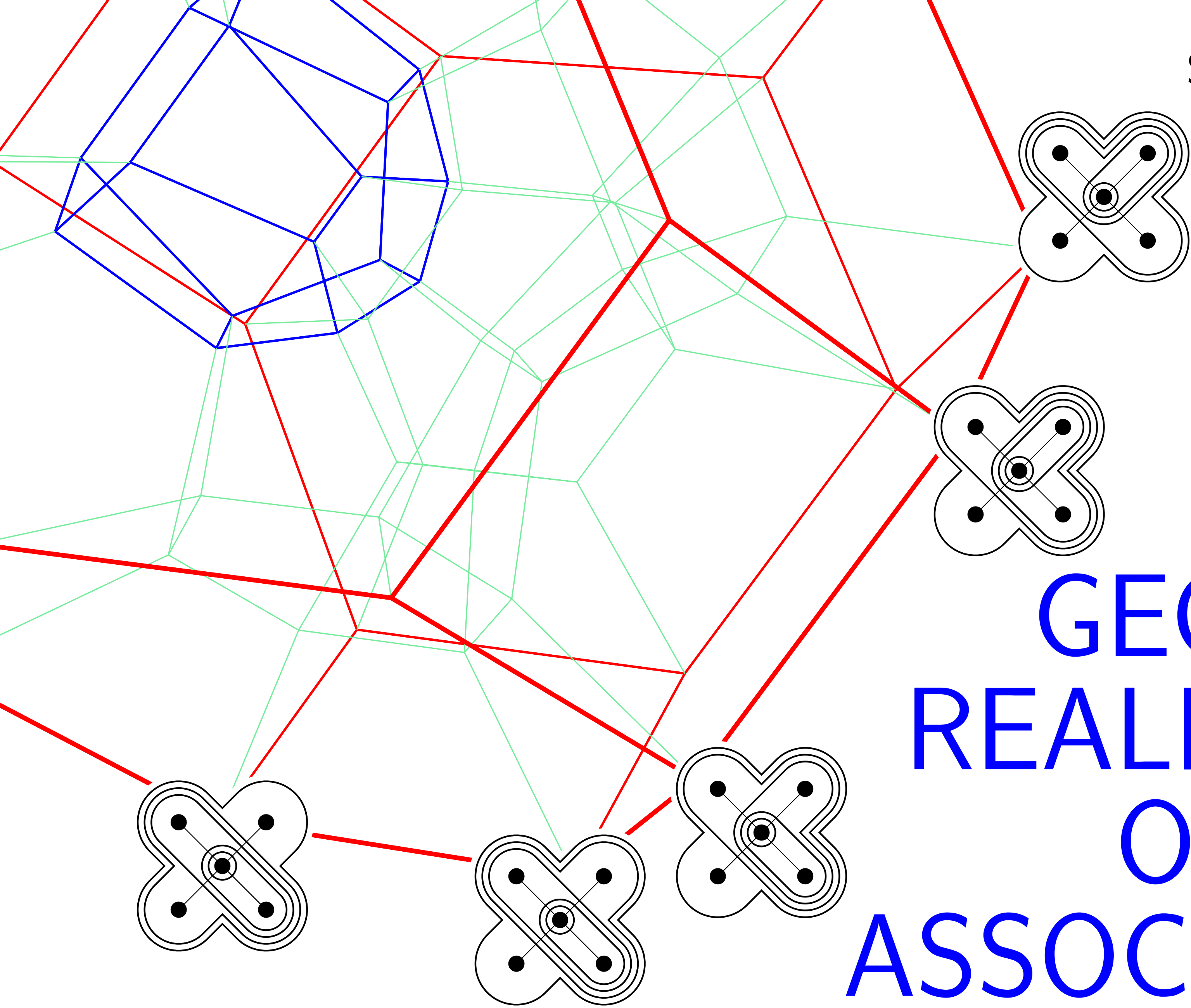


Séminaire Philippe Flajolet
12 février 2015

MANY GEOMETRIC REALIZATIONS OF GRAPH ASSOCIAHEDRA

T. MANNEVILLE (LIX)
V. PILAUD (CNRS & LIX)

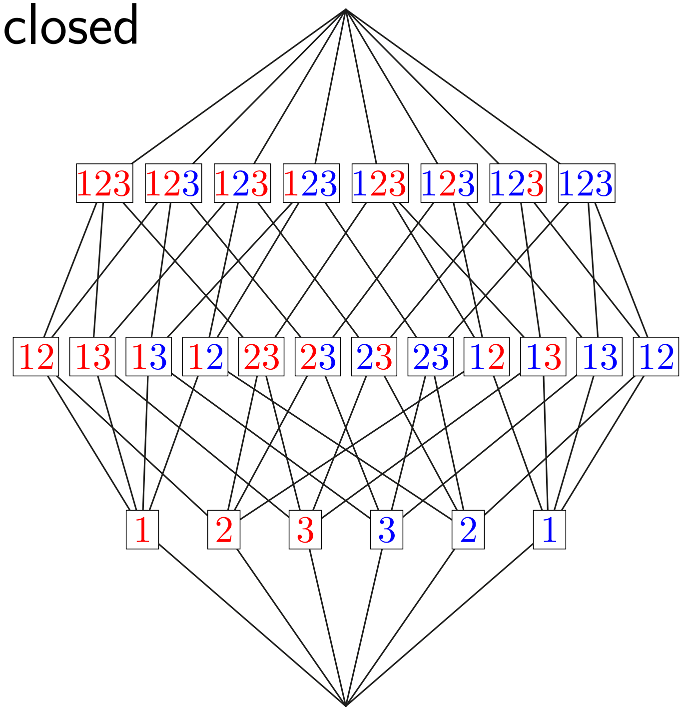
POLYTOPES & COMBINATORICS

SIMPLICIAL COMPLEX

simplicial complex $=$ collection of subsets of X downward closed

exm:

$$X = [n] \cup [n]$$
$$\Delta = \{I \subseteq X \mid \forall i \in [n],\ \{i, i\} \not\subseteq I\}$$

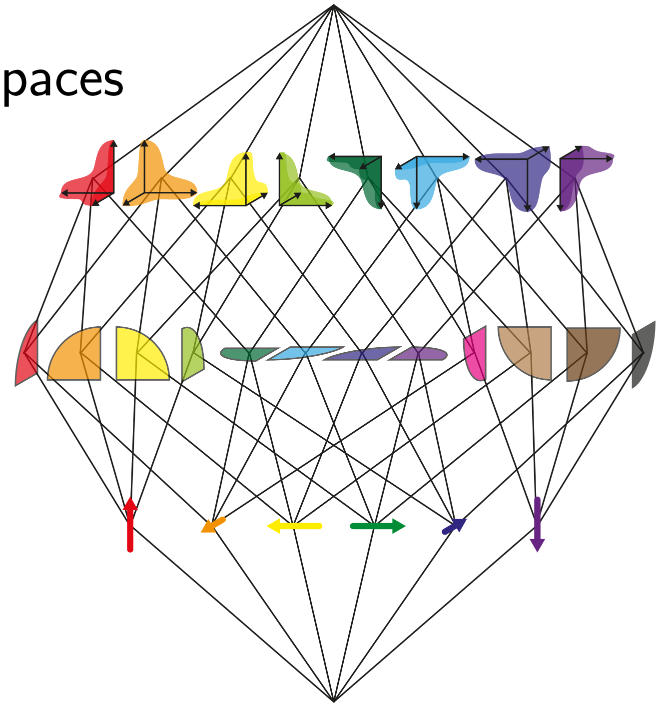


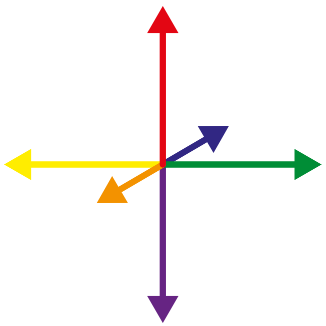
FANS

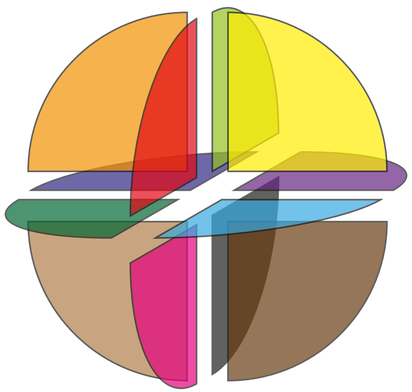
polyhedral cone = positive span of a finite set of $\mathbb{R}^d$

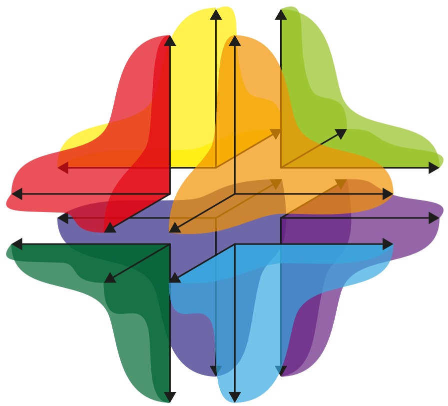
= intersection of finitely many linear half-spaces

fan = collection of polyhedral cones closed by faces and where any two cones intersect along a face









simplicial fan = maximal cones generated by d rays

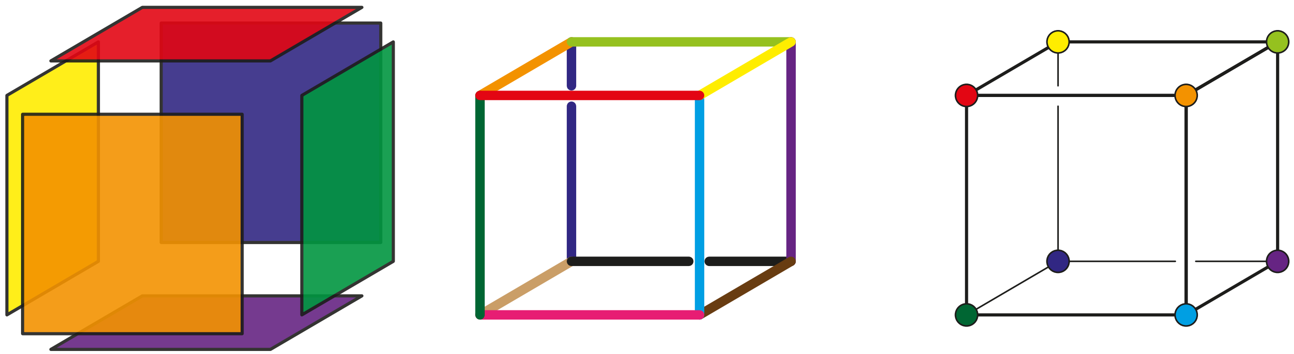
POLYTOPES

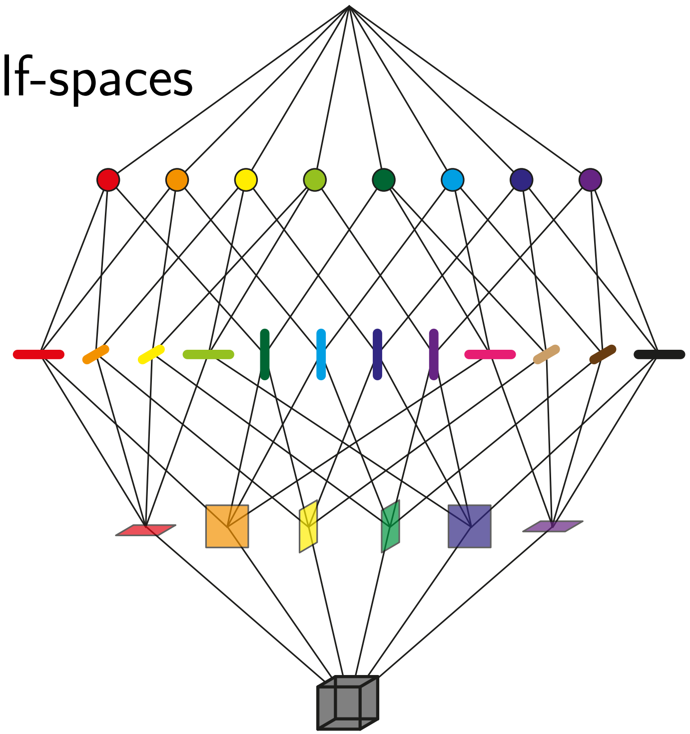
polytope = convex hull of a finite set of $\mathbb{R}^d$

= bounded intersection of finitely many affine half-spaces

face = intersection with a supporting hyperplane

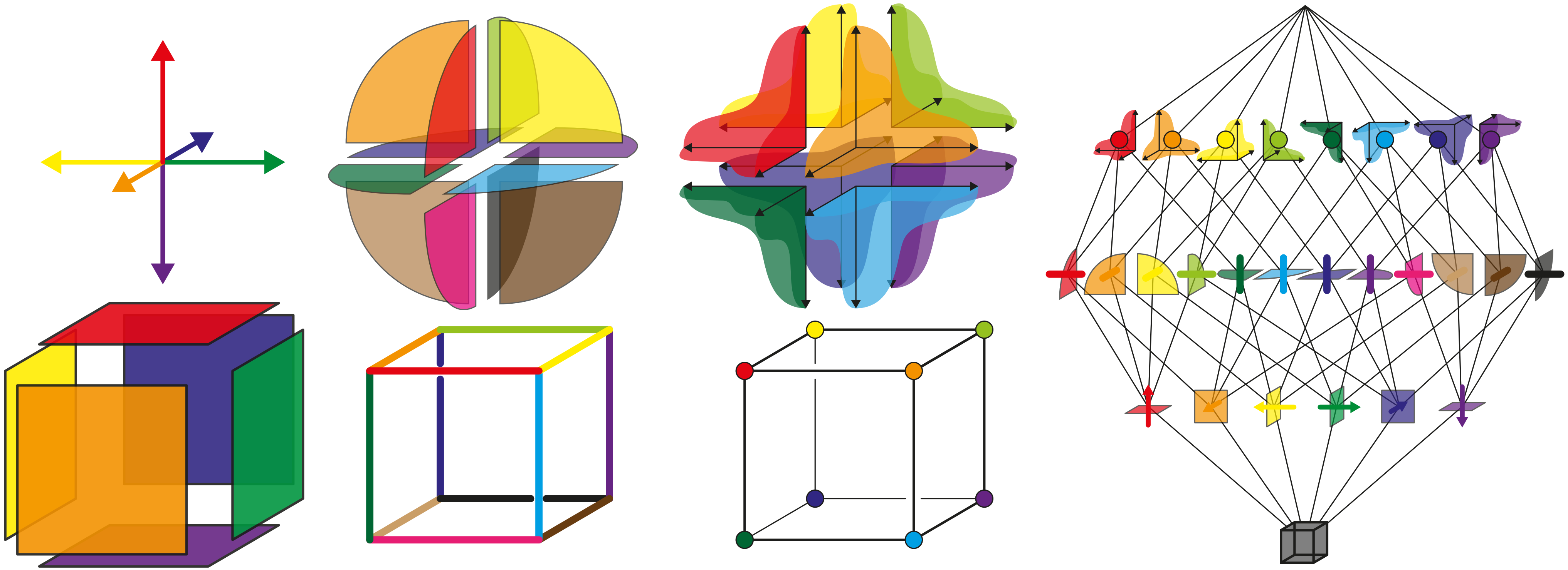
face lattice = all the faces with their inclusion relations





simple polytope = facets in general position = each vertex incident to d facets

SIMPLICIAL COMPLEXES, FANS, AND POLYTOPES



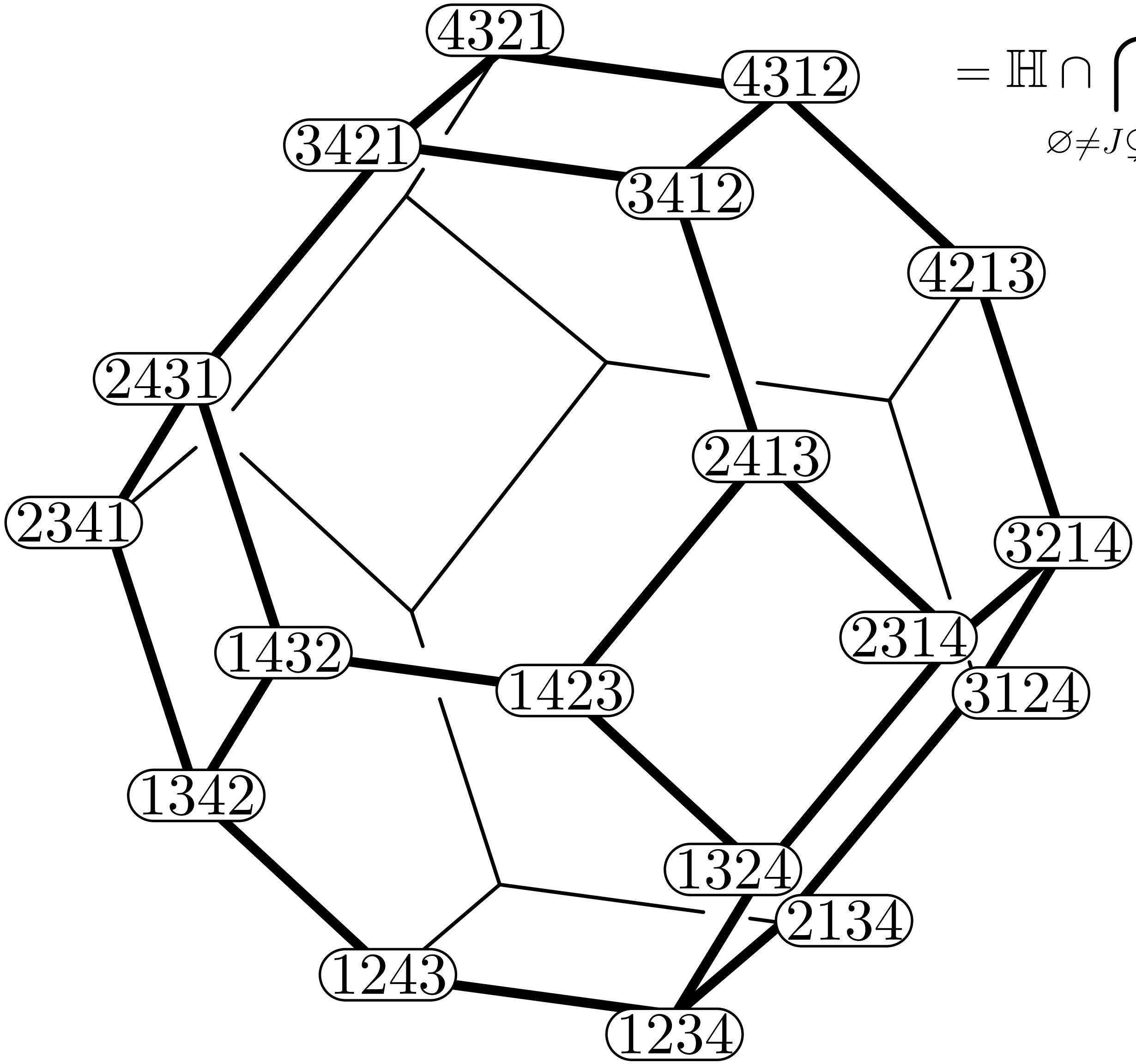
P polytope, F face of P

normal cone of F = positive span of the outer normal vectors of the facets containing F

normal fan of P = $\{$ normal cone of F $\mid$ F face of P $\}$

simple polytope $\Longrightarrow$ simplicial fan $\Longrightarrow$ simplicial complex

PERMUTAHEDRON

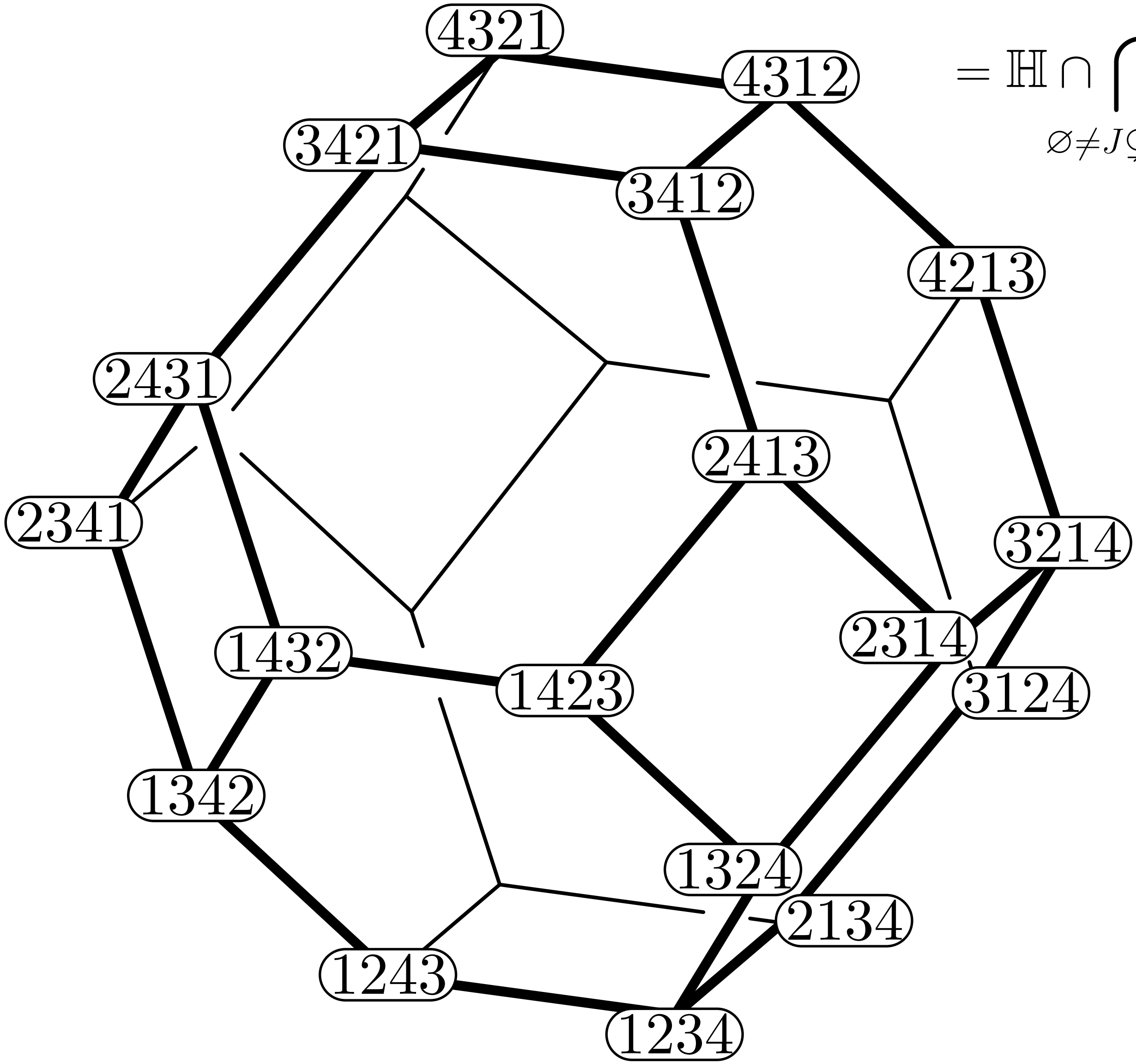


Permutohedron Perm(n)

$$= \operatorname{conv}\left\{(\sigma(1), \dots, \sigma(n+1)) \mid \sigma \in \Sigma_{n+1}\right\}$$

$$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n+1]} \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right\}$$

PERMUTAHEDRON



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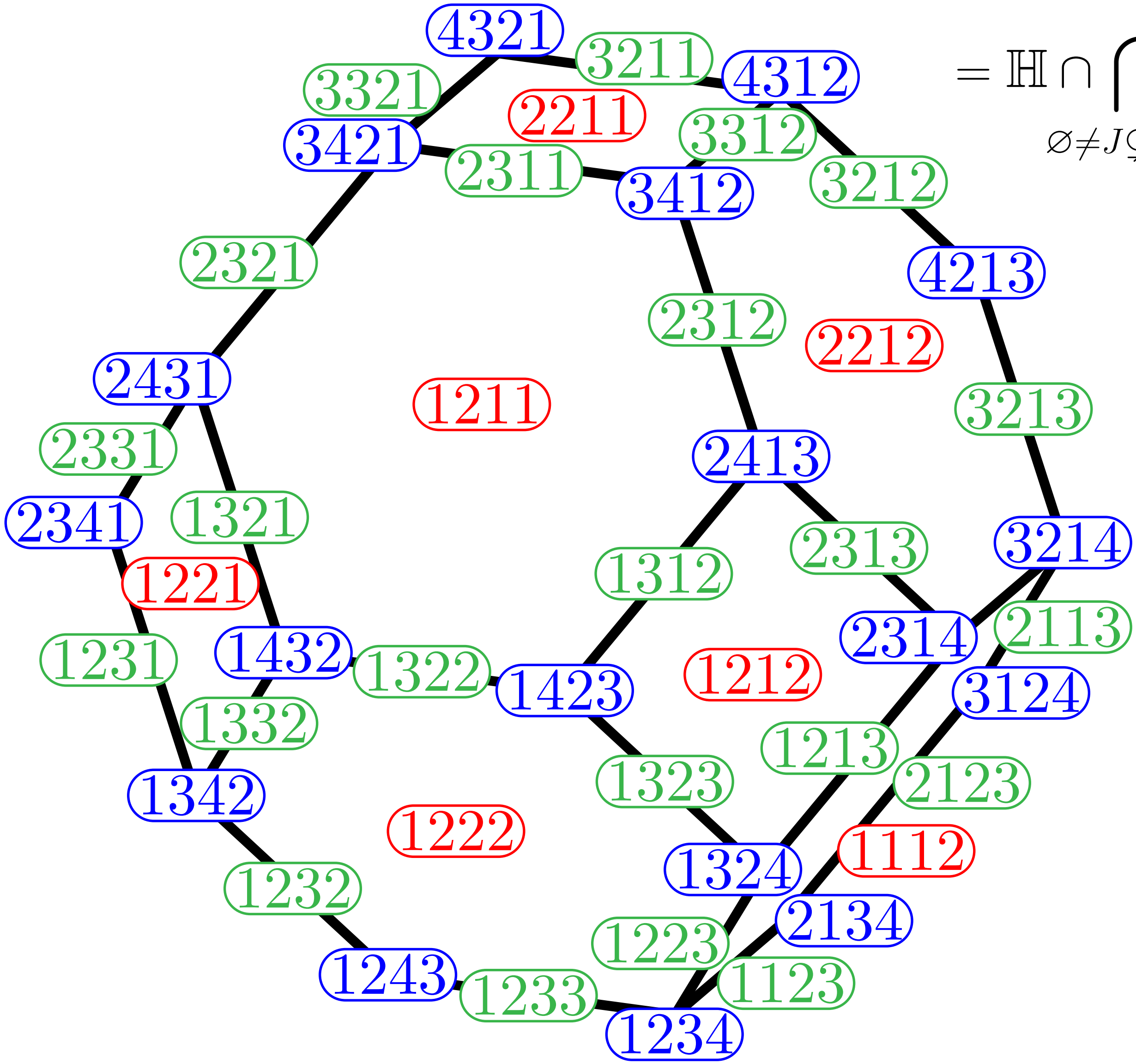
connections to

- weak order
- reduced expressions
- braid moves
- cosets of the symmetric group

PERMUTAHEDRON



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connections to

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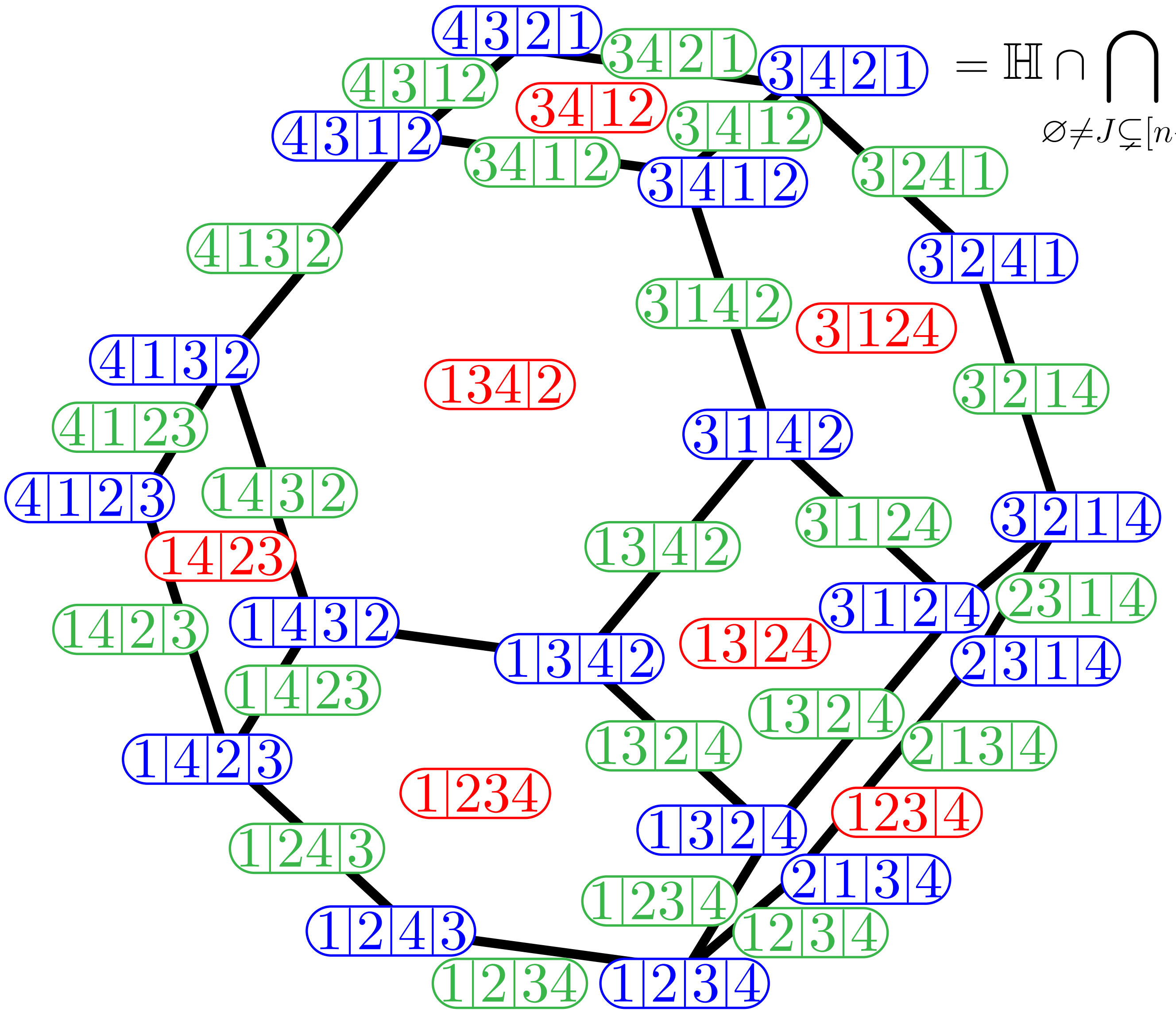
k-faces of Perm(n)
$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

PERMUTAHEDRON



Permutohedron $\mathrm{Perm}(n)$

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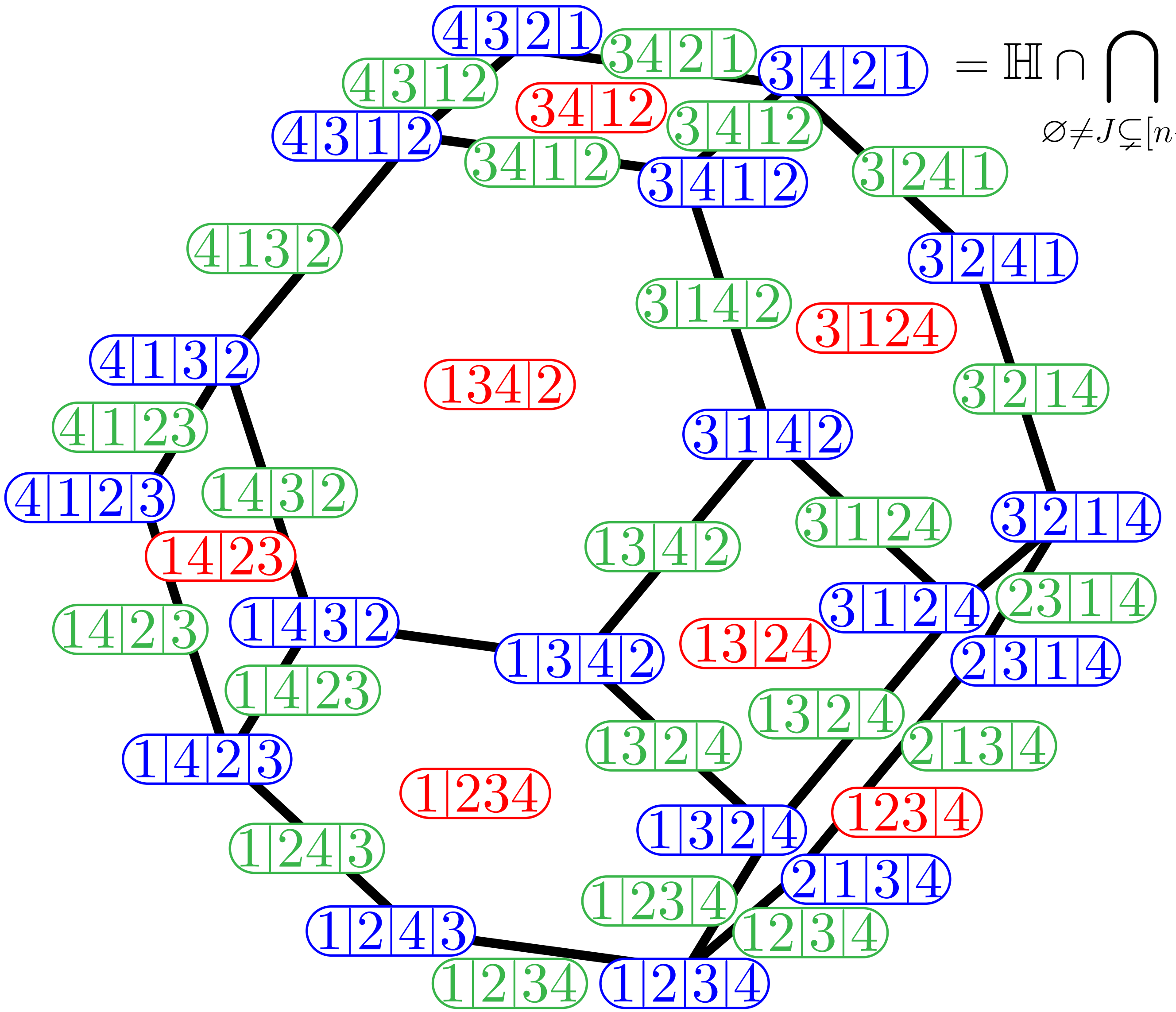
k-faces of $\mathrm{Perm}(n)$

$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

$\equiv$ ordered partitions of $[n+1]$ into $n+1-k$ parts

PERMUTAHEDRON



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connections to

- weak order
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k-faces of Perm(n)

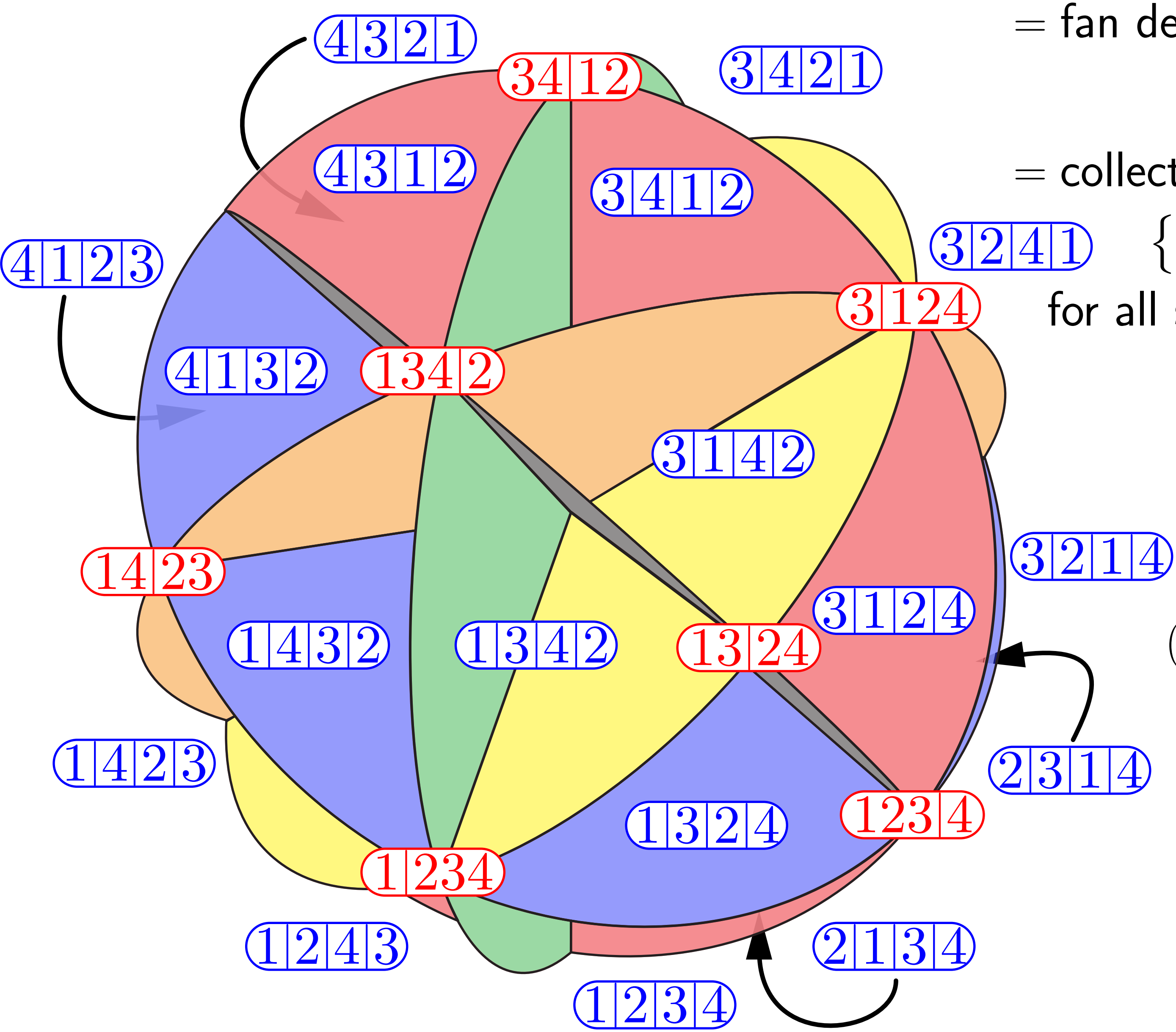
$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

$\equiv$ ordered partitions of $[n+1]$ into $n+1-k$ parts

$\equiv$ collections of $n-k$ nested subsets of $[n+1]$

COXETER ARRANGEMENT

Coxeter fan



= fan defined by the hyperplane arrangement

$$\{\mathbf{x} \in \mathbb{R}^{n+1} \mid x_i = x_j\}_{1 \le i < j \le n+1}$$

= collection of all cones

$$\{\mathbf{x} \in \mathbb{R}^{n+1} \mid x_i < x_j \text{ if } \pi(i) < \pi(j)\}$$

for all surjections $\pi : [n+1] \to [n+1-k]$

$(n-k)$-dimensional cones

$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

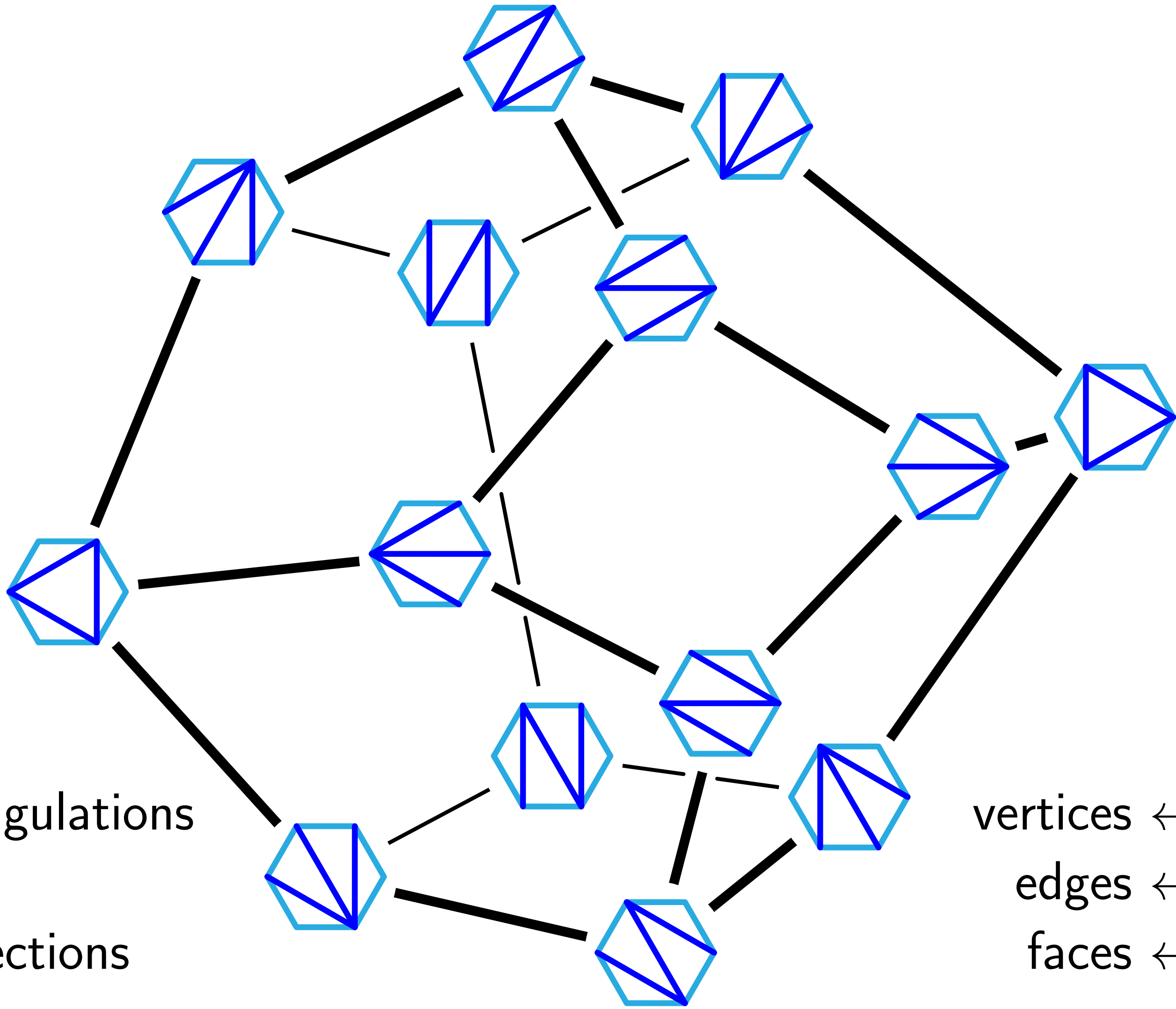
$\equiv$ ordered partitions of $[n+1]$ into $n+1-k$ parts

$\equiv$ collections of $n-k$ nested subsets of $[n+1]$

ASSOCIAHEDRA

ASSOCIAHEDRON

Associahedron = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n+3)$-gon, ordered by reverse inclusion

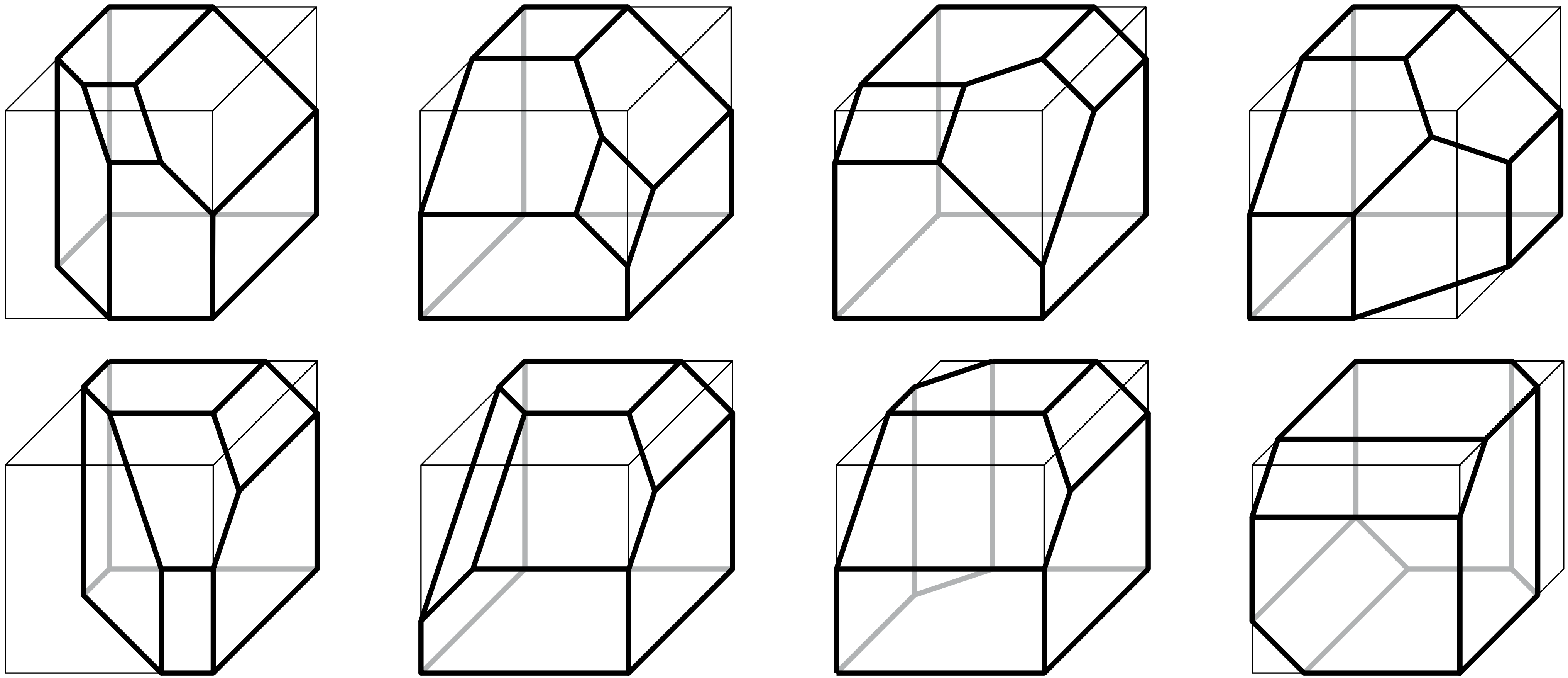


vertices $\leftrightarrow$ triangulations
edges $\leftrightarrow$ flips
faces $\leftrightarrow$ dissections

vertices $\leftrightarrow$ binary trees
edges $\leftrightarrow$ rotations
faces $\leftrightarrow$ Schröder trees

VARIOUS ASSOCIAHEDRA

Associahedron = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n+3)$-gon, ordered by reverse inclusion



(Pictures by Ceballos-Santos-Ziegler)

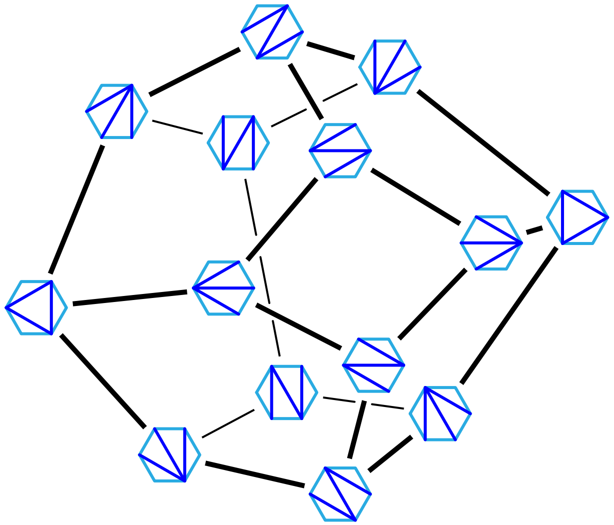
Tamari ('51) — Stasheff ('63) — Haimann ('84) — Lee ('89) —

... — Gel'fand-Kapranov-Zelevinski ('94) — ... — Chapoton-Fomin-Zelevinsky ('02) — ... — Loday ('04) — ...

— Ceballos-Santos-Ziegler ('11)

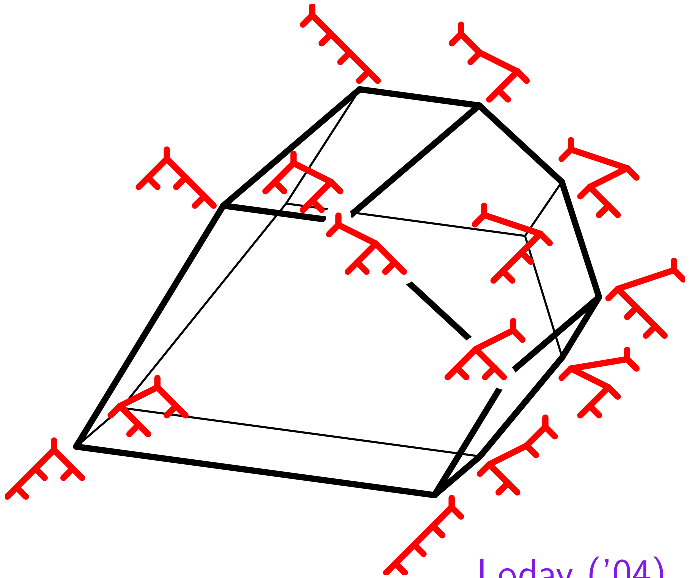
THREE FAMILIES OF REALIZATIONS

SECONDARY POLYTOPE



Gelfand-Kapranov-Zelevinsky ('94)
Billera-Filliman-Sturmfels ('90)

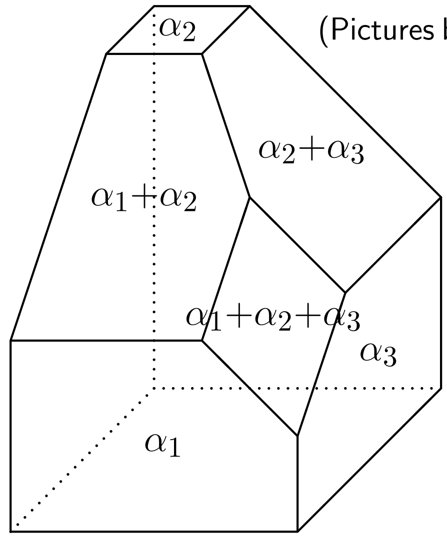
LODAY'S ASSOCIAHEDRON



Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)

CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON

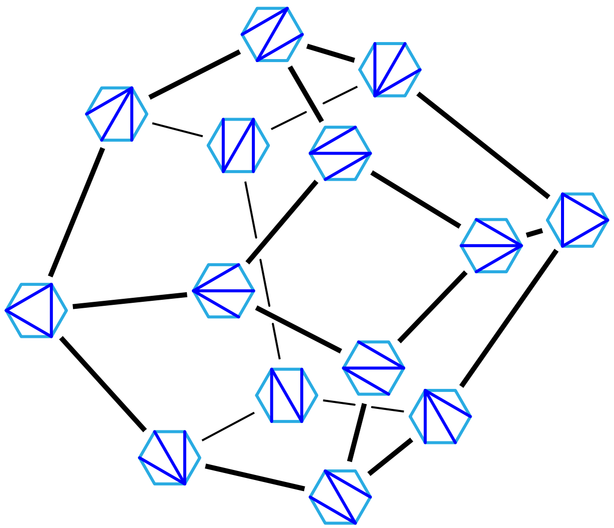
(Pictures by CFZ)



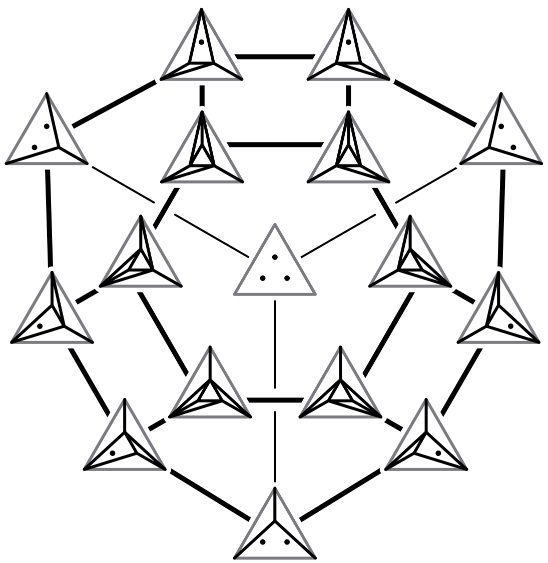
Chapoton-Fomin-Zelevinsky ('02)
Ceballos-Santos-Ziegler ('11)

THREE FAMILIES OF REALIZATIONS

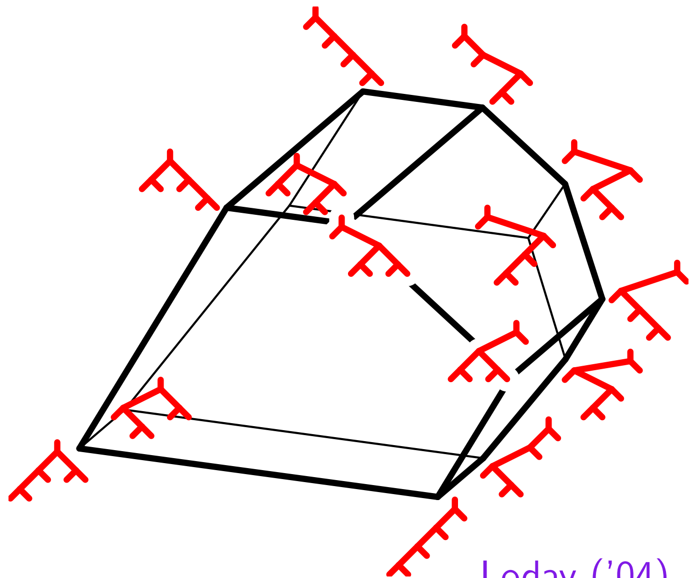
SECONDARY POLYTOPE



Gelfand-Kapranov-Zelevinsky ('94)
Billera-Filliman-Sturmfels ('90)



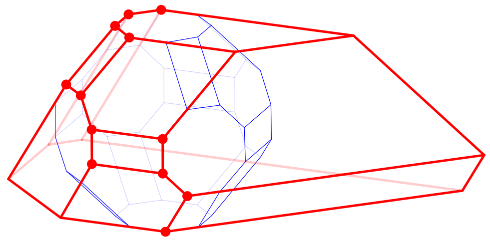
LODAY'S ASSOCIAHEDRON

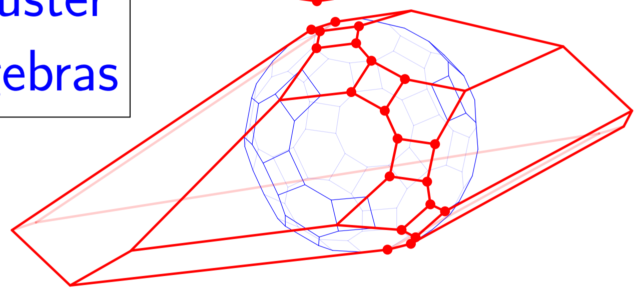


Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)

Hopf algebra

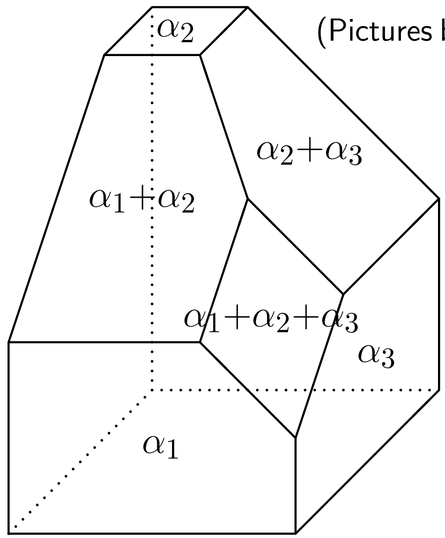
Cluster algebras





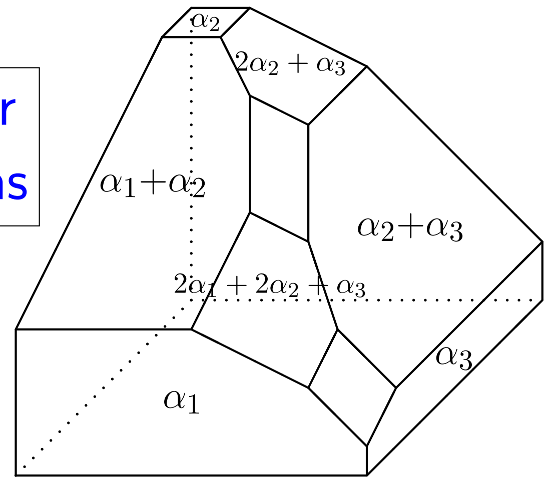
CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON

(Pictures by CFZ)



Chapoton-Fomin-Zelevinsky ('02)
Ceballos-Santos-Ziegler ('11)

Cluster algebras



GRAPH ASSOCIAHEDRA

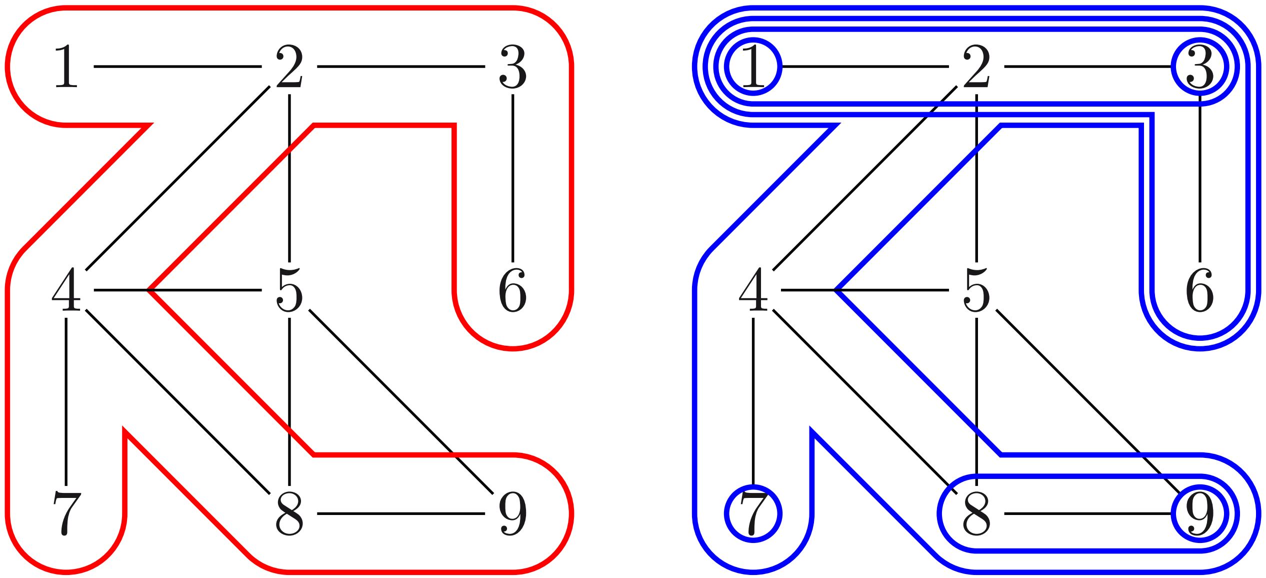
NESTED COMPLEX AND GRAPH ASSOCIAHEDRON

G graph on ground set V

Tube of G = connected induced subgraph of G

Compatible tubes = nested, or disjoint and non-adjacent

Tubing on G = collection of pairwise compatible tubes of G



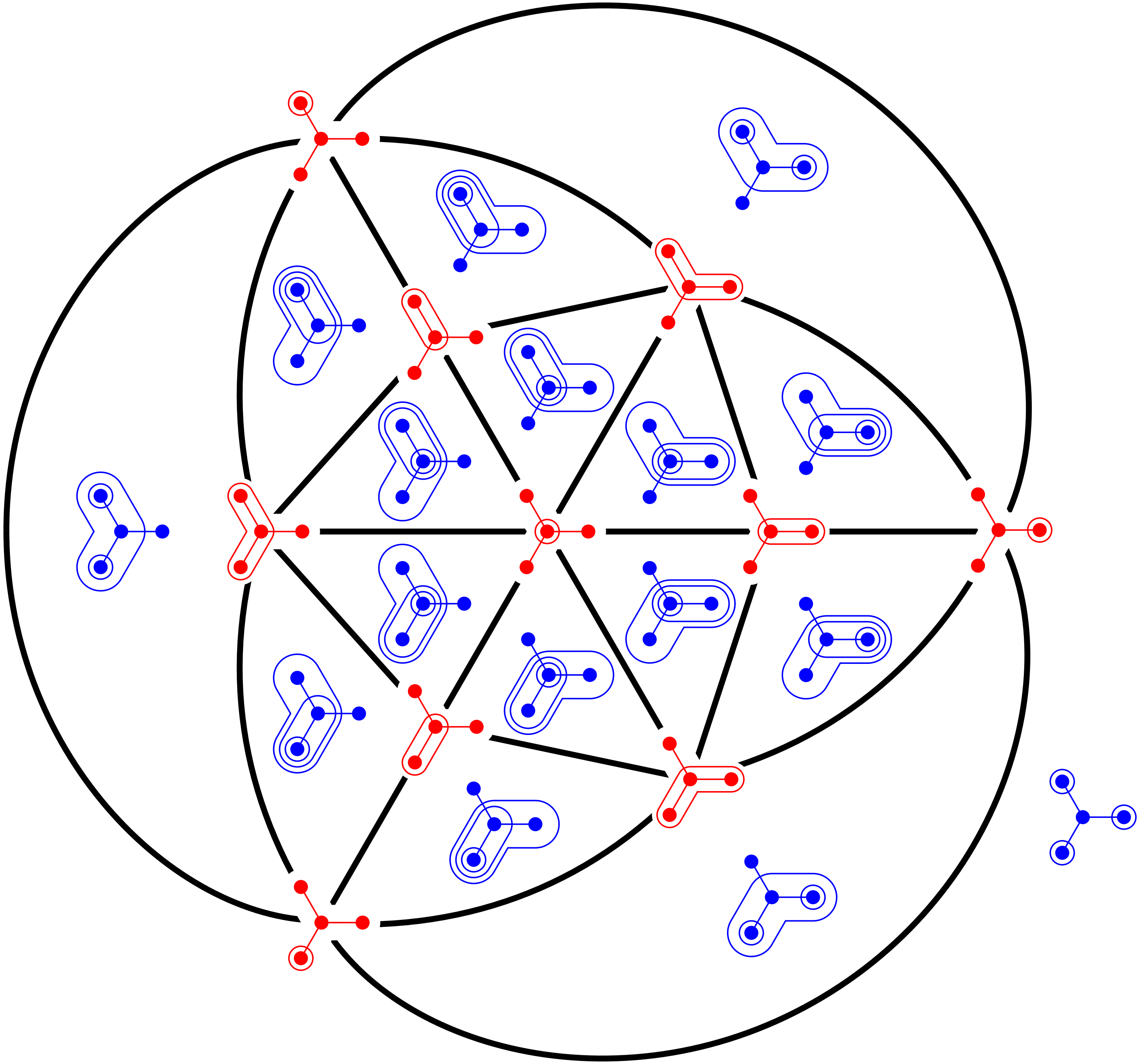
Nested complex $\mathcal{N}(\mathrm{G})$ = simplicial complex of tubings on G

= clique complex of the compatibility relation on tubes

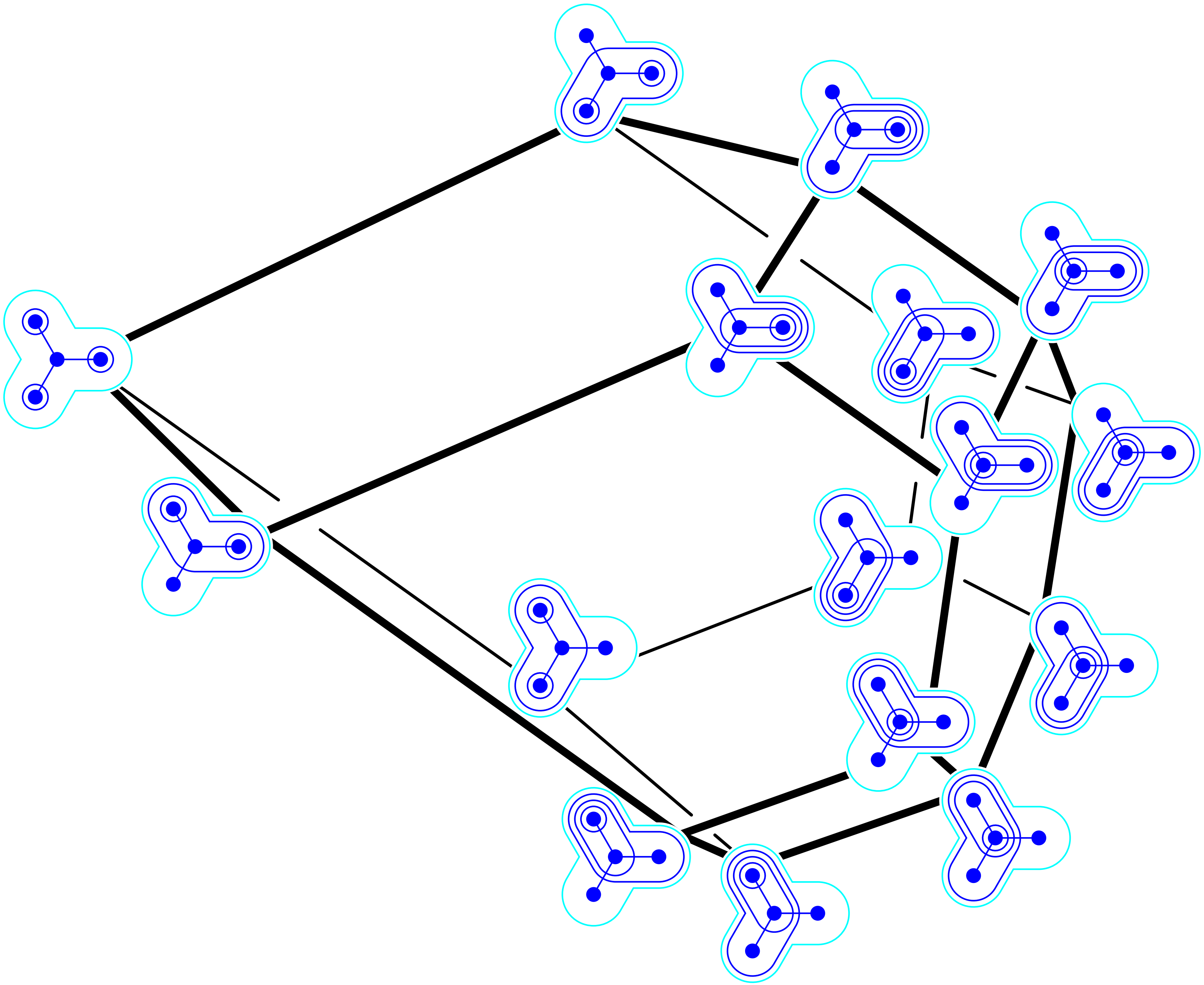
G-associahedron = polytopal realization of the nested complex on G

Carr-Devadoss, Coxeter complexes and graph associahedra ('06)

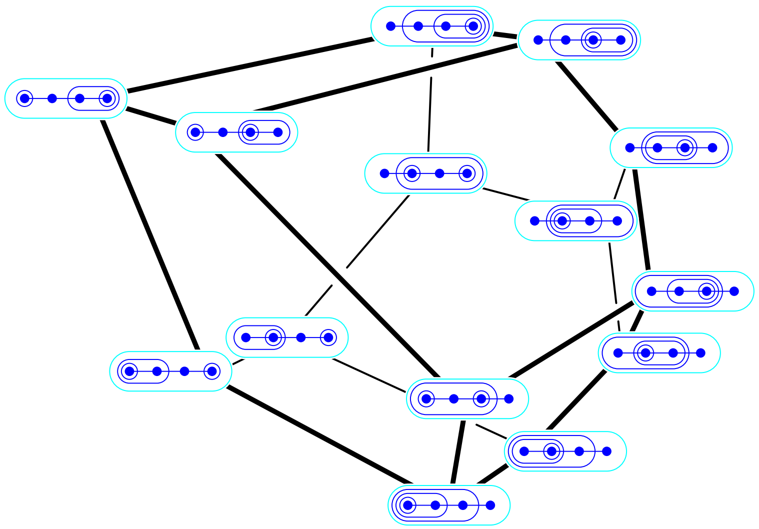
EXM: NESTED COMPLEX

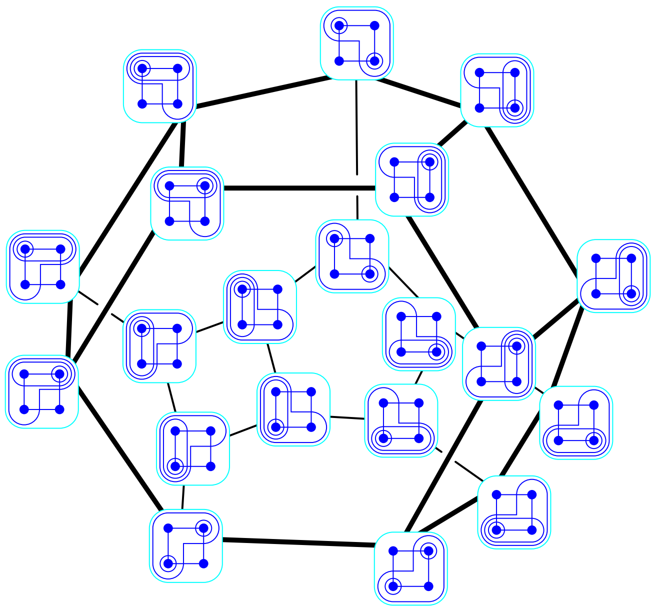


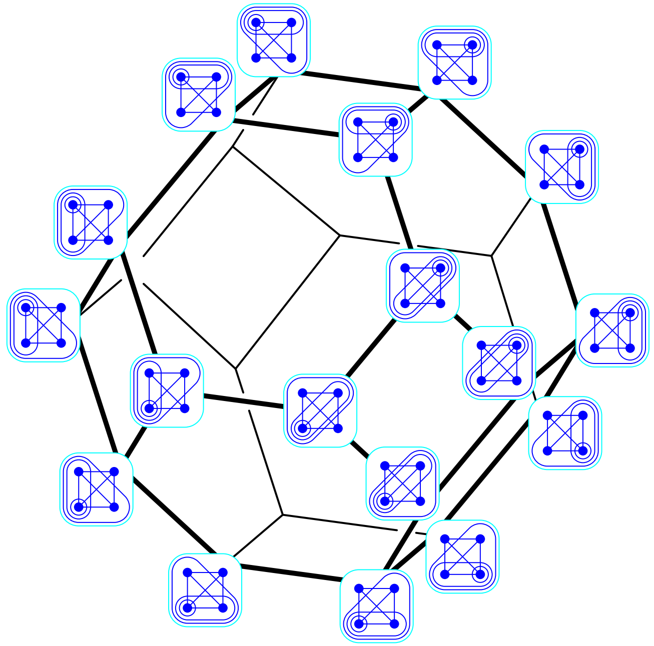
EXM: GRAPH ASSOCIAHEDRON



SPECIAL GRAPH ASSOCIAHEDRA



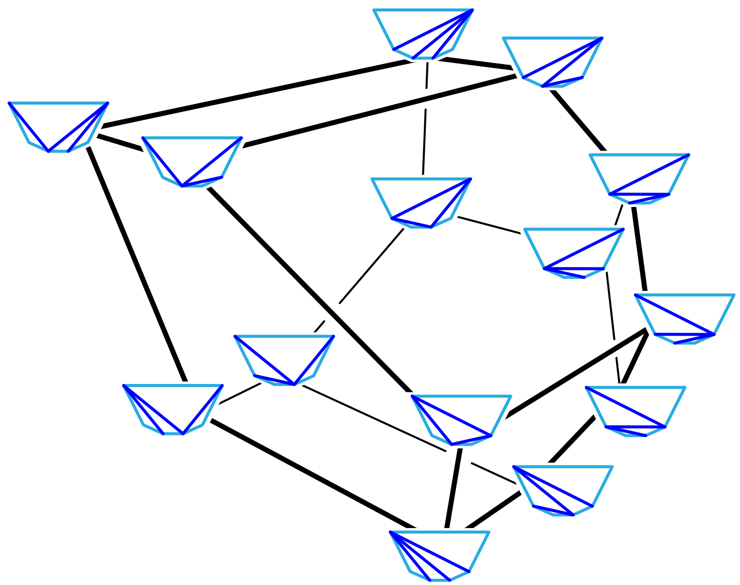


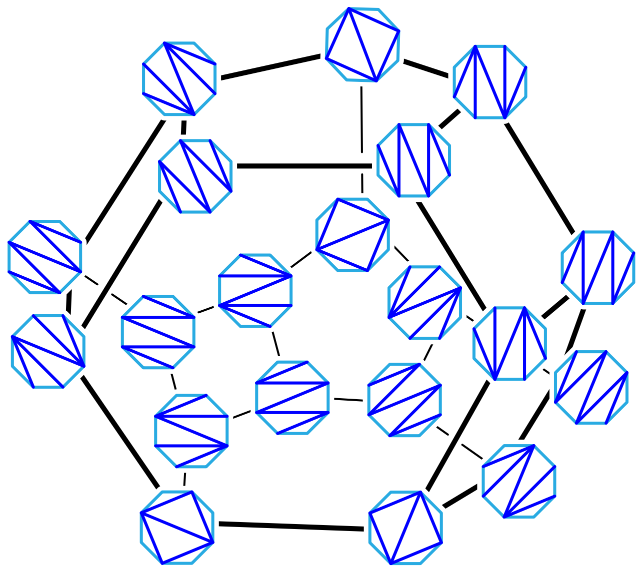


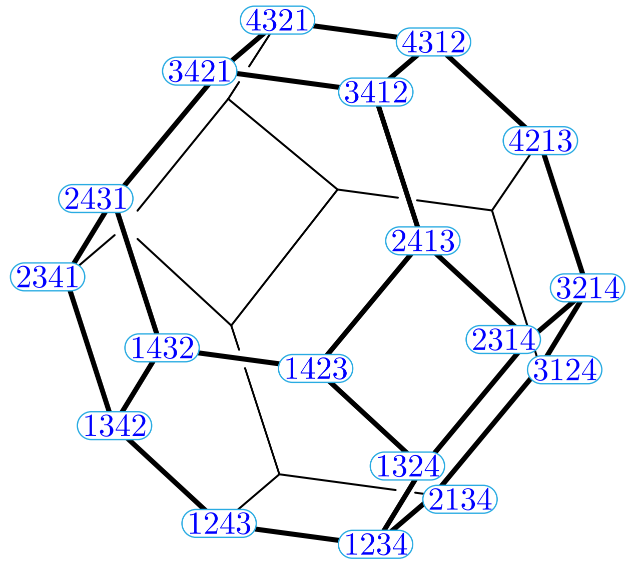
Path associahedron
= associahedron

Cycle associahedron
= cyclohedron

Complete graph associahedron
= permutahedron







COMPATIBILITY FANS FOR GRAPHICAL NESTED COMPLEXES

Thibault Manneville & VP

arXiv:1501.07152

COMPATIBILITY FANS FOR ASSOCIAHEDRA

T° an initial triangulation

δ, δ' two internal diagonals

compatibility degree between δ and δ'

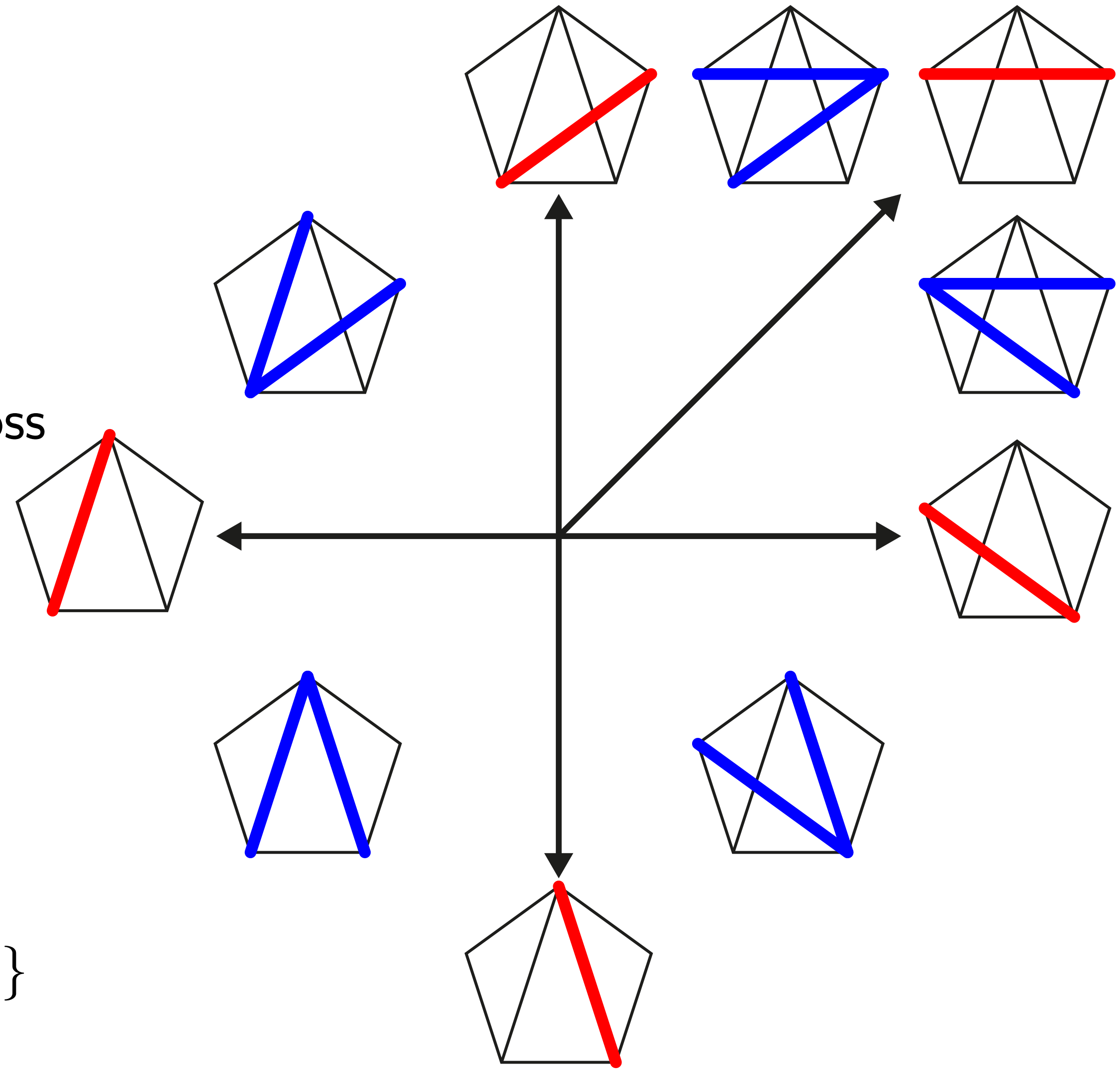
$$(\delta \,\|\, \delta') = \begin{cases} -1 & \text{if } \delta = \delta' \\ 0 & \text{if } \delta \text{ and } \delta' \text{ do not cross} \\ 1 & \text{if } \delta \text{ and } \delta' \text{ cross} \end{cases}$$

compatibility vector of δ wrt T°:

$$\mathbf{d}(\mathrm{T}^\circ, \delta) = \big[(\delta^\circ \,\|\, \delta)\big]_{\delta^\circ \in \mathrm{T}^\circ}$$

compatibility fan wrt T°

$$\mathcal{D}(\mathrm{T}^\circ) = \{\mathbb{R}_{\geq 0}\, \mathbf{d}(\mathrm{T}^\circ, \mathrm{D}) \mid \mathrm{D} \text{ dissection}\}$$



Fomin-Zelevinsky, Y-Systems and generalized associahedra ('03)

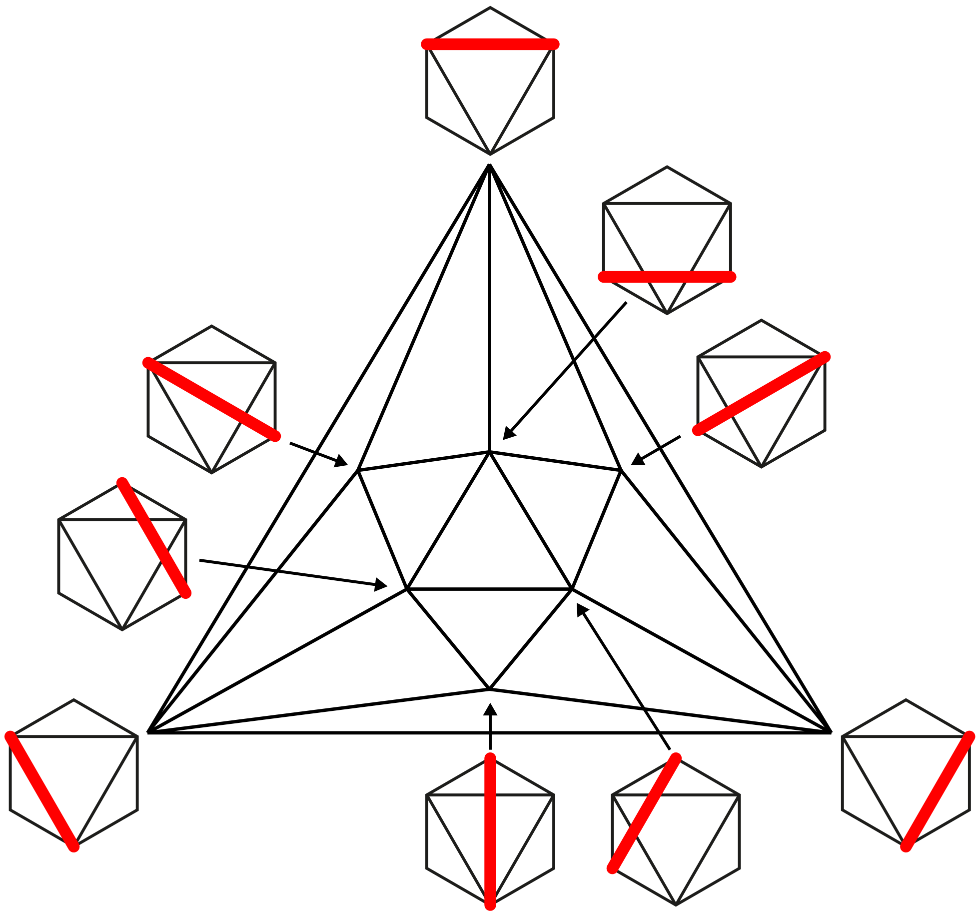
Fomin-Zelevinsky, Cluster algebras II: Finite type classification ('03)

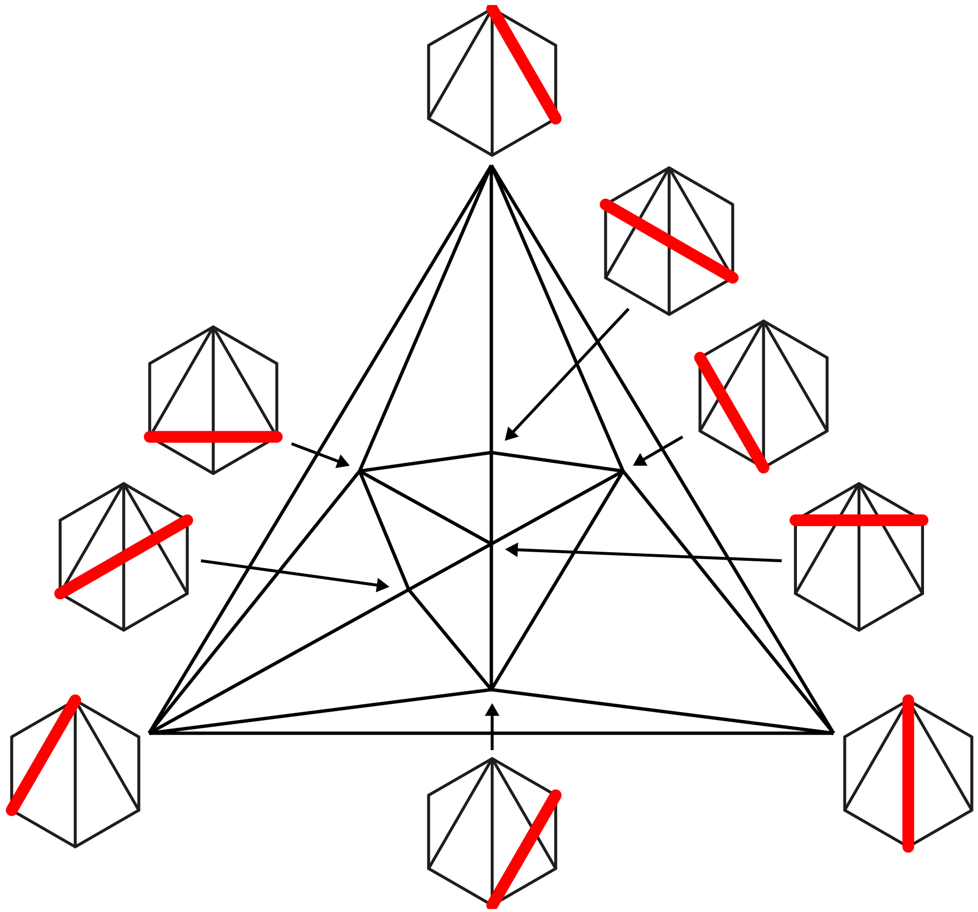
Chapoton-Fomin-Zelevinsky, Polytopal realizations of generalized associahedra ('02)

Ceballos-Santos-Ziegler, Many non-equivalent realizations of the associahedron ('11)

COMPATIBILITY FANS FOR ASSOCIAHEDRA

Different initial triangulations T° yield different realizations





THM. For any initial triangulation T°, the cones $\{\mathbb{R}_{\geq 0}\,\mathbf{d}(\mathrm{T}^{\circ}, \mathrm{D}) \mid \mathrm{D} \text{ dissection}\}$ form a complete simplicial fan. Moreover, this fan is always polytopal.

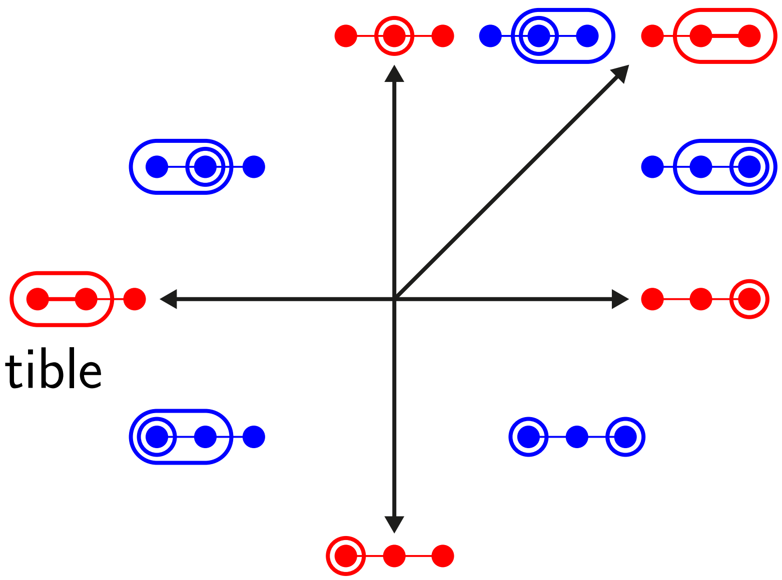
Ceballos-Santos-Ziegler, Many non-equivalent realizations of the associahedron ('11)

COMPATIBILITY FANS FOR GRAPHICAL NESTED COMPLEXES

T° an initial maximal tubing on G
t, t' two tubes of G

compatibility degree between t and t'

$$(\mathrm{t} \,\|\, \mathrm{t}') = \begin{cases} -1 & \text{if } \mathrm{t} = \mathrm{t}' \\ 0 & \text{if } \mathrm{t}, \mathrm{t}' \text{ are compatible} \\ |\{\text{neighbors of } \mathrm{t} \text{ in } \mathrm{t}' \smallsetminus \mathrm{t}\}| & \text{otherwise} \end{cases}$$



compatibility vector of t wrt T°:

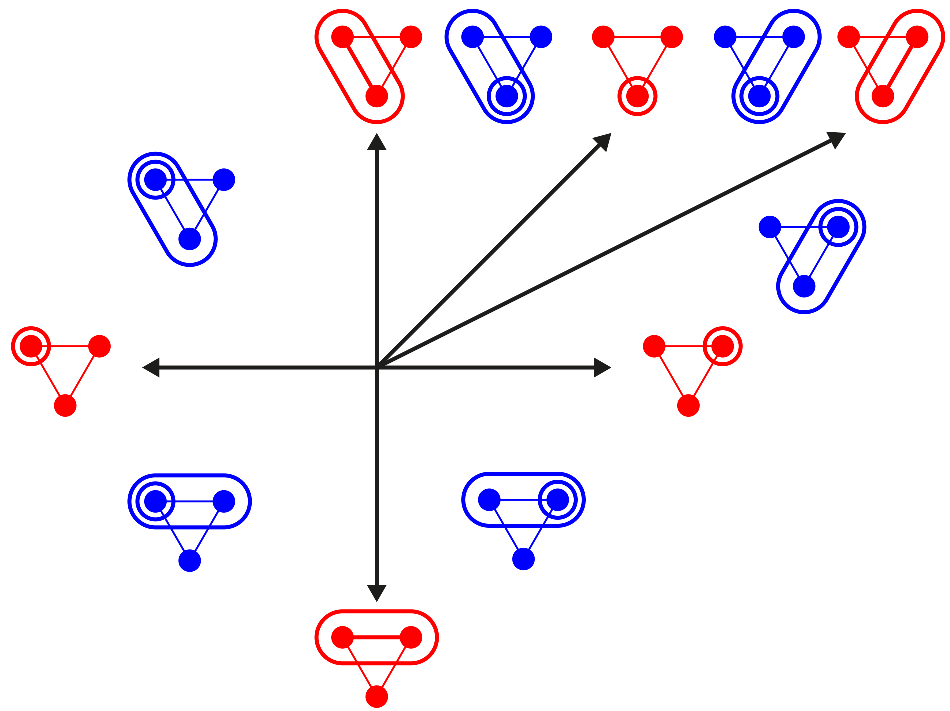
$$\mathbf{d}(\mathrm{T}^\circ, \mathrm{t}) = \big[(\mathrm{t}^\circ \,\|\, \mathrm{t})\big]_{\mathrm{t}^\circ \in \mathrm{T}^\circ}$$

THM. For any initial maximal tubing T° on G, the collection of cones

$$\mathcal{D}(\mathrm{G}, \mathrm{T}^\circ) = \{\mathbb{R}_{\geq 0}\, \mathbf{d}(\mathrm{T}^\circ, \mathrm{T}) \mid \mathrm{T} \text{ tubing on } \mathrm{G}\}$$

forms a complete simplicial fan, called compatibility fan of G.

Manneville-P., Compatibility fans for graphical nested complexes

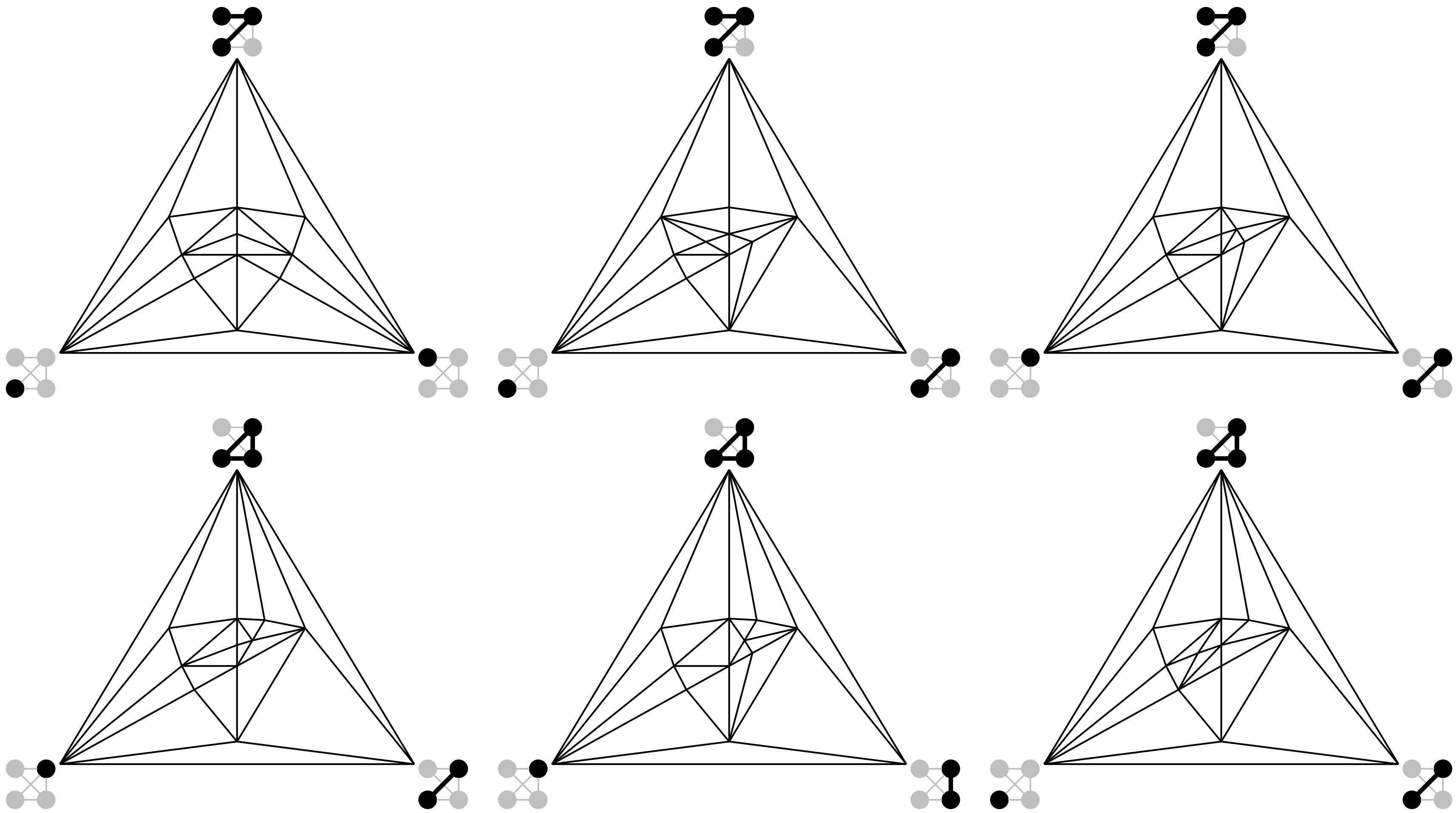


GRAPH CATALAN MANY SIMPLICIAL FAN REALIZATIONS

THM. When none of the connected components of G is a spider,

$$\# \text{ linear isomorphism classes of compatibility fans of } \mathrm{G}$$
$$= \# \text{ orbits of maximal tubings on } \mathrm{G} \text{ under graph automorphisms of } \mathrm{G}.$$

Manneville-P., Compatibility fans for graphical nested complexes



POLYTOPALITY?

QU. Are all compatibility fans polytopal?

POLYTOPALITY?

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Polytopality of a complete simplicial fan $\iff$ Feasibility of a linear program

Exm: We check that the compatibility fan on the complete graph K_7 is polytopal by solving a linear program on 126 variables and $17\,640$ inequalities

POLYTOPALITY?

QU. Are all compatibility fans polytopal?

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Exm: We check that the compatibility fan on the complete graph K_7 is polytopal by solving a linear program on 126 variables and $17\,640$ inequalities

$\Longrightarrow$ All compatibility fans on complete graphs of ≤ 7 vertices are polytopal...
$\Longrightarrow$ All compatibility fans on graphs of ≤ 4 vertices are polytopal...

POLYTOPALITY?

QU. Are all compatibility fans polytopal?

Polytopality of a complete simplicial fan $\iff$ Feasibility of a linear program

Exm: We check that the compatibility fan on the complete graph K_7 is polytopal by solving a linear program on 126 variables and $17\,640$ inequalities

$\Longrightarrow$ All compatibility fans on complete graphs of ≤ 7 vertices are polytopal...
$\Longrightarrow$ All compatibility fans on graphs of ≤ 4 vertices are polytopal...

To go further, we need to understand better the linear dependences between the compatibility vectors of the tubes involved in a flip

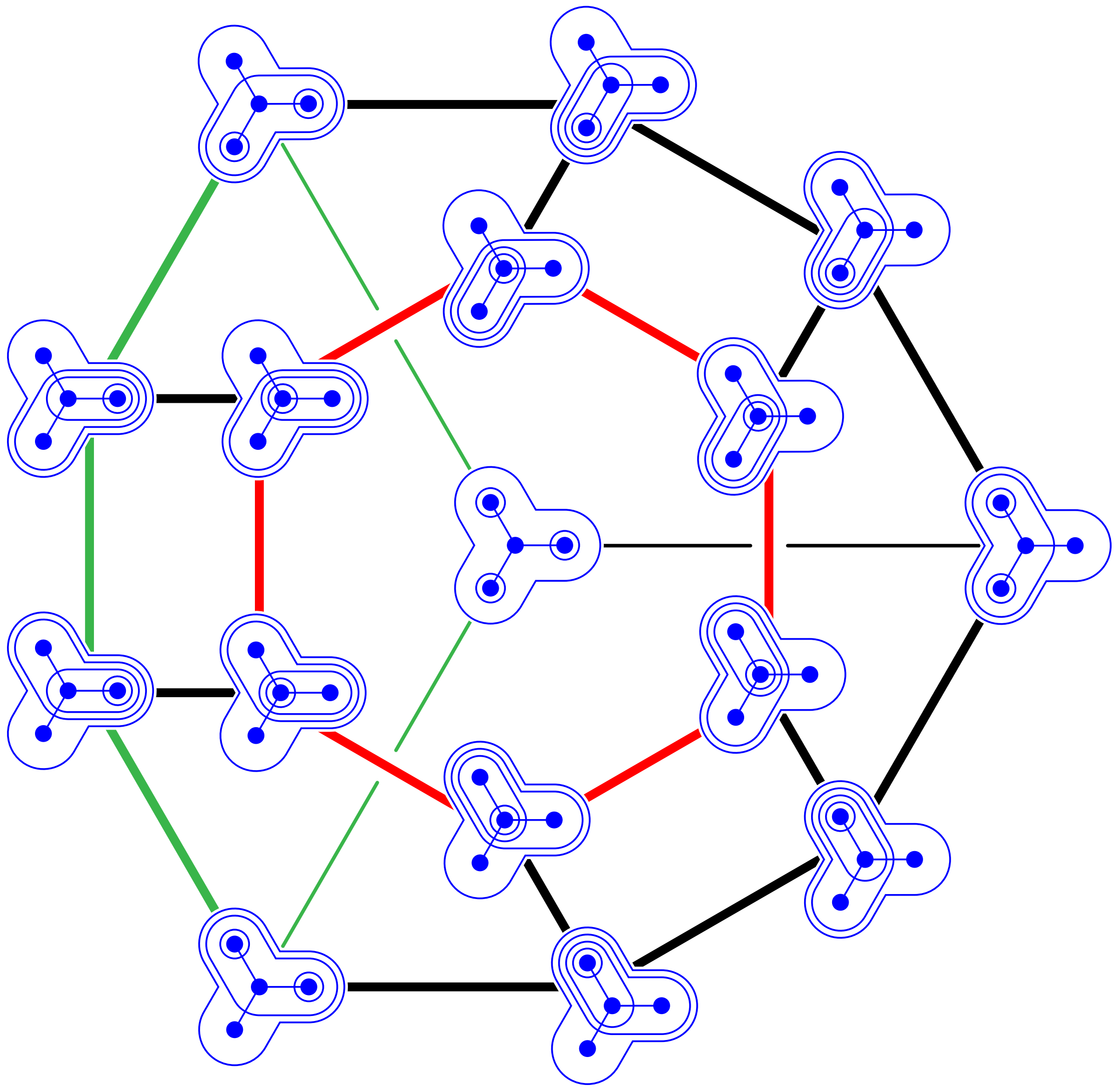
THM. All compatibility fans on the paths and cycles are polytopal

Ceballos-Santos-Ziegler, Many non-equivalent realizations of the associahedron ('11)
Manneville-P., Compatibility fans for graphical nested complexes

POLYTOPALITY?

QU. Are all compatibility fans polytopal?

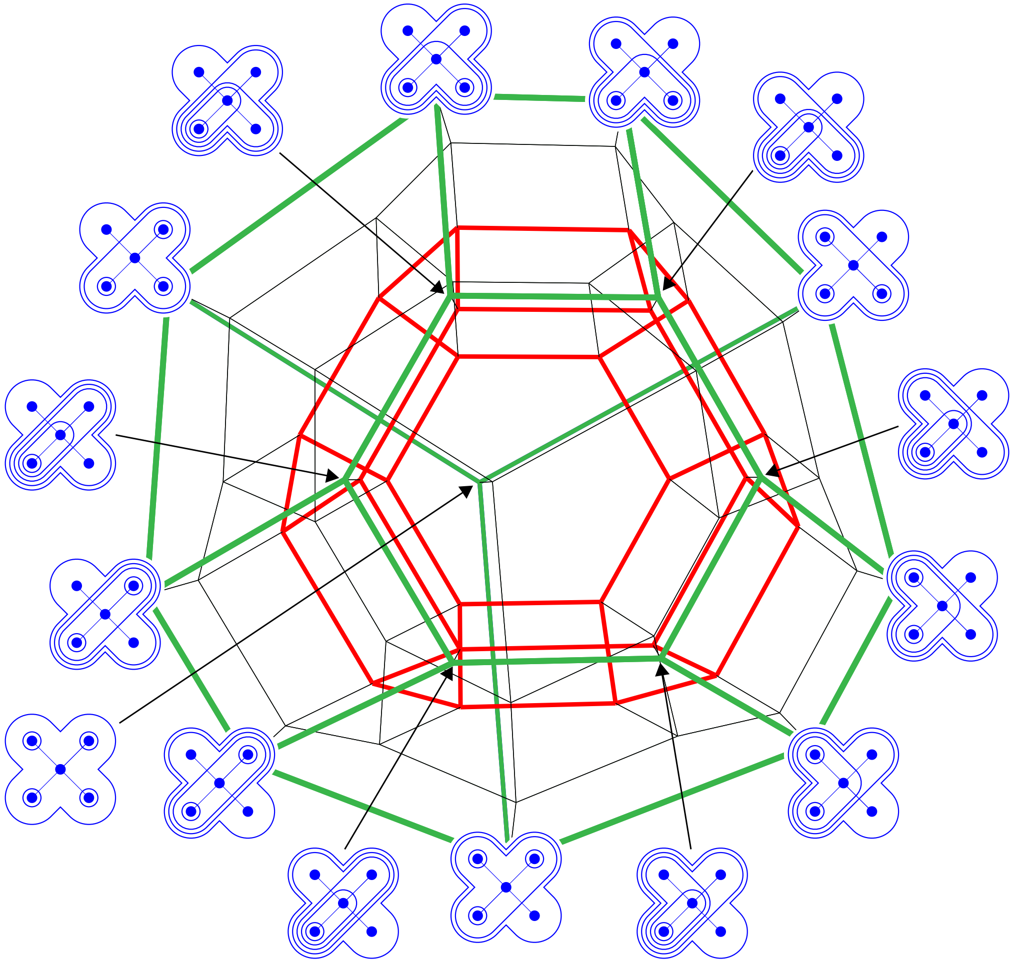
Remarkable realizations of the stellohedra

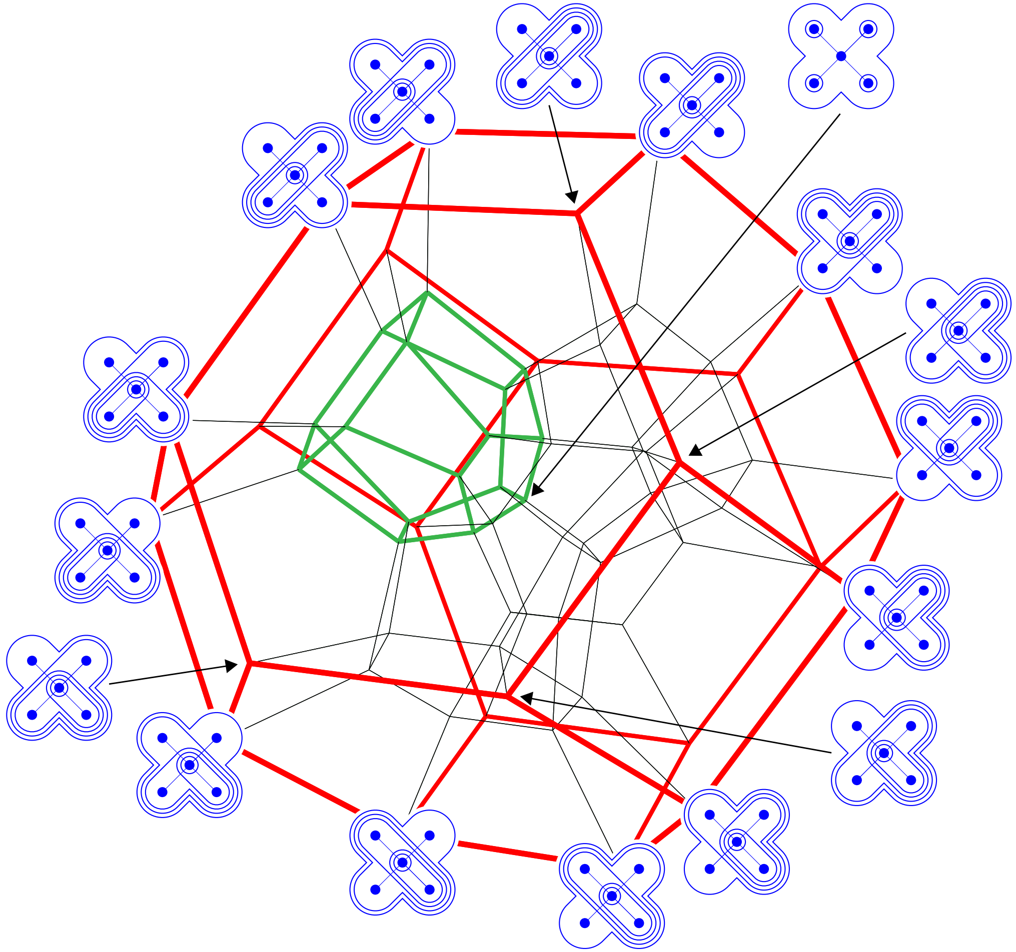


POLYTOPALITY?

QU. Are all compatibility fans polytopal?

Remarkable realizations of the stellohedra





Convex hull of the orbits under coordinate permutations of the set $\left\{ \sum_{i>k} i\, \mathbf{e}_i \;\middle|\; 0 \le k \le n \right\}$

SIGNED TREE ASSOCIAHEDRA

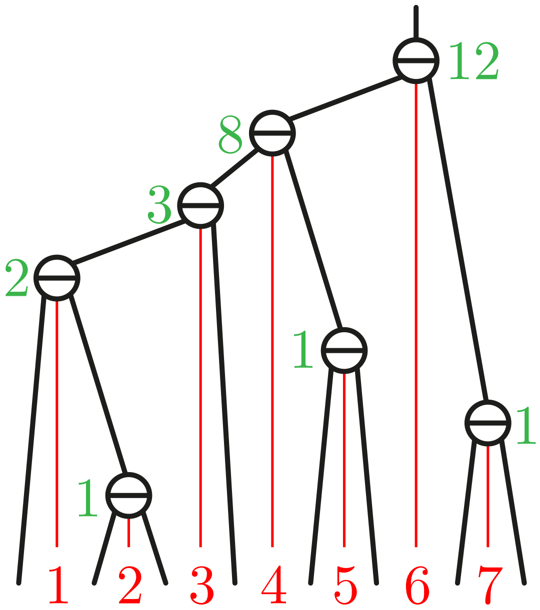
arXiv:1309.5222

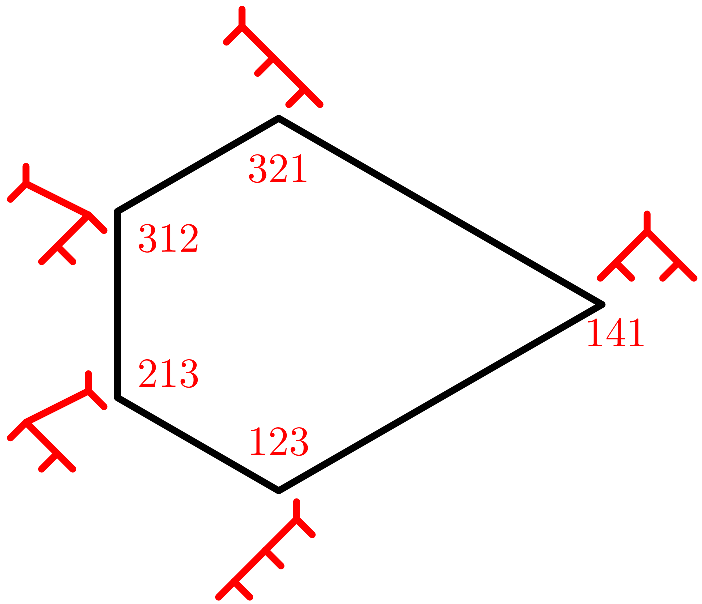
LODAY'S ASSOCIAHEDRON

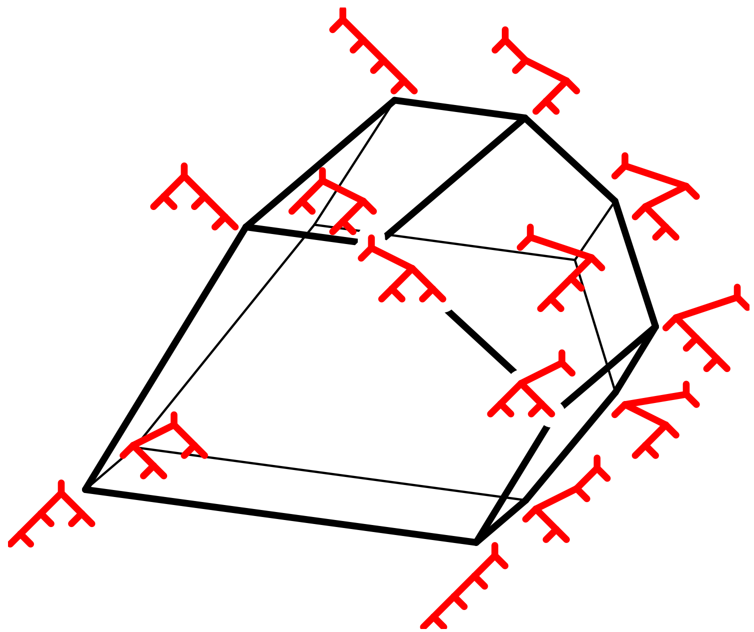
$$\text{Asso}(n) := \operatorname{conv}\{\mathbf{L}(\mathrm{T}) \mid \mathrm{T} \text{ binary tree}\} = \mathbb{H} \cap \bigcap_{1 \le i \le j \le n+1} \mathbf{H}^{\ge}(i,j)$$

$$\mathbf{L}(\mathrm{T}) := \big[\ell(\mathrm{T},i) \cdot r(\mathrm{T},i)\big]_{i \in [n+1]} \qquad \mathbf{H}^{\ge}(i,j) := \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i \le k \le j} x_i \ge \binom{j-i+2}{2} \right\}$$

Loday, Realization of the Stasheff polytope ('04)







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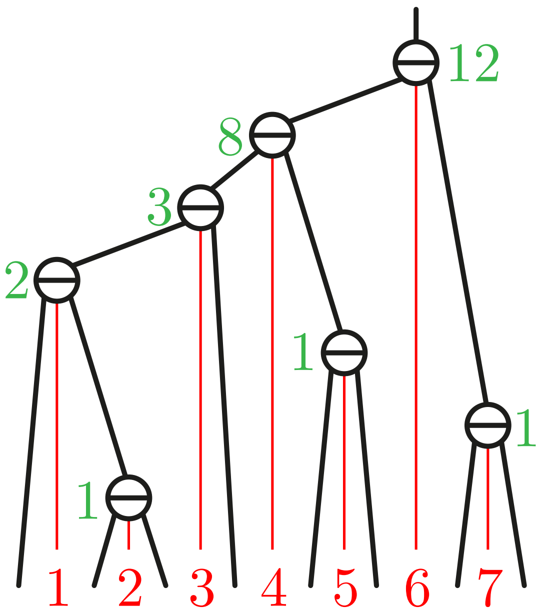
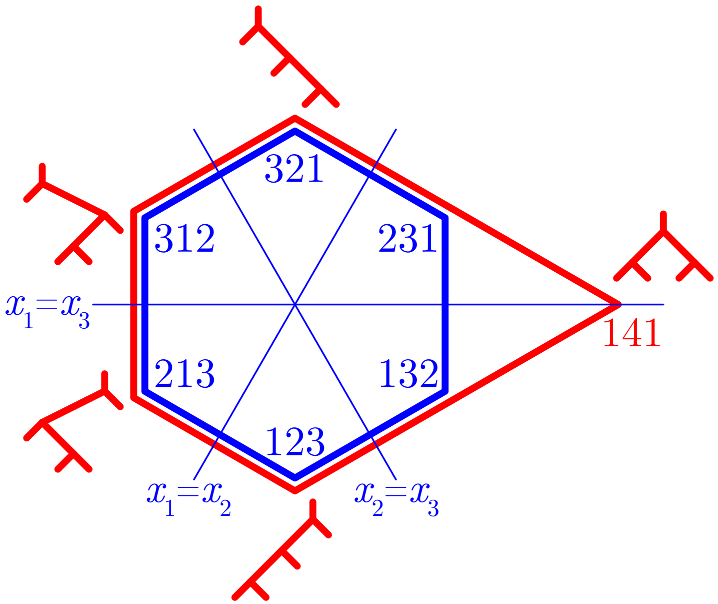
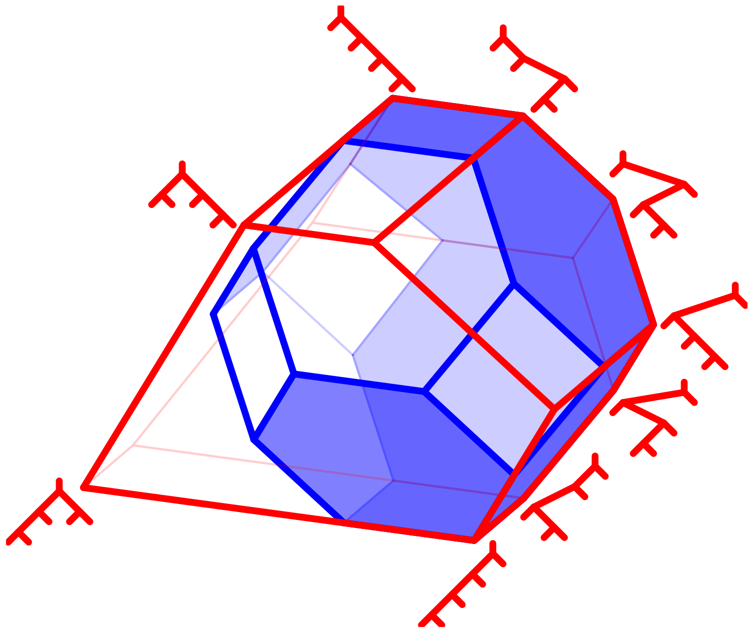




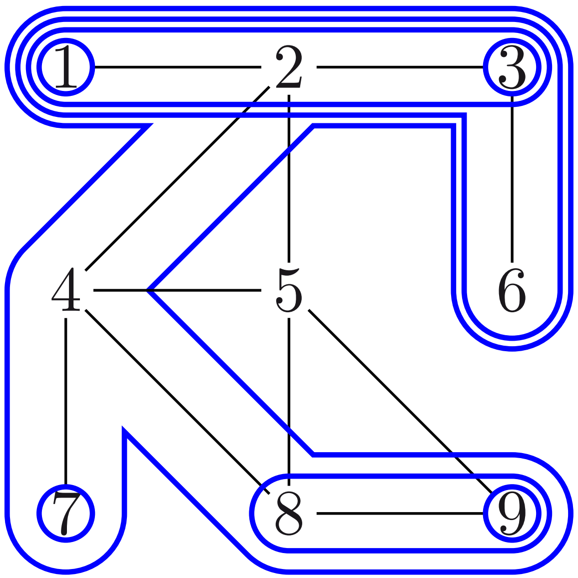


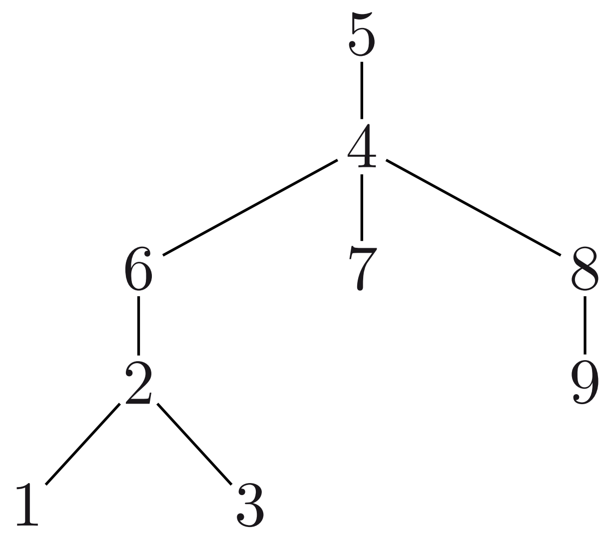
- Asso(n) obtained by deleting inequalities in the facet description of the permutahedron
- normal cone of $\mathbf{L}(\mathrm{T})$ in $\mathsf{Asso}(n) = \{\mathbf{x} \in \mathbb{H} \mid x_i < x_j \text{ for all } i \to j \text{ in } \mathrm{T}\}$

  $= \bigcup_{\sigma \in \mathcal{L}(\mathrm{T})}$ normal cone of σ in $\mathsf{Perm}(n)$

SPINES

spine of a tubing T = Hasse diagram of the inclusion poset of T





tube t of the tubing T $\longmapsto$ node $\mathrm{s(t)}$ of the spine S labeled by $\mathrm{t} \smallsetminus \bigcup \{\mathrm{t'} \mid \mathrm{t'} \in \mathrm{T}, \mathrm{t'} \subsetneq \mathrm{t}\}$

tube $\mathrm{t(s)} := \bigcup \{\mathrm{s'} \mid \mathrm{s'} \le \mathrm{s} \text{ in } \mathrm{S}\}$ of the tubing T $\longleftarrow$ node s of the spine S

S spine on G $\iff$ for each node s of S with children $\mathrm{s}_1 \dots \mathrm{s}_k$, the tubes $\mathrm{t(s_1)} \dots \mathrm{t(s}_k\mathrm{)}$ lie in distinct connected components of $\mathrm{G}\big[\mathrm{t(s)} \smallsetminus \mathrm{s}\big]$

SPINES

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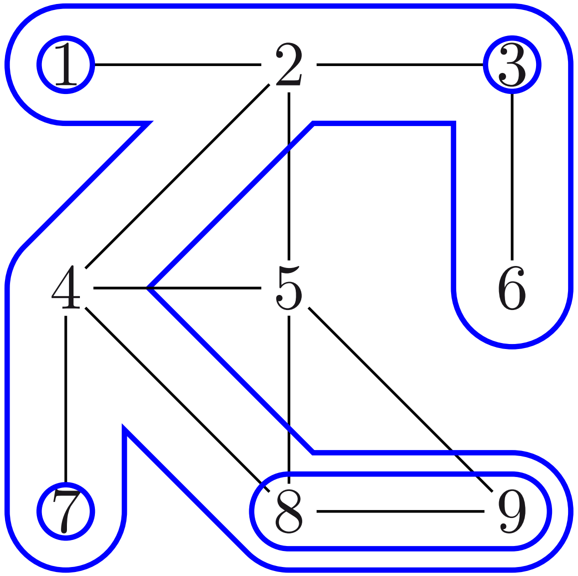

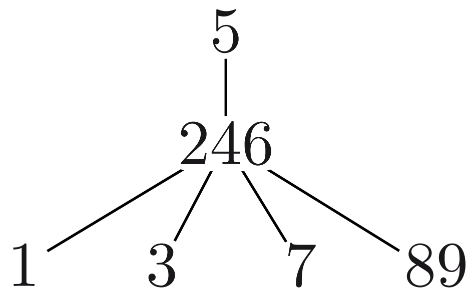

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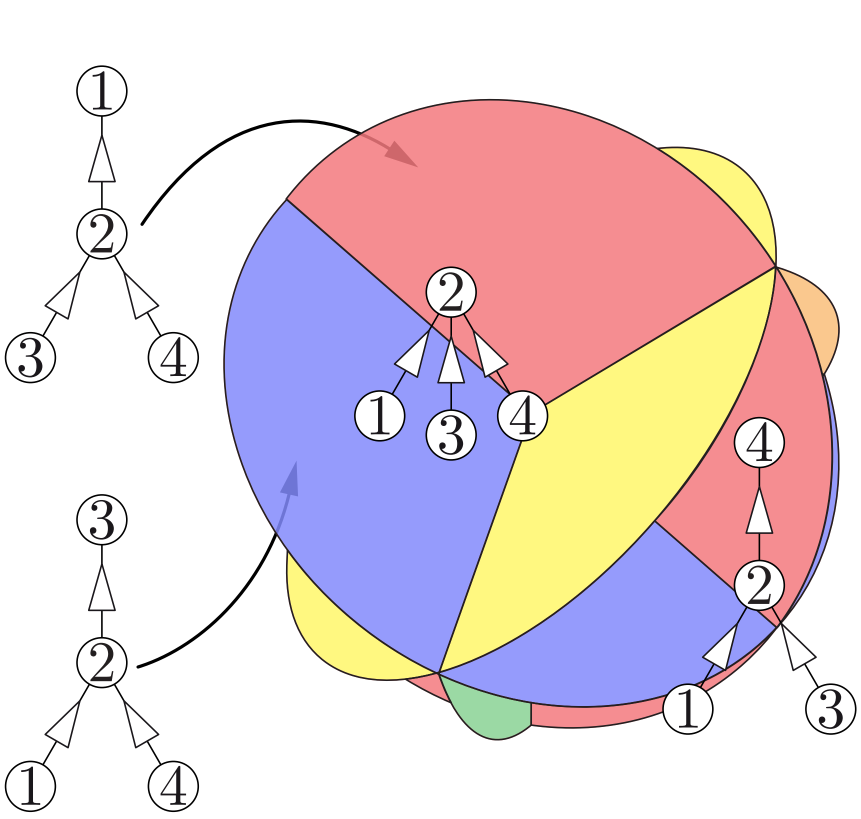
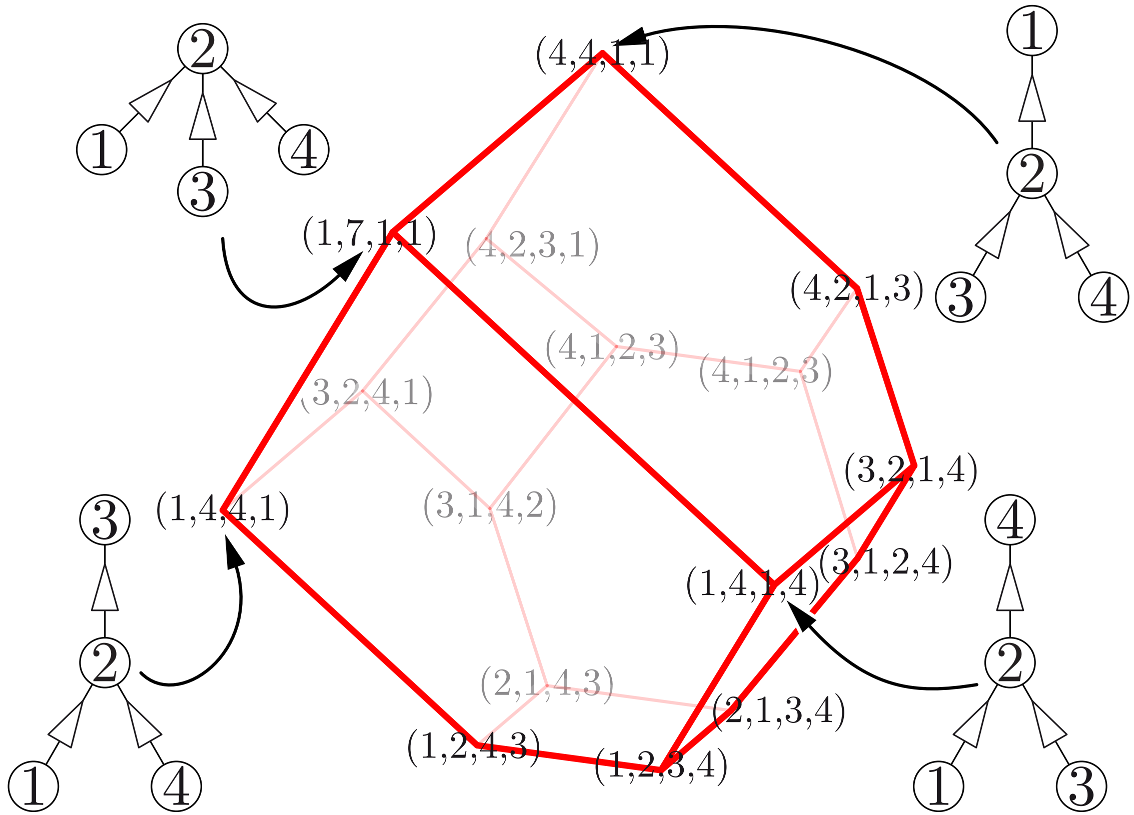
NESTED FANS AND GRAPH ASSOCIAHEDRA

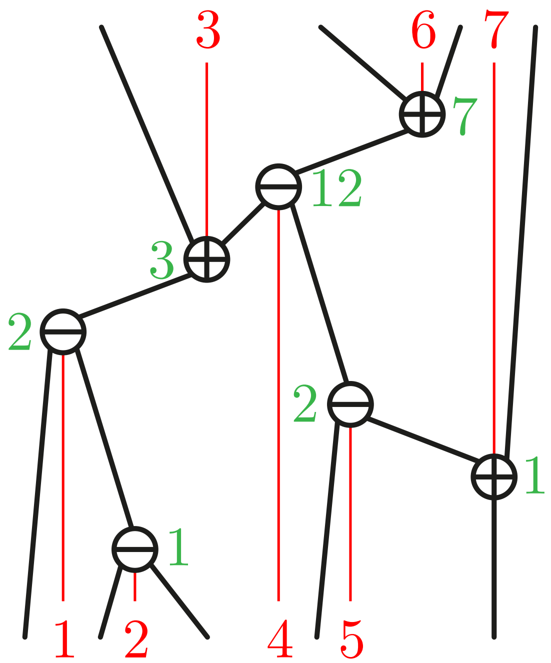
THM. The collection of cones $\big\{ \{\mathbf{x} \in \mathbb{H} \mid x_i < x_j \text{ for all } i \to j \text{ in } \mathrm{T}\} \;\big|\; \mathrm{T} \text{ tubing on } \mathrm{G} \big\}$ forms a complete simplicial fan, called the nested fan of G. This fan is always polytopal.

Carr-Devadoss, Coxeter complexes and graph associahedra ('06)
Postnikov, Permutohedra, associahedra, and beyond ('09)
Zelevinsky, Nested complexes and their polyhedral realizations ('06)





HOHLWEG-LANGE'S ASSOCIAHEDRA

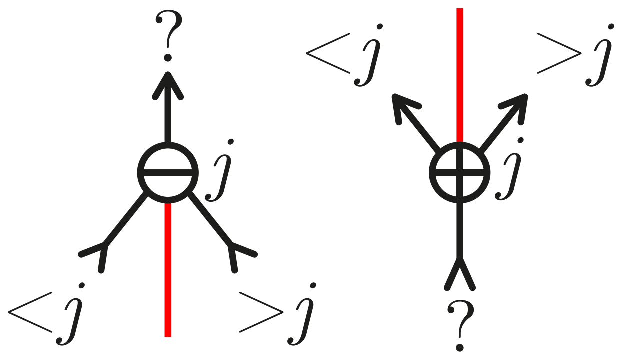


for an arbitrary signature $\varepsilon \in \pm^{n+1}$,

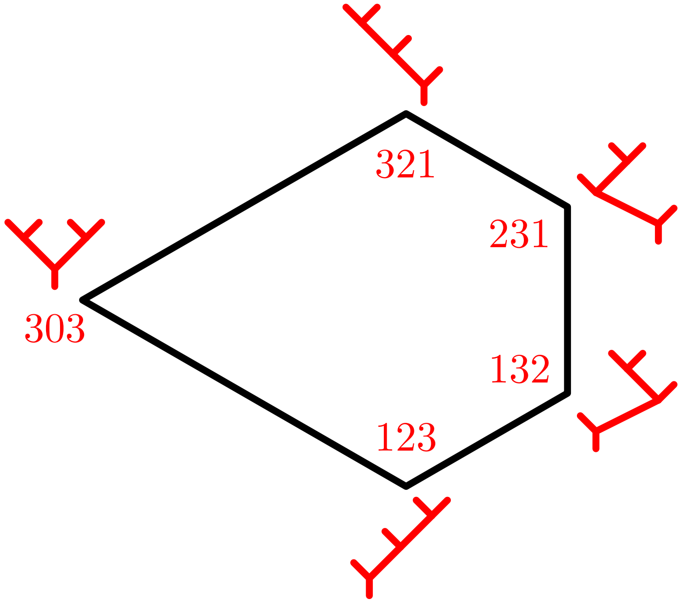
$$\mathsf{Asso}(\varepsilon) \coloneqq \operatorname{conv}\left\{\mathbf{HL}(\mathrm{T}) \mid \mathrm{T}\ \varepsilon\text{-Cambrian tree}\right\}$$

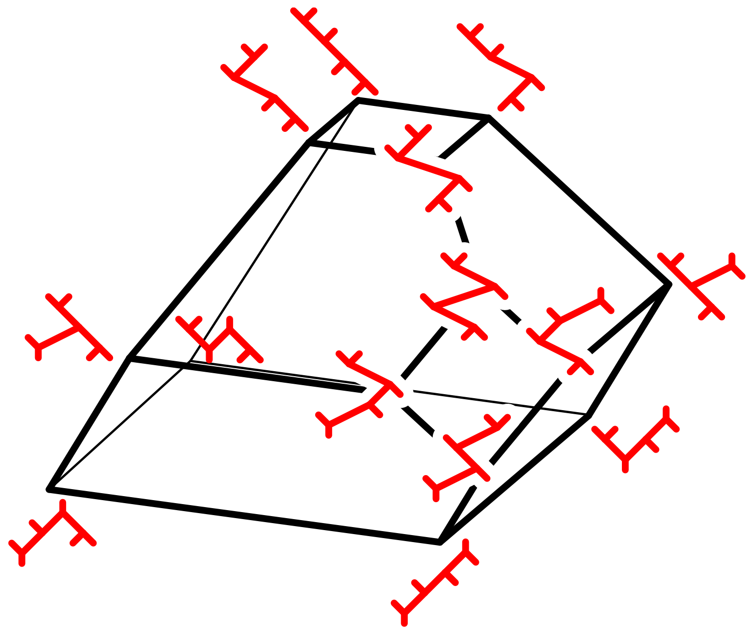
with $$\mathbf{HL}(\mathrm{T})_j \coloneqq \begin{cases} \ell(\mathrm{T},j)\cdot r(\mathrm{T},j) & \text{if } \varepsilon(j) = - \\ n+2-\ell(\mathrm{T},j)\cdot r(\mathrm{T},j) & \text{if } \varepsilon(j) = + \end{cases}$$

Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)
Lange-P., *Using spines to revisit a construction of the associahedron* ('13^{+})



Rule for Cambrian trees





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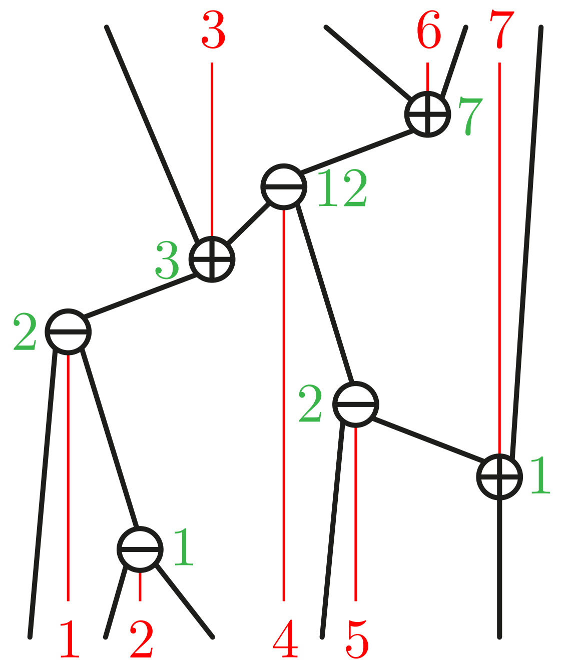

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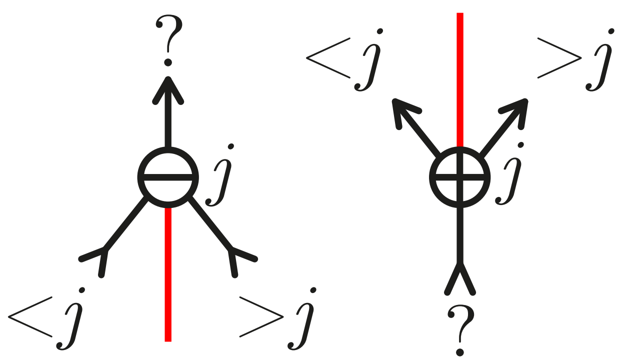

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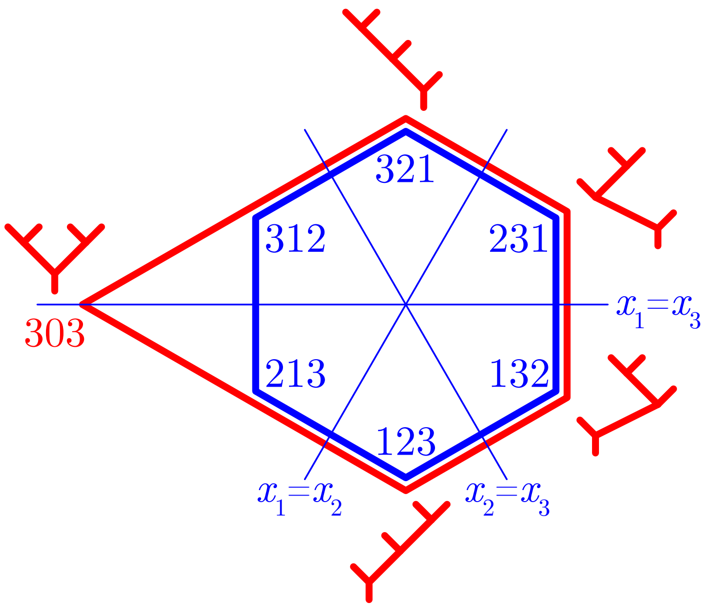

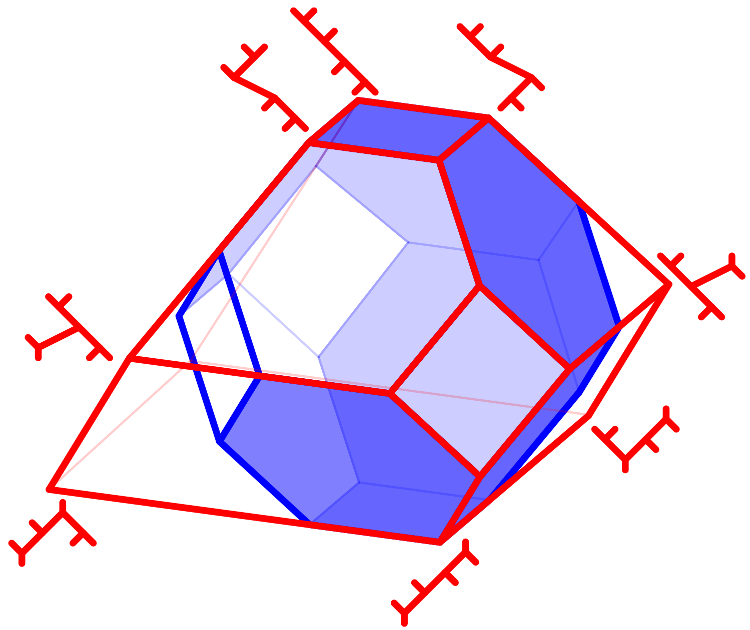

- Asso(n) obtained by deleting inequalities in the facet description of the permutahedron
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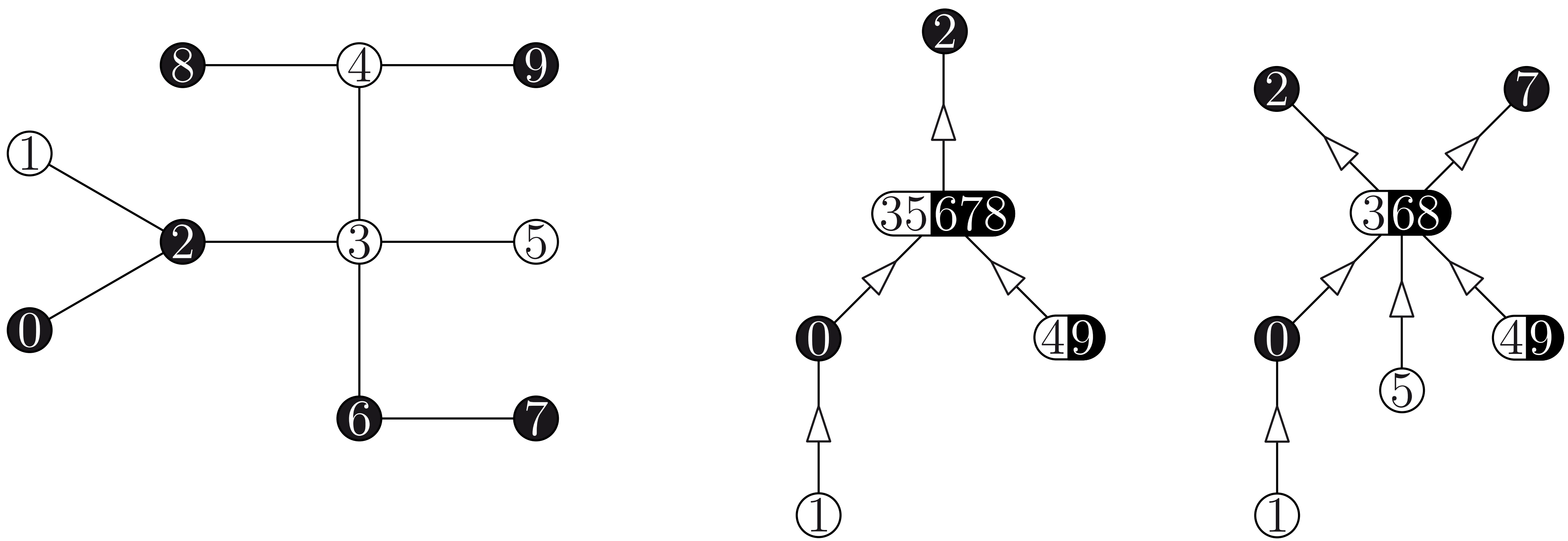
SIGNED SPINES ON SIGNED TREES

T tree on the signed ground set $\mathrm{V} = \mathrm{V}^- \sqcup \mathrm{V}^+$ (negative in white, positive in black)

Signed spine on T = directed and labeled tree S st

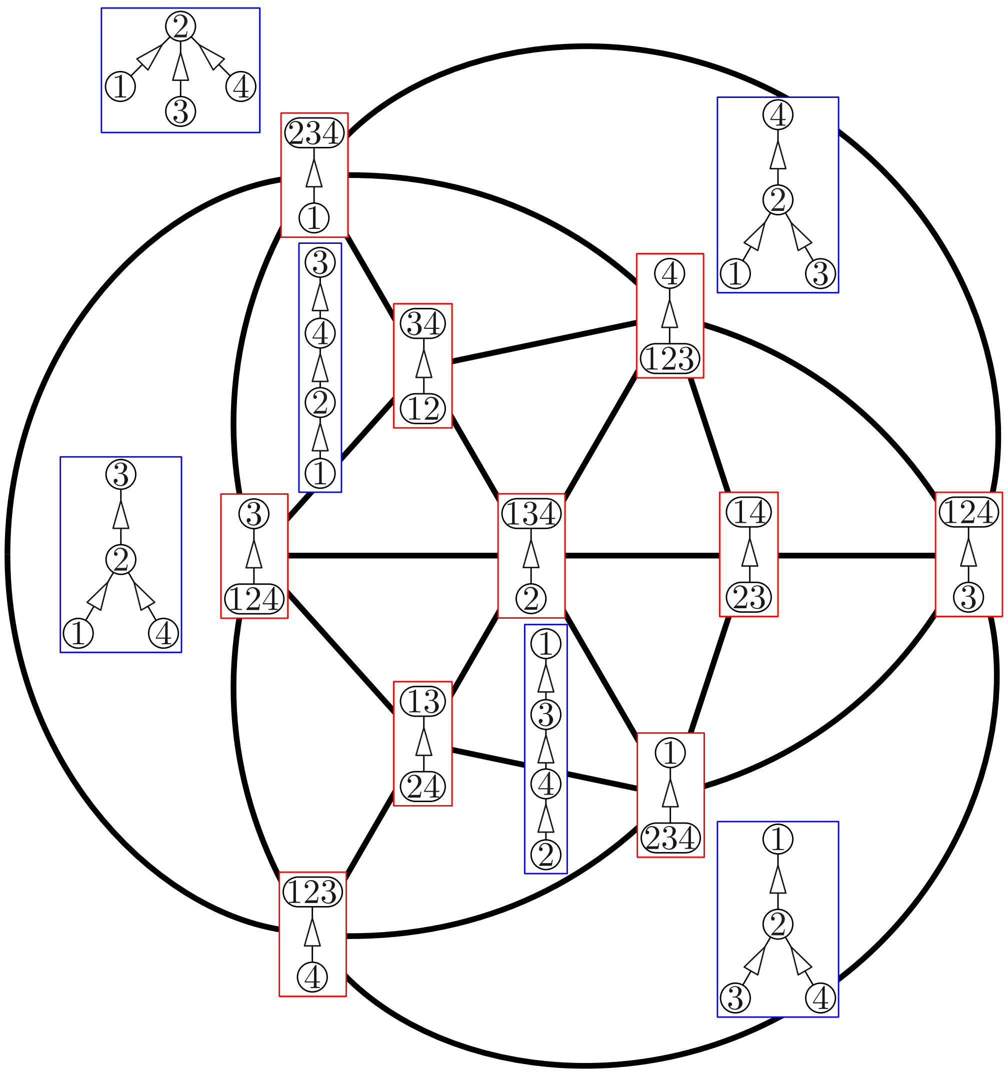
(i) the labels of the nodes of S form a partition of the signed ground set V

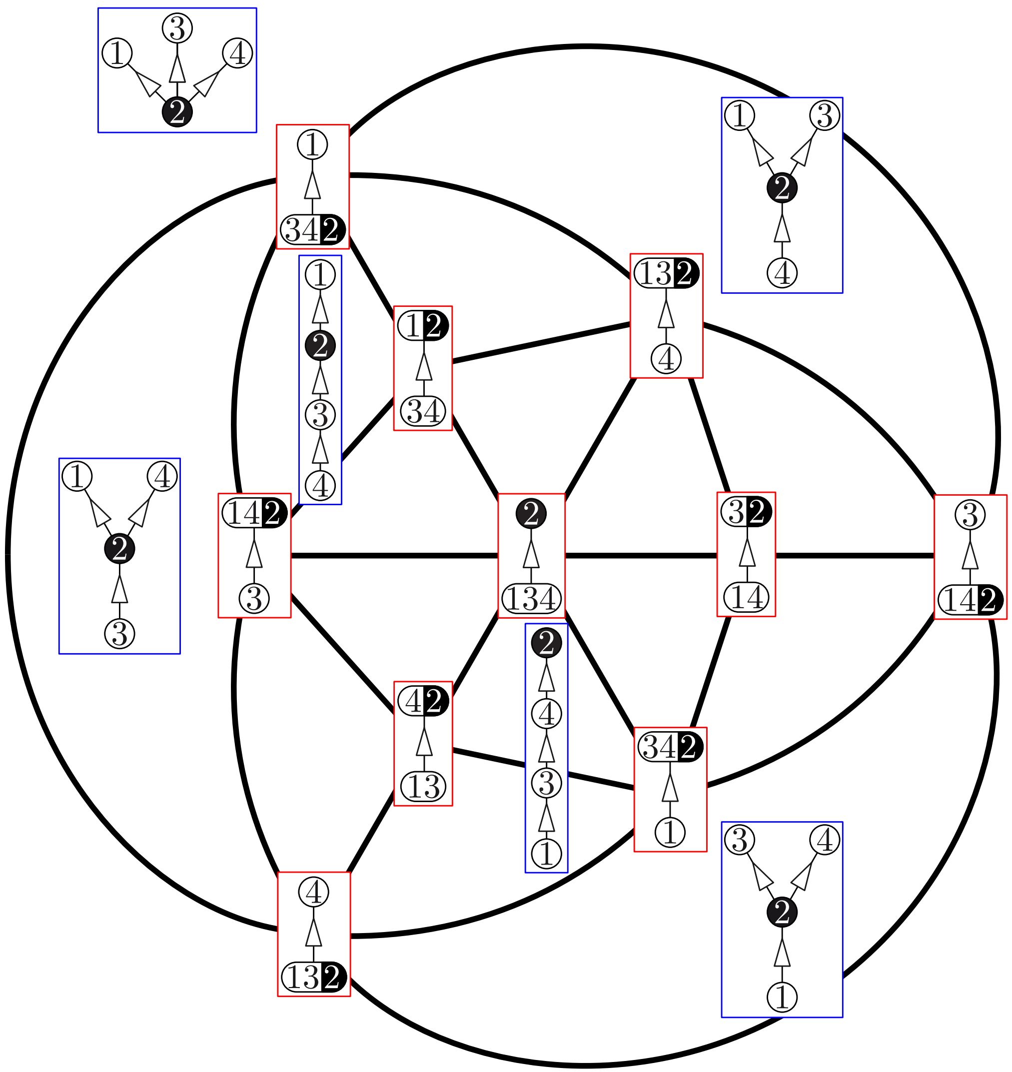
(ii) at a node of S labeled by $U = U^- \sqcup U^+$, the source label sets of the different incoming arcs are subsets of distinct connected components of $\mathrm{T} \smallsetminus U^-$, while the sink label sets of the different outgoing arcs are subsets of distinct connected components of $\mathrm{T} \smallsetminus U^+$



SPINE COMPLEX

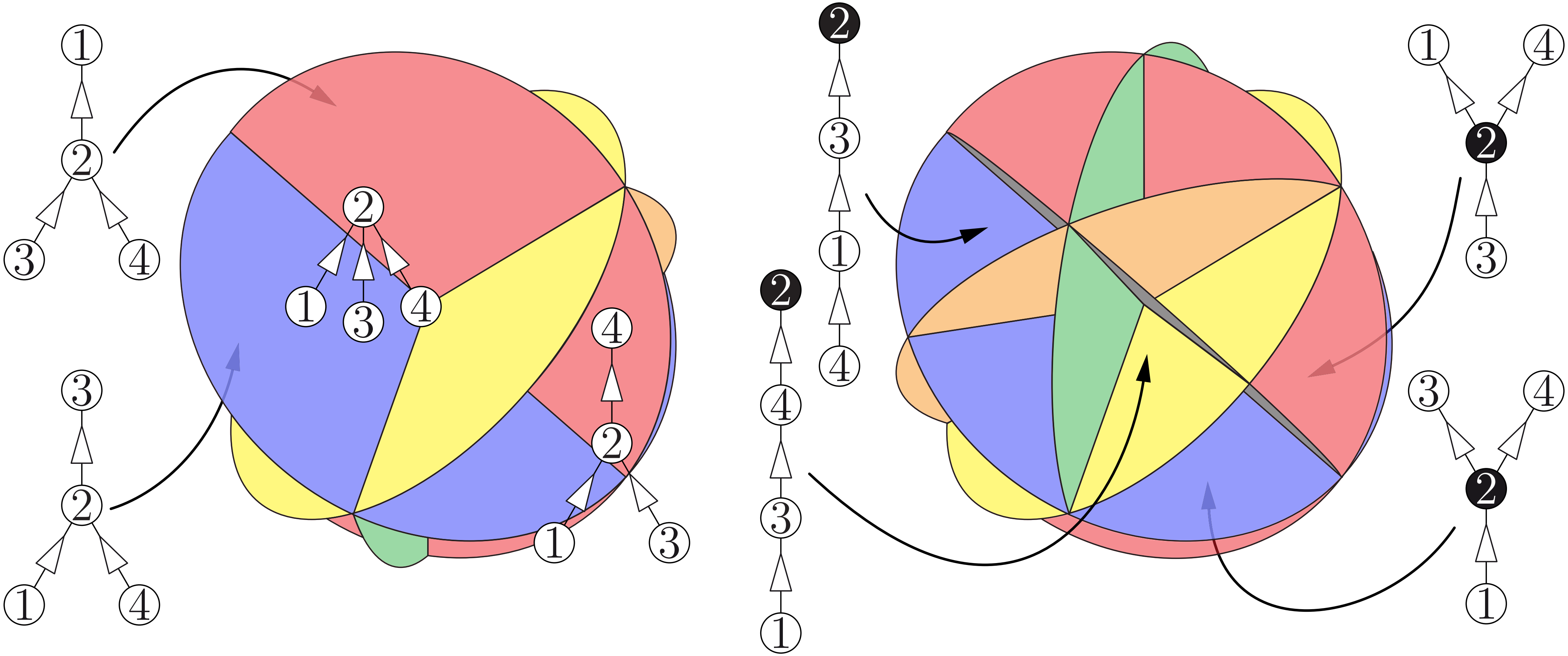
Signed spine complex $\mathcal{S}(\mathrm{T})$ = simplicial complex whose inclusion poset is isomorphic to the poset of edge contractions on the signed spines of T





SPINE FAN

For S spine on T, define $\mathrm{C}(\mathrm{S}) \coloneqq \{\mathbf{x} \in \mathbb{H} \mid x_u \leq x_v, \text{ for all arcs } u \to v \text{ in } \mathrm{S}\}$



THEO. The collection of cones $\mathcal{F}(\mathrm{T}) \coloneqq \{\mathrm{C}(\mathrm{S}) \mid \mathrm{S} \in \mathcal{S}(\mathrm{T})\}$ defines a complete simplicial fan on $\mathbb{H}$, which we call the spine fan

SIGNED TREE ASSOCIAHEDRA

THM. The spine fan $\mathcal{F}(\mathrm{T})$ is the normal fan of the signed tree associahedron $\mathsf{Asso}(\mathrm{T})$, defined equivalently as

(i) the convex hull of the points

$$\mathbf{a}(\mathrm{S})_v = \begin{cases} \big| \{\pi \in \Pi(\mathrm{S}) \mid v \in \pi \text{ and } r_v \notin \pi\} \big| & \text{if } v \in \mathrm{V}^- \\ |\mathrm{V}| + 1 - \big| \{\pi \in \Pi(\mathrm{S}) \mid v \in \pi \text{ and } r_v \notin \pi\} \big| & \text{if } v \in \mathrm{V}^+ \end{cases}$$

for all maximal signed spines $\mathrm{S} \in \mathcal{S}(\mathrm{T})$

(ii) the intersection of the hyperplane $\mathbb{H}$ with the half-spaces

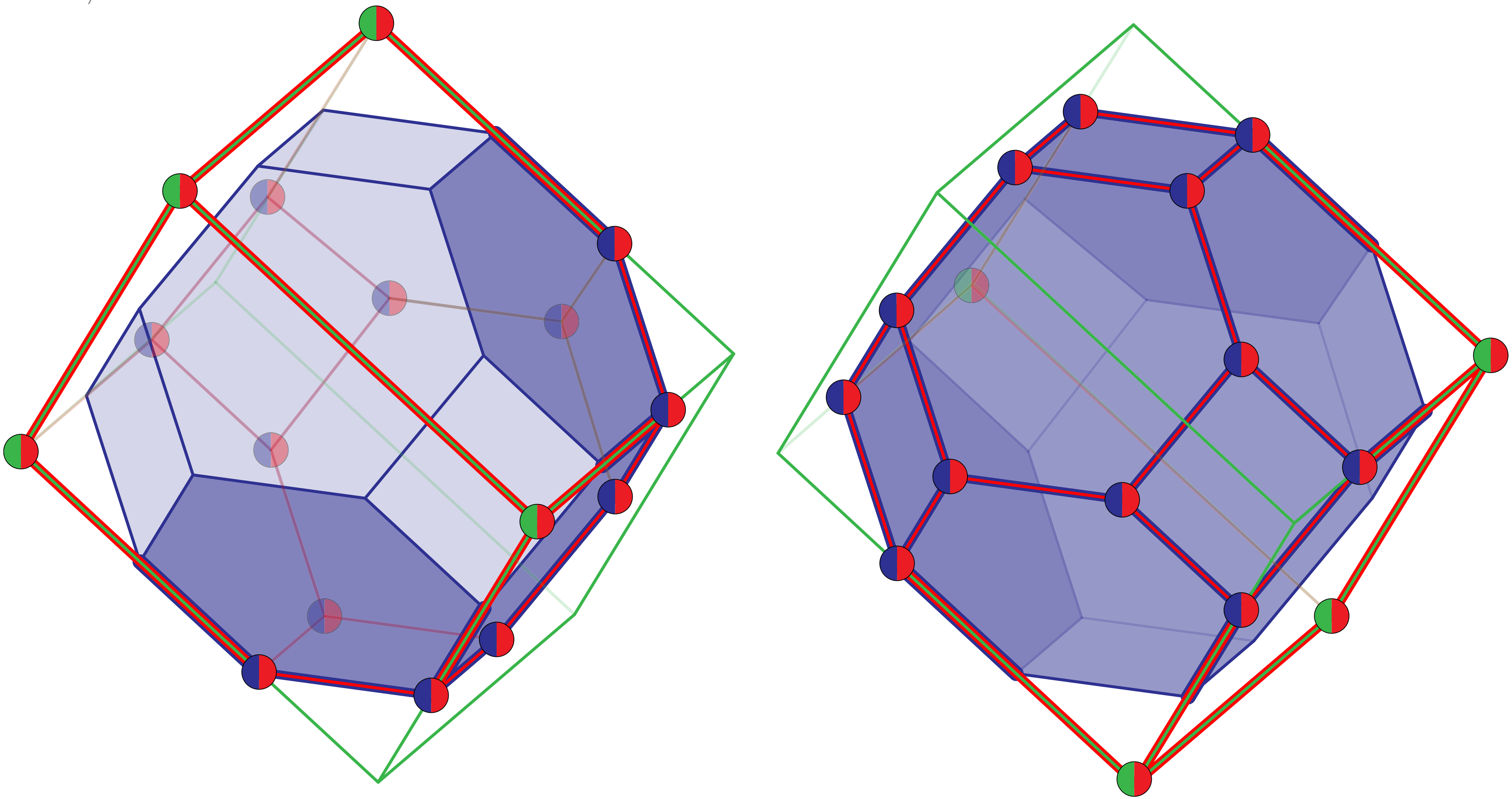
$$\mathbf{H}^{\geq}(B) := \left\{ \mathbf{x} \in \mathbb{R}^{\mathrm{V}} \;\middle|\; \sum_{v \in B} x_v \geq \binom{|B|+1}{2} \right\}$$

for all signed building blocks $B \in \mathcal{B}(\mathrm{T})$

SIGNED TREE ASSOCIAHEDRA

The signed tree associahedron $\mathrm{Asso}(\mathrm{T})$ is sandwiched between the permutahedron $\mathrm{Perm}(\mathrm{V})$ and the parallelepiped $\mathrm{Para}(\mathrm{T})$

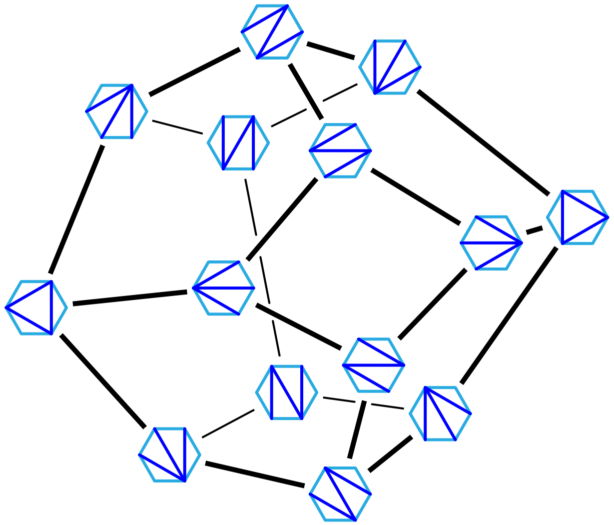
$$\sum_{u \neq v \in \mathrm{V}} [e_u, e_v] = \mathrm{Perm}(\mathrm{T}) \quad \subset \quad \mathrm{Asso}(\mathrm{T}) \quad \subset \quad \mathrm{Para}(\mathrm{T}) = \sum_{u - v \in \mathrm{T}} \pi(u - v) \cdot [e_u, e_v]$$



WHAT SHOULD I TAKE HOME FROM THIS TALK?

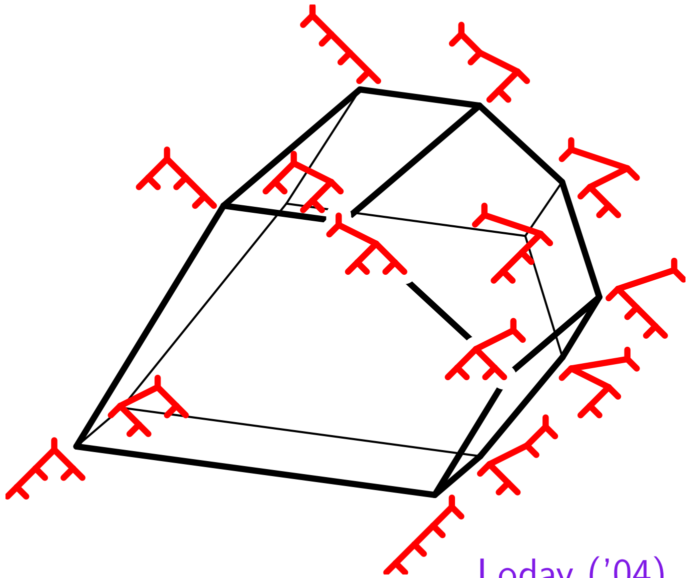
THREE FAMILIES OF REALIZATIONS

SECONDARY POLYTOPE



Gelfand-Kapranov-Zelevinsky ('94)
Billera-Filliman-Sturmfels ('90)

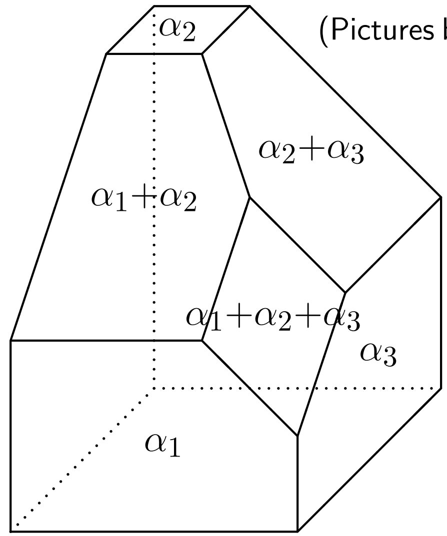
LODAY'S ASSOCIAHEDRON



Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)

CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON

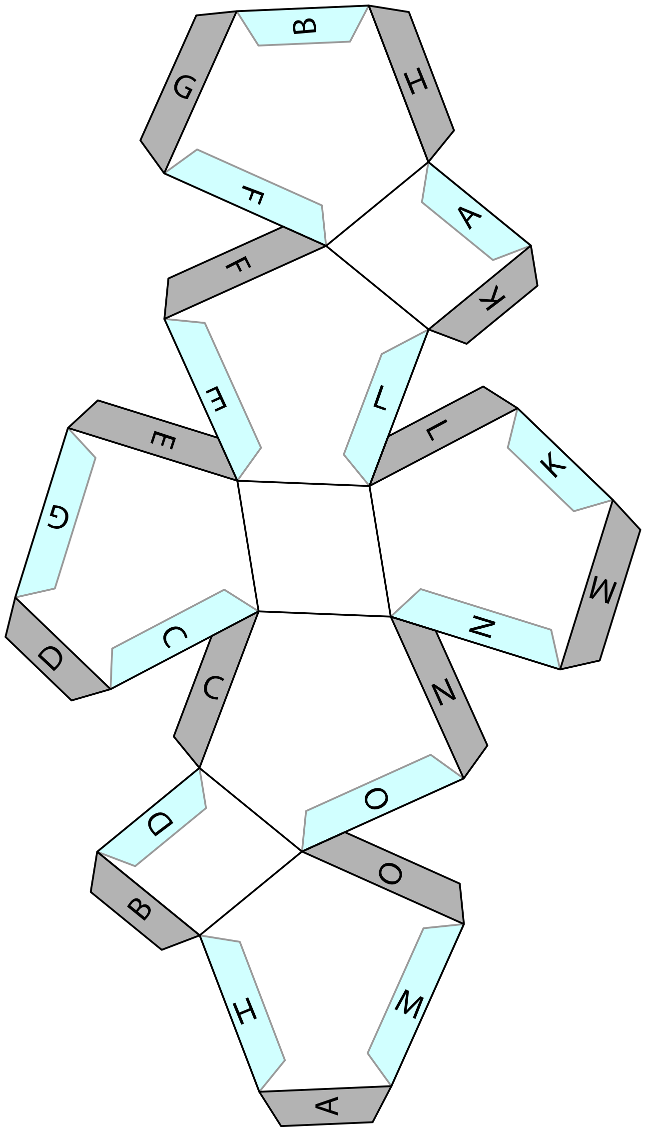
(Pictures by CFZ)



Chapoton-Fomin-Zelevinsky ('02)
Ceballos-Santos-Ziegler ('11)

TAKE HOME YOUR ASSOCIAHEDRA!

SECONDARY POLYTOPE



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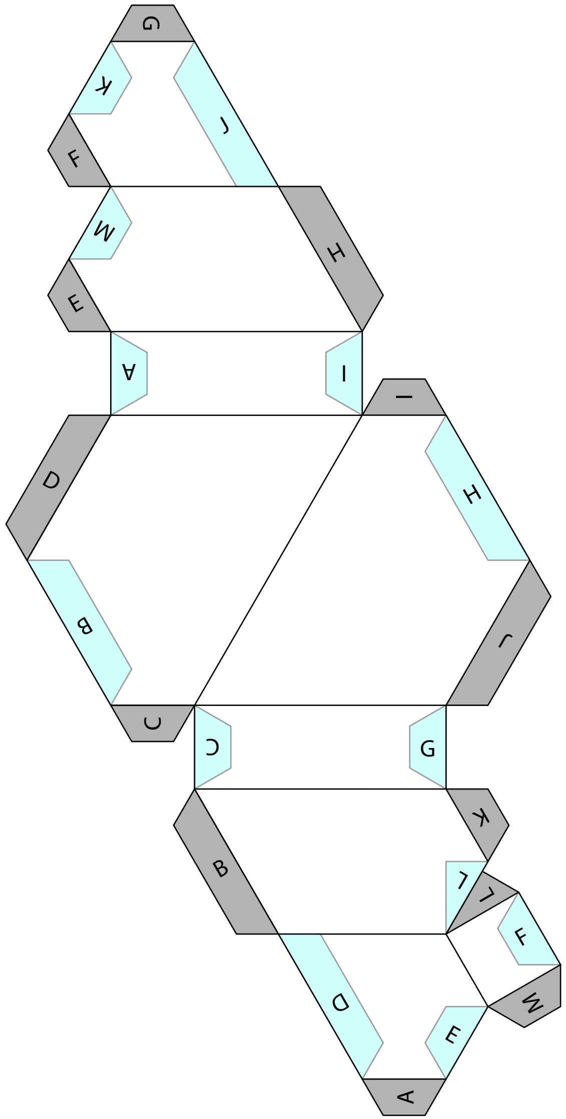

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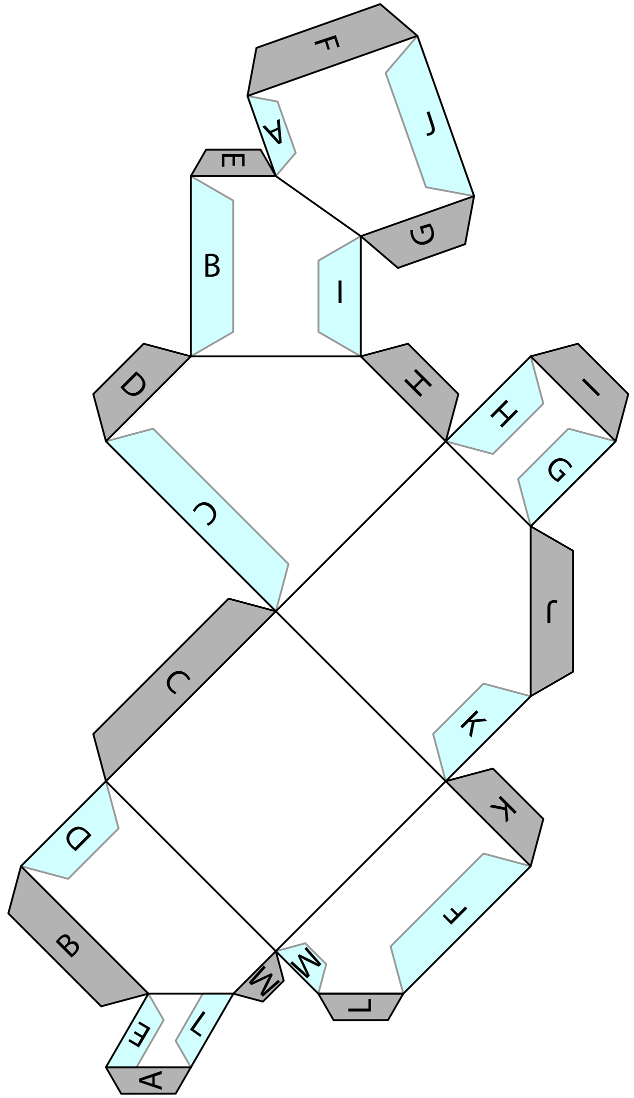
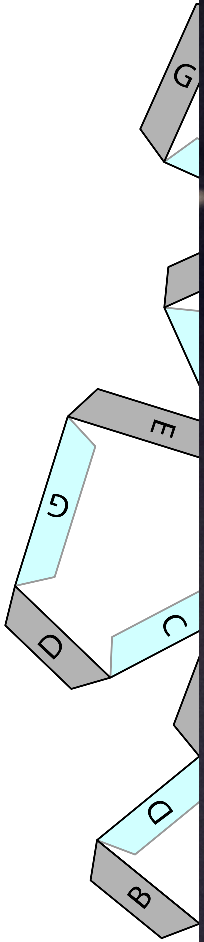
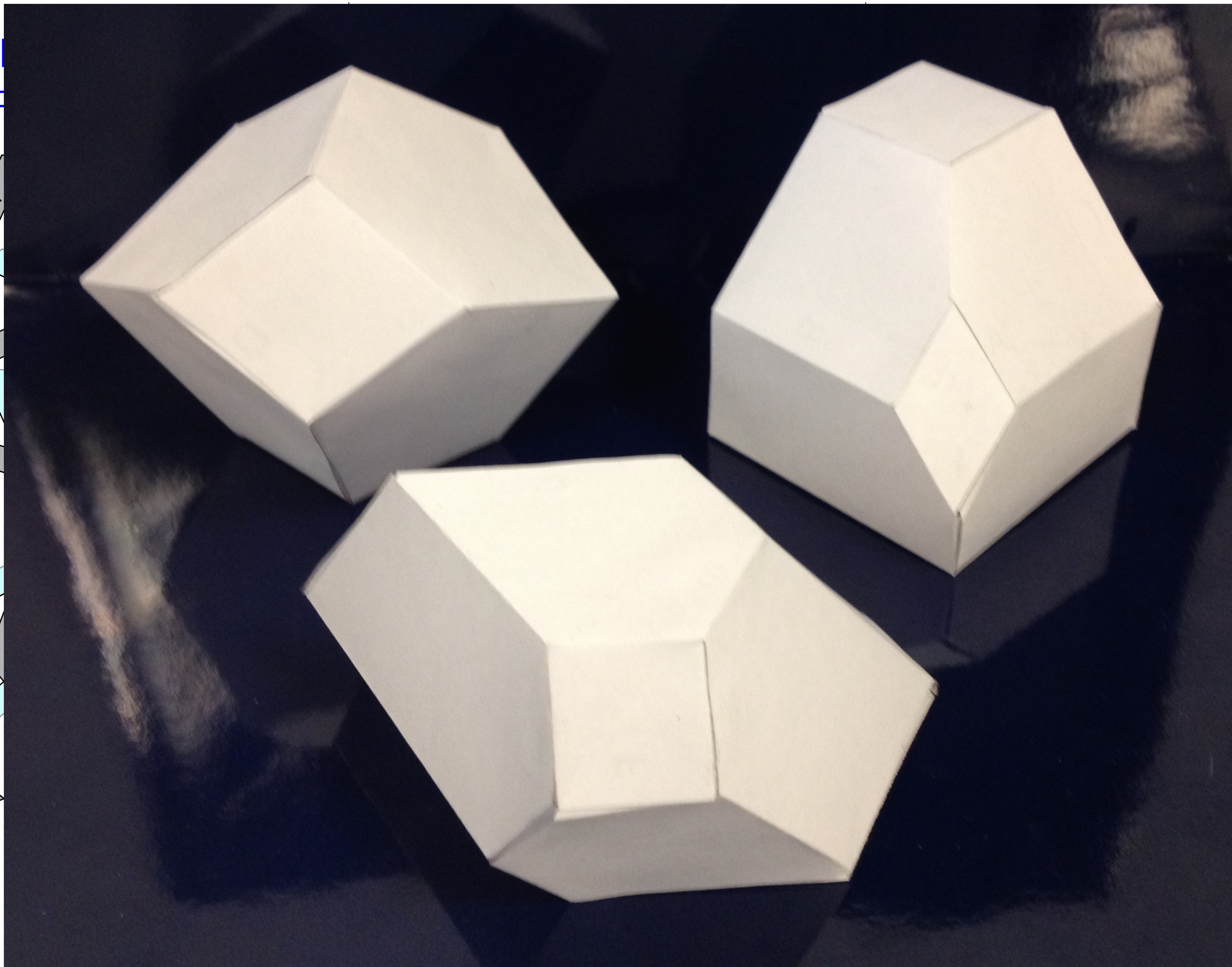
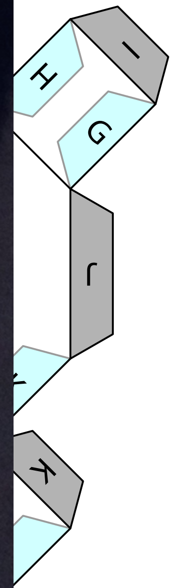


TAKE HOME YOUR ASSOCIAHEDRA!



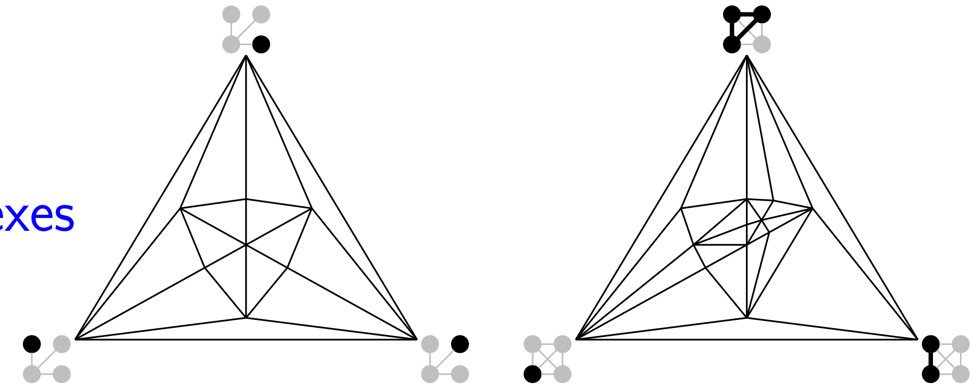




Thibault Manneville & VP

Compatibility fans for graphical nested complexes

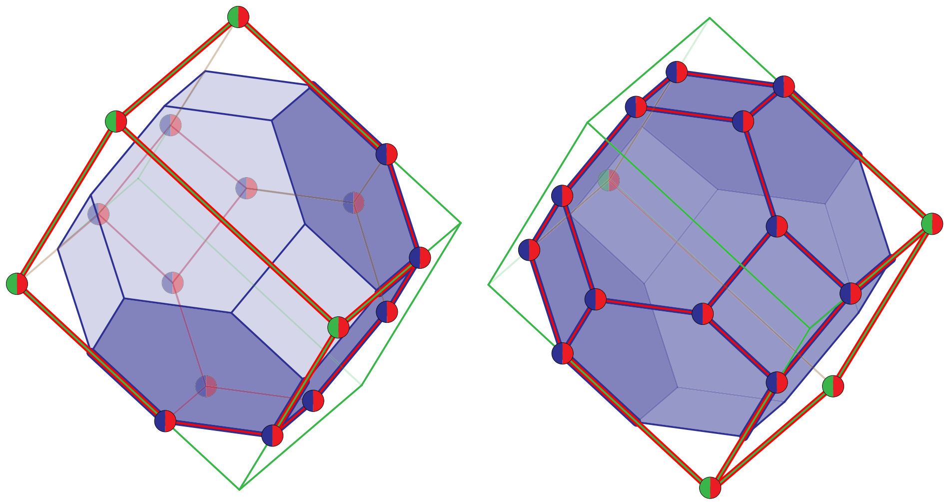
arXiv:1501.07152



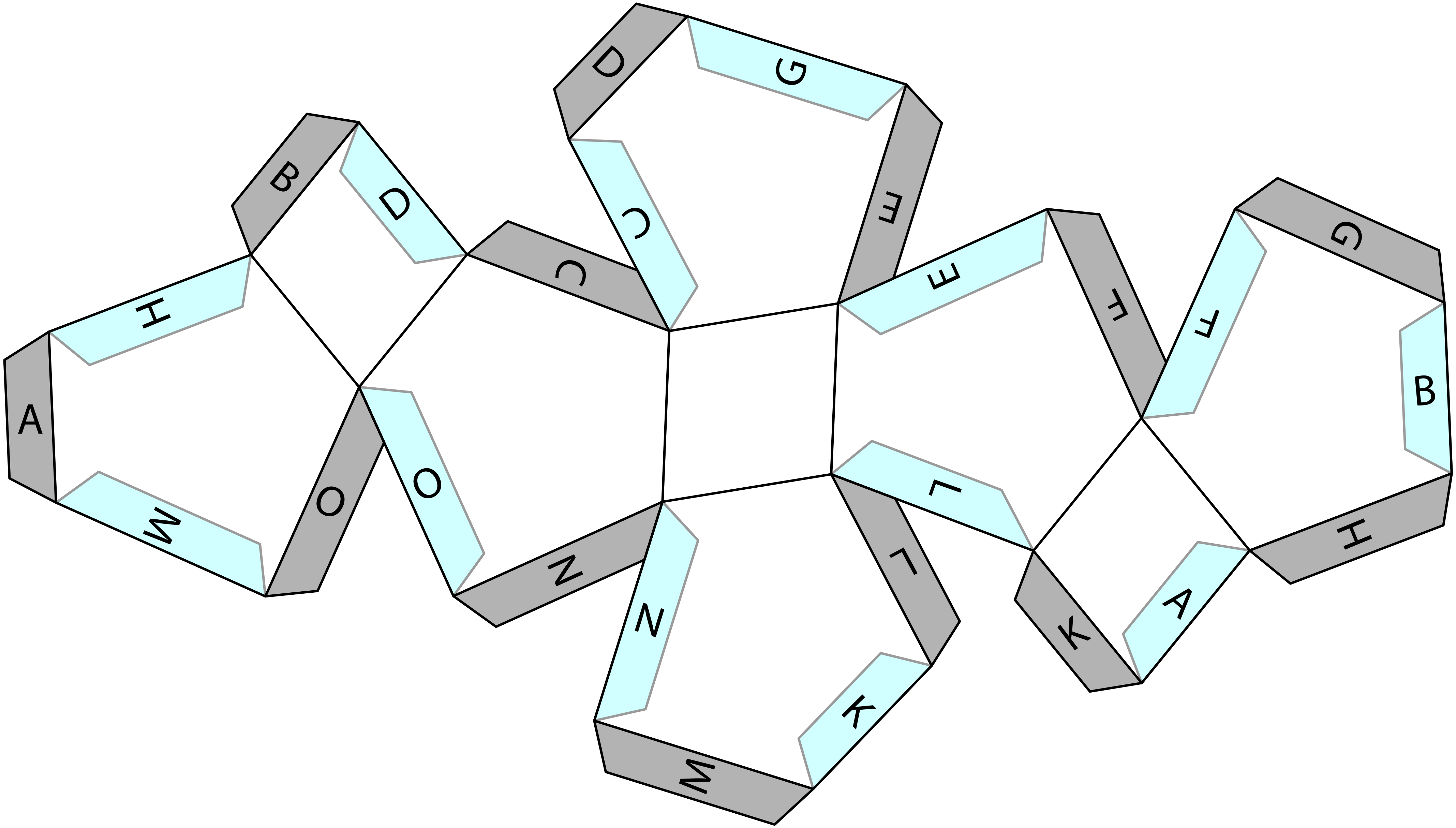
VP

Signed tree associahedra

arXiv:1309.5222



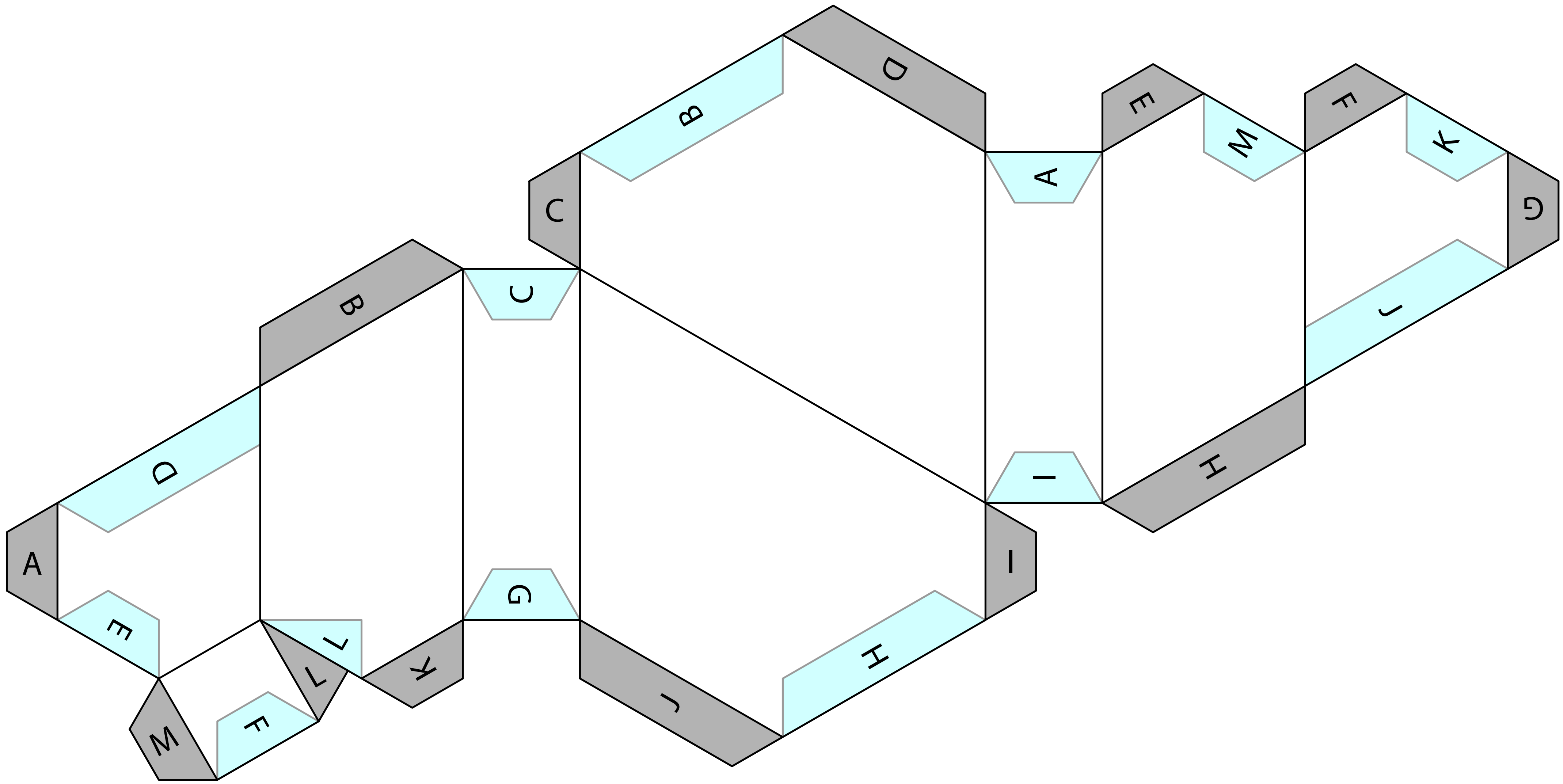
THANK YOU



SECONDARY POLYTOPE

Gelfand-Kapranov-Zelevinsky ('94)

Billera-Filliman-Sturmfels ('90)



LODAY'S ASSOCIAHEDRON

Loday ('04)
Hohlweg-Lange ('07)
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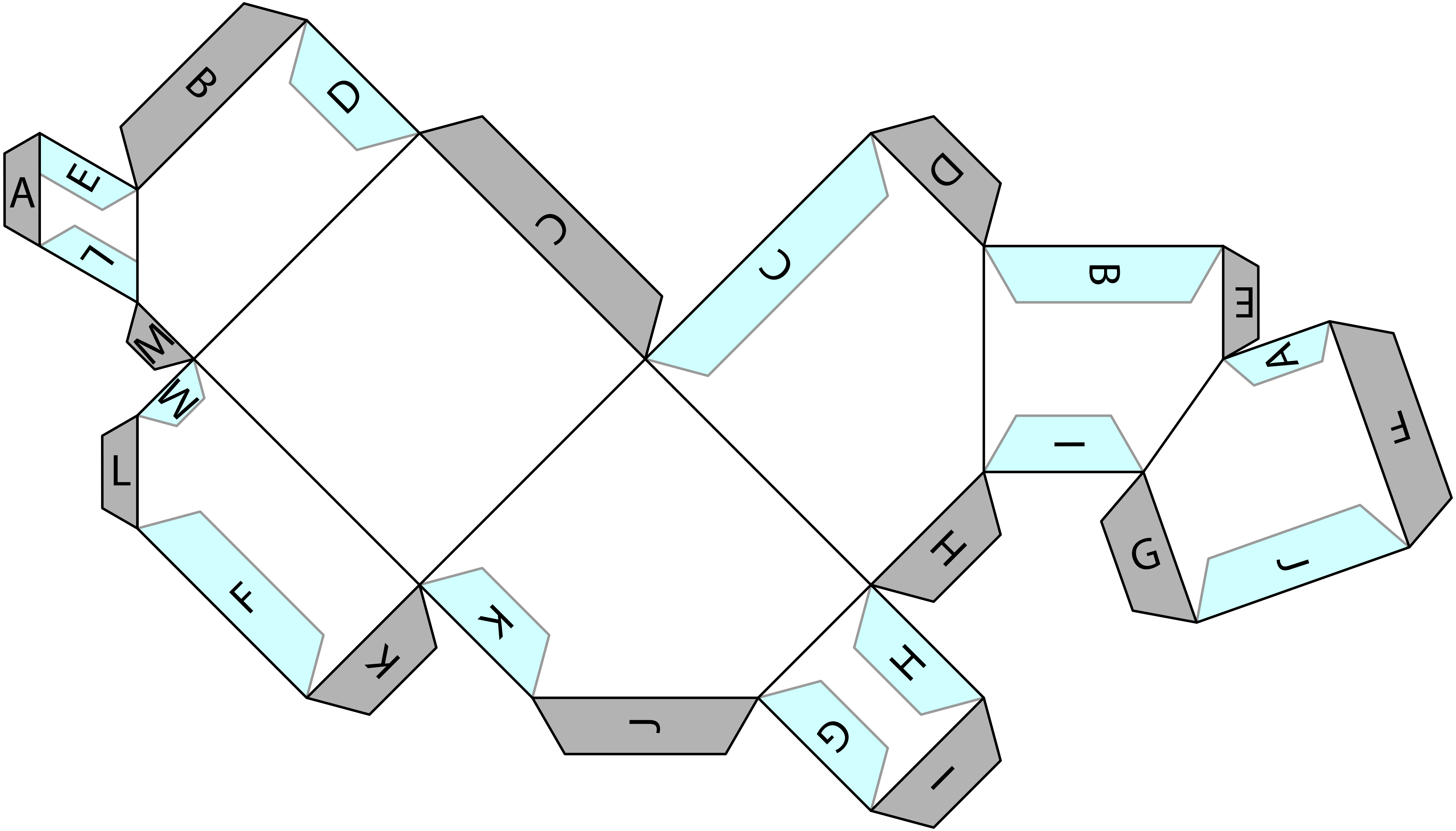

CHAPOTON-FOMIN-ZELEVINSKY'S ASSOCIAHEDRON

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