Autour de la combinatoire des arrangements de Shi dans les groupes de Coxeter

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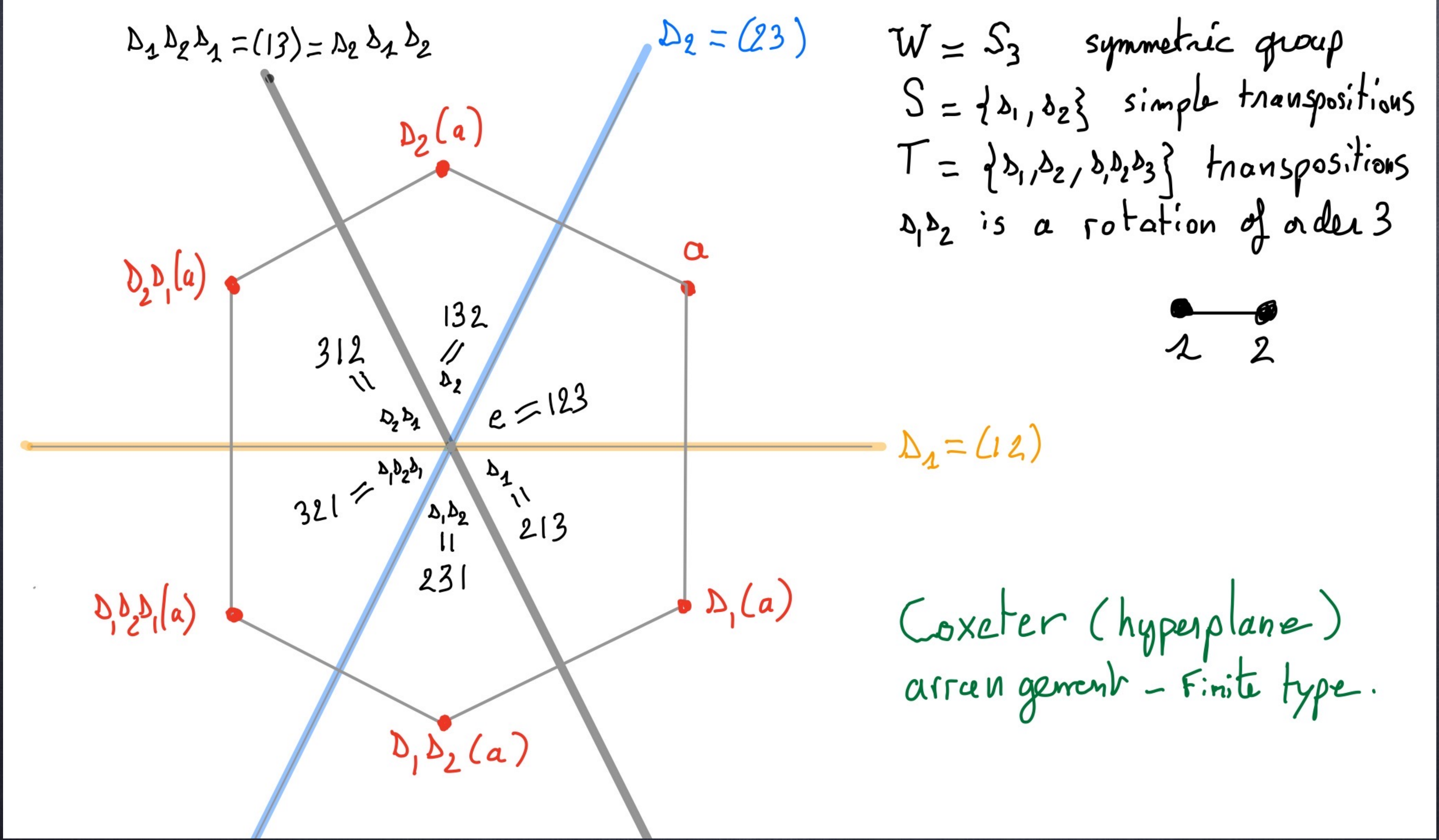
Coxeter groups

Coxeter system (W, S): W group generated by S, the `simple reflections'; $T = \{wsw^{-1} \mid s \in S, w \in W\}$, the `reflections'



$W = S_3$ symmetric group

$S = \{s_1, s_2\}$ simple transpositions

$T = \{s_1, s_2, s_1 s_2 s_3\}$ transpositions

$s_1 s_2$ is a rotation of order 3

Coxeter (hyperplane) arrangement – Finite type.

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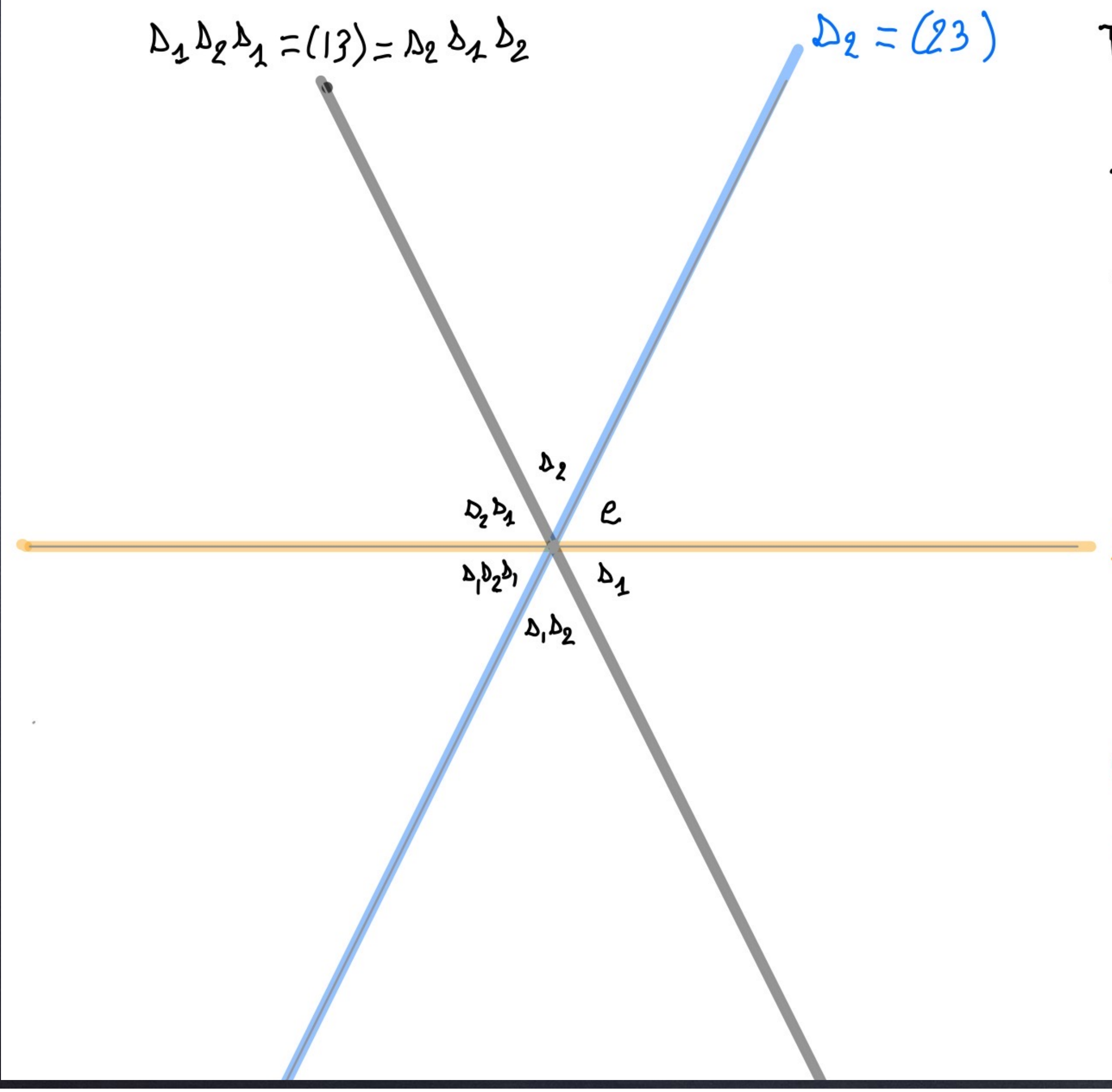

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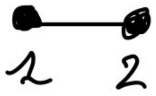

Coxeter (hyperplane)
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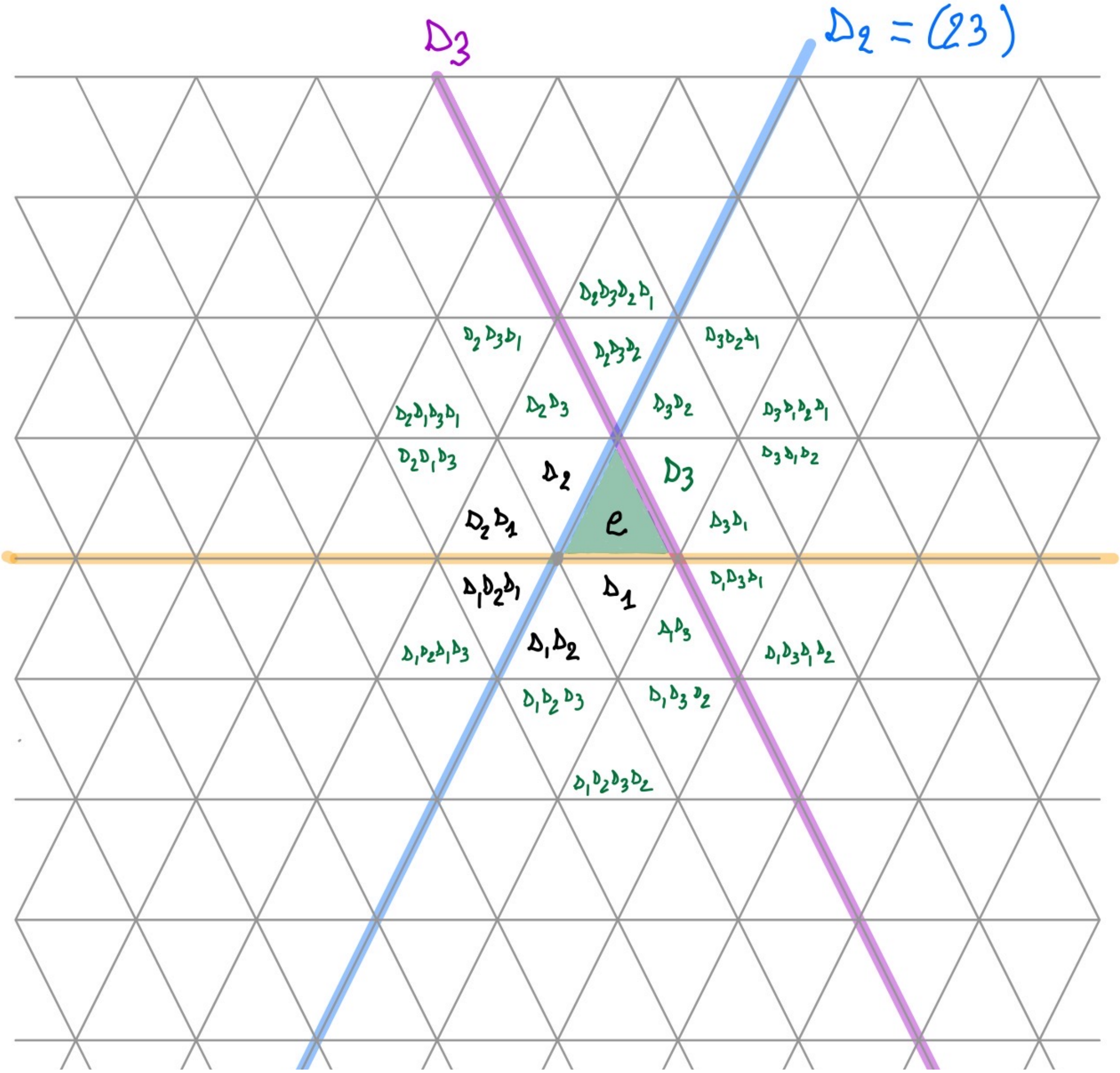
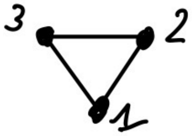


$W = \tilde{S}_3$ AFFINE symmetric group

$S = \{s_1, s_2, s_3\}$

T infinite

s_is_j is a rotation of order 3 $(i \neq j)$



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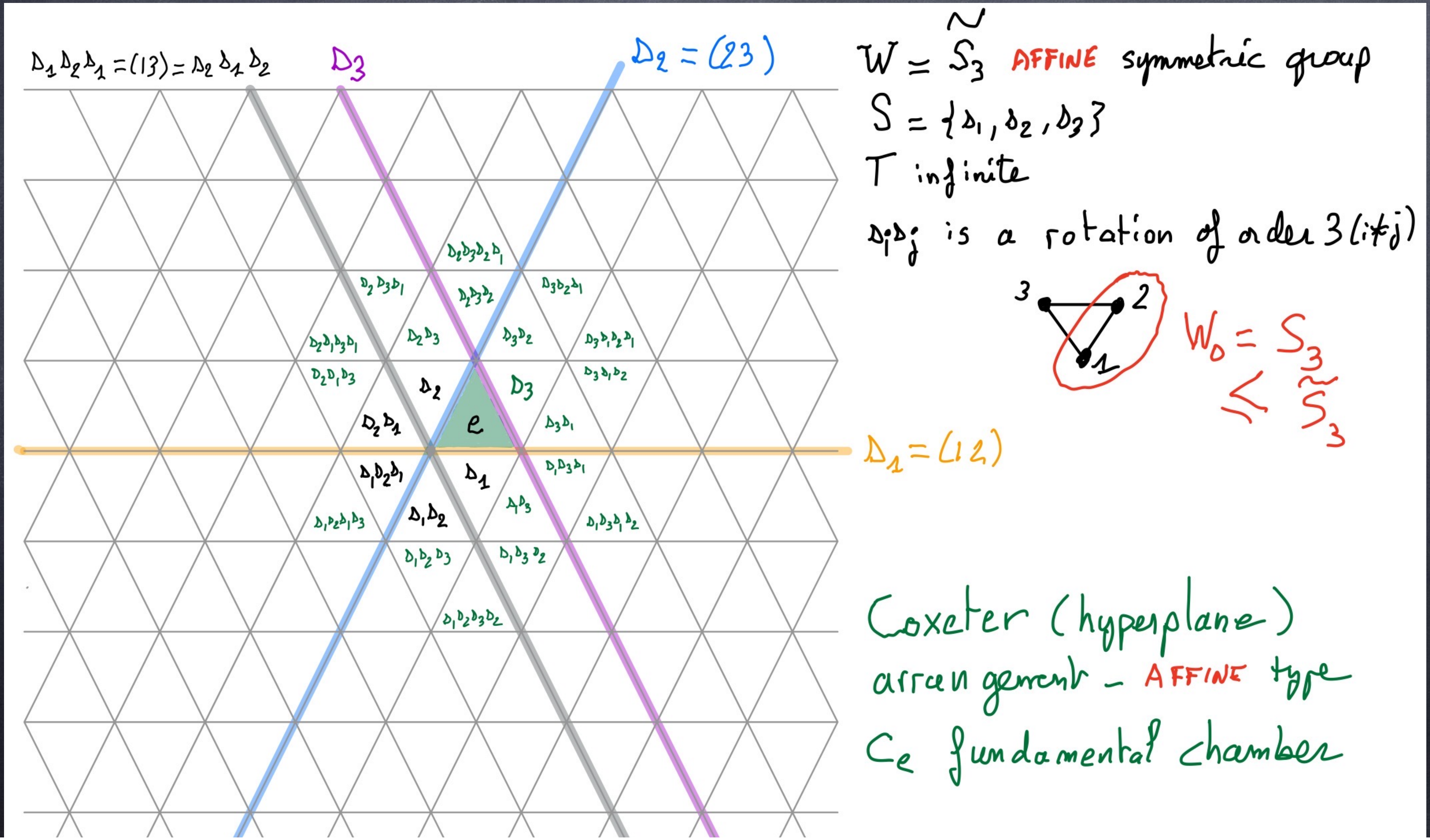

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$W_0 = S_3 \leq \tilde{S}_3$

Coxeter (hyperplane) arrangement - AFFINE type

C_e fundamental chamber

Coxeter groups

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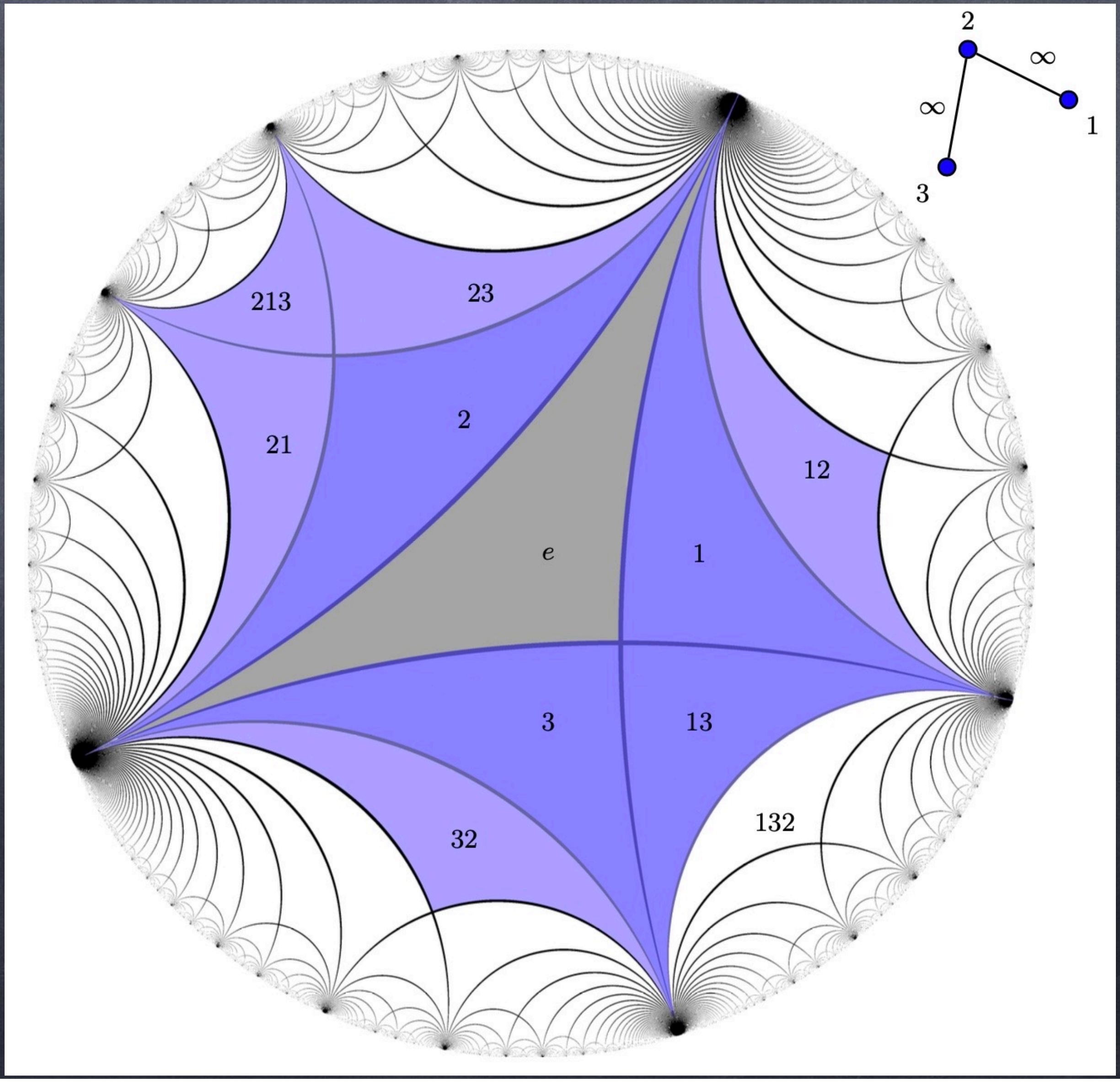
Indefinite Coxeter system (not finite, nor affine)

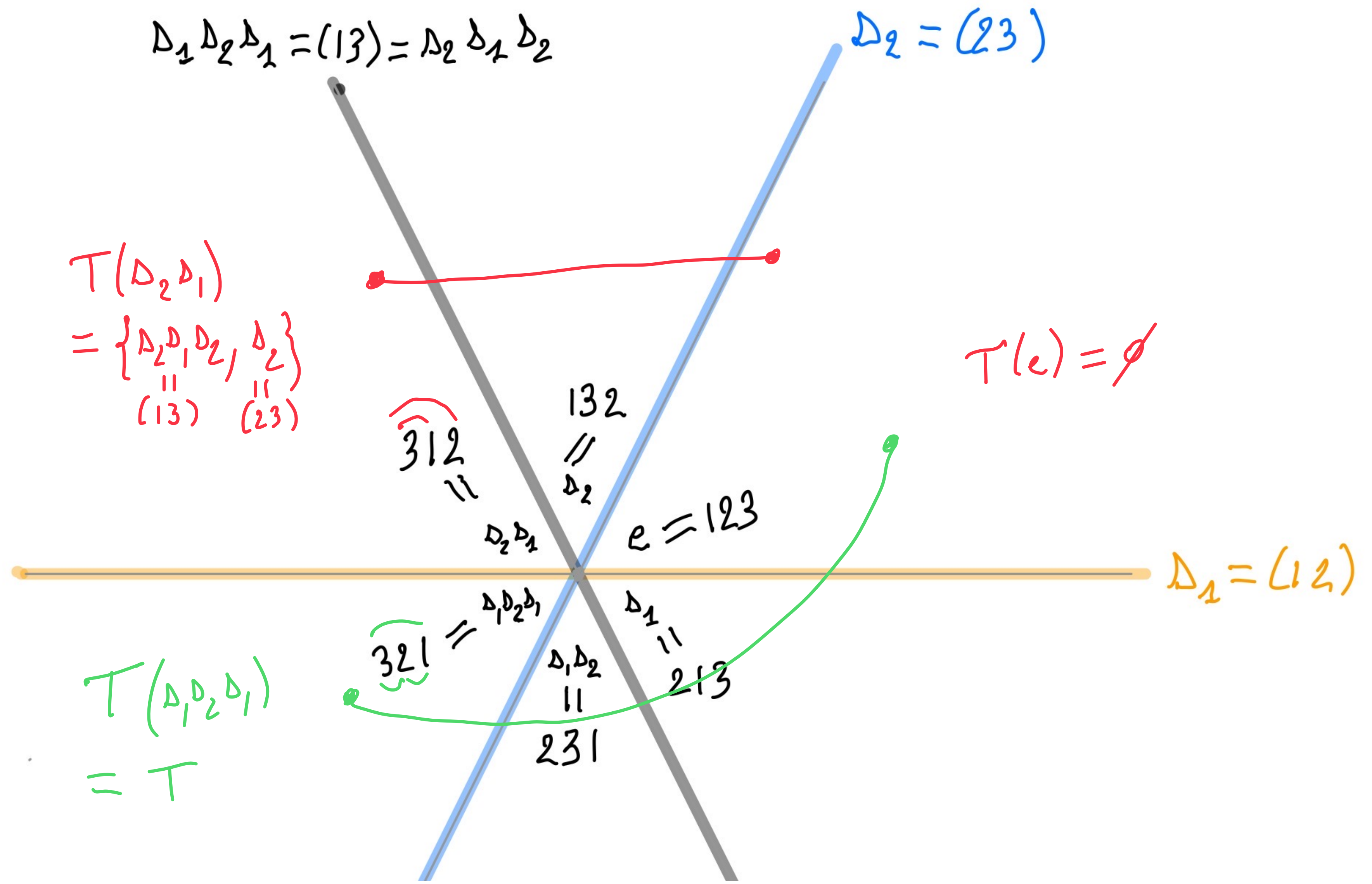
General philosophy: to generalize combinatorial methods of S_n to arbitrary W.



Inversions and descents

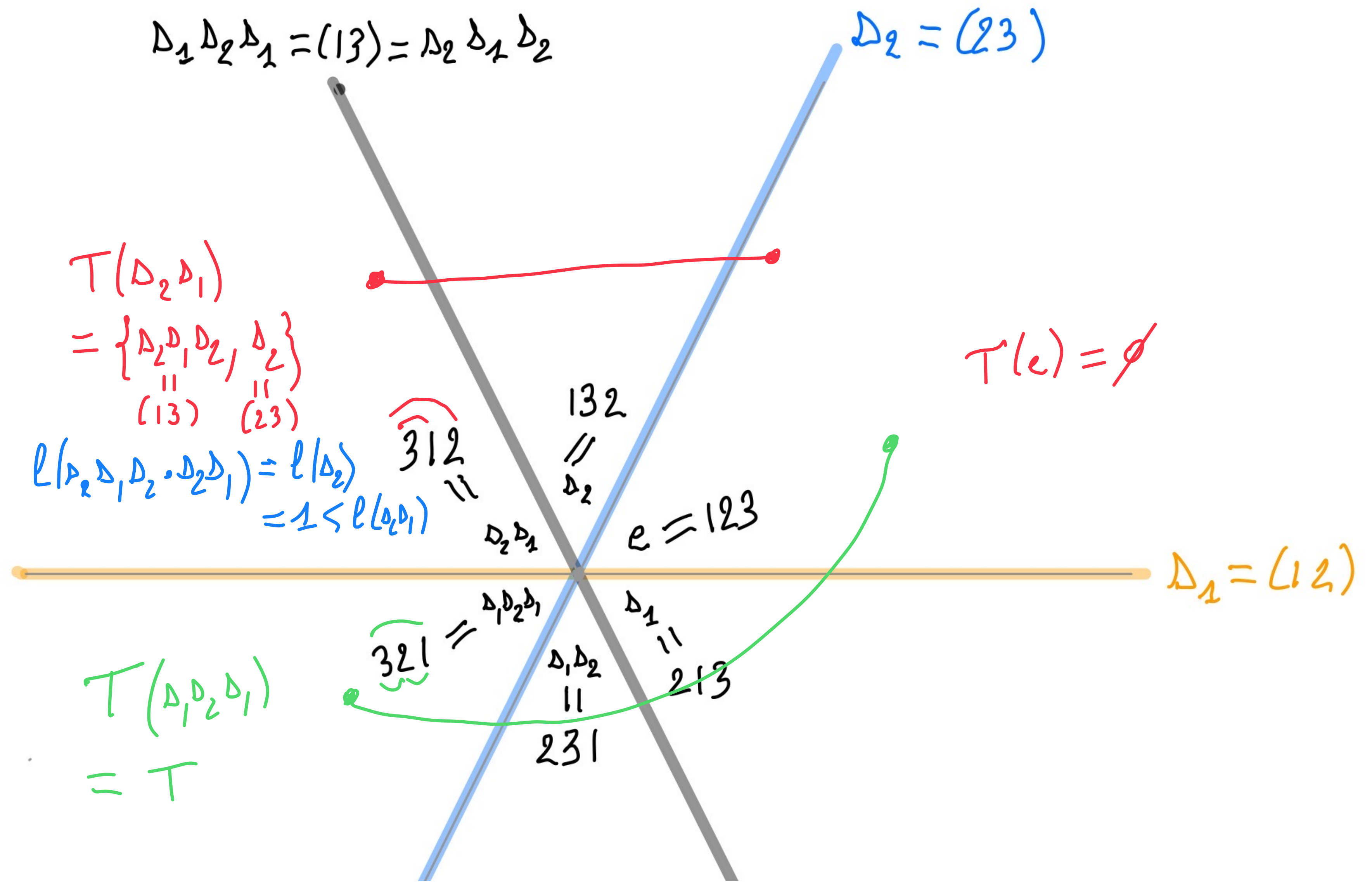
Inversion set of $w \in W$: $T(w) = \{t \in T \mid H_t$ separates w form $e\}$

Length of $w \in W$: $\ell(w) = |T(w)|$ = length of a reduced word



Inversions and descents

Proposition. $T(w) = \{t \in T \mid \ell(tw) < \ell(w)\}$



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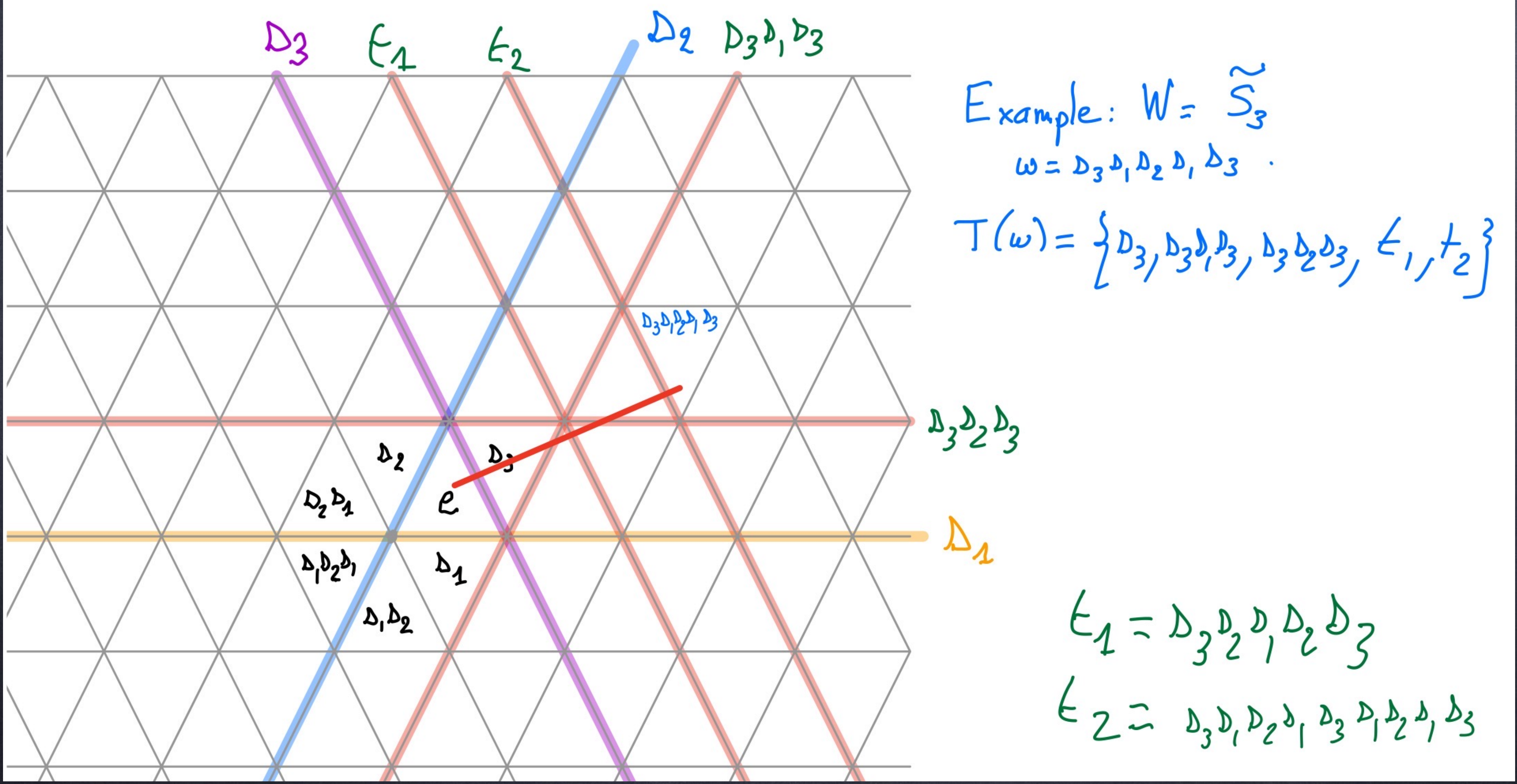

Example: $W = \tilde{S}_3$

$w = s_3 s_1 s_2 s_1 s_3$.

$T(w) = \{s_3, s_3s_1s_3, s_3s_2s_3, t_1, t_2\}$

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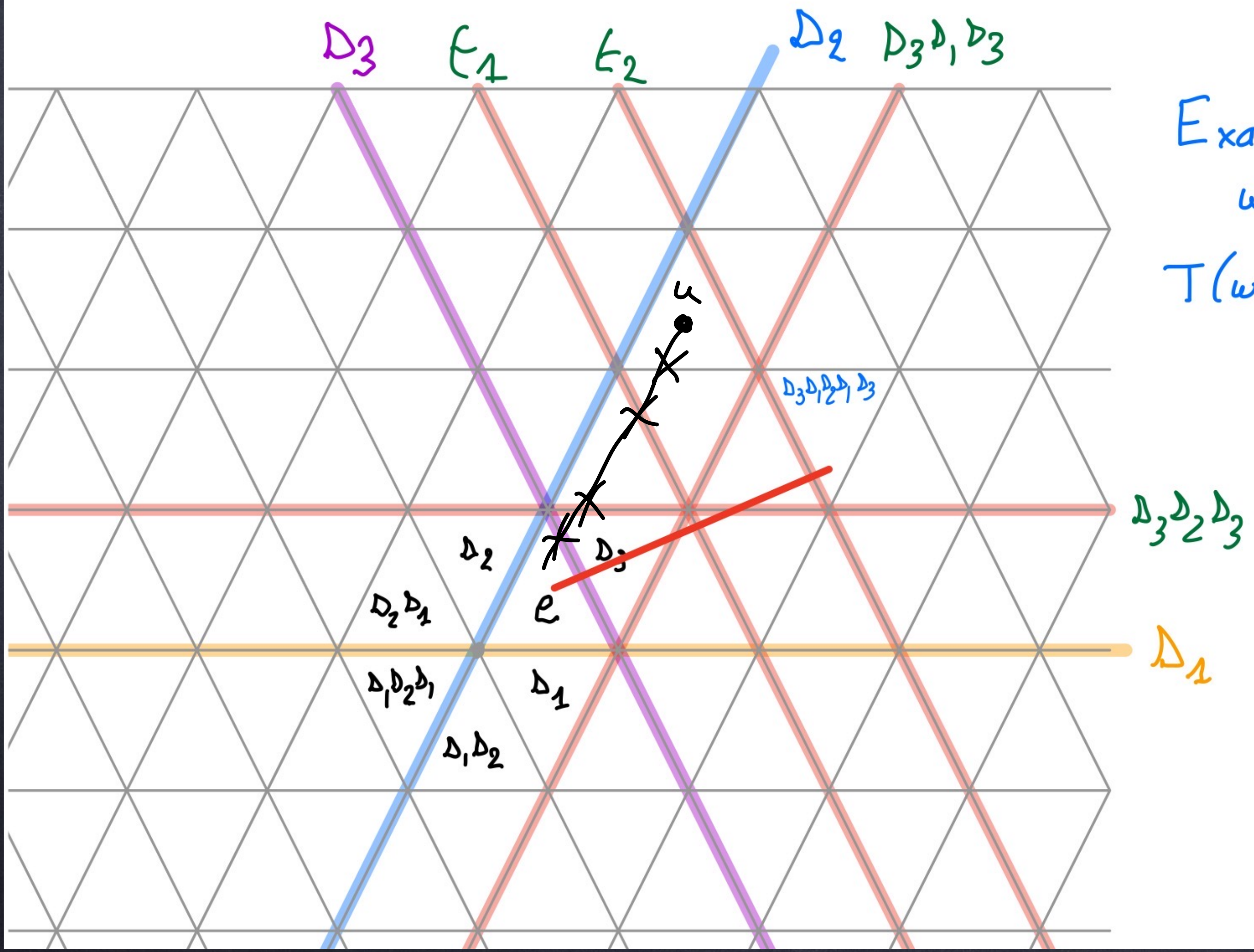

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$T(w) = \{s_3, \boxed{s_3 s_1 s_3}, s_3 s_2 s_3, t_1, t_2\}$

$u = s_3 s_1 s_3 \, w$

$\ell(u) = 4 < 5 = \ell(w)$

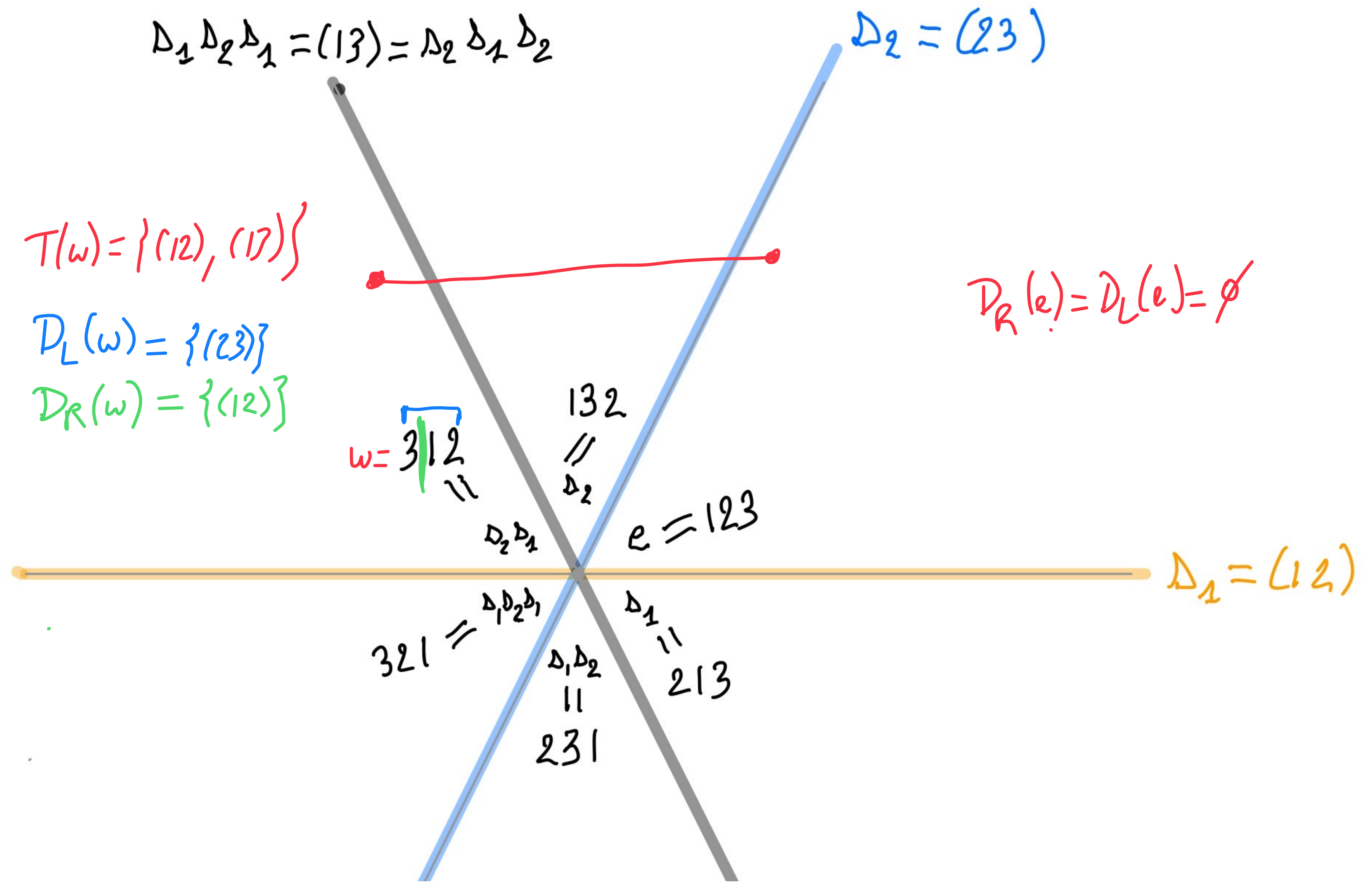
$t_1 = s_3 s_2 s_1 s_2 s_3$

$t_2 = s_3 s_1 s_2 s_1 s_3 s_1 s_2 s_1 s_3$

Inversions and descents

Let $w \in W$: $D_L(w) = \{s \in S \mid \ell(sw) < \ell(w)\}$ (left descents)

$D_R(w) = \{s \in S \mid \ell(ws) < \ell(w)\}$ (right descents)



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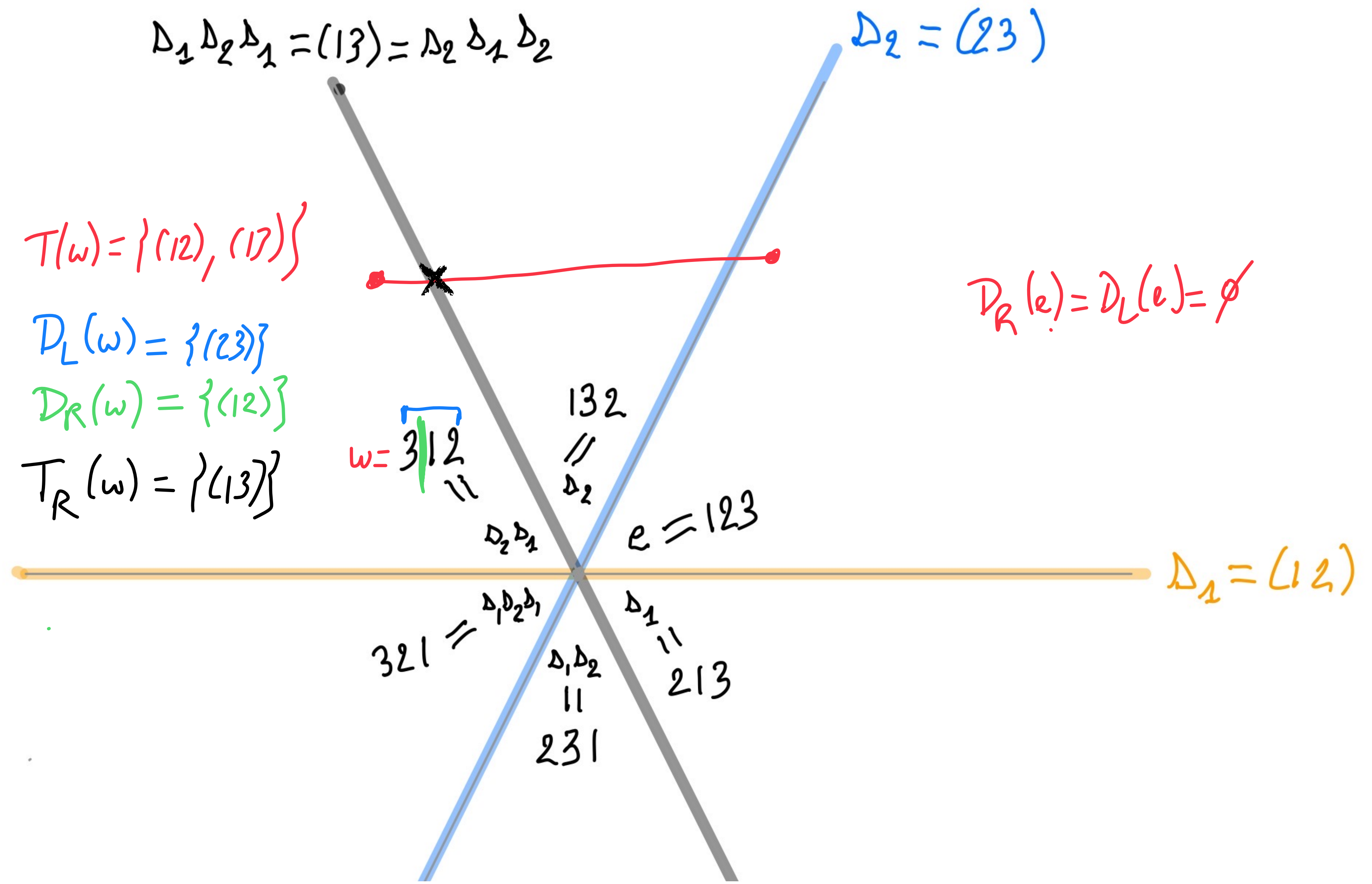

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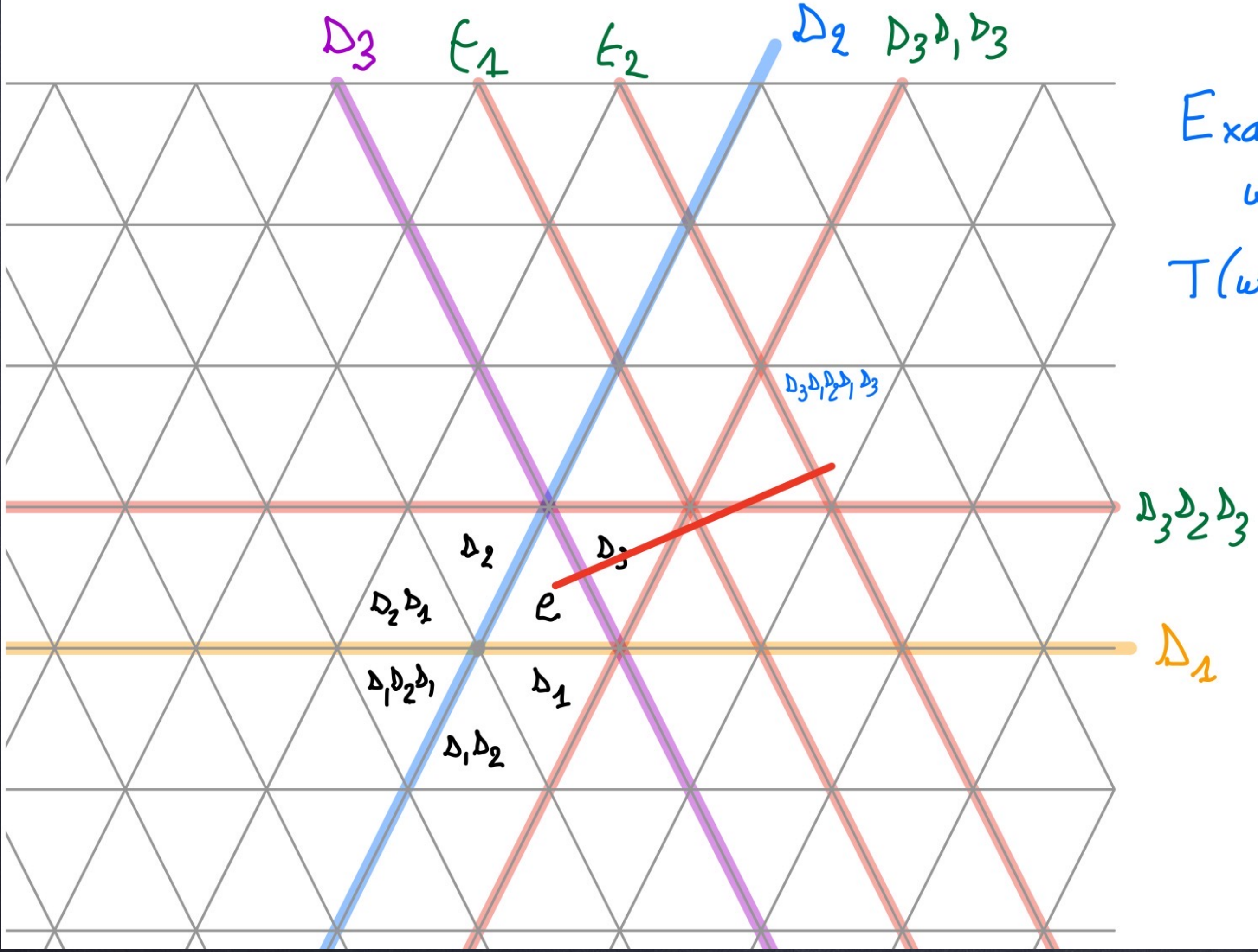

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$D_L(w) = \{s_3\} = D_R(w)$

$t_1 = s_3s_2s_1s_2s_3$

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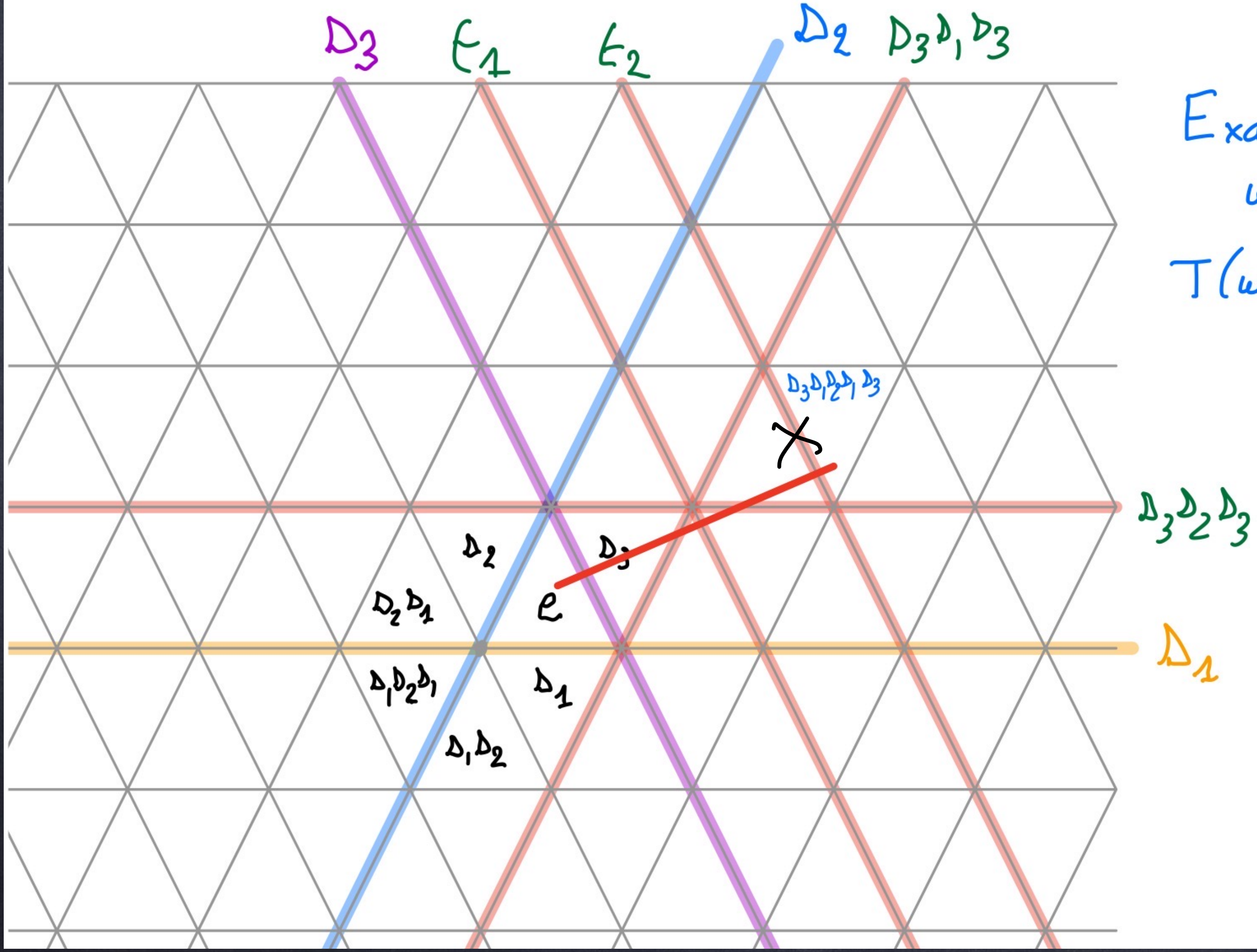

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$D_L(w) = \{s_3\} = D_R(w)$

$T_R(w) = \{t_2\}$

$t_1 = s_3s_2s_1s_2s_3$

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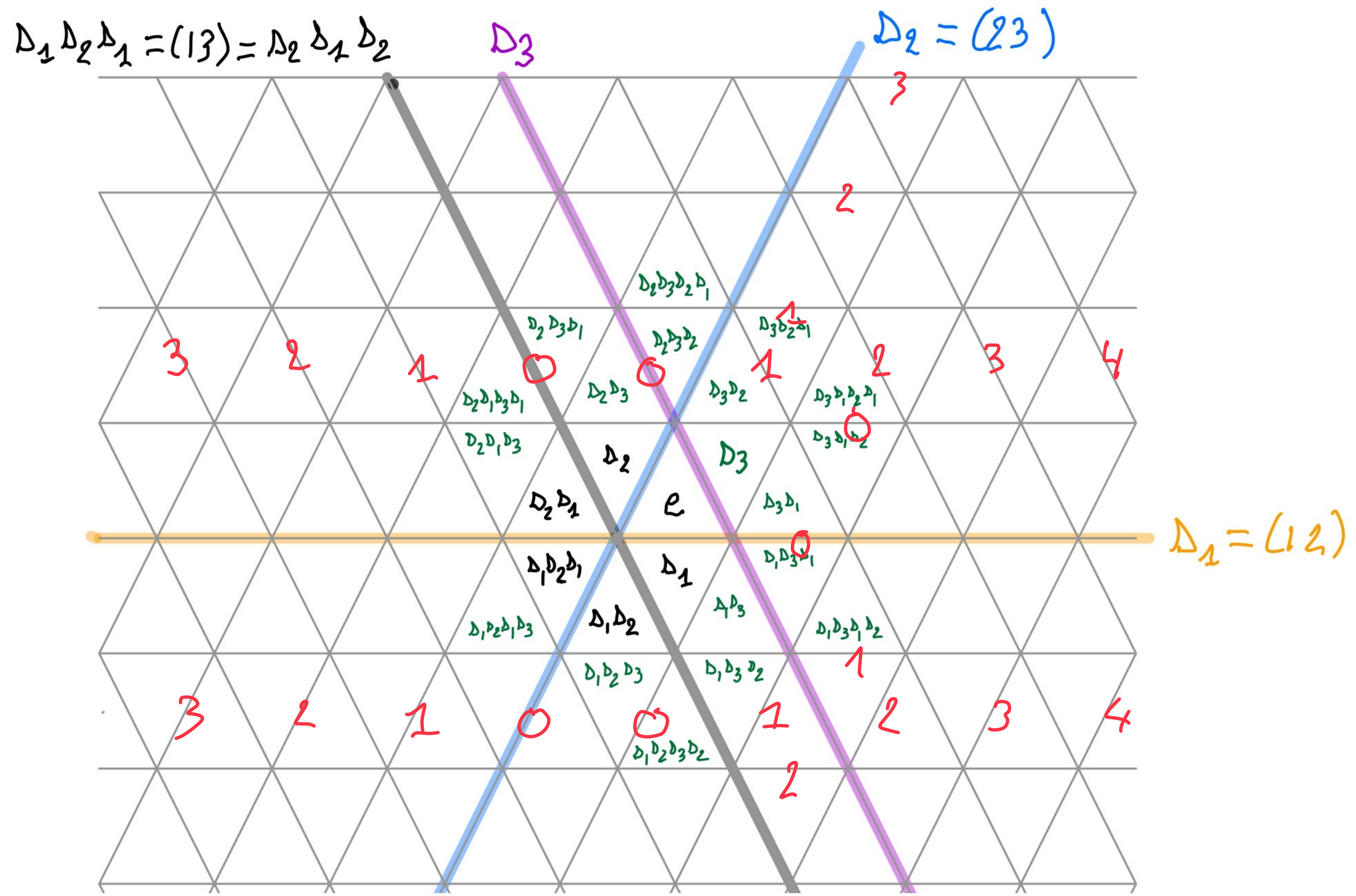
Shi arrangements

Infinite-depth of reflections (Brink-Howlett 1993, Fu 2012):

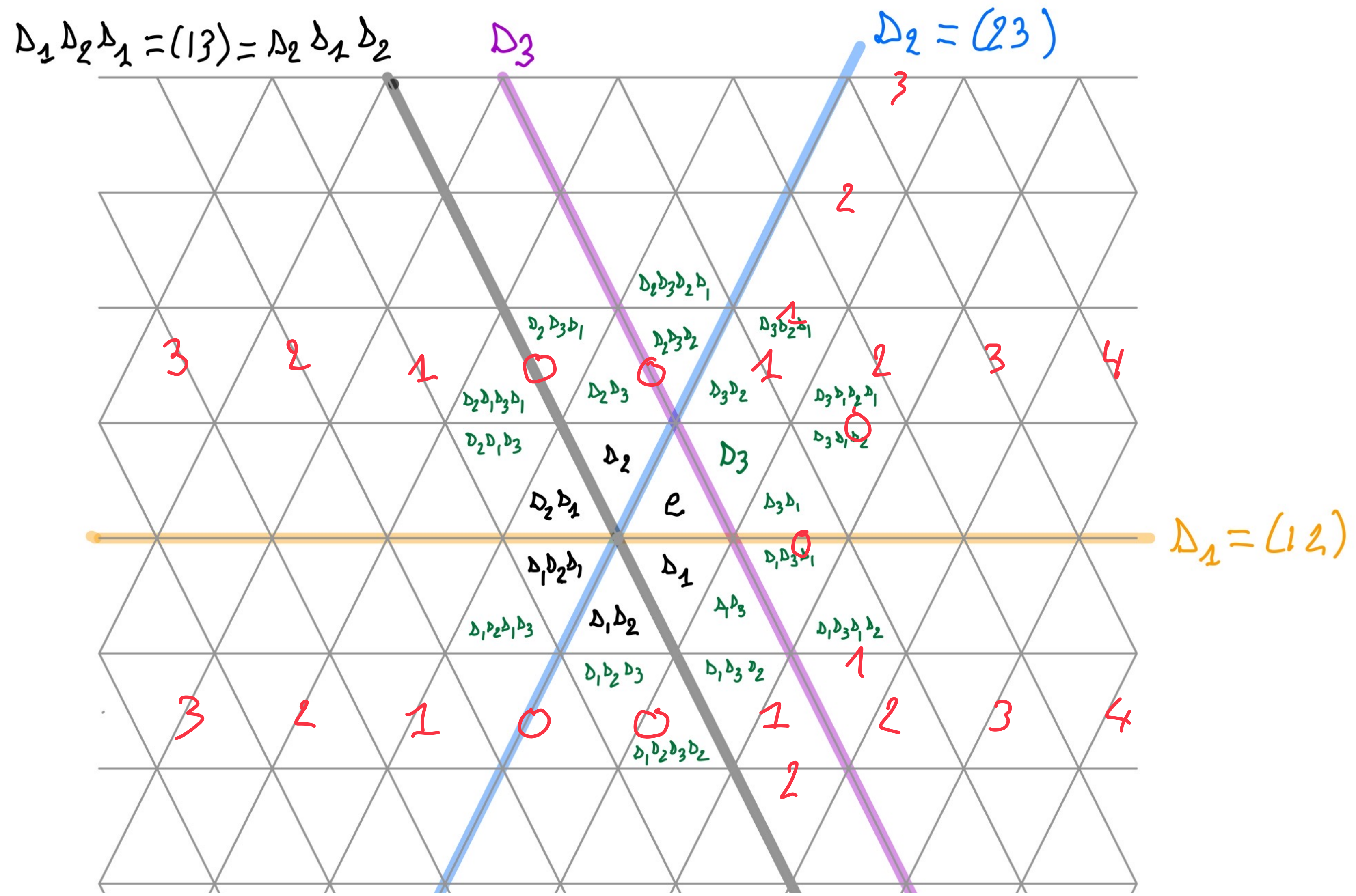
$\mathrm{dp}_{\infty}(t)$ = # distinct parallels H_r to H_t that separates H_t from e.



Shi arrangements

m−small reflections ($m \in \mathbb{N}$): $\Sigma_m = \{t \in T \mid \mathrm{dp}_\infty(t) \le m\}$.

Theorem (Brink-Howlett 1993, Fu 2012) Σ_m is a finite set.



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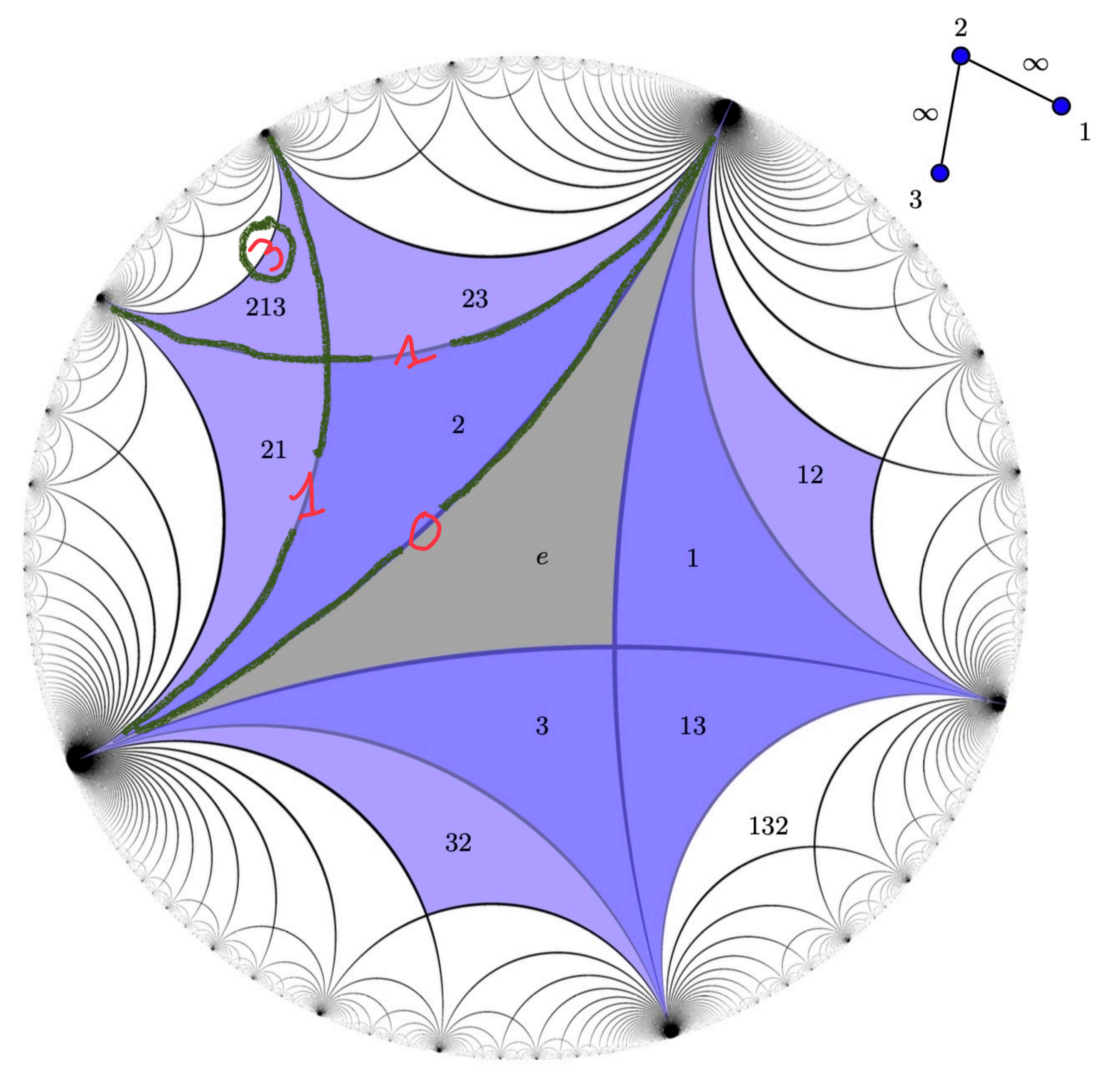
Finite Coxeter groups

$\Sigma_m = \Sigma_0 = \mathscr{A}$ finite for all $m \in \mathbb{N}$

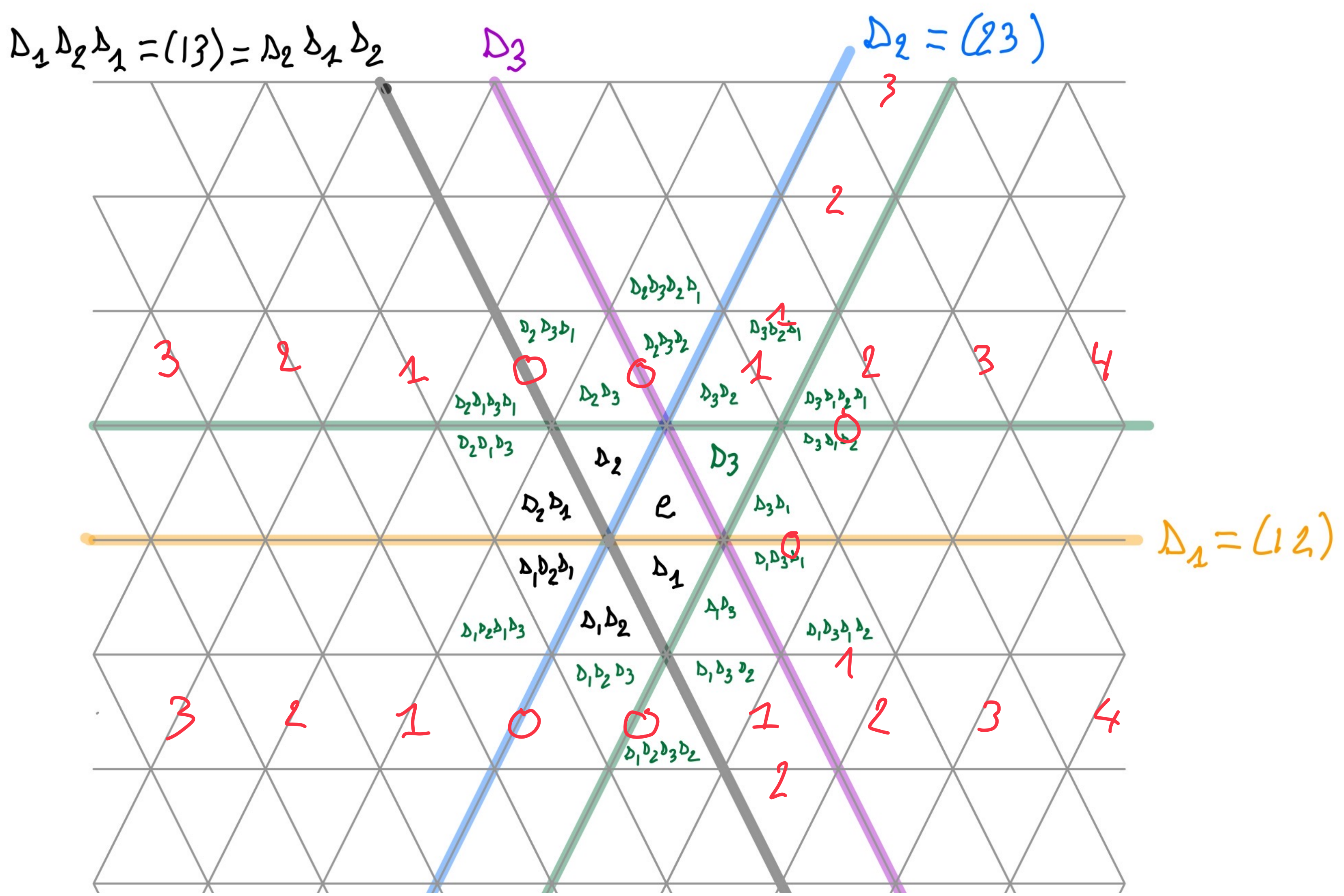
Affine Coxeter groups: easy (transitivity parallelism)

Indefinite (hyperbolic etc.): work to do (combinatorics of Coxeter groups)!



Shi arrangements: Shi_0

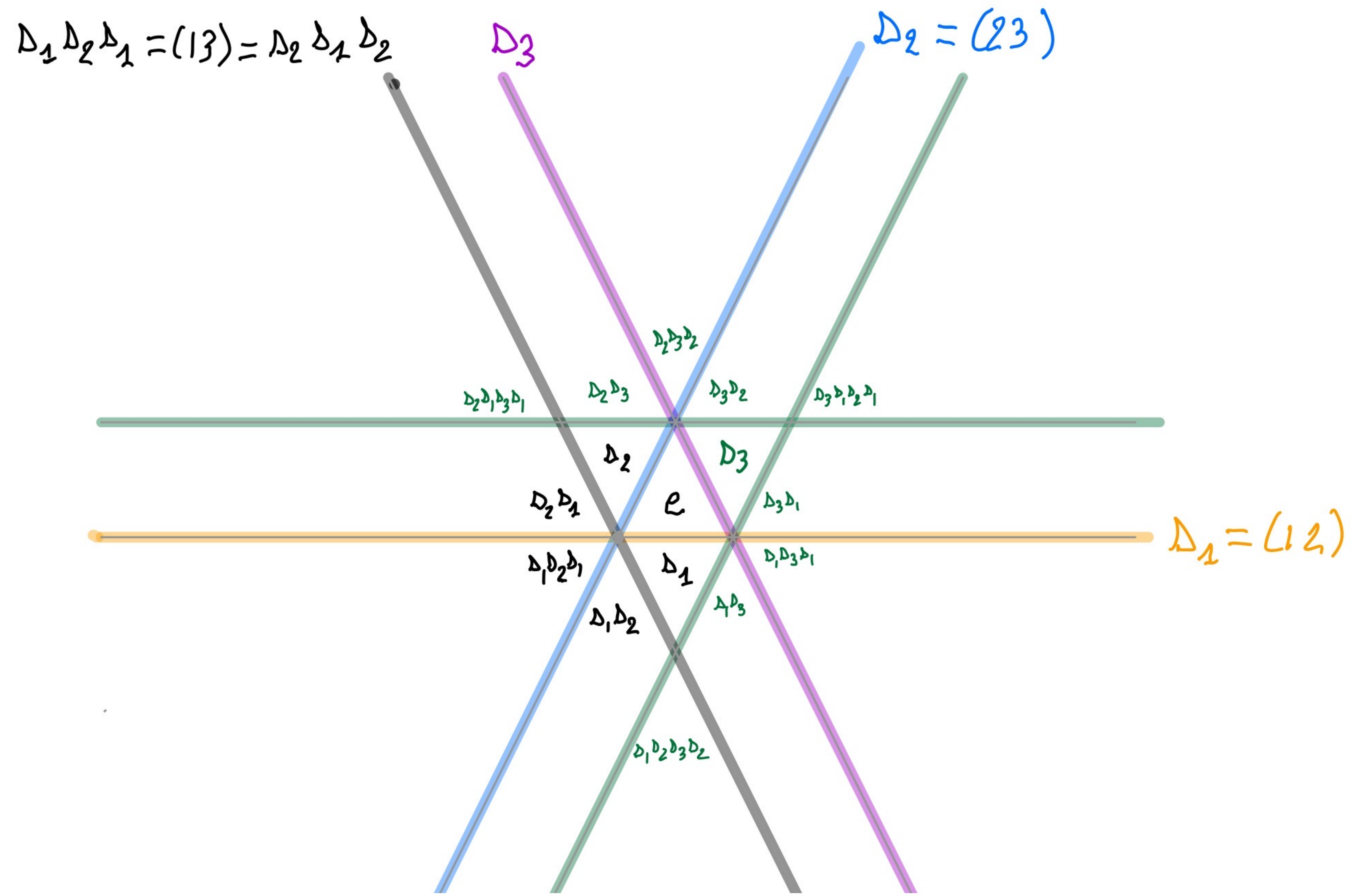
Example: $\mathrm{Shi}_0 = \{H_t \mid t \in \Sigma_0\}$ (Shi 1988 in affine types).



Shi arrangements: Shi_0

Theorem (Shi 88) Enumeration of the number of regions in affine type.

Example: For $\tilde{S}_n$, there are $(n+1)^{n-1}$ regions.



Shi arrangements: Shi_0

Theorem (Shi 88) Enumeration of admissible signs in affine type.

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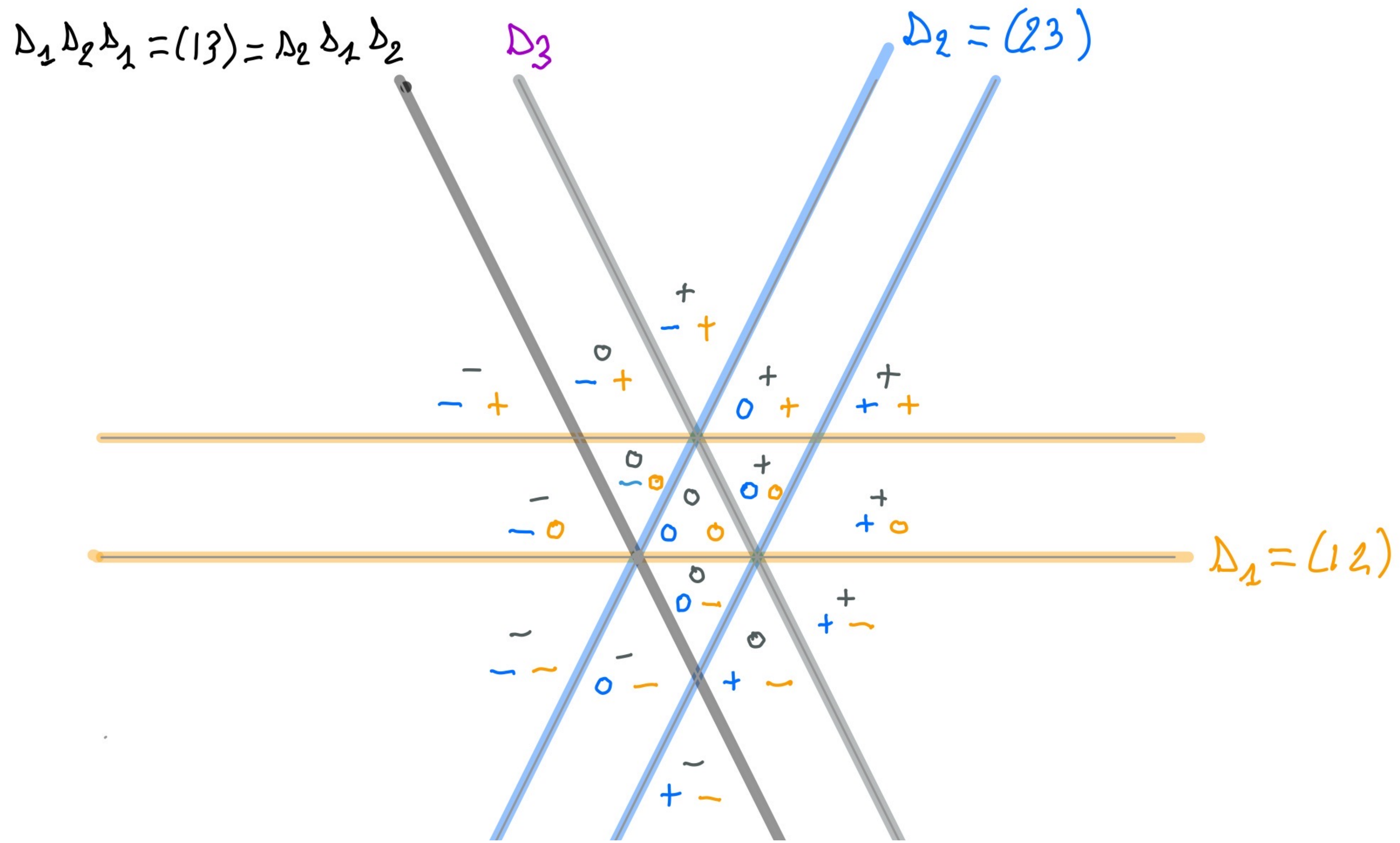
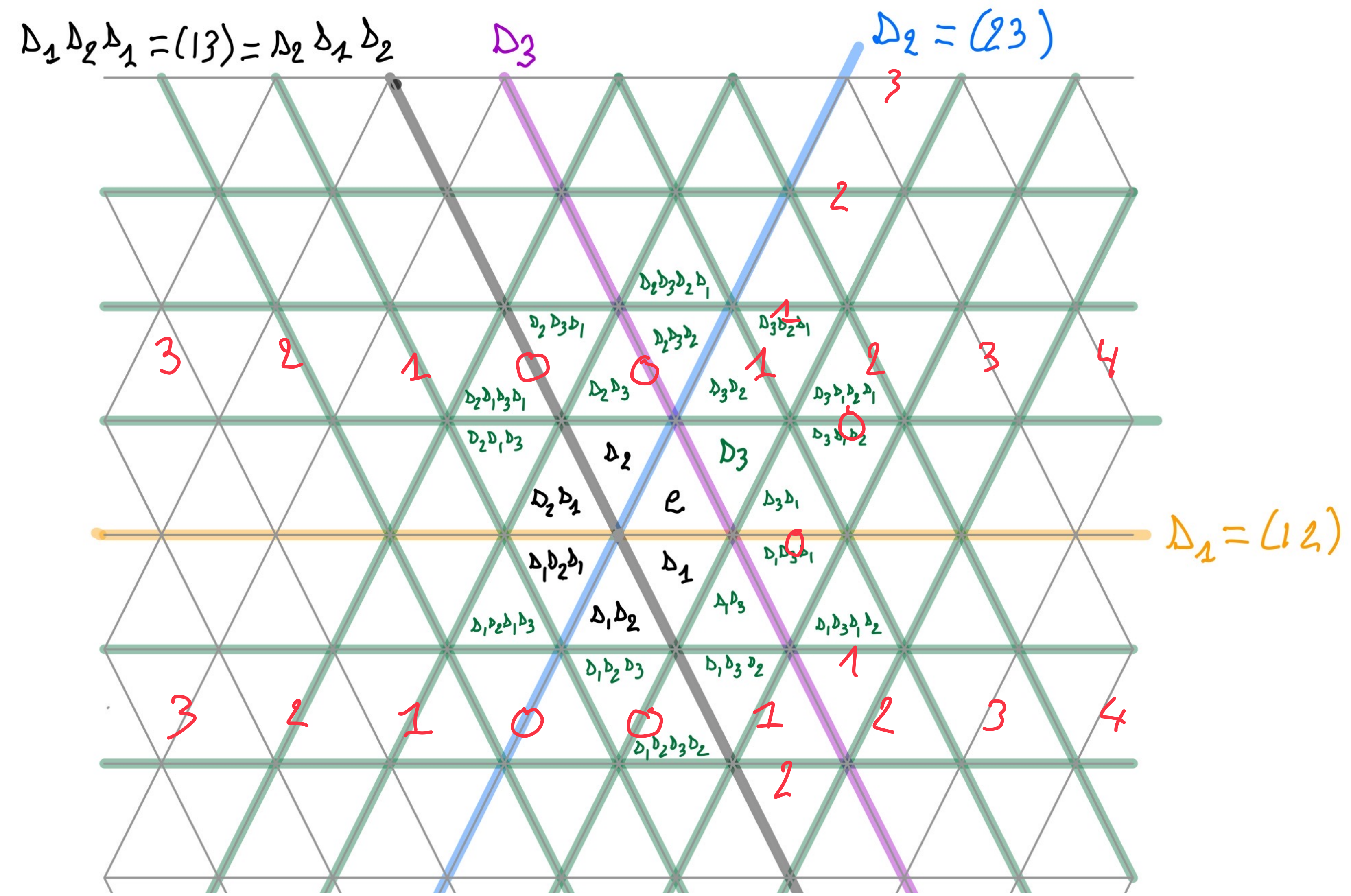


Shi arrangements: Shi$_m$

$m-$Shi arrangements ($m \in \mathbb{N}$): Shi$_m = \{H_t \mid t \in \Sigma_m\}$.

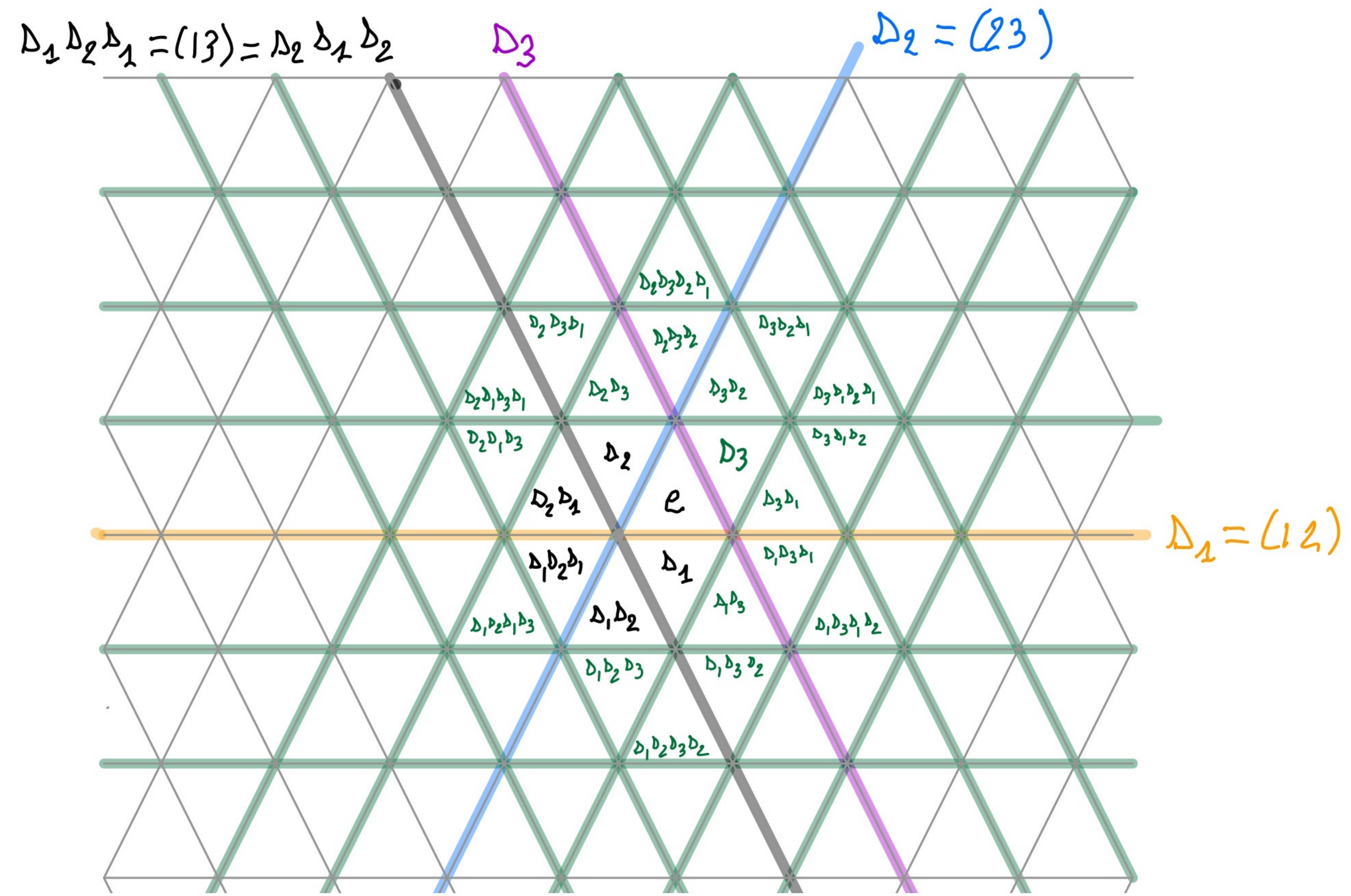
Example: Shi$_2 = \{H_t \mid t \in \Sigma_2\}$



Shi arrangements: Shi_m

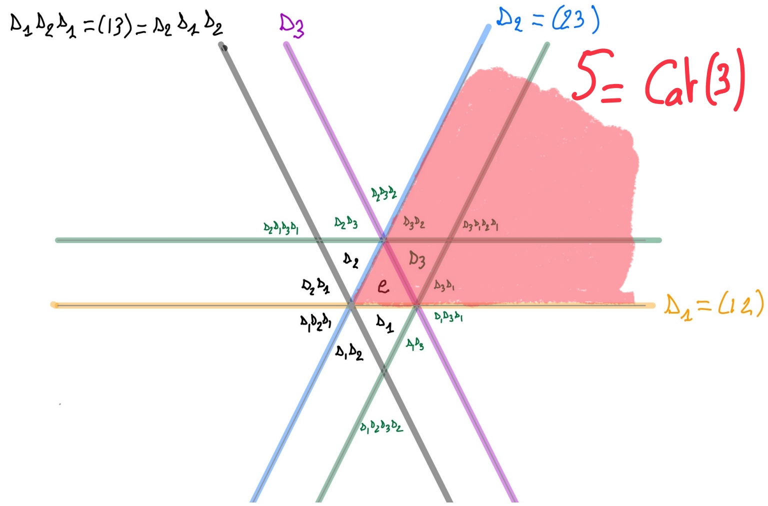
Theorem (Yosiniga 04, Thiel 16) Enumeration in affine type.

Example: For $\tilde{S}_n$, there are $((m+1)n+1)^{n-1}$ regions.



Shi arrangements

- Catalan combinatorics (Catalan numbers in affine types)
- Kazhdan-Lusztig cells (Affine types, Shi 86)
- Provides a finite Garside family in the Artin-Tits monoïd, a step forward answering the word problem in Artin (braid) groups (Dehornoy-Dyer-CH 15)
- Automata recognizing the language of reduced words (Eriksson, Headley, Brink-Howlett 90')
- Provides the basic tools to prove automaticity (Brink-Howlett 94) and bi-automaticity (Osajda-Przyticky, 22) of Coxeter systems

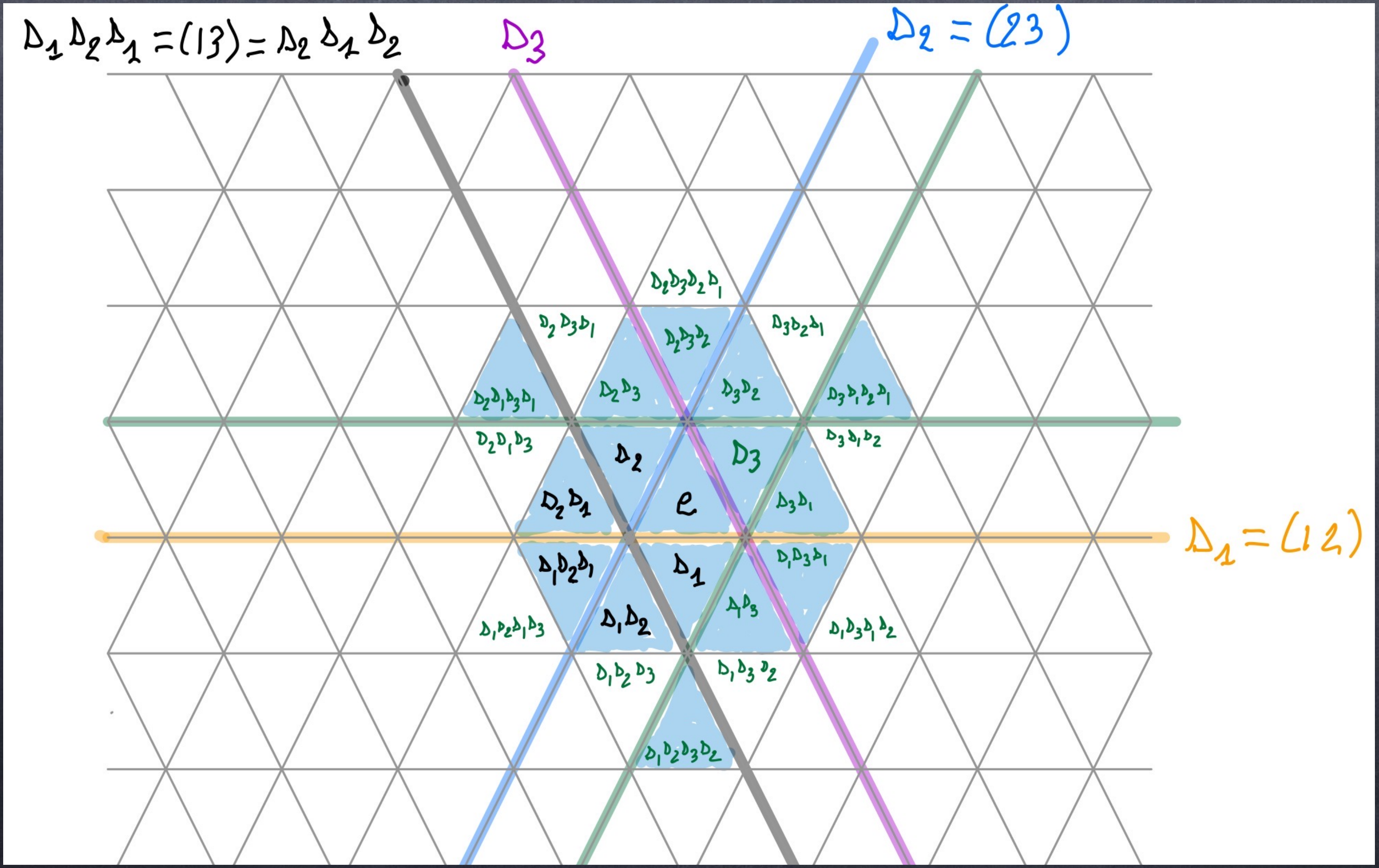


Shi arrangements

affine types: enumeration

Shi's method (88) extended by Thiel for all $m \in \mathbb{N}$ (2015):

1. Shi$_m$ is gated, i.e., each region contains a unique minimal length element. Denote by $\mathcal{G}_m$ the set of gates.



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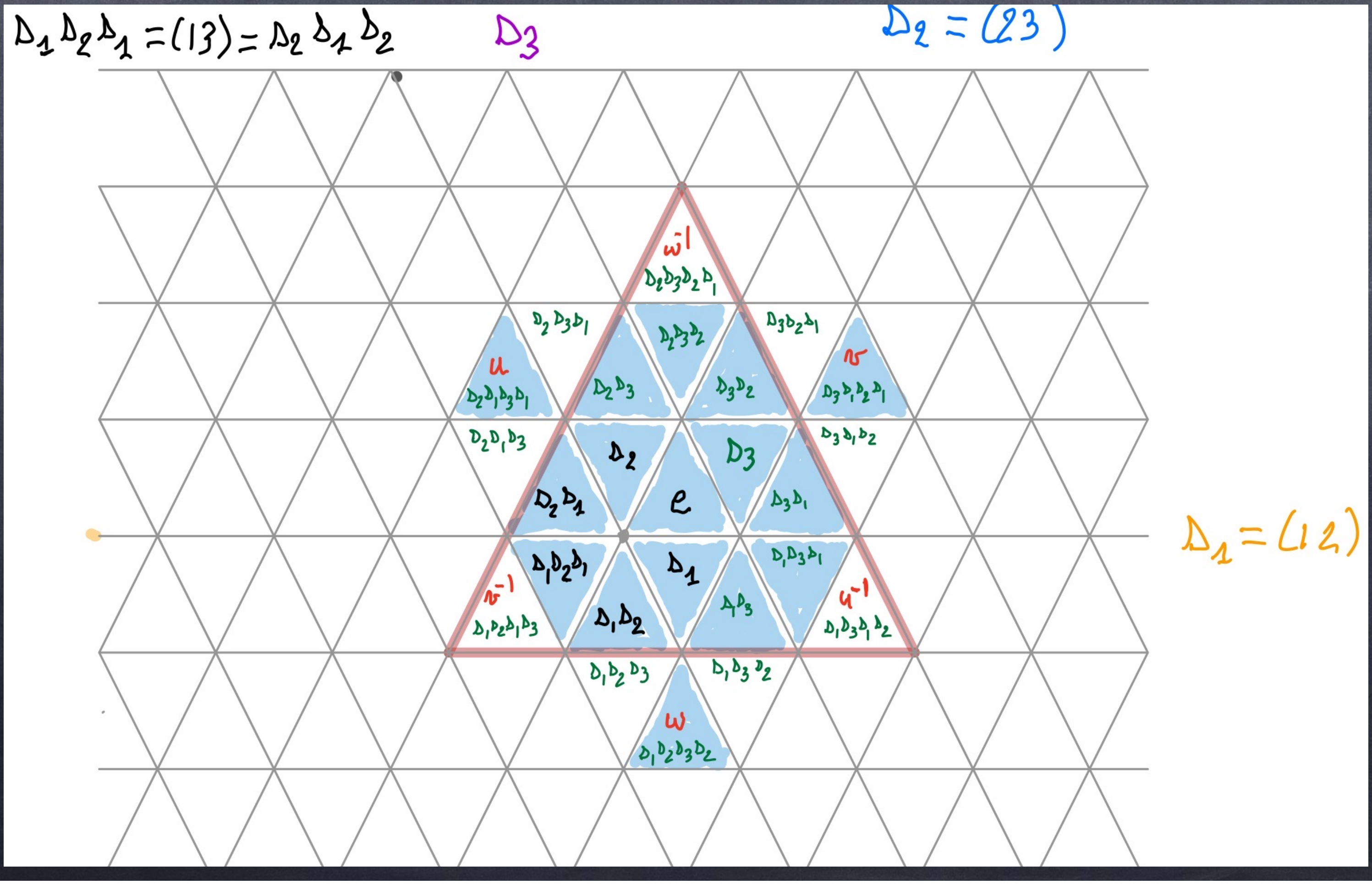
2. Shi$_m$ has the convex property: $P = \bigcup_{w \in \mathcal{G}_m} w^{-1}(C_e)$ is convex



Shi arrangements

affine types: enumeration

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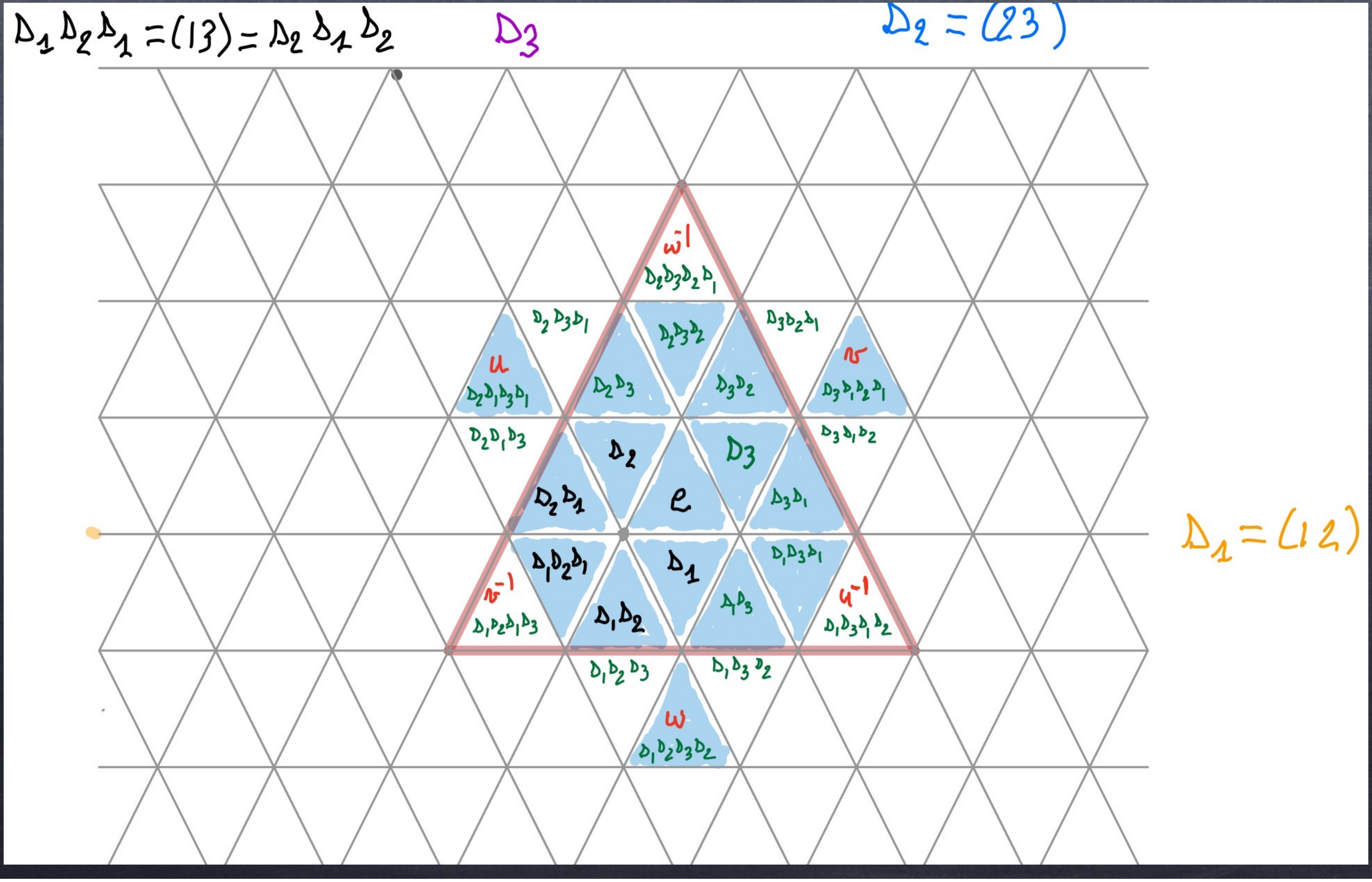
3. P is a dilatation of (factor $((m+1)n+1)$) in $\tilde{S}_n$) of C_e

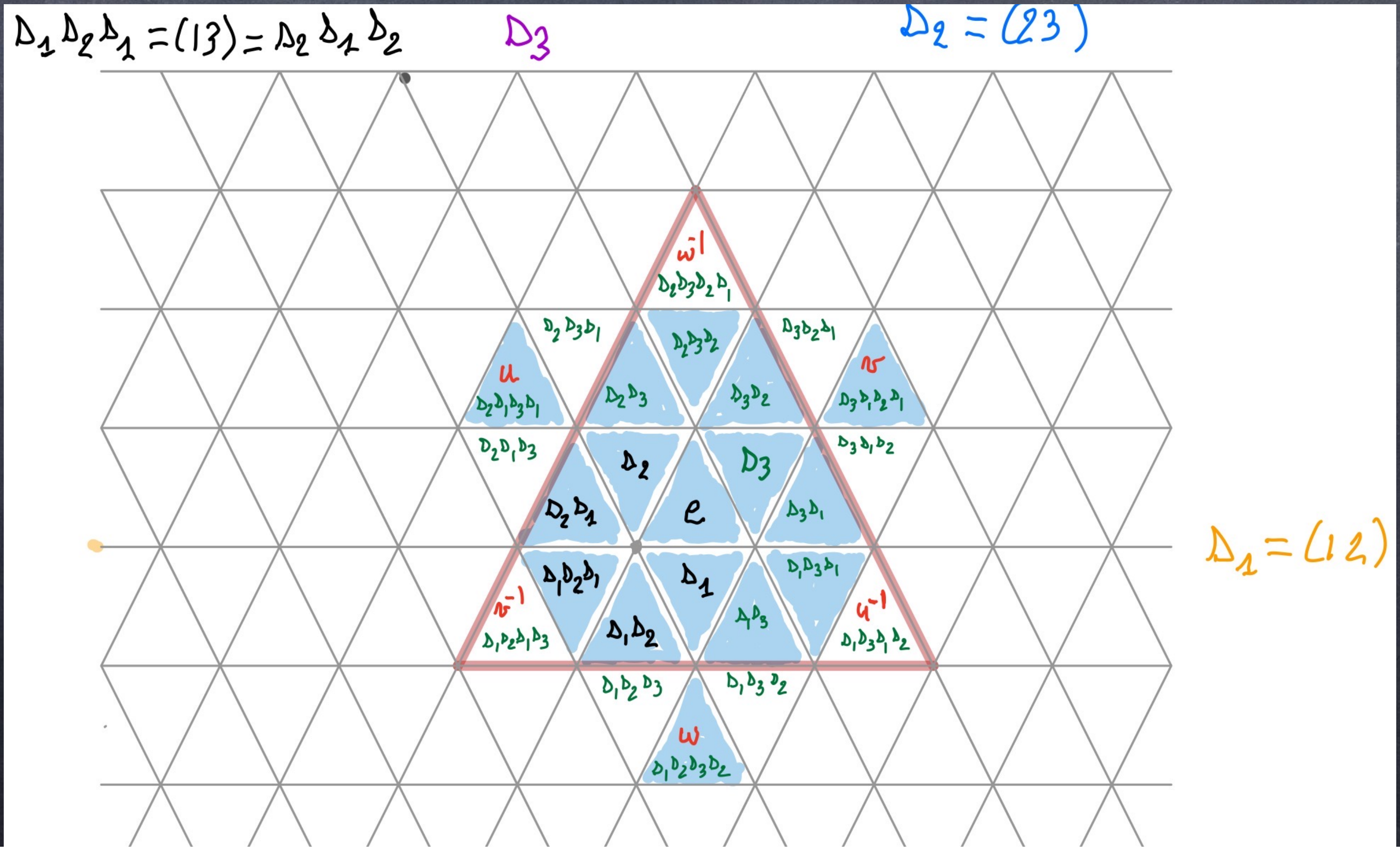


Shi arrangements

affine types: enumeration

Aim: generalize these results to all Coxeter systems

1. **Minimal elements** (conjectured by Dyer-CH 16)
2. **Convexity property** (conjectured by CH-Nadeau-Williams 16)



Low elements

Candidates for $\mathscr{G}_m$

Short inversions of $w \in W$: $T^1(w) = \{t \mid \ell(tw) = \ell(w) - 1\}$.

Theorem (Dyer, 93) $T^1(w)$ characterizes uniquely the inversion set $T(w)$

Examples: $W = S_3$;

- $w = s_1s_2 = \underline{2\overset{\frown}{31}}$; $T^1(w) = T(w) = \{\underline{(12)}, \underline{(13)}\}$
- $w = s_1s_2s_1 = \overset{\frown}{32}\underline{21}$; $T^1(w) = \{\underline{(12)}, \underline{(23)}\}$; $T(w) = T$ and $(13) = (12)(23)(12)$

Examples: $W = S_4$;

- $w = s_2s_1s_3s_2 = 3412$; $T^1(w) = T(w) = \{(23), (13), (24), (14)\}$
- $w = s_1s_2s_3s_2s_1 = 4\overset{\text{NO}}{\boxed{\cancel{2}\cancel{3}}}1$; $T^1(w) = \{(12), (13), (34), (24)\}$ and $T(w) = T^1(w) \cup \{(14)\}$, where $(14) = (12)(24)(12)$.

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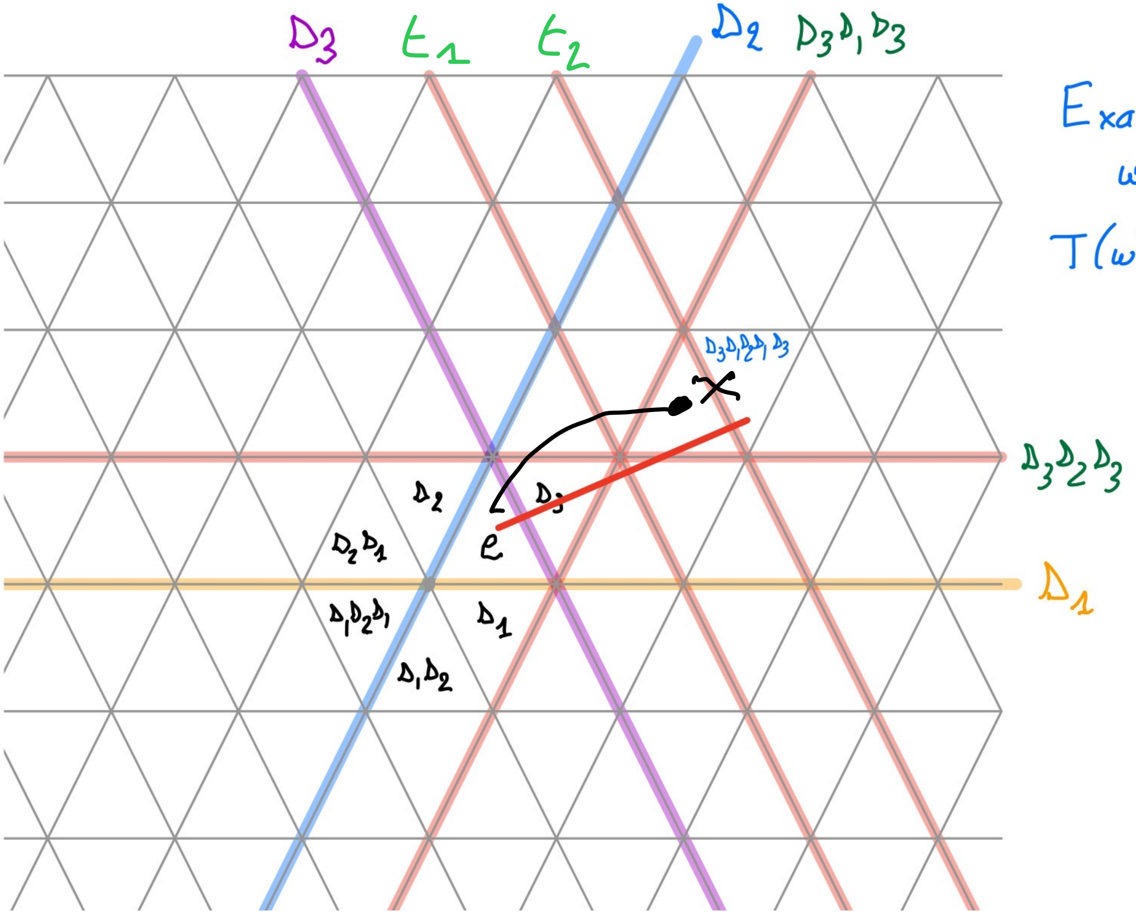

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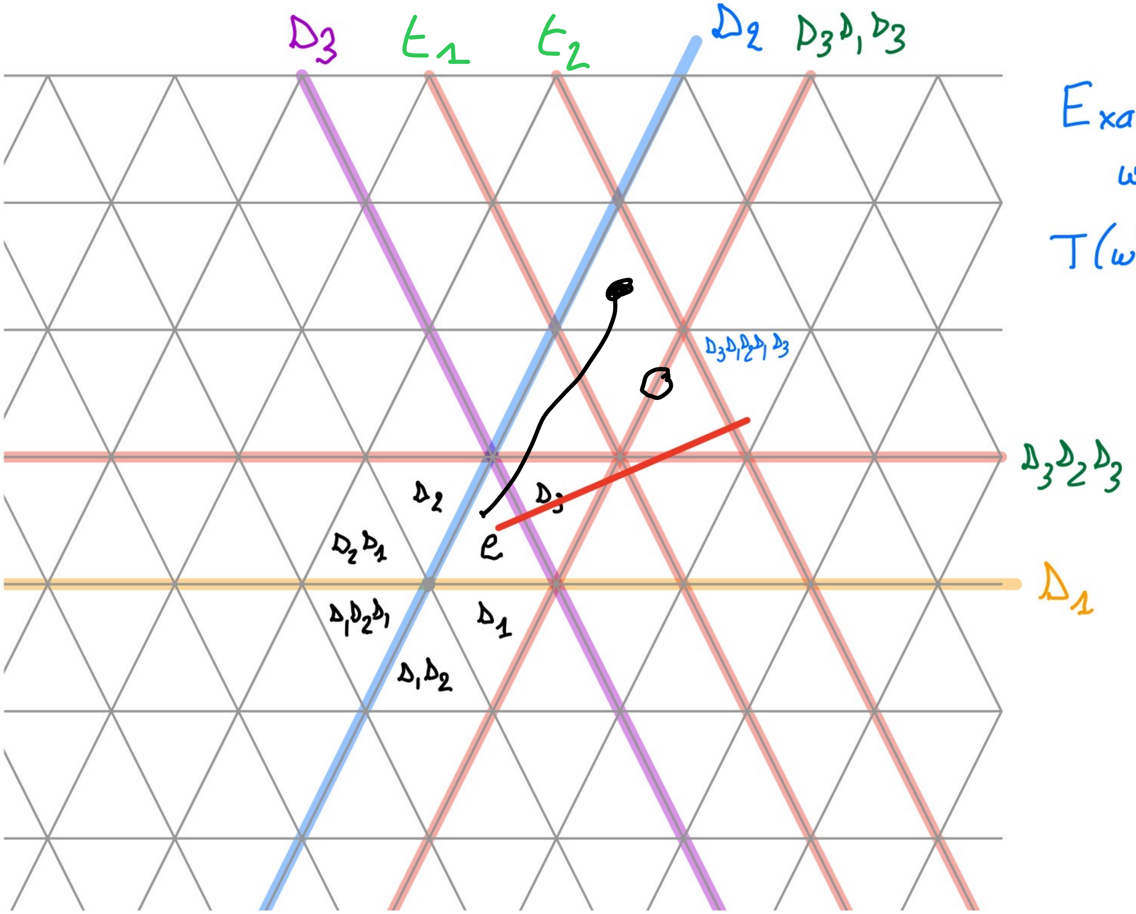

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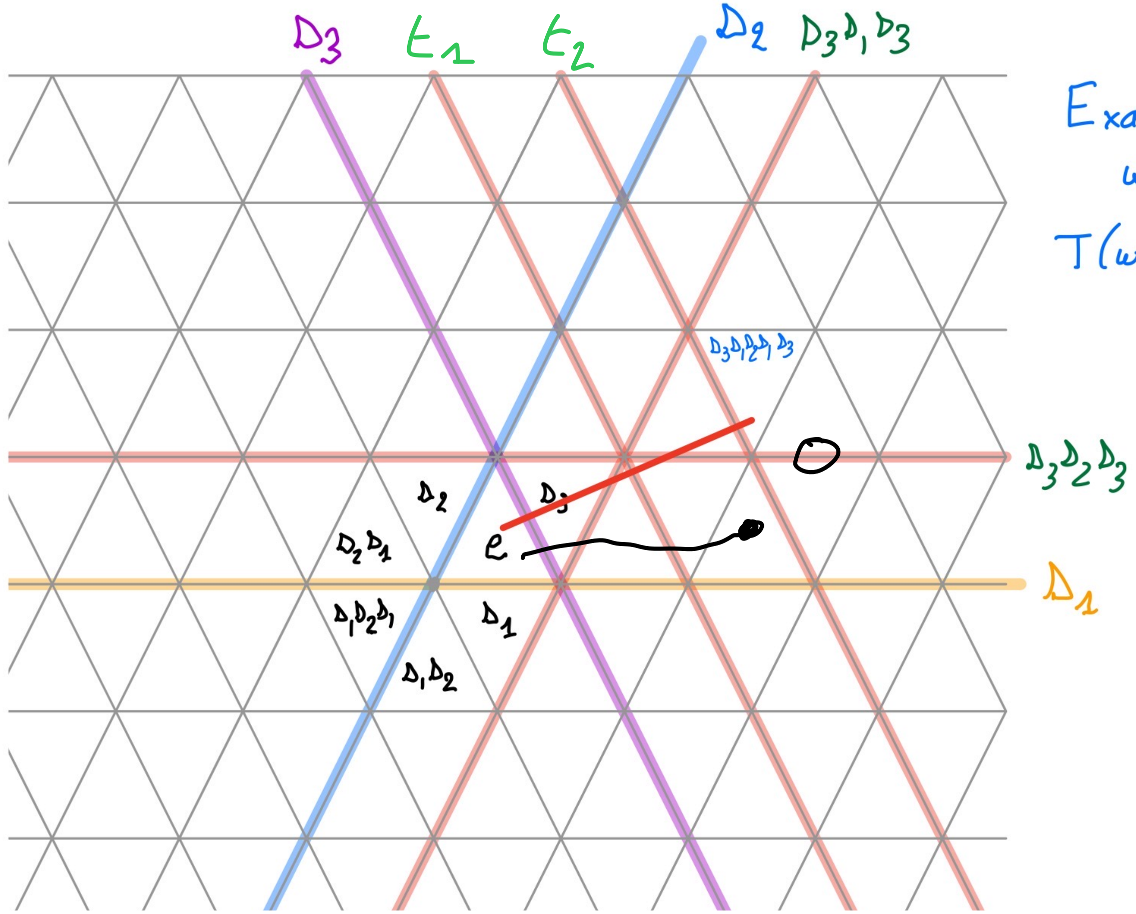

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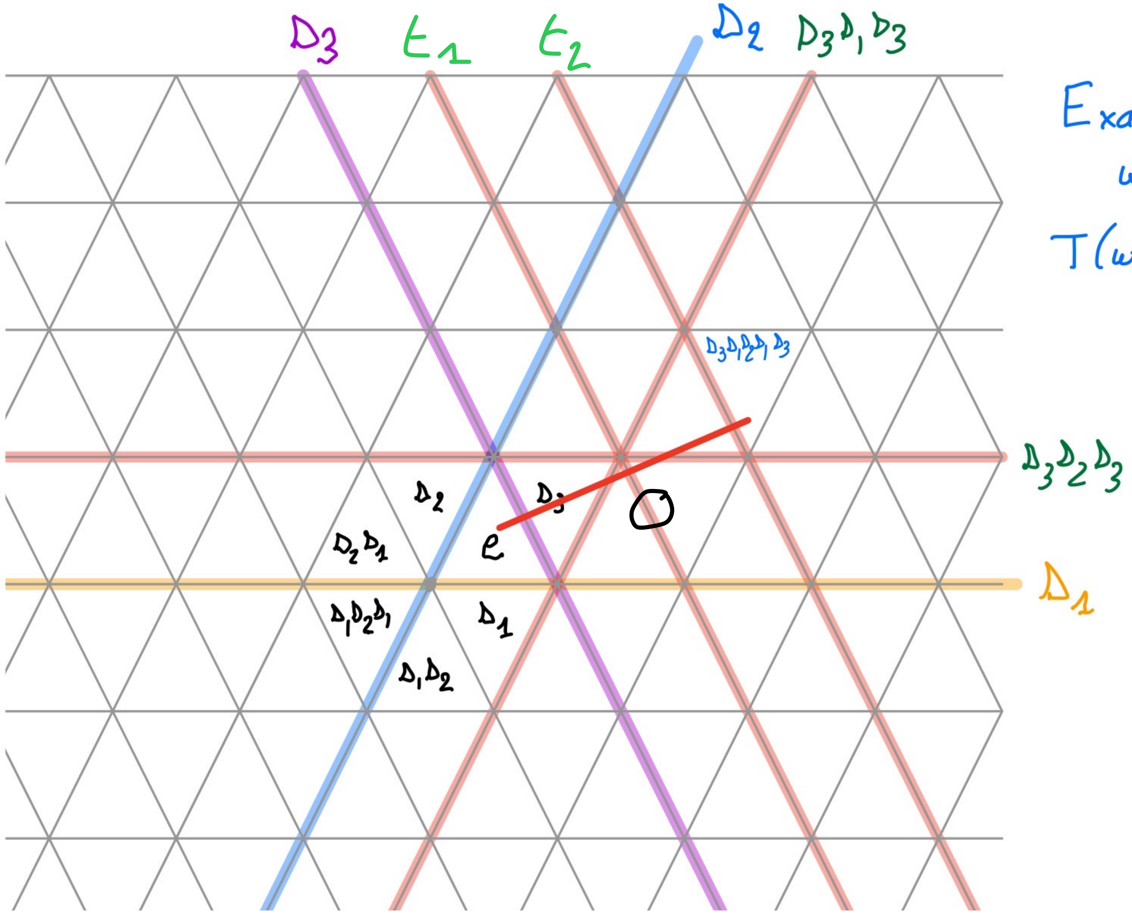

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$t_1 \notin T^1(w)$

since $t_1 w = e$

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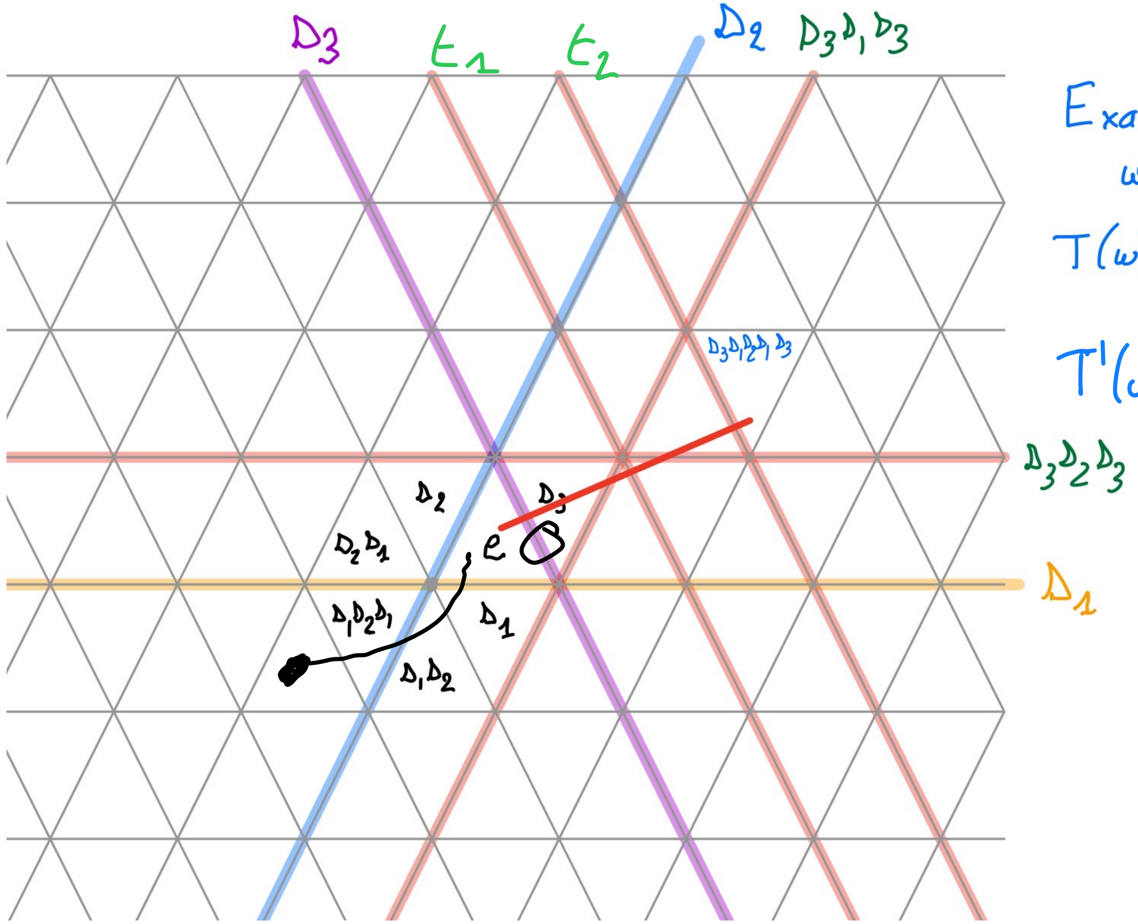

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$T^1(w) = \{s_3, s_3s_1s_3, s_3s_2s_3, t_2\}$

Notice that:

$t_1 = s_3s_2s_3 \cdot s_3s_1s_3 \cdot s_3s_2s_3$

Low elements

Candidates for $\mathscr{G}_m$

$m-$Low elements: $L_m = \{w \in W \mid T^1(w) \subseteq \Sigma_m\}.$

Remarks:

- Introduced in the context of the word problem of Artin (braid) groups (Dehornoy, Dyer, CH 2015): they produce Garside families in the corresponding Artin monoïd (Dyer-CH 2016; Dyer 2022).
- W finite: $L_m = L_0 = W.$
- In general: $L_m \subseteq \mathscr{G}_m$ (therefore L_m is finite) and there is at most one $m-$low element in each $m-$Shi region of Shi$_m$ (Dyer-CH 2016; CH-Nadeau-Williams 2016).

Aim: to prove the equality $L_m = \mathscr{G}_m.$

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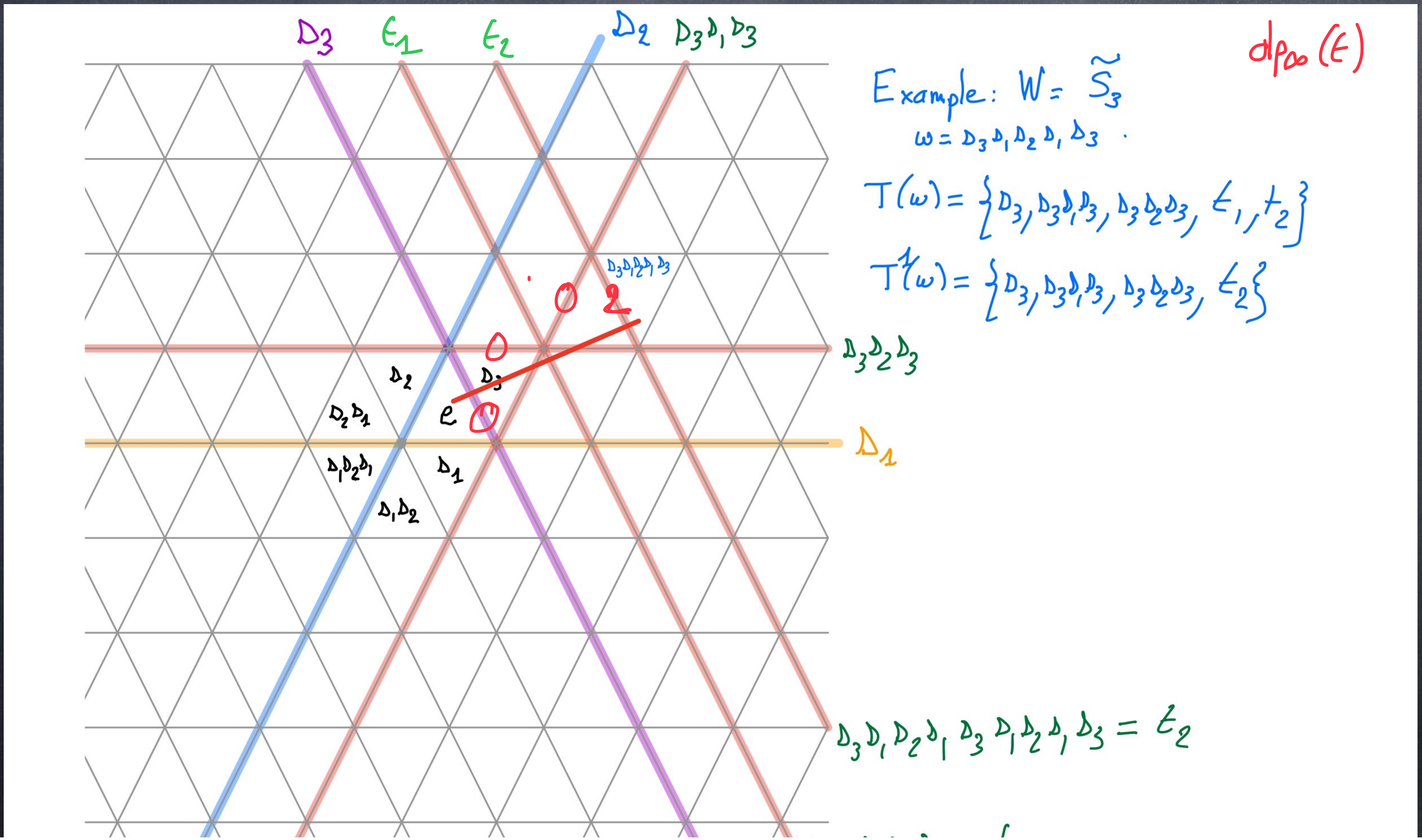

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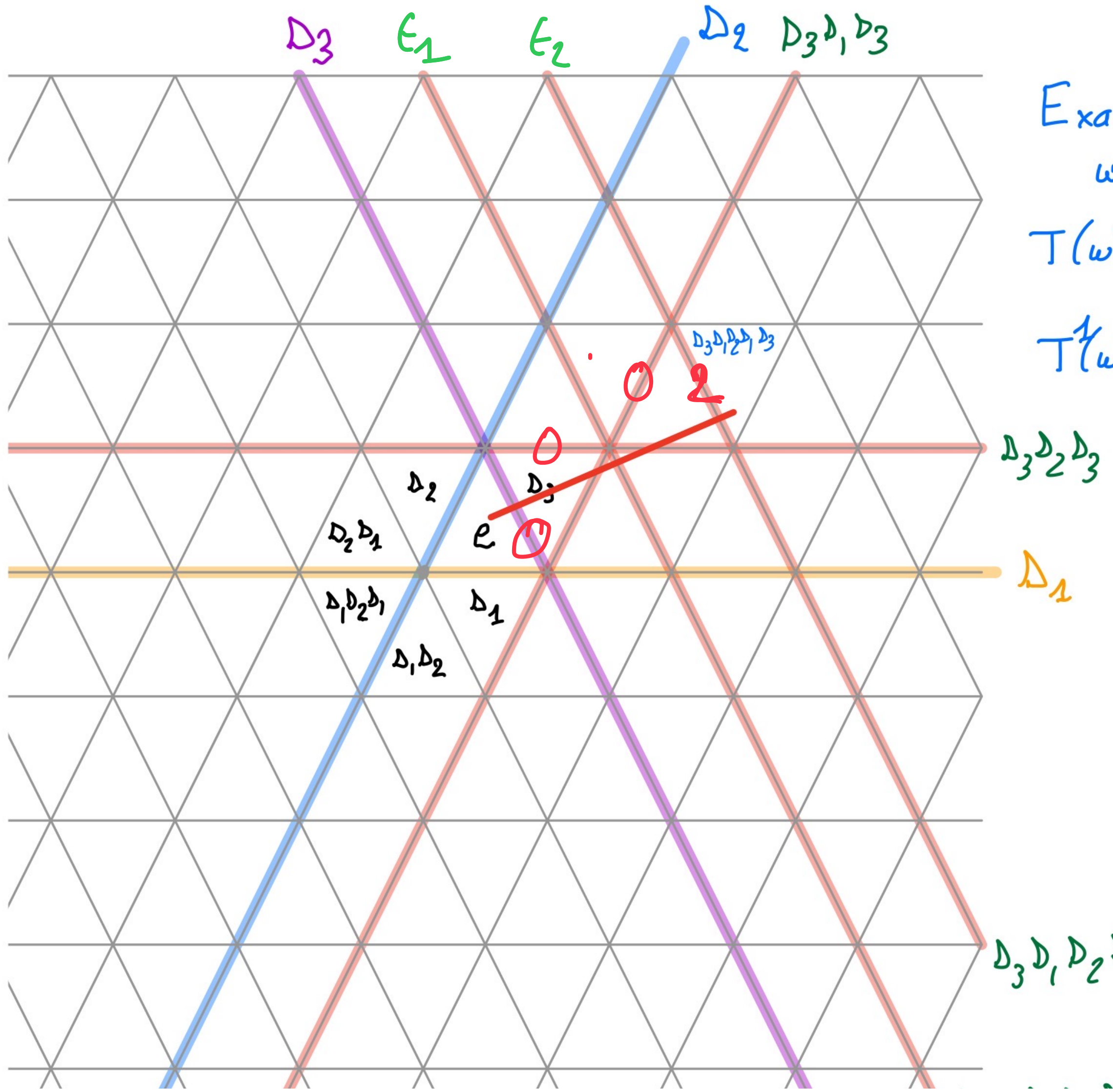
Low elements

Candidates for $\mathcal{G}_m$

m−Low elements: $L_m = \{w \in W \mid T^1(w) \subseteq \Sigma_m\}.$



$dp_\infty(t)$

Example: $W = \tilde{S}_3$

$w = s_3s_1s_2s_1s_3$.

$T(w) = \{s_3, s_3s_1s_3, s_3s_2s_3, t_1, t_2\}$

$T^1(w) = \{s_3, s_3s_1s_3, s_3s_2s_3, t_2\}$

Here: $w \in L_2$

since $dp_\infty(t_2) = 2$

and $dp_\infty(\text{others}) = 0$.

So $T^1(w) \subseteq \Sigma_2$.

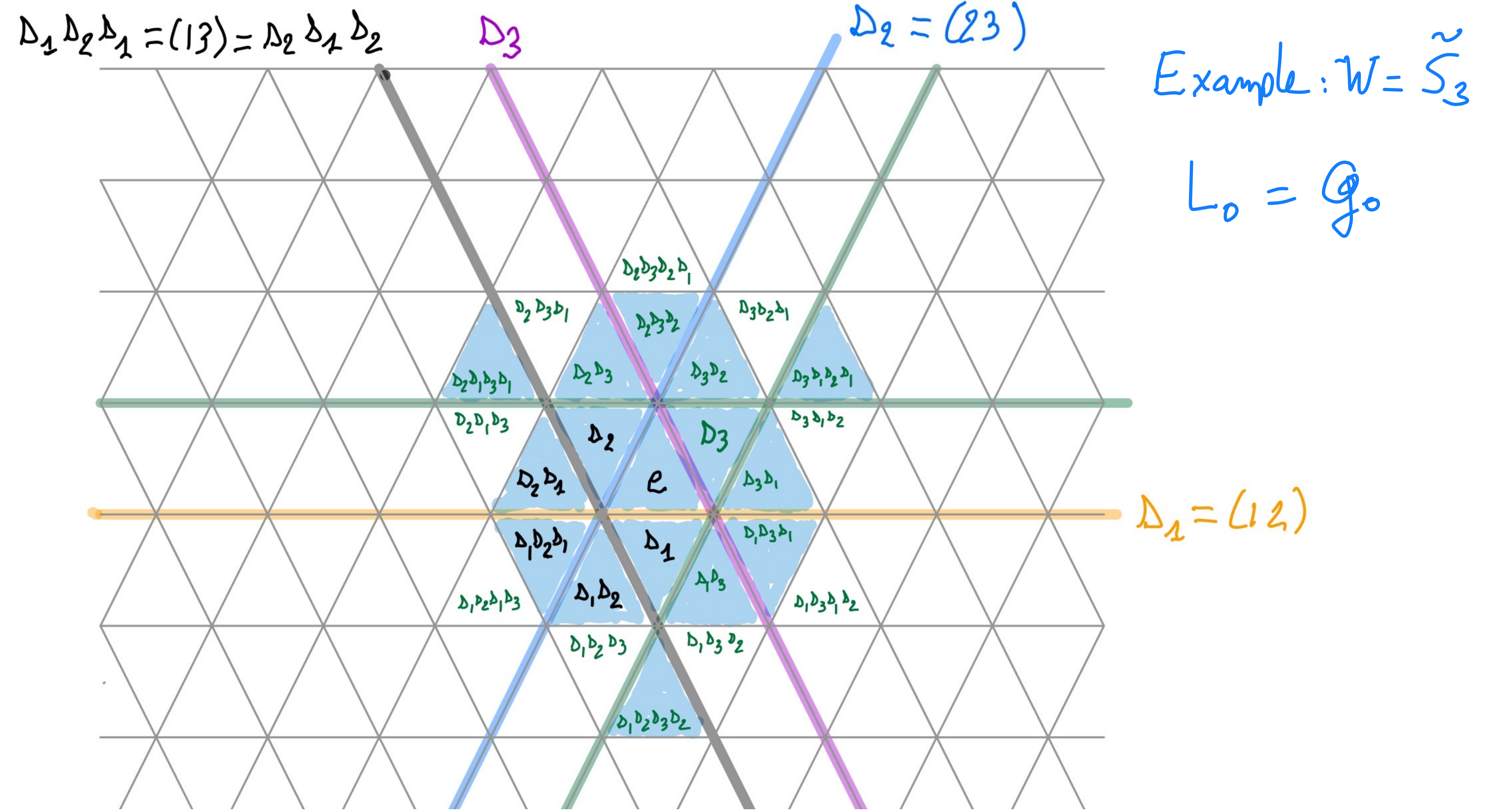
BUT: $w \notin L_0$ and $w \notin L_1$

Low elements

Candidates for $\mathcal{G}_m$

Fact. Let $w \in \mathcal{G}_m$, then $T_R(w) \subseteq \Sigma_m$. But is $T^1(w) \subseteq \Sigma_m$?

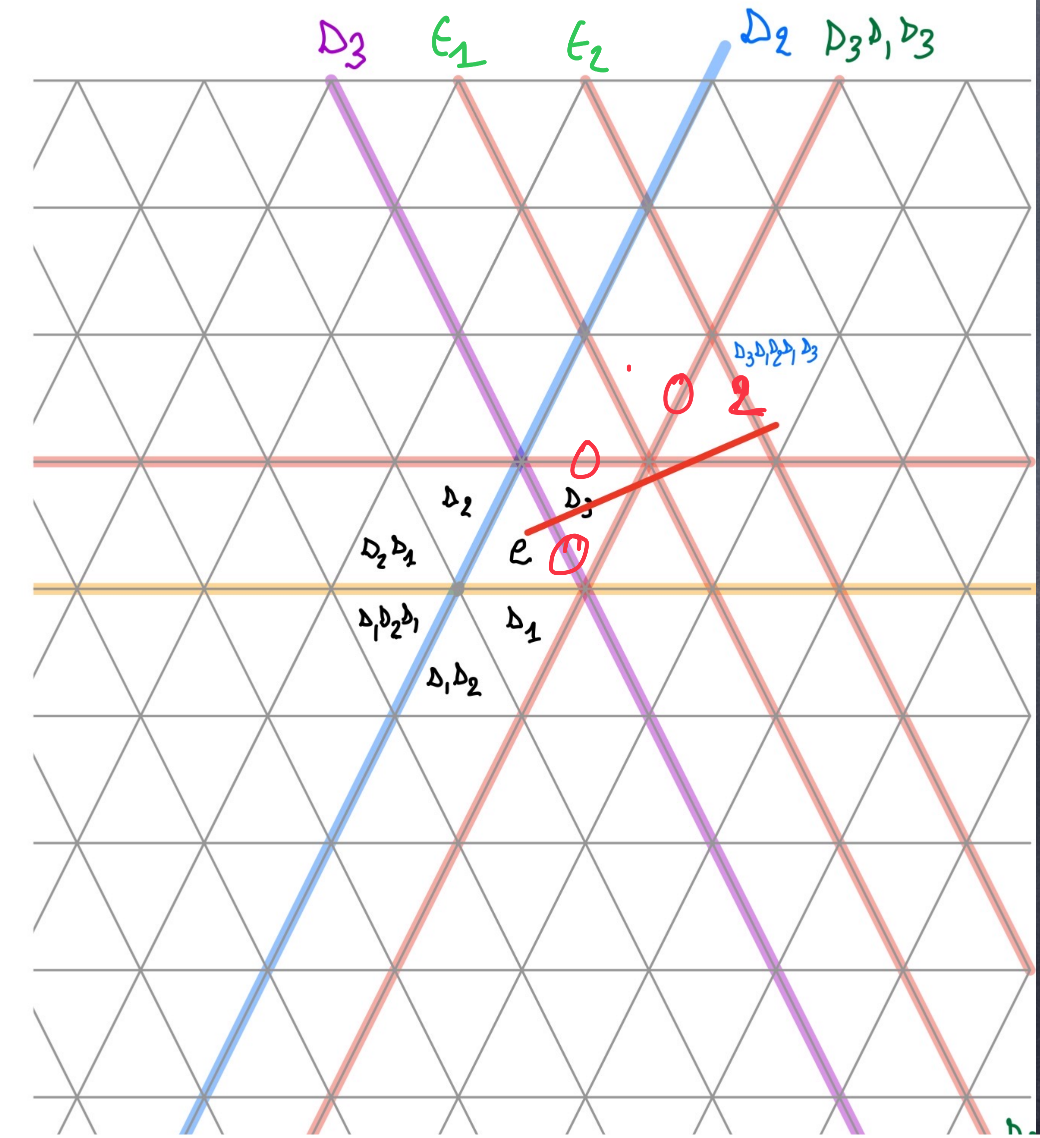
Recall that: $T_R(w) = \{wsw^{-1} \mid s \in D_R(w)\}$ descent-walls.



Low elements

Candidates for $\mathscr{G}_m$

Aim: to prove the equality $L_m = \mathscr{G}_m$.



Strategy: to prove that that the function dp$_\infty : T^1(w) \to \mathbb{N}$ reaches its maximum on $T_R(w)$ (i.e., the `descent-walls').

Surprisingly it goes down to a combinatorial problem to be solved even in finite Coxeter groups

The short inversion poset

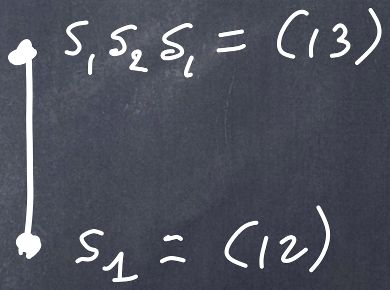
Let $w \in W$, the short inversion poset $(T^1(w), \preceq_w)$ is the transitive and reflexive closure of the relation $\dot{\prec}_w$: we write $s \,\dot{\prec}_w\, t$ if:

- $s \in T(t)$ or there is $r \in T \setminus T(w)$ with $\langle s, t \rangle \subseteq \langle s, r \rangle$ and $r \in T(t)$.

Theorem (Ch-Dyer 16, Dyer 22) For $w \in W$: $s \preceq_w t \implies \mathrm{dp}_\infty(s) \leq \mathrm{dp}_\infty(t)$

Examples: $W = S_3$;

- $w = s_1 s_2 = 231$; $T^1(w) = T(w) = \{\underline{(12)}, \underline{(13)}\}$



Here: $s_1 = (12) \in T(s_1 s_2 s_1) = T(321)$.

- $w = s_1 s_2 s_1 = 321$; $T^1(w) = \{\underline{(12)}, \underline{(23)}\}$

Here: $(12) \notin T(23)$; $(23) \notin T(12)$
and $T \setminus T(w) = \emptyset$

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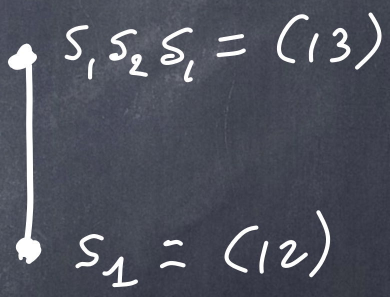
Examples: $W = S_3$;

- $w = s_1s_2 = 231$; $T^1(w) = T(w) = \{\underline{(12)}, \underline{(13)}\}$



Here: $s_1 = (12) \in T(s_1s_2s_1) = T(321)$.

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Here: $(12) \notin T(23)$; $(23) \notin T(12)$
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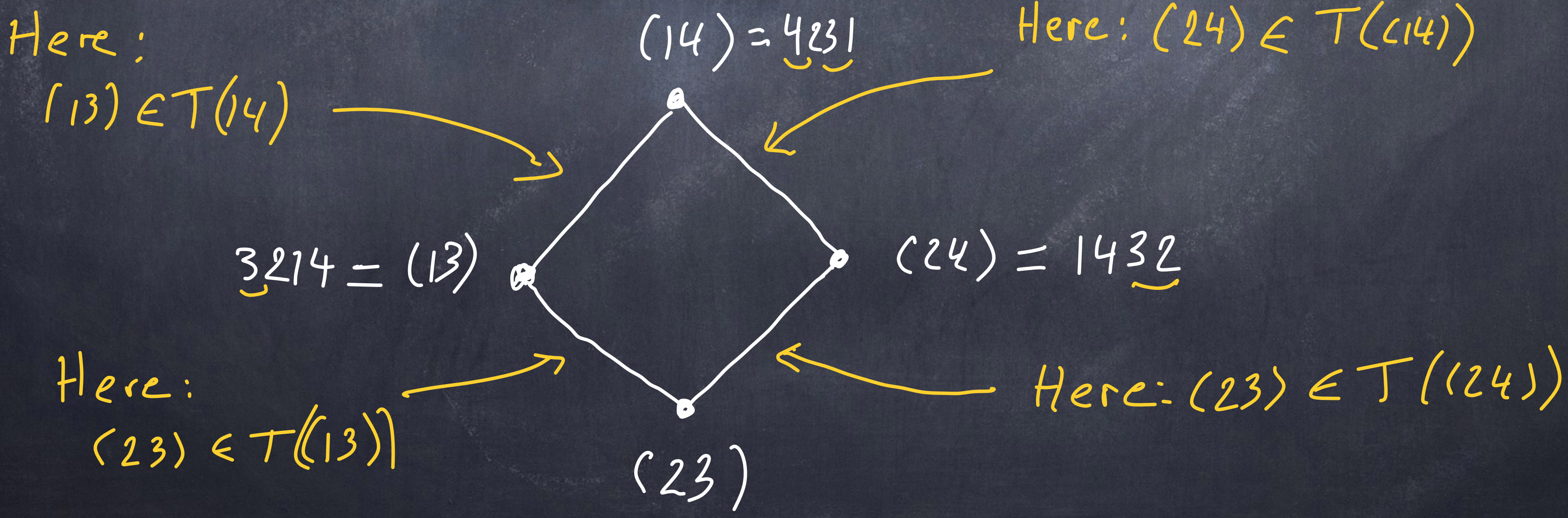
- $s \in T(t)$ or there is $r \in T \backslash T(w)$ with $\langle s, t \rangle \subseteq \langle s, r \rangle$ and $r \in T(t)$.

This condition is empty for S_n and $\tilde{S}_n$

Examples: $W = S_4$;

- $w = s_2 s_1 s_3 s_2 = 3412$; $T^1(w) = T(w) = \{(23), (13), (24), (14)\}$



Here: (13) ∈ T((14))

Here: (24) ∈ T((14))

Here: (23) ∈ T((13))

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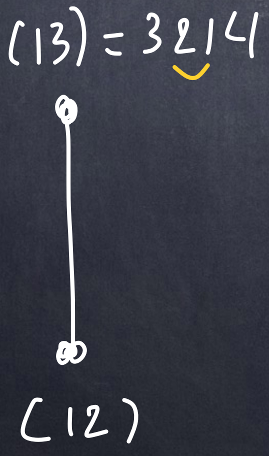

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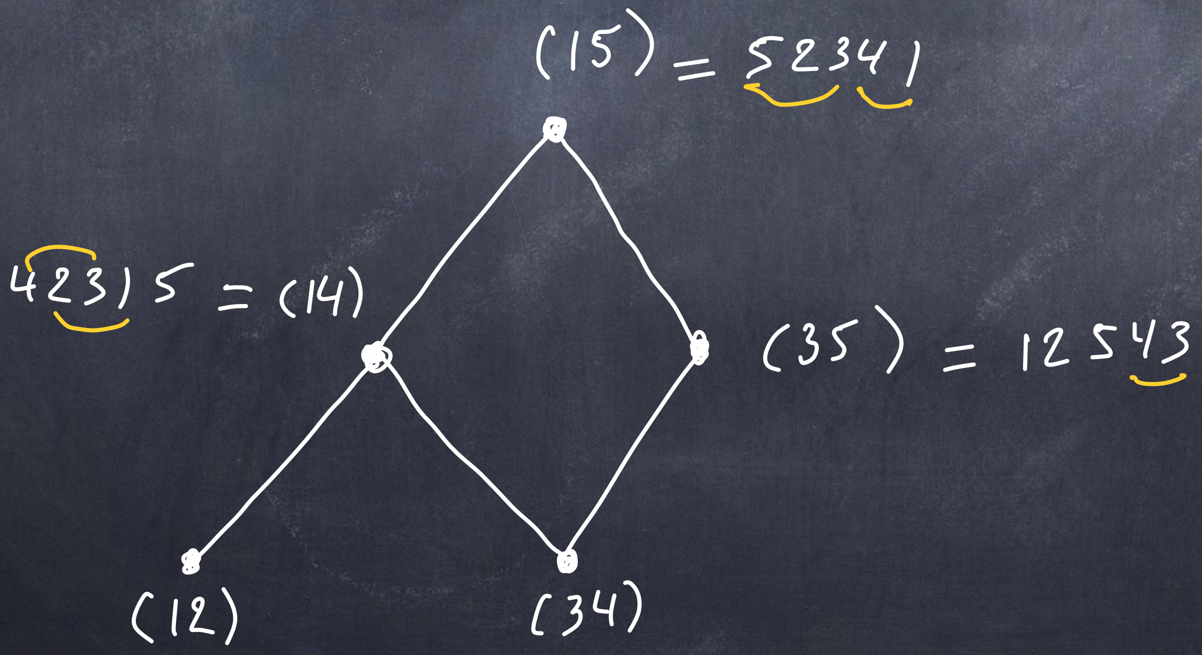
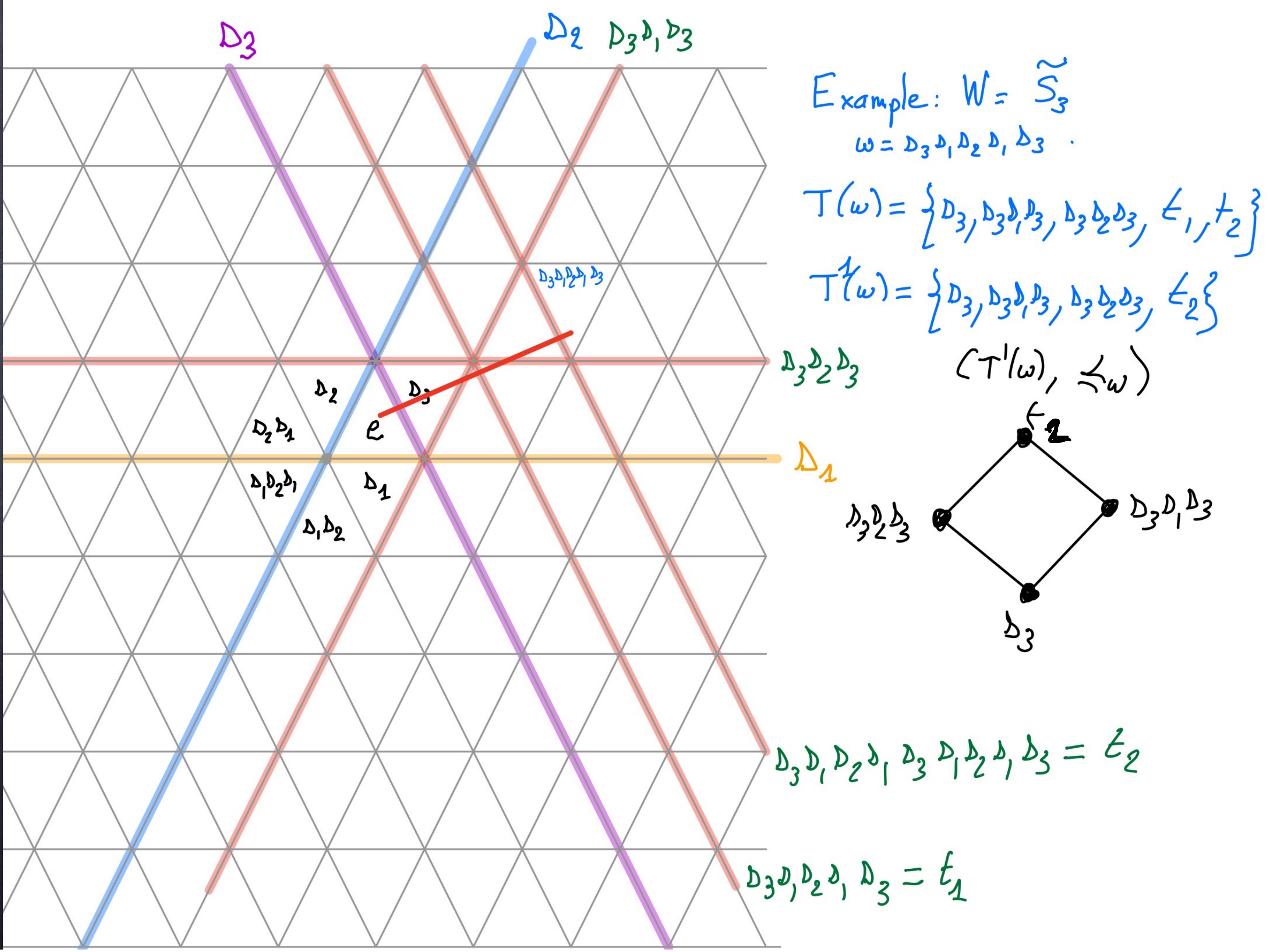
Examples: $W = S_5$;

- $w = 24513$; $T^1(w) = T(w) = \{(15),(14),(12),(35),(34)\}$



The short inversion poset



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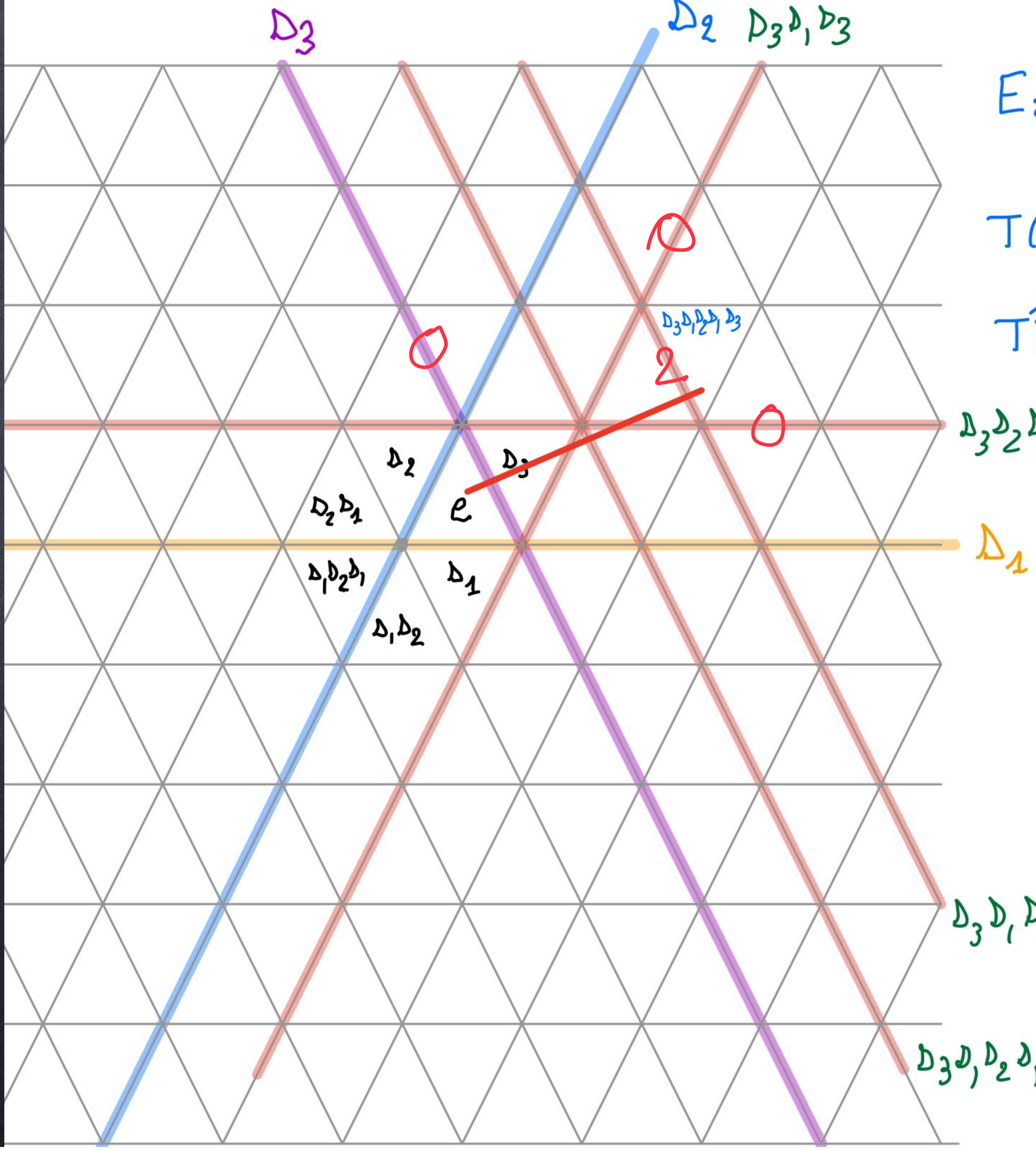
$(T^1(w), \preceq_w)$

The short inversion poset



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$T(w) = \{s_3, s_3s_1s_3, s_3s_2s_3, t_1, t_2\}$

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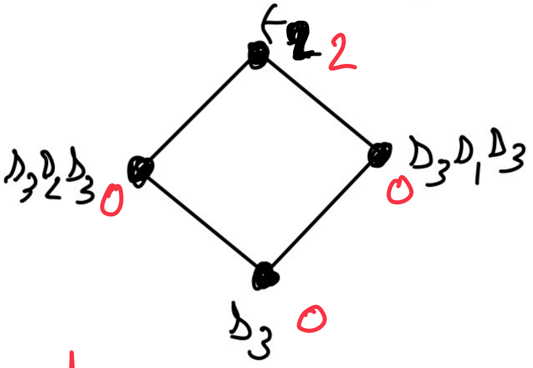
$(T^1(w), \preceq_w)$



dp_∞

The short inversion poset

Minimal and maximal elements

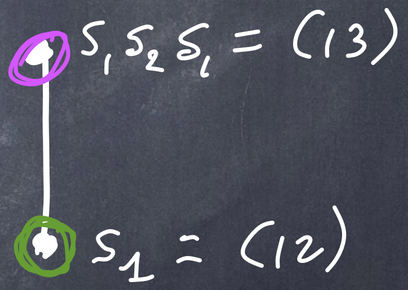
Theorem (Dyer, CH, Fishel, Mark '23) Let $w \in W$, for any $r \in T^1(w)$, there is $s \in D_L(w)$ and $t \in T_R(w)$ such that $s \preceq_w r \preceq_w t$.

Where: $D_L(w) = T(w) \cap S$ and $T_R(w) = \{wsw^{-1} \mid s \in D_R(w)\}$

Examples: $W = S_3$;

- $w = s_1 s_2 = 231$; $T^1(w) = T(w) = \{(12), (13)\}$

$D_L(w) = \{(12)\}$ and $T_R(w) = \{(13)\}$



- $w = s_1 s_2 s_1 = 321$; $T^1(w) = \{(12), (23)\}$

$D_L(w) = T^1(w) = T_R(w)$

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Examples: $W = S_4$;

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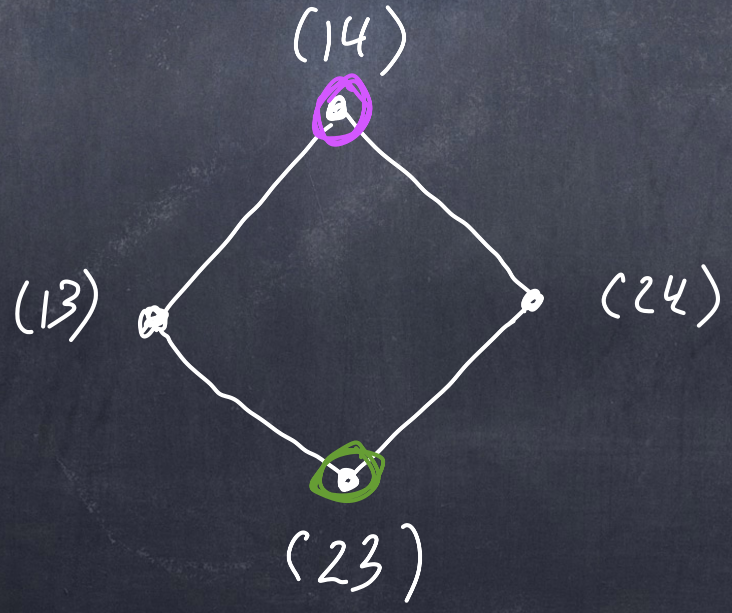
$D_L(w) = \{(23)\}$



$D_R(w) = \{(23)\}$

$T_R(w) = \{(14)\}$

The short inversion poset

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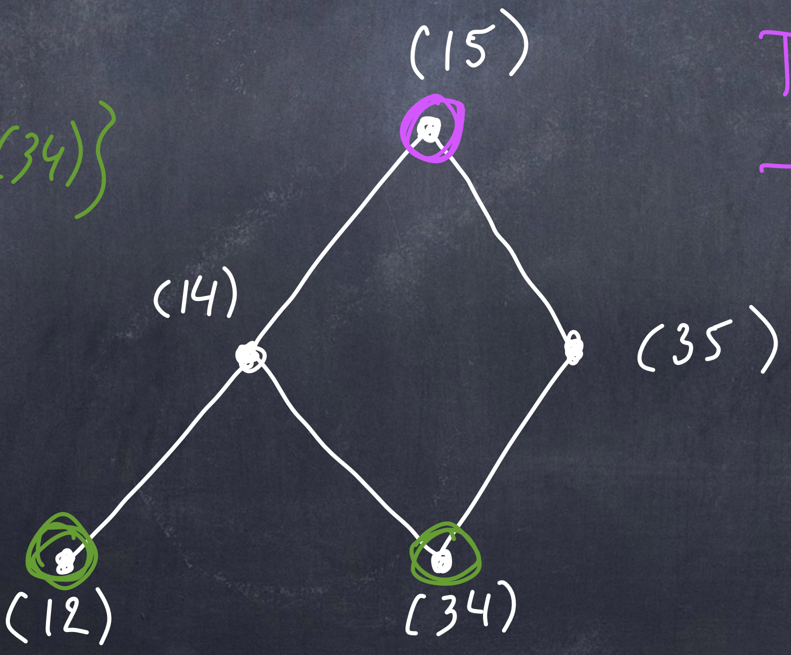

$D_R(w) = \{(34)\}$

$T_R(w) = \{(15)\}$

The short inversion poset

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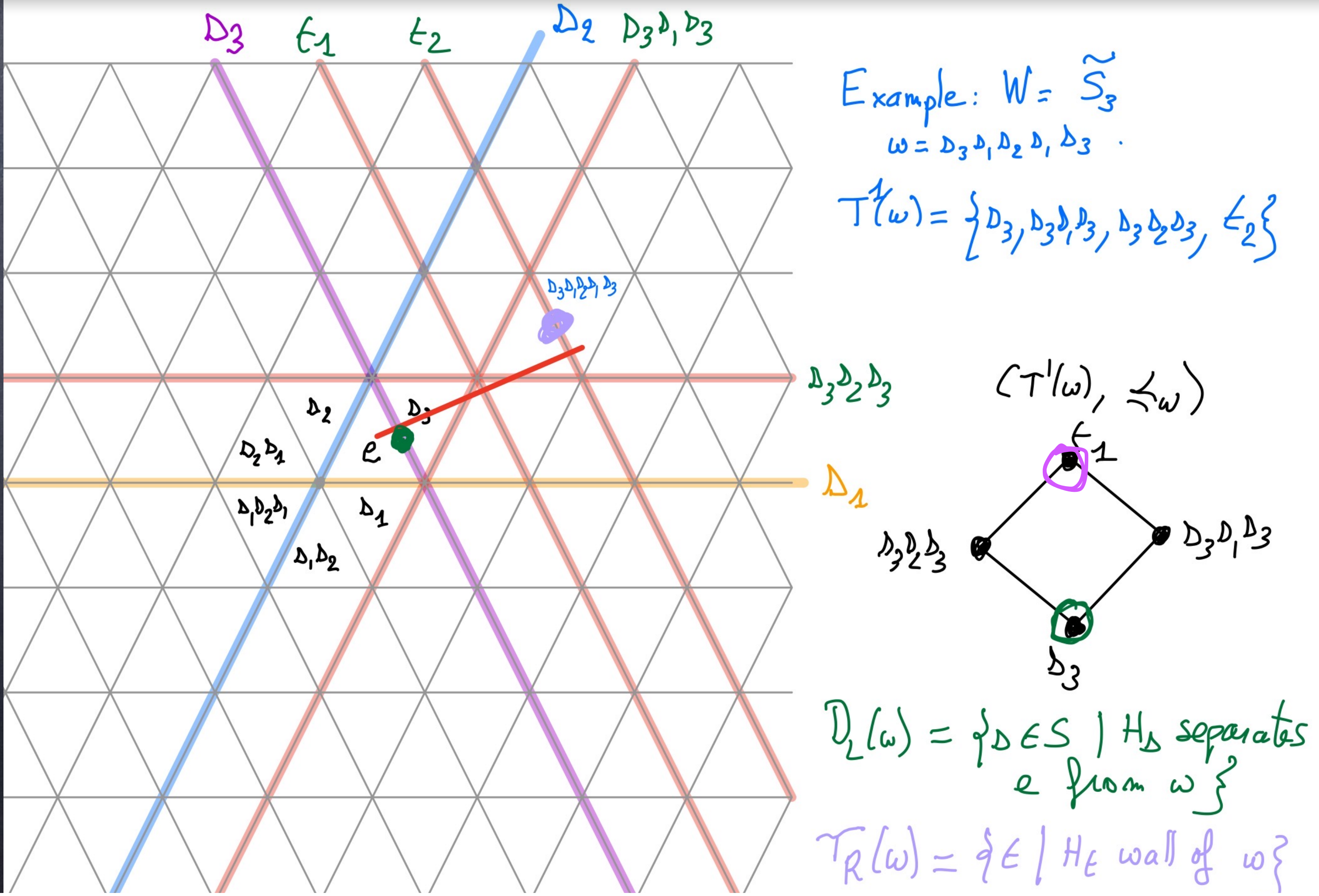

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$w = s_3 s_1 s_2 s_1 s_3$.

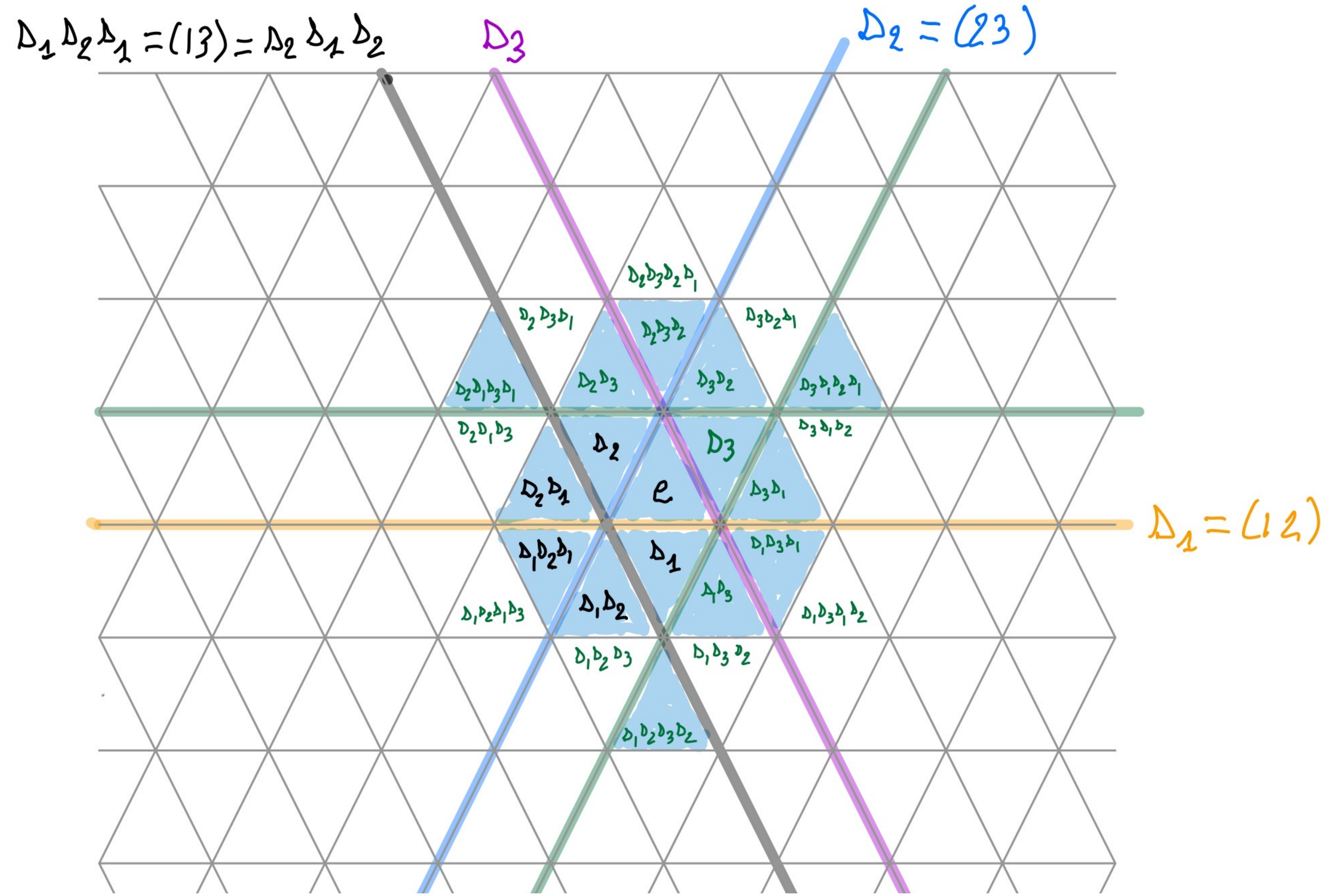
$T^1(w) = \{s_3, s_3s_1s_3, s_3s_2s_3, t_2\}$

$D_L(w) = \{s \in S \mid H_s$ separates e from $w\}$

$T_R(w) = \{t \mid H_t$ wall of $w\}$

Shi arrangement in general

Theorem (Dyer, CH, Fishel, Mark '23) For (W, S) and $m \in \mathbb{N}$, one has $\mathcal{G}_m = L_m$.



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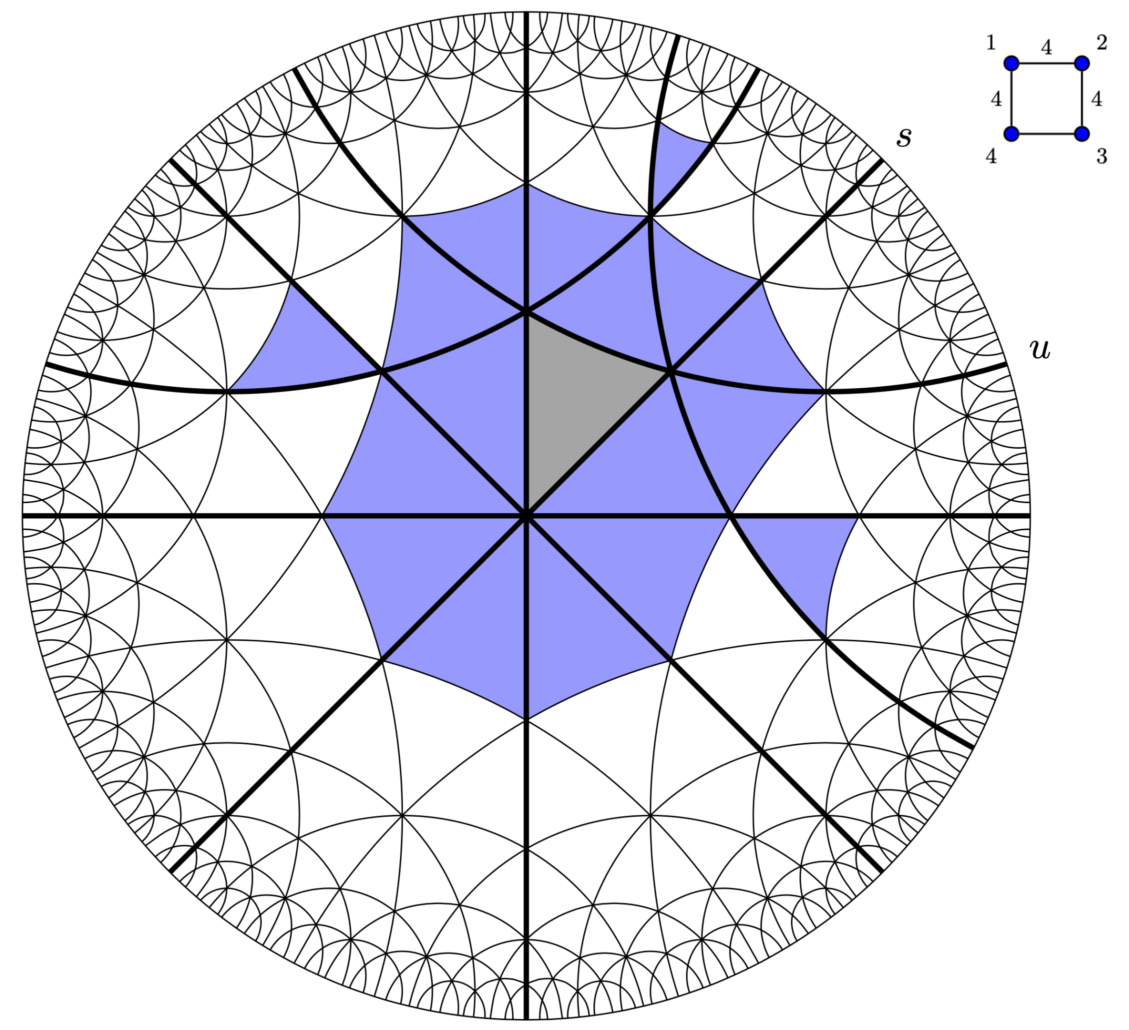
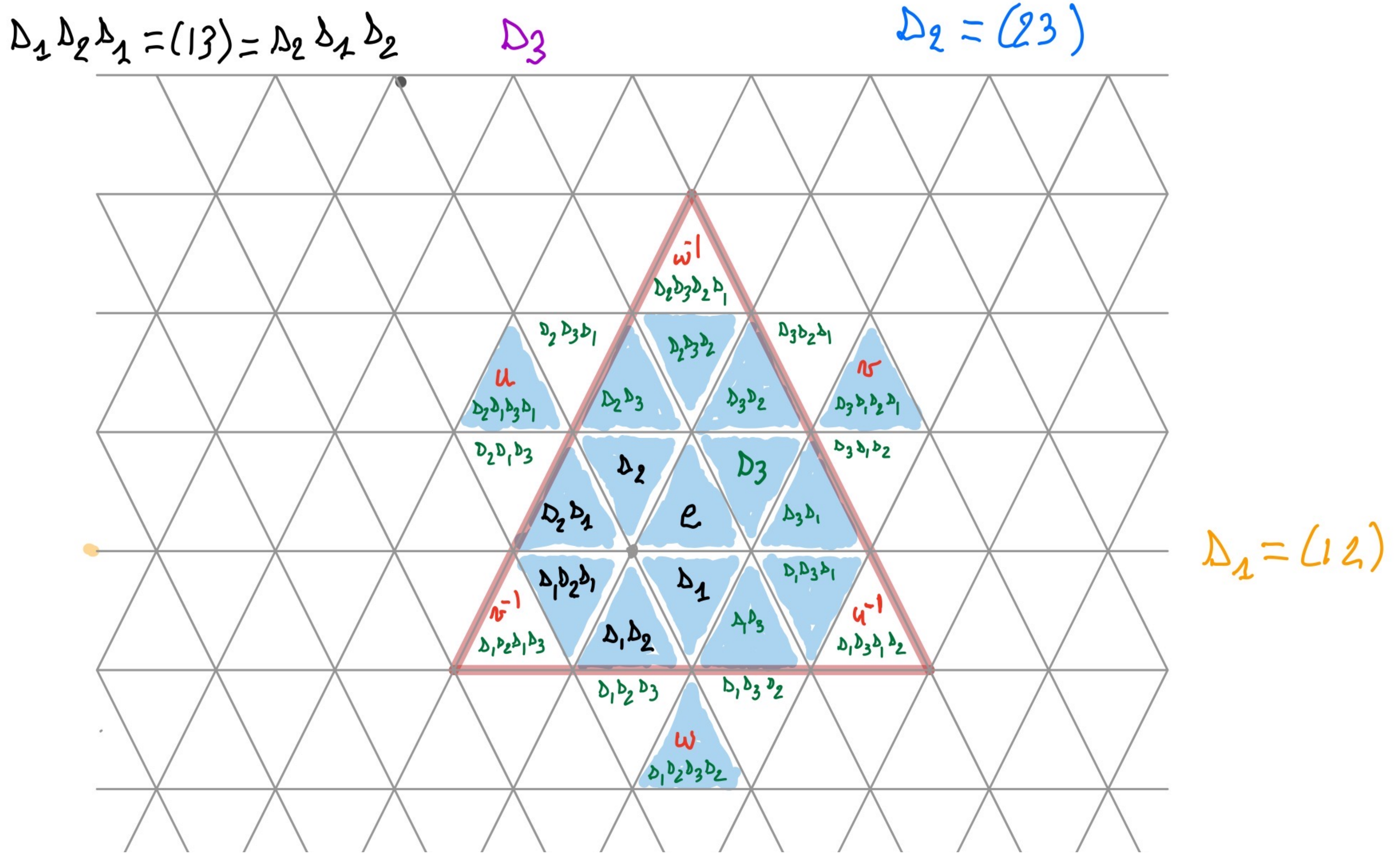
Hyperbolic Coxeter system



Shi arrangement in general

Theorem (Dyer, CH, Fishel, Mark '23) For (W, S), Shi_0 has the convex property.

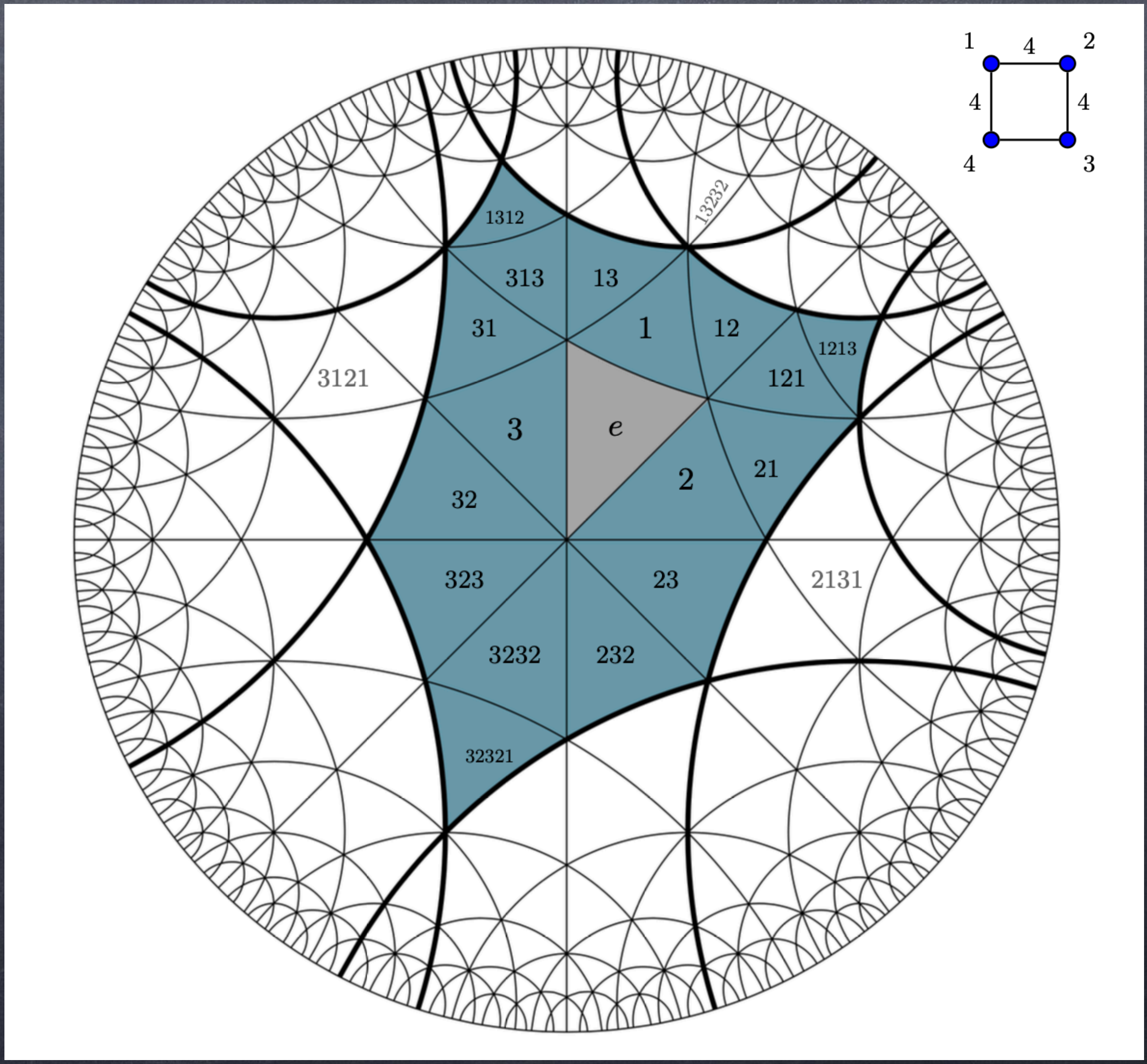


Shi arrangement in general

Theorem (Dyer, CH, Fishel, Mark ’23) For (W, S), Shi$_0$ has the convex property.

Counting in indefinite Coxeter system: the convex is not anymore a dilatation of the fundamental chamber!

Enumeration is unknown!



Shi arrangement in general

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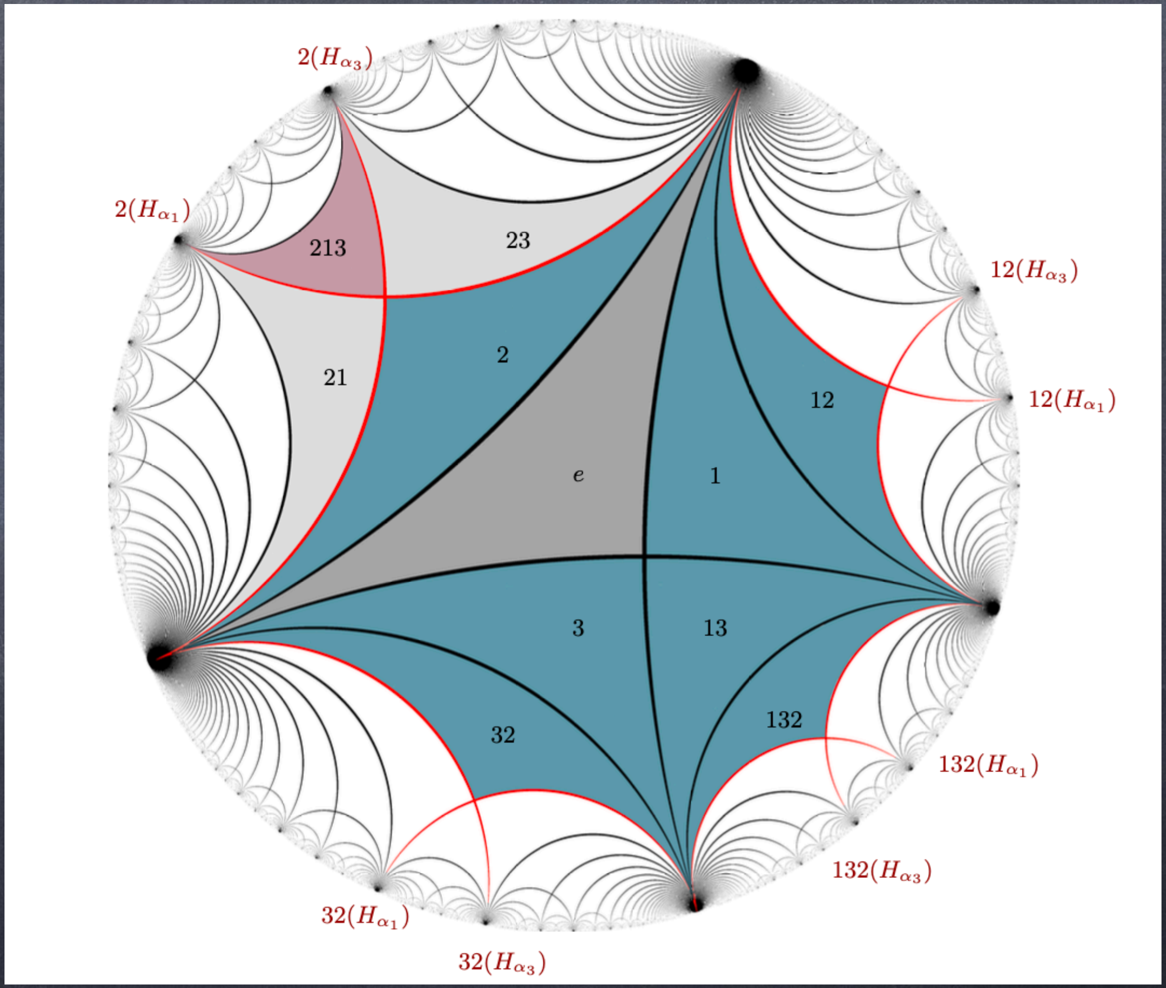
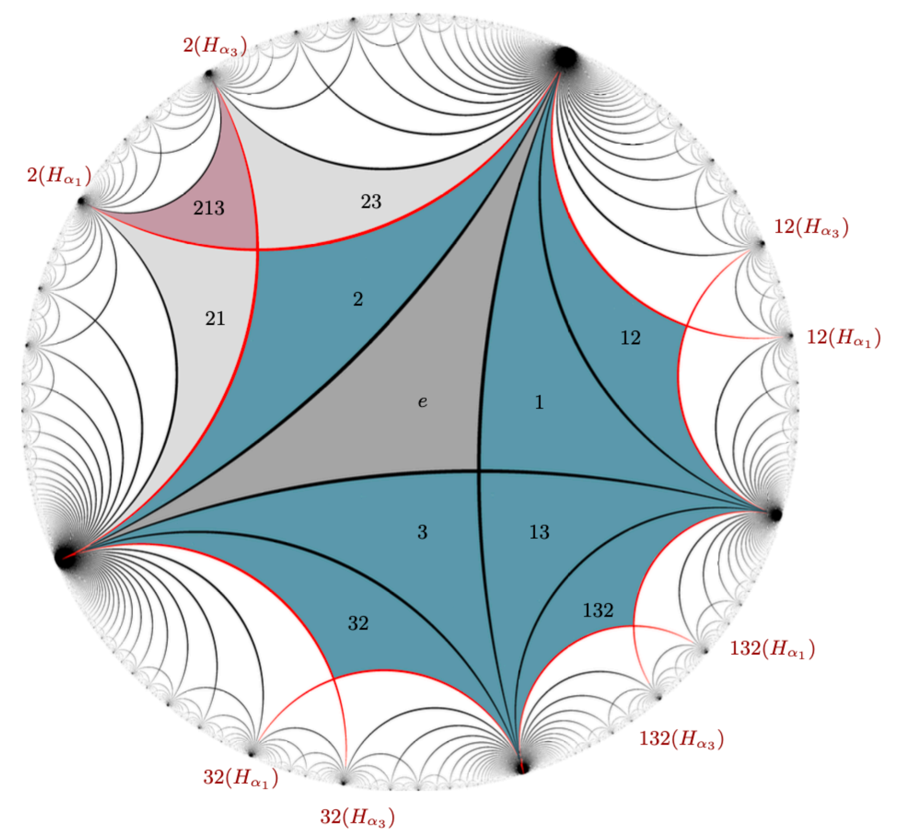
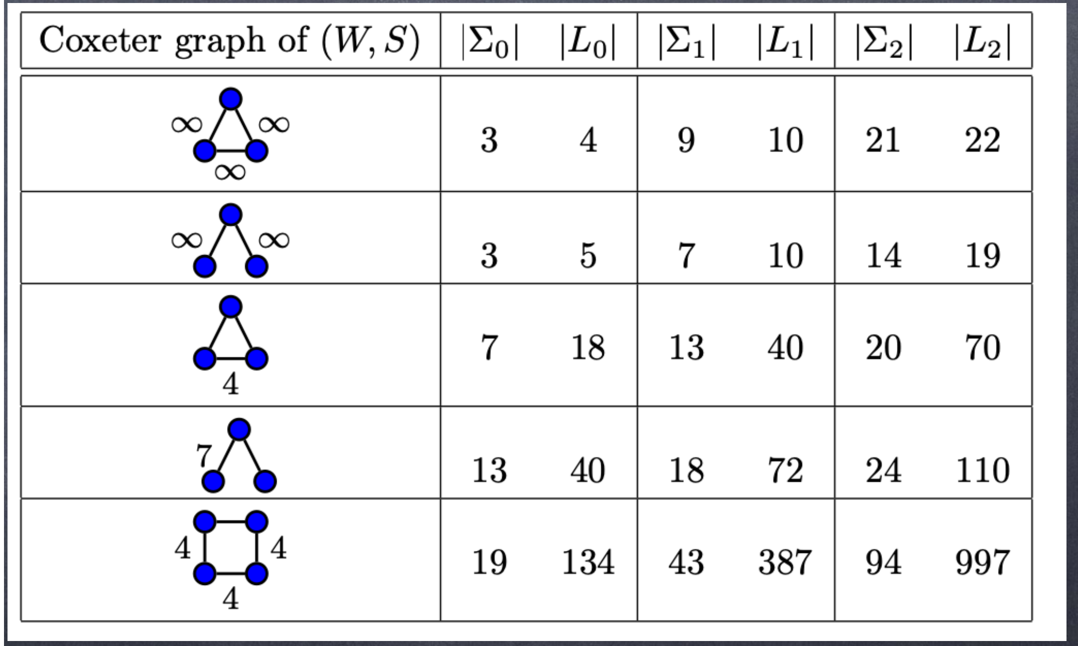
Shi$_m$ $(m \geq 1)$ does not have in general the convexity property



Final remarks

- Study Shi arrangement in general (enumeration, classification of Shi$_m$ with the convexity property)



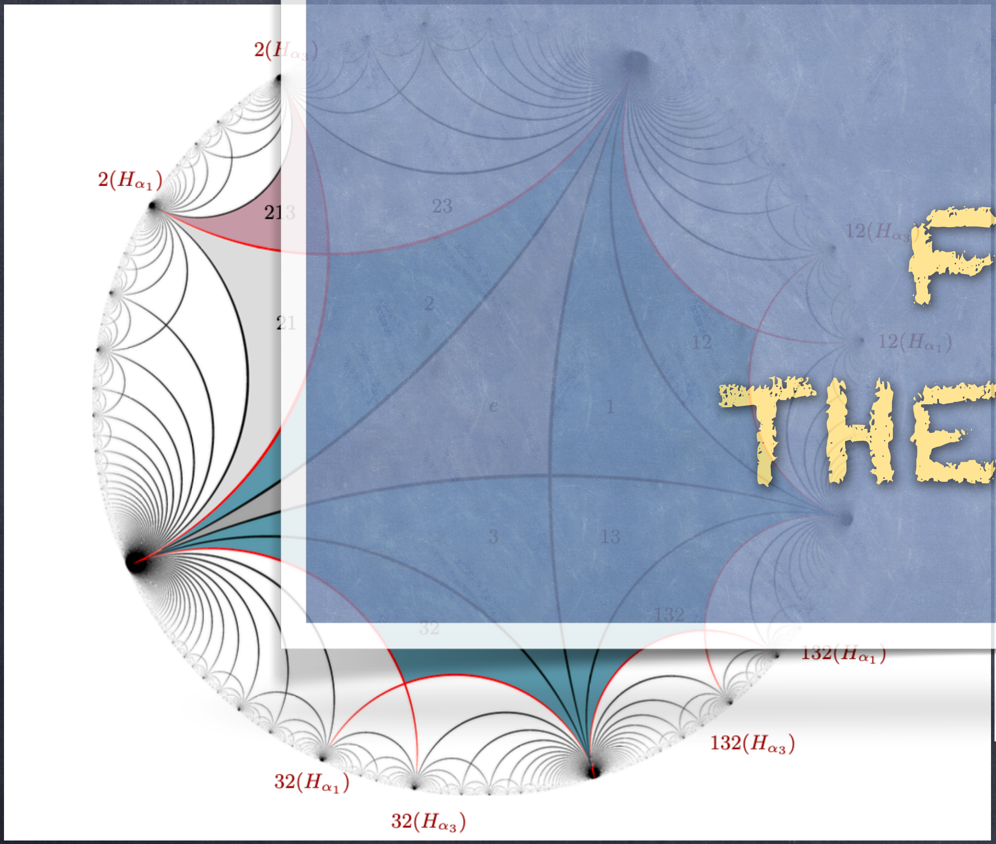


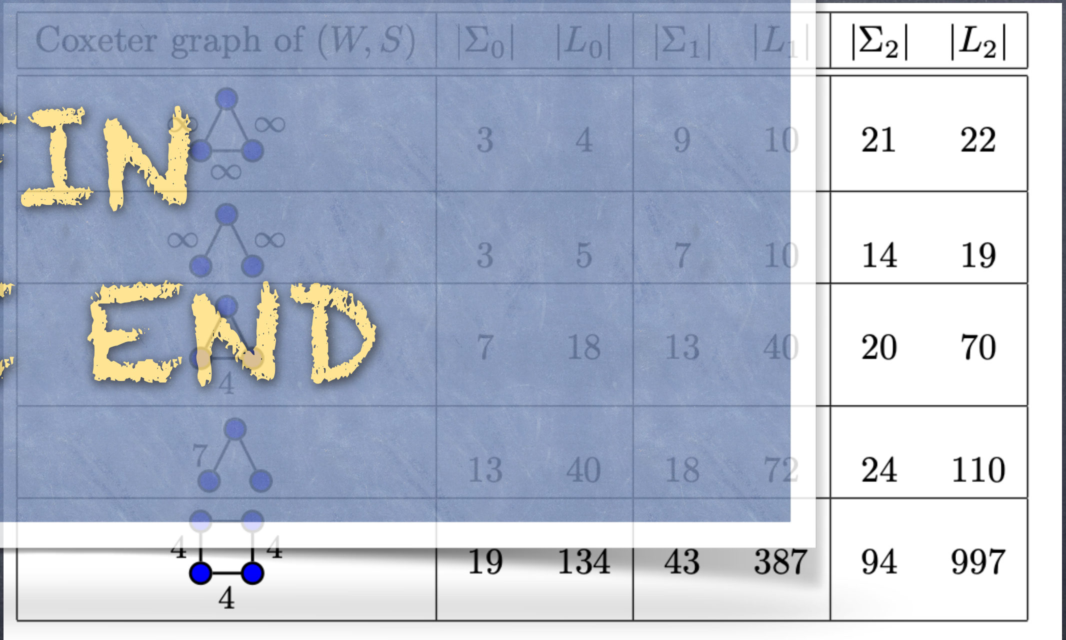
Coxeter graph of (W, S)	$\lvert\Sigma_0\rvert$	$\lvert L_0\rvert$	$\lvert\Sigma_1\rvert$	$\lvert L_1\rvert$	$\lvert\Sigma_2\rvert$	$\lvert L_2\rvert$
∞ ∞ ∞	3	4	9	10	21	22
∞ ∞	3	5	7	10	14	19
4	7	18	13	40	20	70
7	13	40	18	72	24	110
4 4 4	19	134	43	387	94	997

- Study the short inversion poset in relation with the weak order: how to find join, meet etc.

Final remarks

- Study Shi arrangement in general (enumeration, classification of Shi_m with the convexity property)





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FIN

THE END

- Study the short inversion poset in relation with the weak order: how to find join, meet etc.