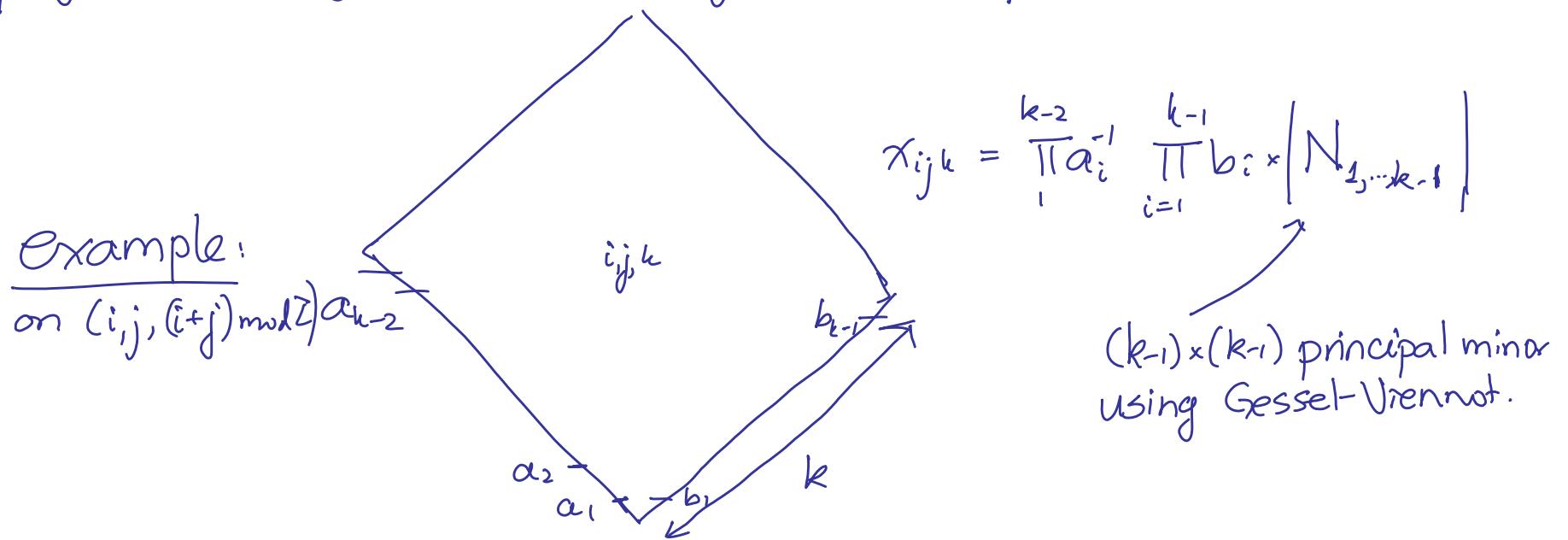
Gessel-Viennot:

$$X = b^{-1} Z_{12}^{1'2'} e d$$

$$\frac{\alpha_1}{\delta} \times \frac{x}{\beta} \times \frac{\beta}{b_2} \times \frac{\delta}{b_1} = \frac{\alpha_1}{b_1 b_2} \cdot x$$

To compute  $x_{ijk}$  in terms of initial data on  $S = \{(i,j, K_0(i,j))\}$

project the pyramid with  $x_{ijk}$  at the top onto  $S$

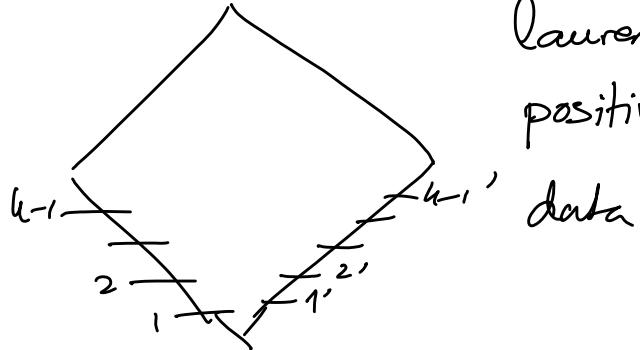


## Network solution

1) Allows to write  $T_{i,j,k}$  in terms of any initial data surface

2)  $T_{i,j,k} = \text{monomial} \times \text{partition function of non-intersecting paths}$

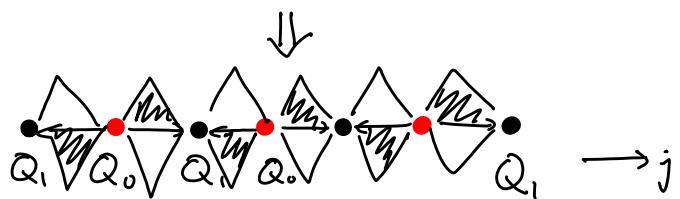
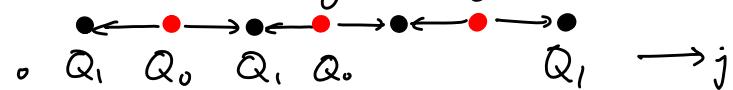
from  $(1, \dots, k-1) \rightarrow (1', \dots, k-1')$ , on a network with positive Laurent monomial weights  $\Rightarrow$  Laurent positivity of  $T_{ijk}$  in terms of initial data



3) Solution allows for more complicated boundary conditions.

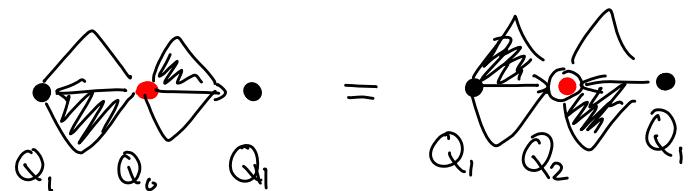
Network solutions  $\rightarrow$  Q-systems: (Type A<sub>1</sub>)  $Q_{1,k} \equiv Q_k$

Initial data of Q-system  $Q_{k+1}Q_{k-1} = Q_k^2 + 1$  is  $(Q_0, Q_1)$



$[V(Q_1, Q_0)U(Q_0, Q_1)]^3$  network matrix on the 2-periodic lattice

- $V(Q_1, Q_0)U(Q_0, Q_1) = U(Q_1, Q_2)V(Q_2, Q_1)$



- $Q_k = [V(Q_1, Q_0)U(Q_0, Q_1)]_{11}^{k-1} \times Q_1$   
 $= \left( \prod_{i=1}^{k-1} U(Q_i, Q_{i+1}) \quad \prod_{i=k}^1 V(Q_i, Q_{i-1}) \right)_{11} \times Q_1$

$$= \begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix} \begin{pmatrix} \frac{Q_k}{Q_1} \\ 0 \end{pmatrix}_{11} \times Q_1 = Q_k.$$

## 4) Quantization



For any cluster algebra with invertible exchange matrix  $B$  we can write a (canonical) quantization. (quantizing the Poisson structure of Gekhtman, Shapiro, Vainshtein; see Berenstein-Zelevinsky).

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- 3)  $q$ -mutation relation  $x_i' = \sum_{j: j \rightarrow i} x_j x_i^{-1} + \sum_{j: i \rightarrow j} x_j x_i^{-1}$

where:

$$[x_1 x_2 \cdots x_k] = q^{-\frac{1}{2} \sum_{i < j} \lambda_{ij}} x_1 x_2 \cdots x_k .$$

Example:  $A_1$  quantum Q-system:

- clusters consist of  $Q_i, Q_{i+1}$  and  $Q_i Q_{i+1} = q Q_{i+1} Q_i$
- mutation is  $Q_i Q_{i+1} = q Q_i^2 + 1$
- write  $U(a,b) = \begin{pmatrix} 1 & 0 \\ b^{-1} & ab^{-1} \end{pmatrix}, V(a,b) = \begin{pmatrix} ab^{-1} & b^{-1} \\ 0 & 1 \end{pmatrix}$

then

$$V(a,b) U(b,a) = U(a,b') V(b',a) \quad \text{if } -ab = q'ba$$

That is,  $V(Q_1, Q_0) U(Q_0, Q_1) = U(Q_1, Q_2) V(Q_2, Q_1)$

- $bb' = qa^2 + 1$
- $ab' = qb'a$

- so the network solution survives the quantization.

$$Q_k = ([V(Q_1, Q_0) U(Q_0, Q_1)]^{k-1})_{\parallel} Q_1$$

paths with non-commuting weights

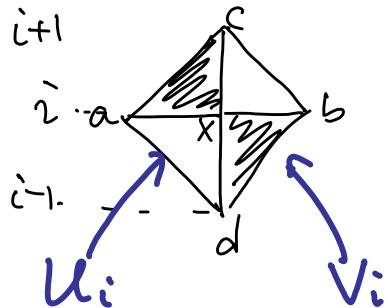
## Quantum Q-system: (type An)

- The quantization of the Q-system is canonical:  $\lambda = NC^{-1}$

$$\lambda_{ij} = N \min(ij) - ij$$

- $Q_{i,k} Q_{j,k+1} = q^{\lambda_{ij}} Q_{j,k+1} Q_{i,k}$
  - $Q_{i,k} Q_{j,k} = Q_{jk} Q_{ik}$
  - $Q_{i,k-1} Q_{i,k+1} = q^{\lambda_{ii}} Q_{i,k}^2 + q^{\frac{1}{2}(\lambda_{i,i+1} + \lambda_{i,i-1})} Q_{i-1,k} Q_{i+1,k}$
- } commutation relations

The network matrices which realize this system are:



$$U_i = \begin{pmatrix} 1 & 0 \\ q^{\frac{1}{2}\lambda_{i+1,i}} x^{-1} & ax^{-1} \end{pmatrix}, \quad V_i = \begin{pmatrix} xb^{-1} & q^{\frac{1}{2}\lambda_{i-1,i}} db^{-1} \\ 0 & 1 \end{pmatrix}$$

## Remarks about quantization

1) The conserved quantities of the  $Q$ -system become quantum conserved quantities:

- $C^{(l)}$  are independent of "time"  $Q$
- $C^{(1)}, \dots, C^{(N-1)}, C^{(N)}$  are in involution

$\nwarrow \#1$

2) The variables  $\{Q_{1k} \mid k \in \mathbb{Z}\}$  generate an algebra

$$\approx \mathcal{U}_q(\hat{\mathfrak{n}}_+) / \langle Q_{NH} = 0 \rangle \quad (\hat{\mathfrak{n}}_+ \in \mathcal{U}_q(\hat{sl}_2))$$

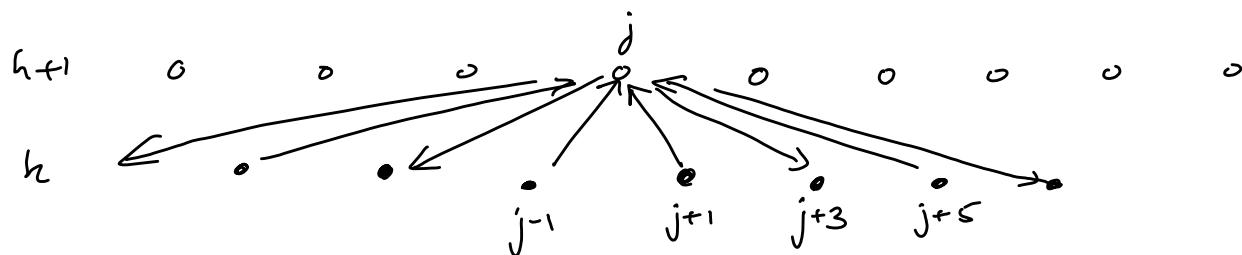
[see 1606.09052]

◦ ◦ ◦

## Quantization of T-system: $A_1$ case

For the T-system, there is no unique quantization, the exchange matrix has infinite rank.

one choice, for type  $A_1$ :



$$x_{j,h+1} x_{j',h} = q^{\lambda_{j,j'}} x_{j,h} x_{j',h+1}$$

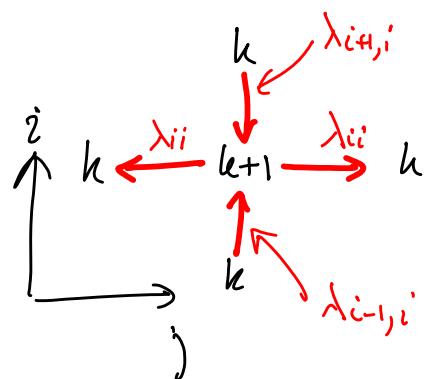
The network solution survives this quantization:

$$a \begin{array}{c} \diagup \\ \text{---} \\ \diagdown \end{array} b = \begin{pmatrix} 1 & 0 \\ b^{-1} & ab^{-1} \end{pmatrix} \quad a \begin{array}{c} \diagup \\ \text{---} \\ \diagdown \end{array} b = \begin{pmatrix} ab^{-1} & b^{-1} \\ 0 & 1 \end{pmatrix}.$$

compatible with 2-periodic solution (Q-system).

## Quantization of $A_r$ T-system:

The quantization compatible with the Q-systems has the following local commutation relations:

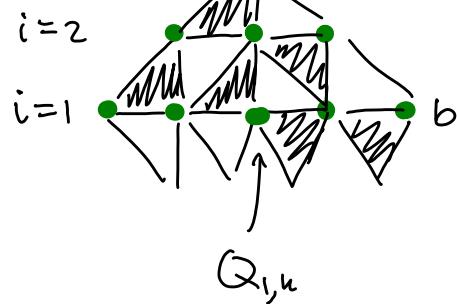


- This quantum T-system is realized by the same network matrices as above.
- Compatibility with exchange matrix determines all other CR: These do not appear in the network solution.

## Quantum networks?

construct Network matrix as before, with

non-commuting entries



For  $i=1$ ,  $Q_{1,h} = \text{weight of path } 1 \rightarrow 1' \times b = N_{11} \times b$

Question:  $Q_{\alpha,h}$ : how to define partition fn. of NI. paths?

$q$ -determinant and the quantum  $Q$ -system: (quantum desnanot -Jacobi)

$$Q_{\alpha, k-1} Q_{\alpha, k+1} = q^{\lambda_{\alpha\alpha}} Q_{\alpha, k}^2 - q^{\frac{1}{2}(\lambda_{\alpha, \alpha+1} + \lambda_{\alpha, \alpha-1})} Q_{\alpha-1, k} Q_{\alpha+1, k}$$

$\Downarrow$

$$Q_{\alpha, k} = \left[ \prod_{1 \leq i < j \leq \alpha} \left( 1 - q^{N+i-j} \frac{u_j}{u_i} \right) \prod_{i=1}^{\alpha} Q(u_i) \right] \frac{1}{(u_1 \cdots u_{\alpha})^k}$$

where  $Q(u) = \sum_{n \in \mathbb{Z}} Q_{1,n} u^n$ .

Question:

a) From the quantum  $\mathbb{Q}$ -system,

$$Q_{\alpha, h} = q\text{-determinant of } W_k^{(\alpha)}$$

b) The quantum network matrix is well-defined, and invariant under mutations.

How to define a partition function of non-intersecting paths with non-commuting weights compatible with (a)?  
(quantum Gessel-Viennot)

Merci!