# Blossoming trees and planar maps
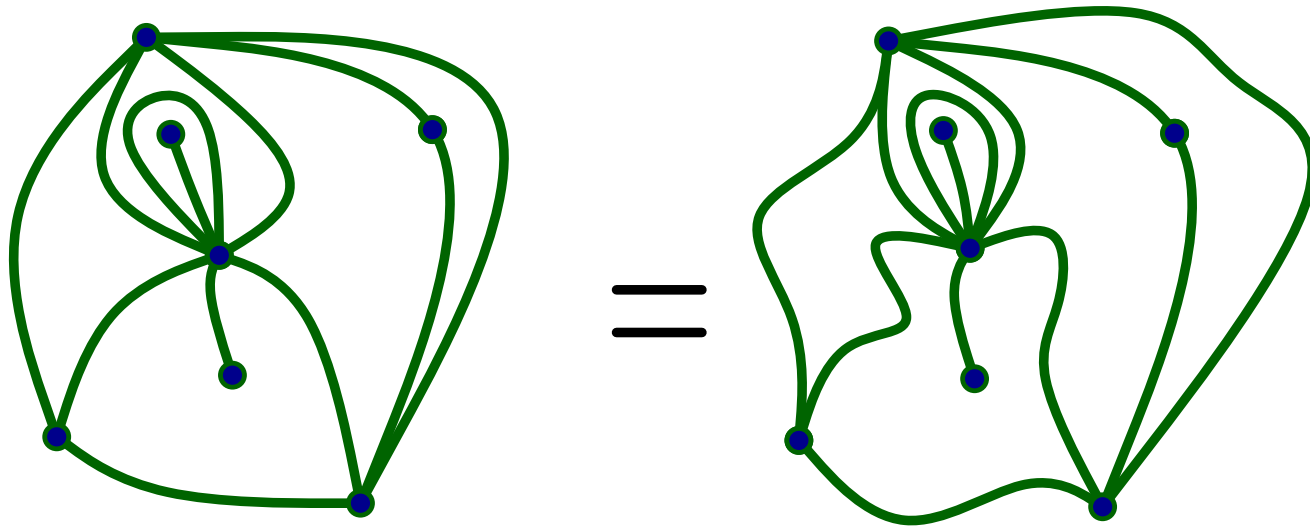
**Marie Albenque (CNRS, LIX, École Polytechnique)**

joint work with Louigi Addario-Berry (McGill University Montréal)
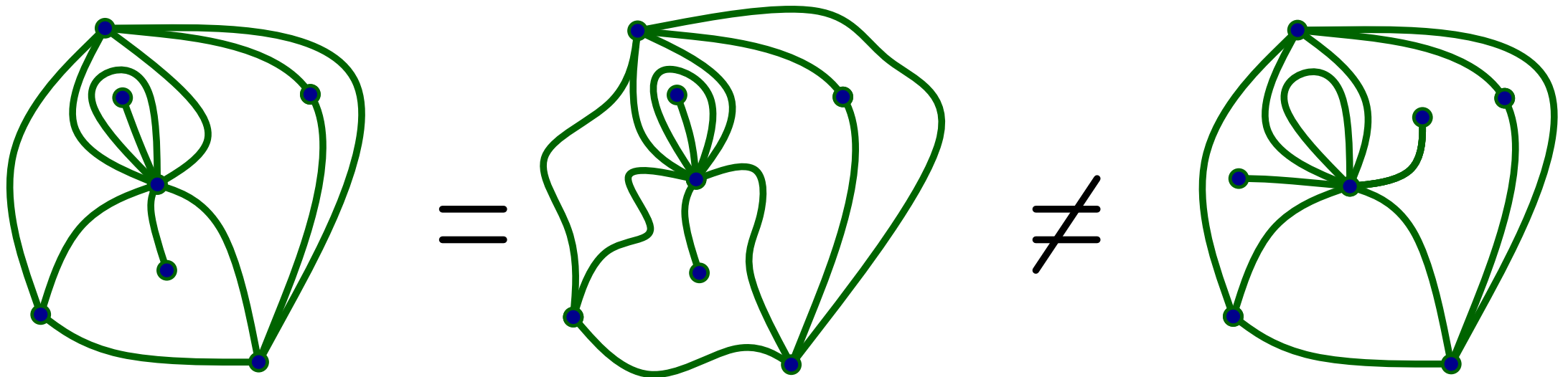and Dominique Poulalhon (LIAFA)

# Planar Maps – Definition.

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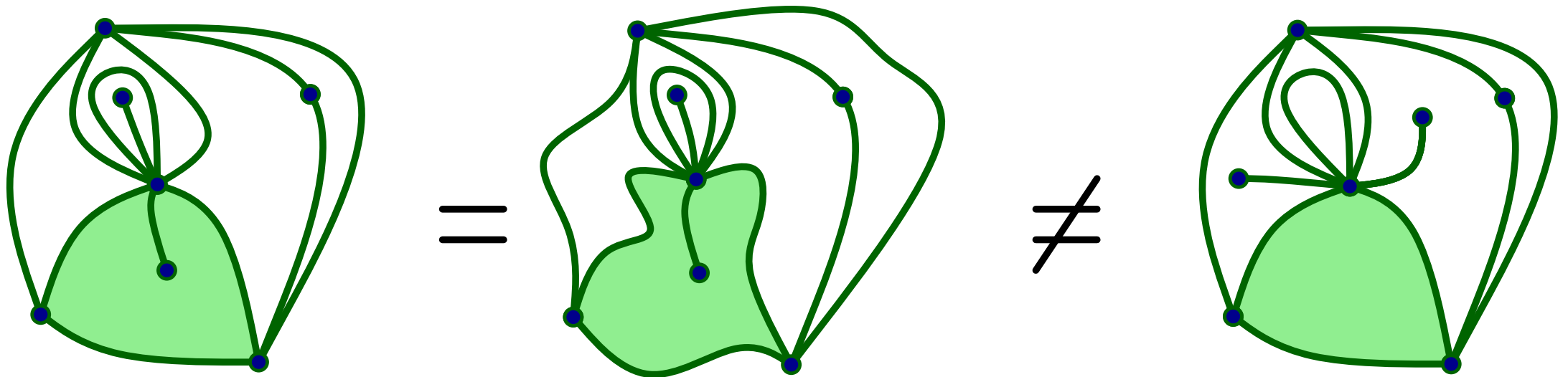
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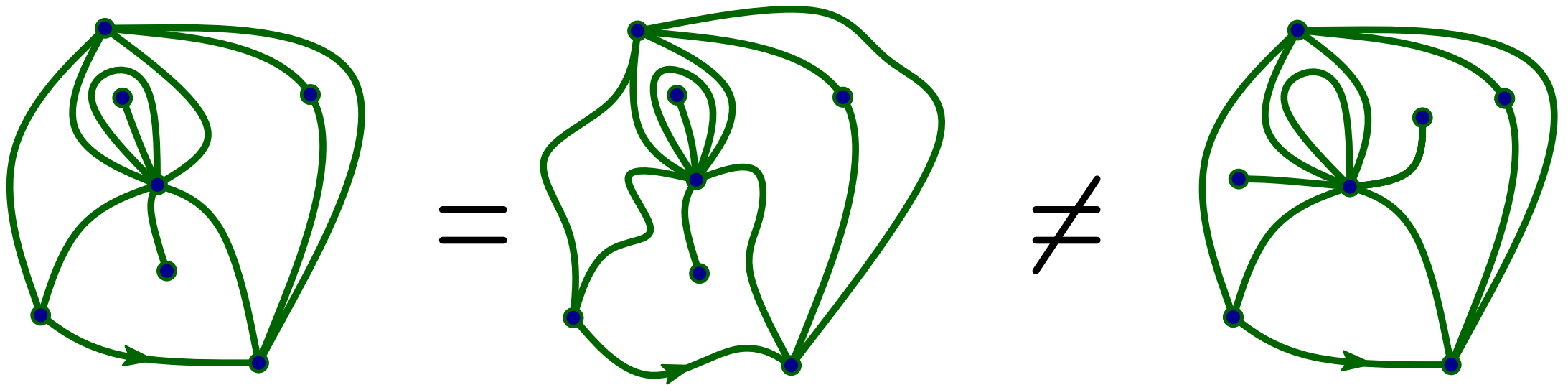


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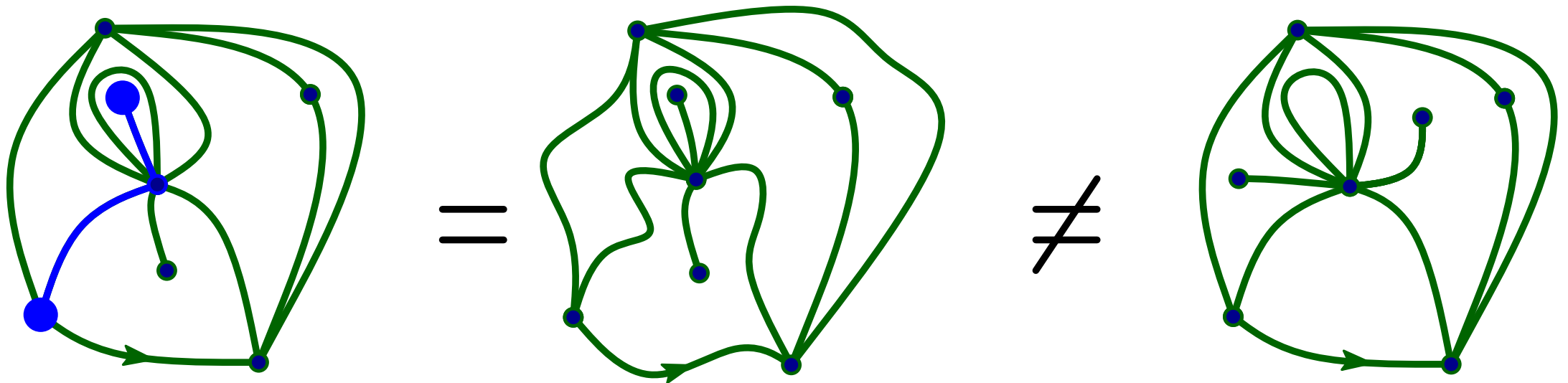


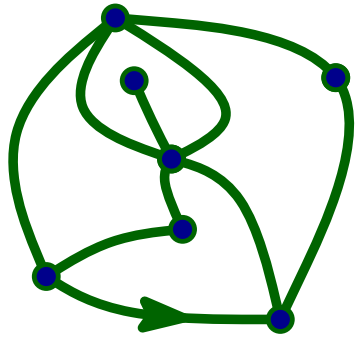planar map = planar graph + cyclic order of neigbours around each vertex.

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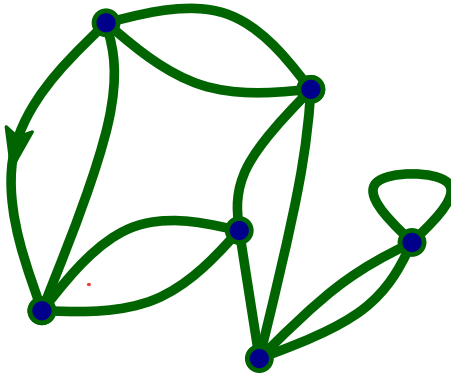
Plane maps are **rooted** : by orienting an edge.

Distance between two vertices = number of edges between them.
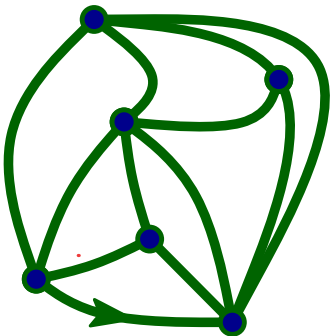Planar map = **Metric space**

# Which maps ?

**Quadrangulations**

**4-regular maps**

**Simple triangulations** (no loops nor multiple edges)

# Why maps ?

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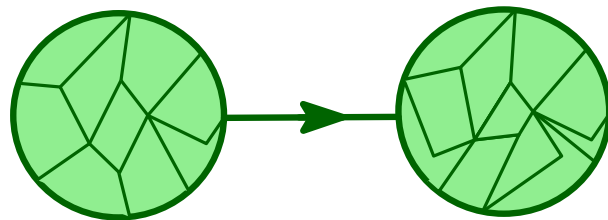What the motivation for studying maps instead of graphs ?

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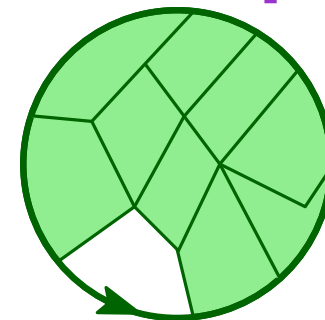
Structure allows recursive decomposition $\Rightarrow$ enumeration [Tutte, '60s].

Two possibilities:



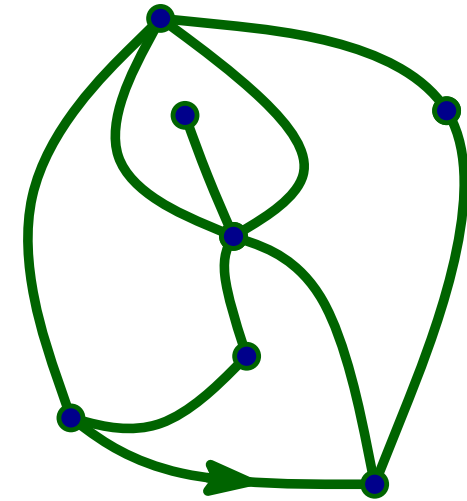The root edge is an isthmus     OR     The root edge is NOT an isthmus

$q_n =$ number of quadrangulations with $n$ faces $= \dfrac{2}{n+2} \dfrac{3^n}{n+1} \dbinom{2n}{n}$

# Random maps

$\mathcal{Q}_n = \{$Quadrangulations of size $n\}$
$\quad = n + 2$ vertices, $n$ faces, $2n$ edges

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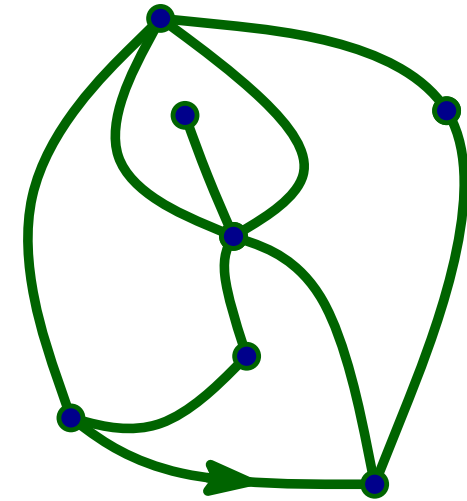$(V(Q_n), d_{gr})$ is a random compact metric space

Simulations by N.Curien

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# Random maps

**What is the behavior of $Q_n$ when $n$ goes to infinity ?**
**typical distances?**
**convergence towards a continuous object ?**

well understood:

• Schaeffer's bijection : quadrangulations $\leftrightarrow$ labeled trees.
Labels in the trees $=$ distances in the map.

• distance between two random points $\sim n^{1/4} +$ law of the distance
[Chassaing-Schaeffer '04]

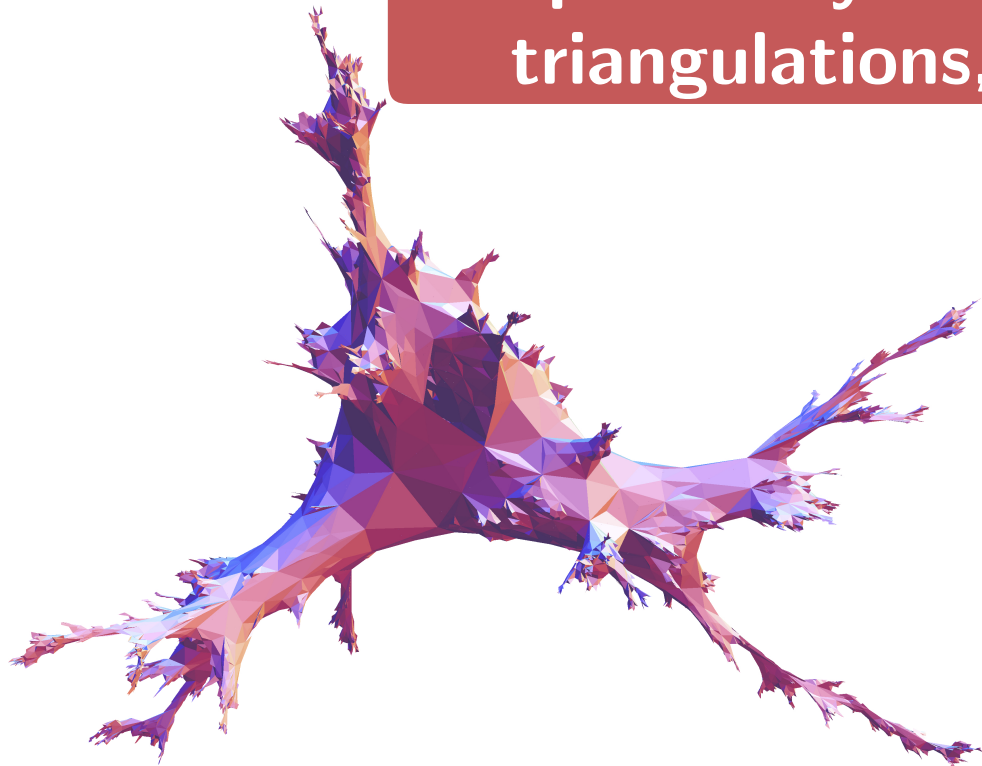• cvgence of normalized quadrangulations $+$ limiting object: Brownian map.
[Marckert-Mokkadem '06], [Le Gall '07], [Miermont '08],
[Miermont 13], [Le Gall 13]

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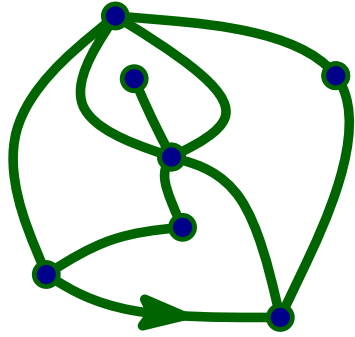
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# Random maps

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**Idea :** The Brownian map is a **universal** limiting object.
All "reasonable models" of maps (properly rescaled) are expected to converge towards it.

**Problem :** These results relie on nice bijections between maps and labeled trees [Schaeffer '98], [Bouttier-Di Francesco-Guitter '04].
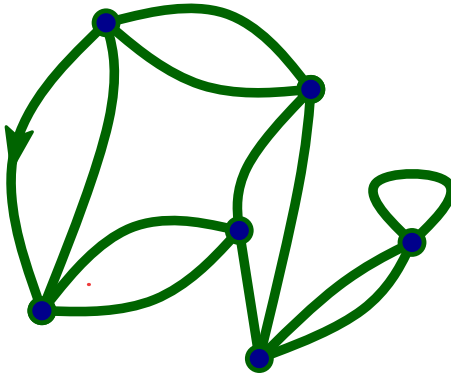
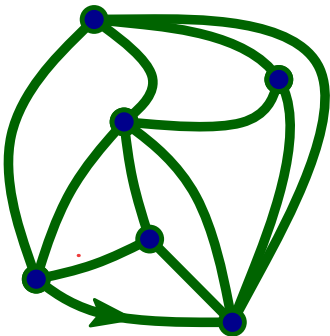# Which maps ?

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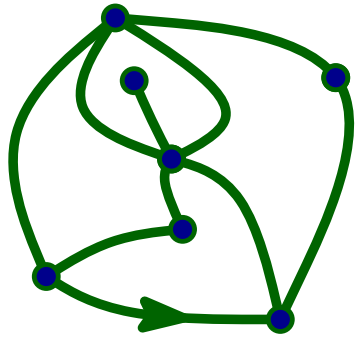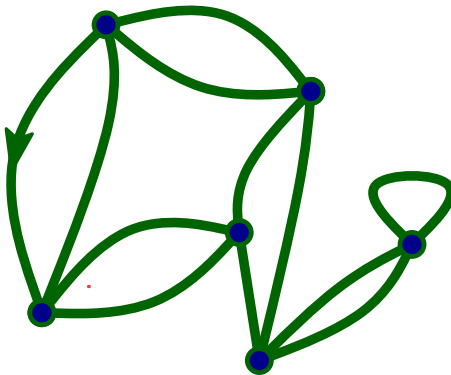[Tutte, 60], [Cori-Vauquelin '81], [Schaeffer '98]

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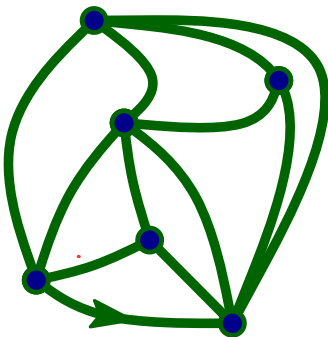[Tutte, 60], [Cori-Vauquelin '81], [Schaeffer '98]

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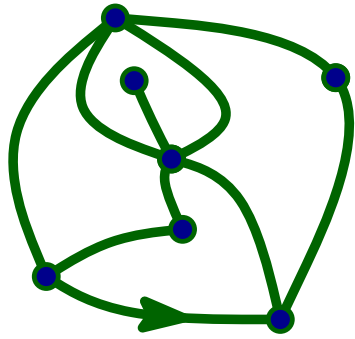
Number of rooted 4-regular maps with $n$ vertices:

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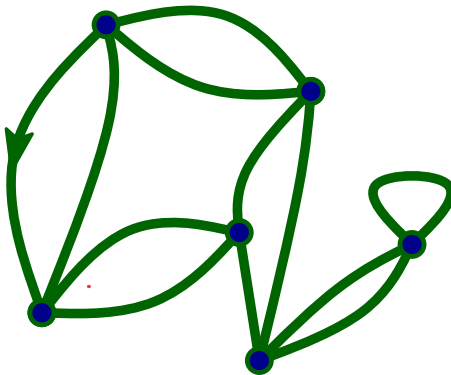## Simple triangulations (no loops nor multiple edges)

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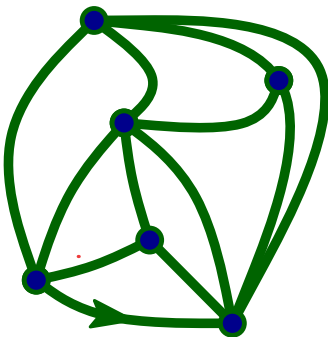## Quadrangulations

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Number of simple triangulations with $n + 2$ vertices:

$$\Delta_n = \frac{2 \cdot (4n-3)!}{n!(3n-1)!}$$

[Tutte, 62], [Poulalhon-Schaeffer '05]

# History : what questions about maps ?

- **Enumerate them** : a lot of different techniques

Recursive decomposition: [Tutte, '60]

Matrix integrals: [t'Hooft, '74], [Brézin, Itzykson, Parisi and Zuber '78]

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# Today: what's the plan ?

What is a blossoming tree ?

Can we unify the constructions involving blossoming trees ?

Can we prove some convergence results to the Brownian map using
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# Today: what's the plan ?

What is a blossoming tree ?      **Wait a second**

Can we unify the constructions involving blossoming trees ?

**Yes, cf also** [Bernardi,Fusy]

Can we prove some convergence results to the Brownian map using blossoming trees ?
i.e. can we put "distances" on trees ?      **Yes ... for some models**

# What is a blossoming tree ?

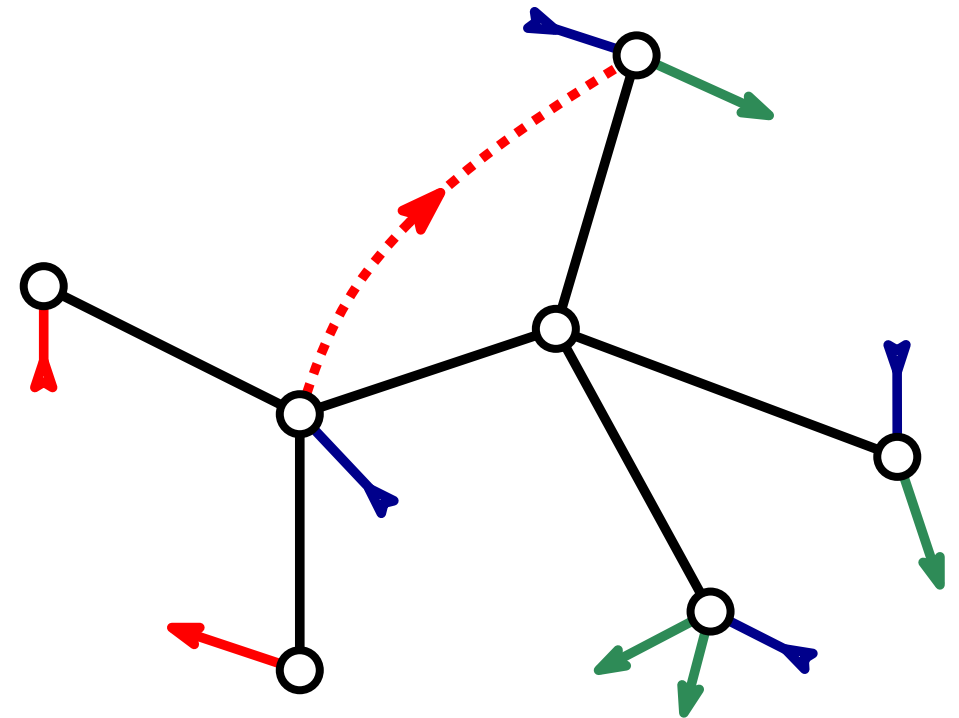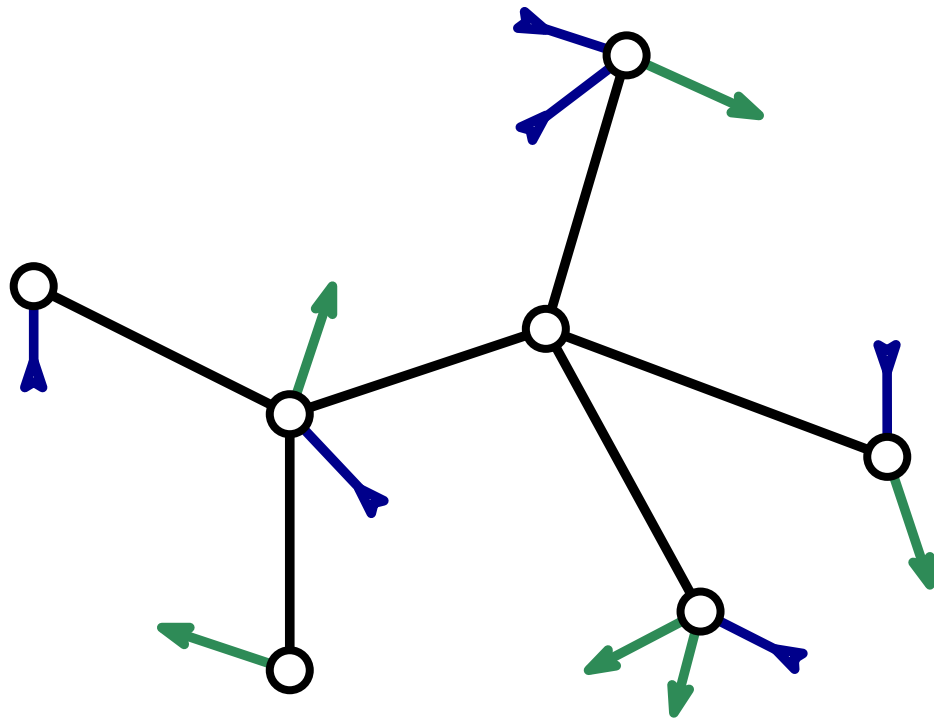A **blossoming tree** is a plane tree where vertices can carry **opening stems** or **closing stems**, such that :
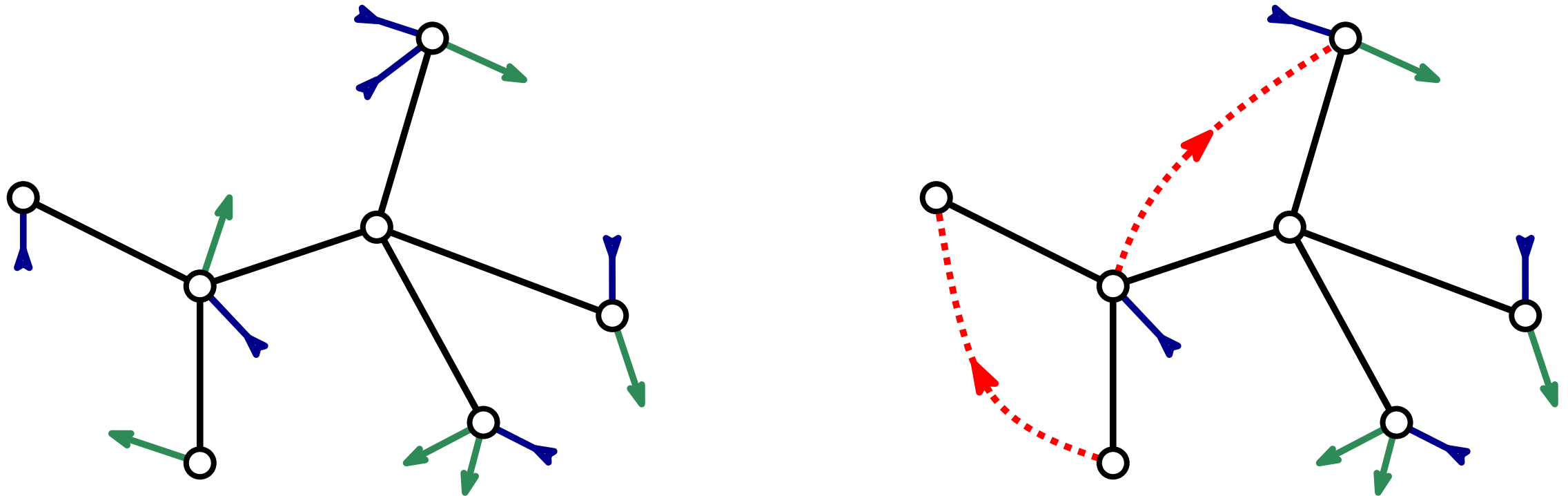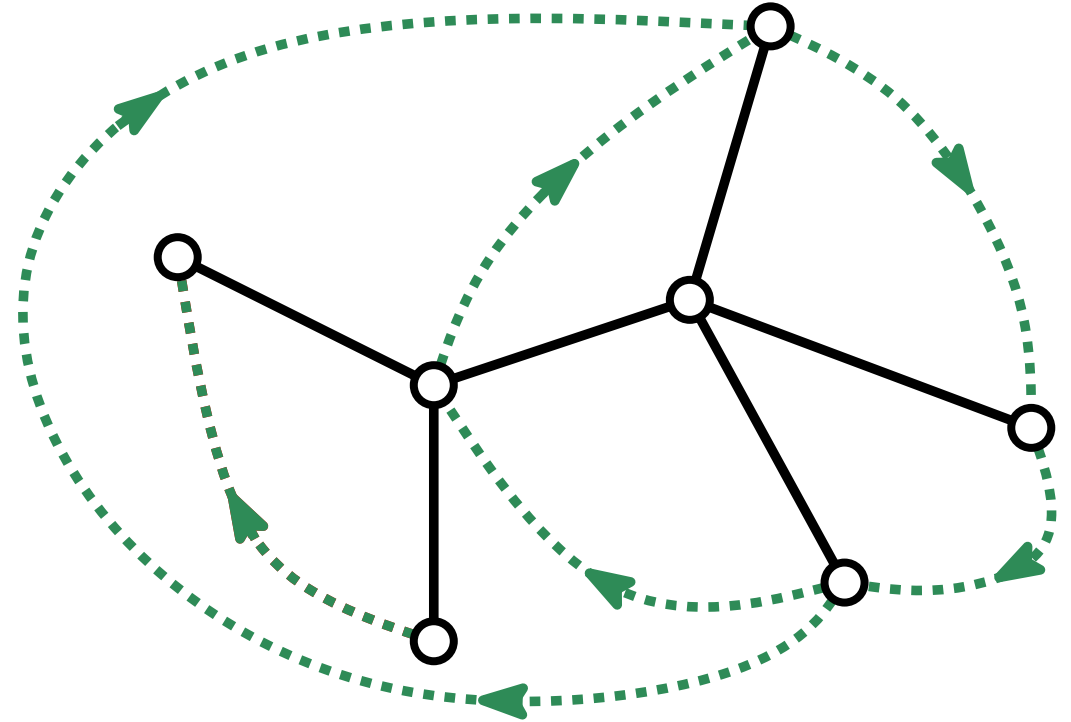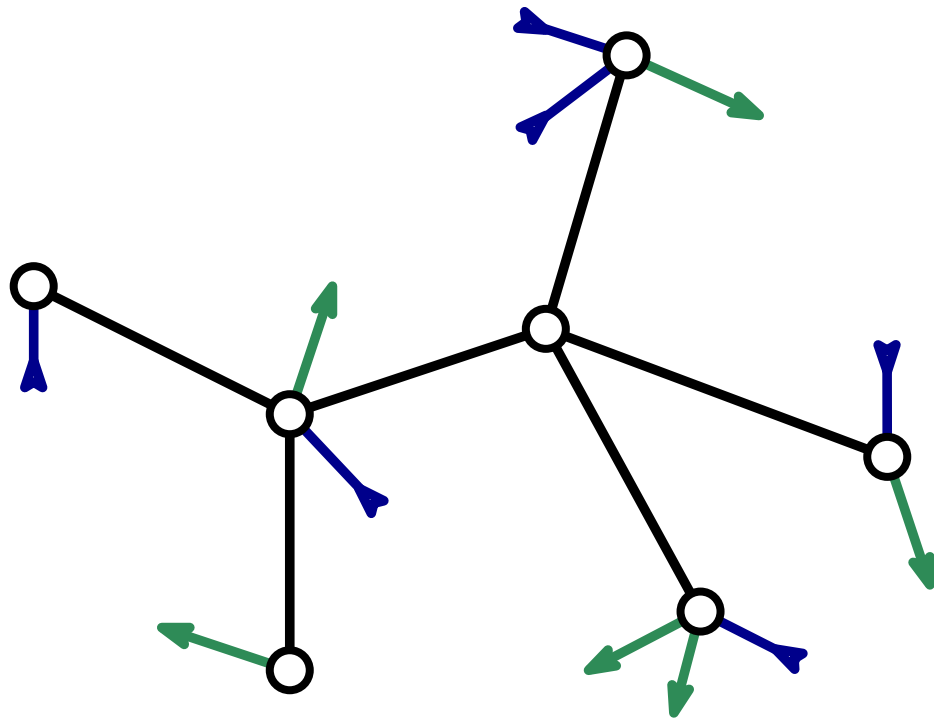
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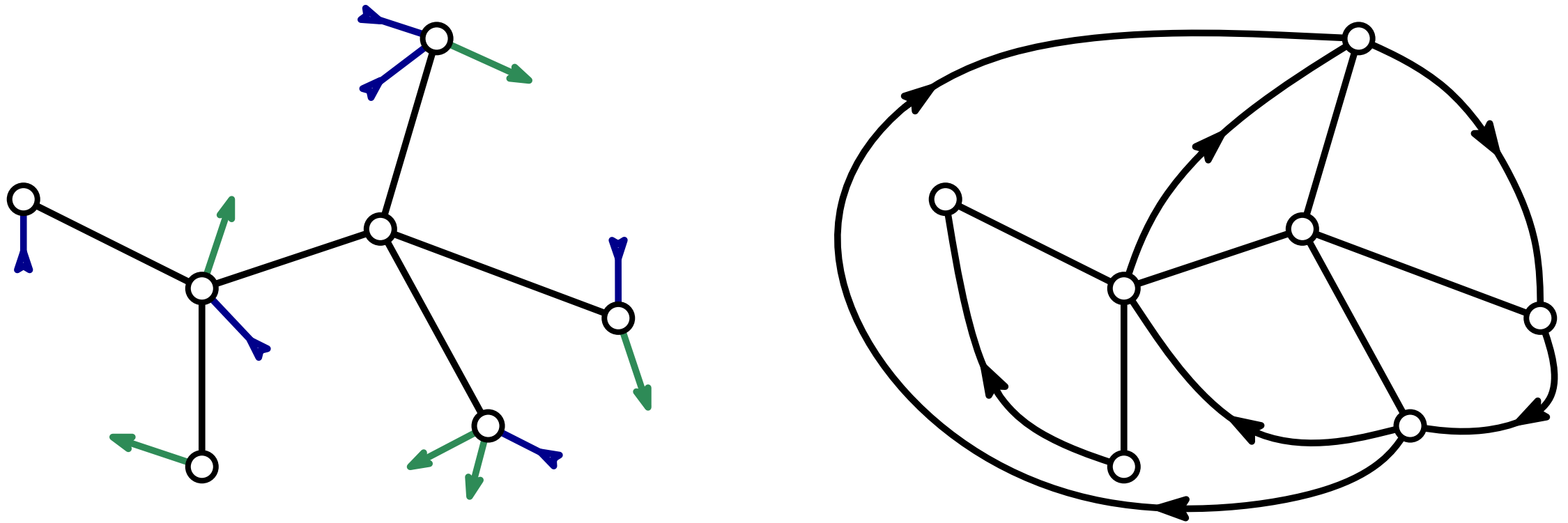
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A plane map can be canonically associated to any blossoming tree by
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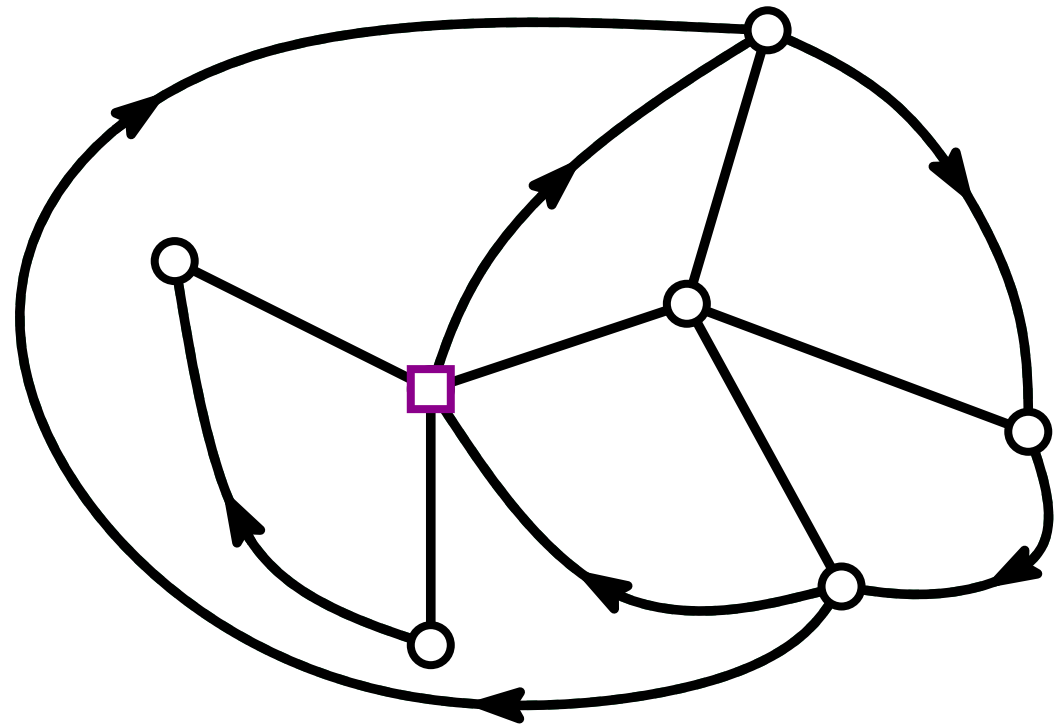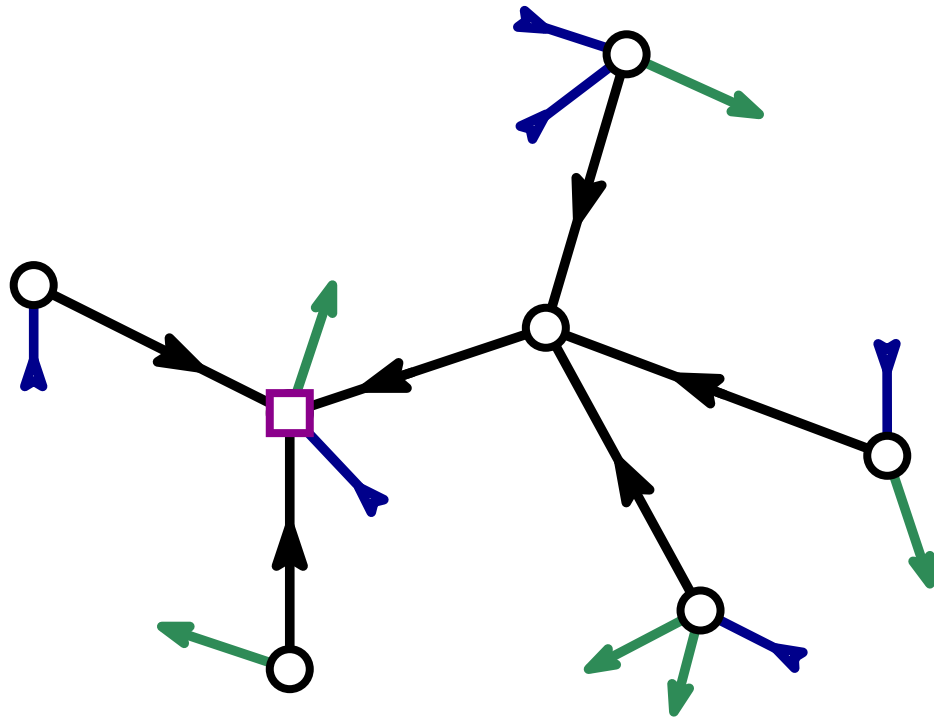


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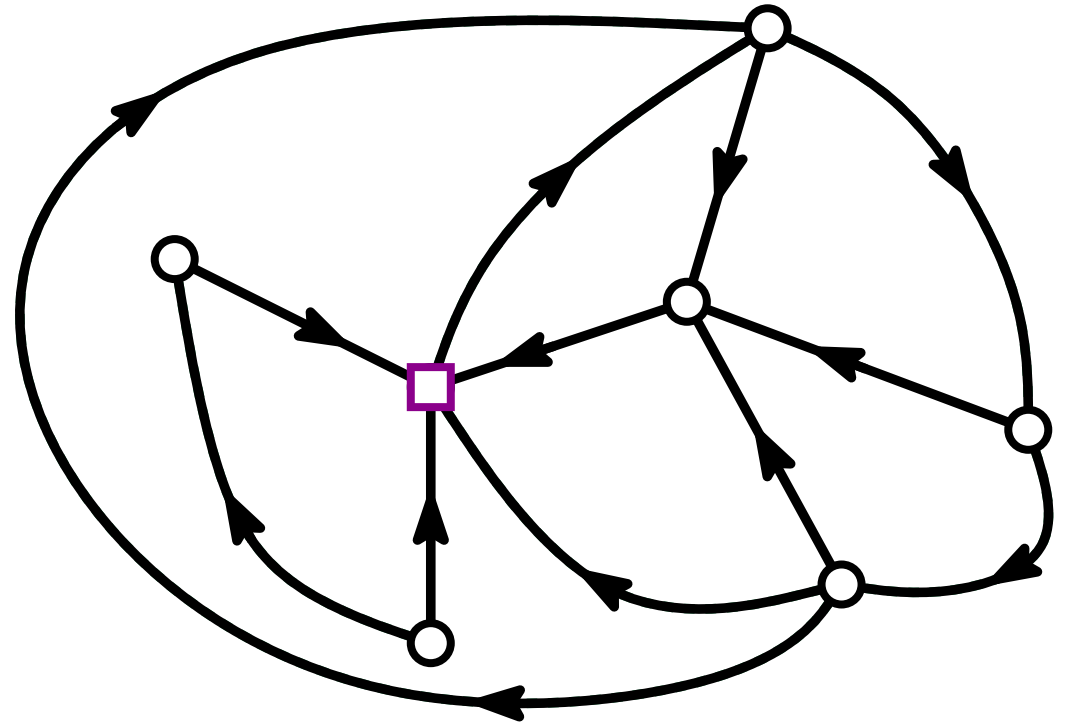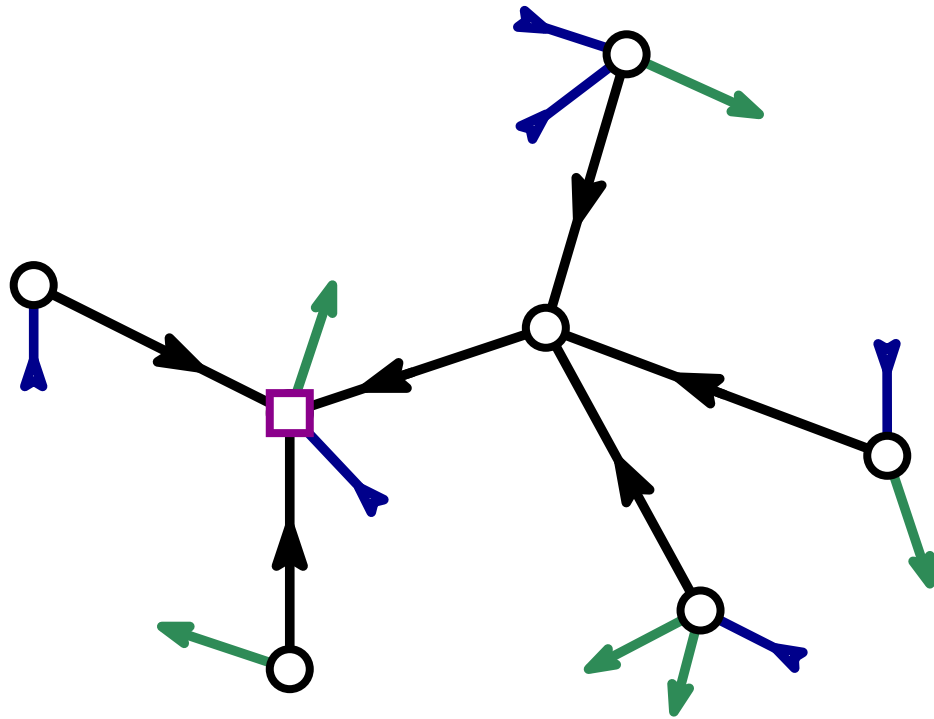


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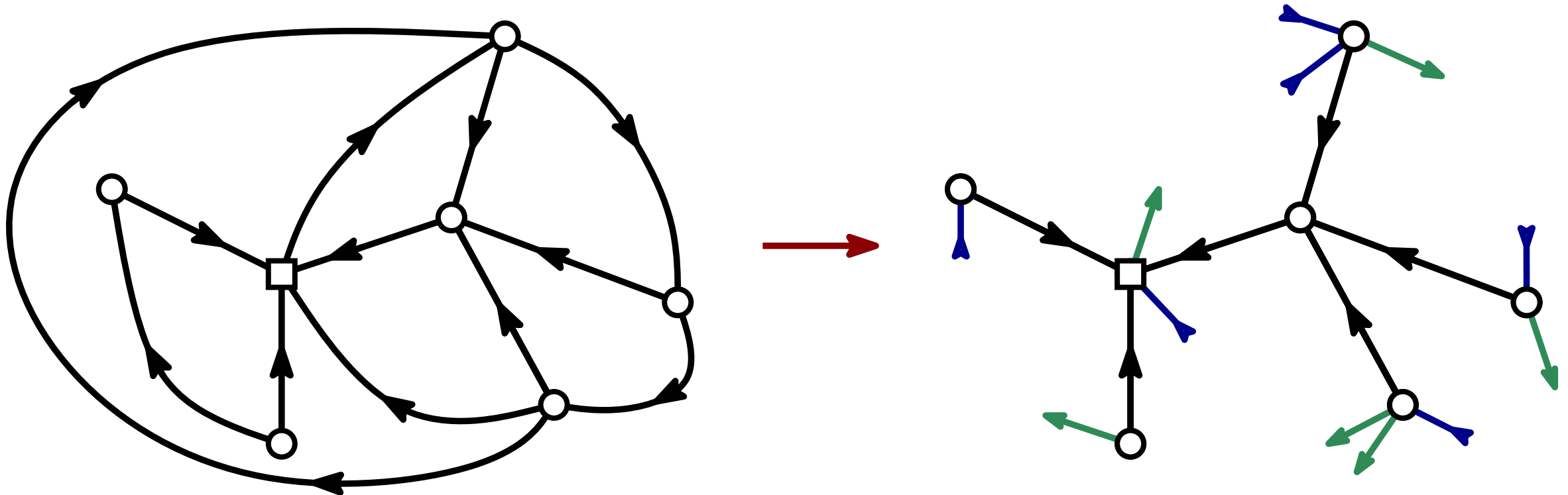⇒ Accessible orientation of the map without ccw cycles.

# Can we transform a plane map into a blossoming tree ?

**Theorem :** [Bernardi '07], [A., Poulalhon 14+]
If a plane map $M$ has a marked vertex $v$ is endowed with an orientation such that :
- there exists a directed path from any vertex to $v$,
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Proof by induction on the number of faces + identification of closure edges ....
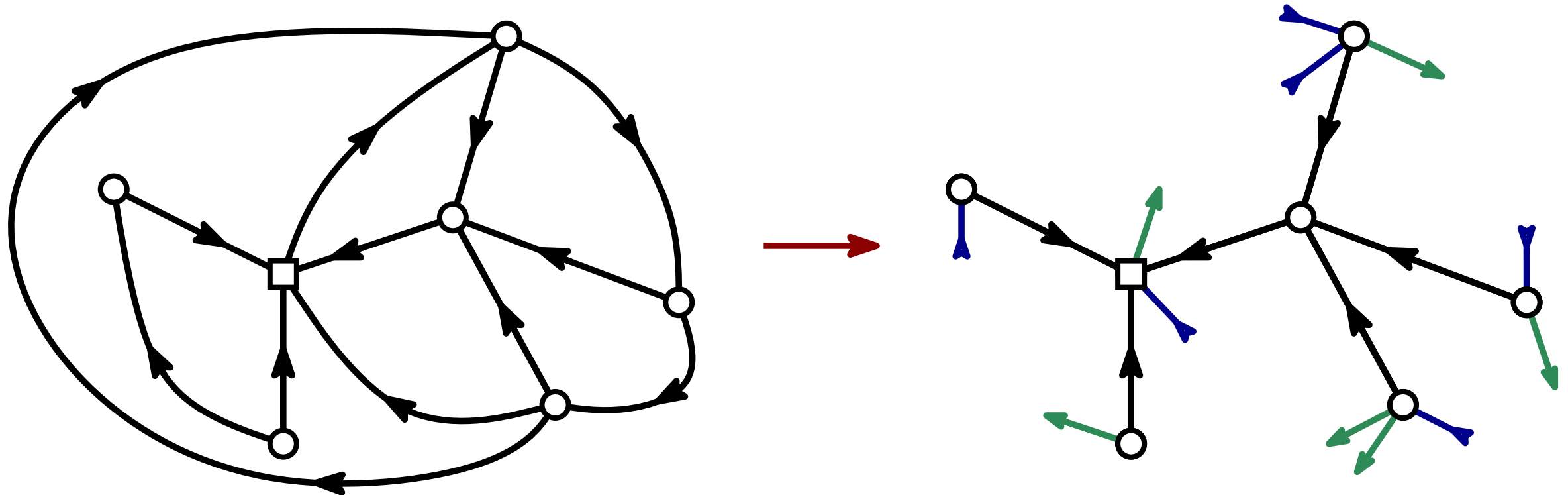
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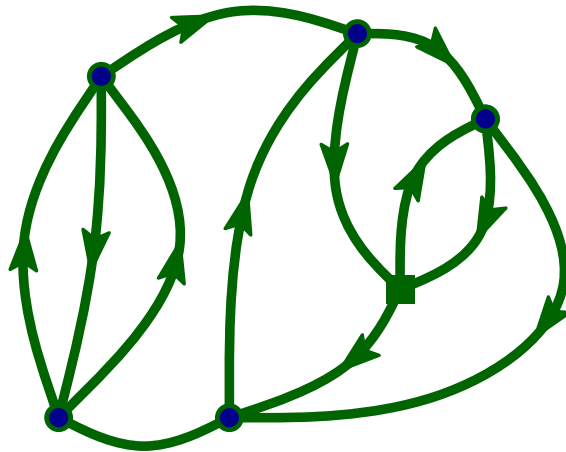
**4-regular maps**



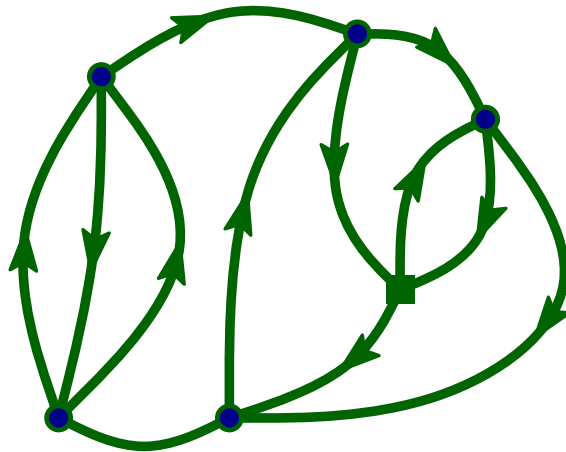2 outgoing edges/vertex
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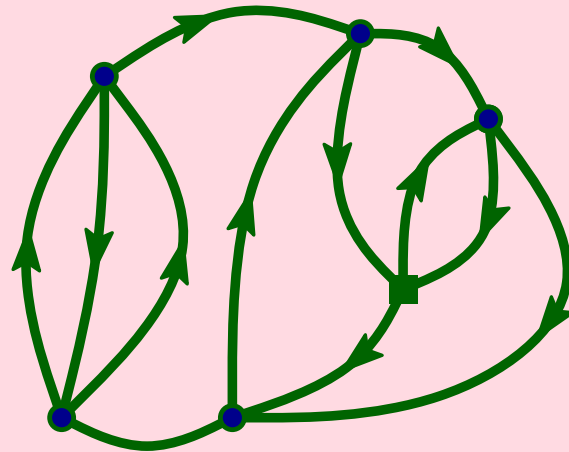


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**A map is 4-regular iff it admits an orientation with indegree 2 and outdegree 2 for each vertex.**

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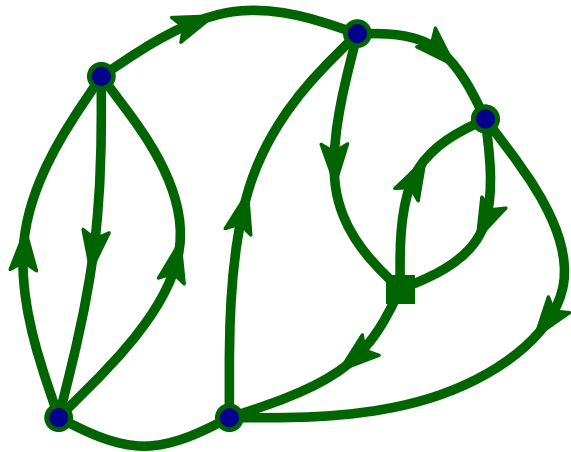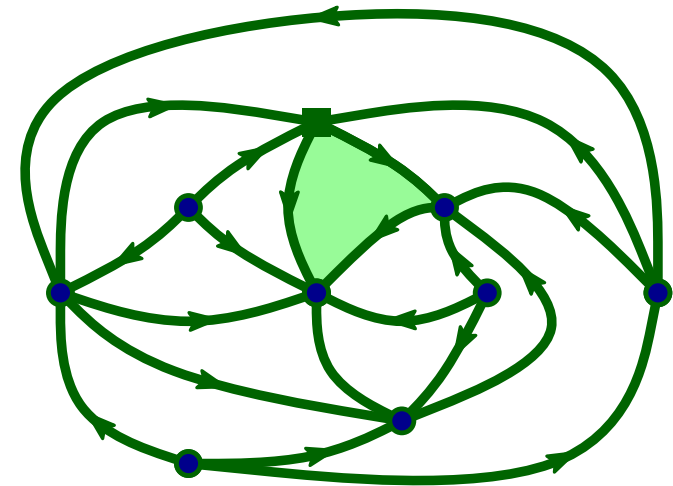


| **4-regular maps** | **Simple triangulations** |
| :---: | :---: |
| 2 outgoing edges/vertex | 3 outgoing edges / non-root vertex |
| 2 ingoing edges/vertex | 1 outgoing edge / root vertex |

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To apply the construction: need to find **canonical orientations**



**4-regular maps**
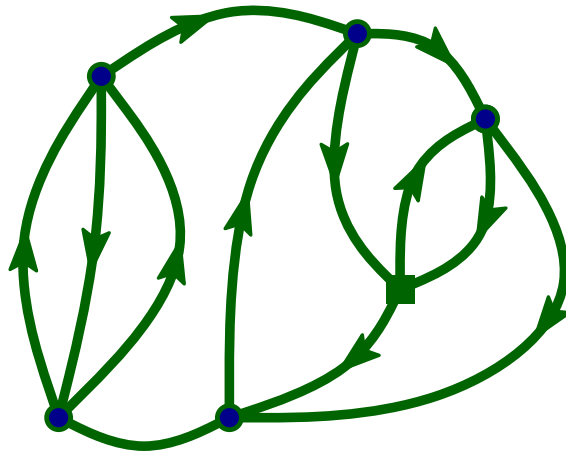
2 outgoing edges/vertex
2 ingoing edges/vertex

**Simple triangulations**
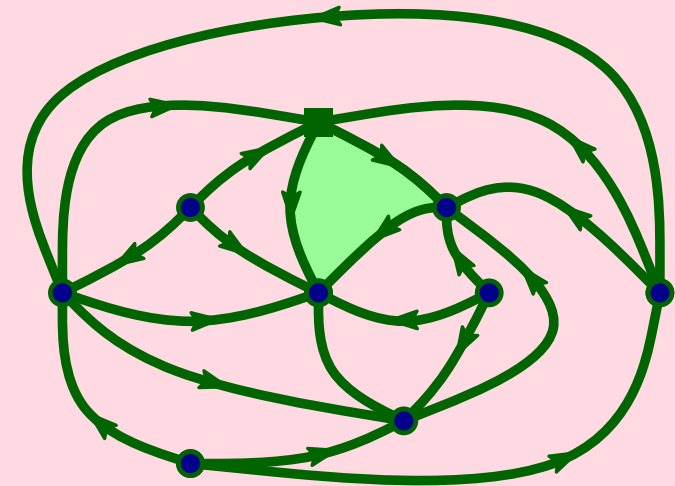
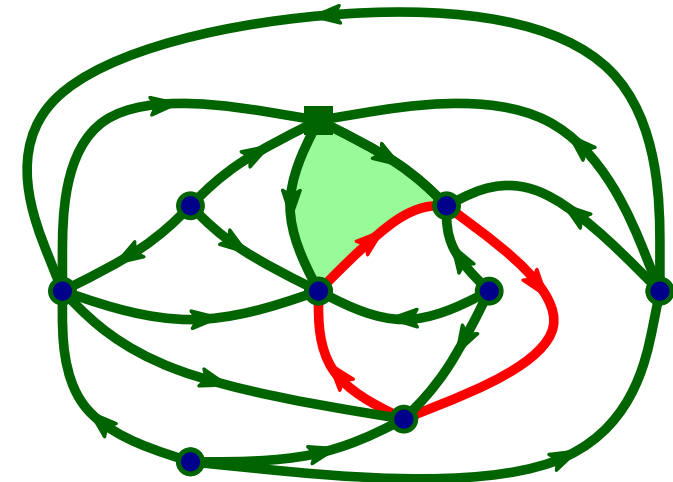3 outgoing edges / non-root vertex
1 outgoing edge / root vertex

**A triangulation is simple iff it admits an orientation with:**
**outdegree 3 for each non-root vertex**
**outdegree 1 for each vertex on the root face.**

# Orientations

## 4-regular maps



2 outgoing edges/vertex
2 ingoing edges/vertex

## Simple triangulations
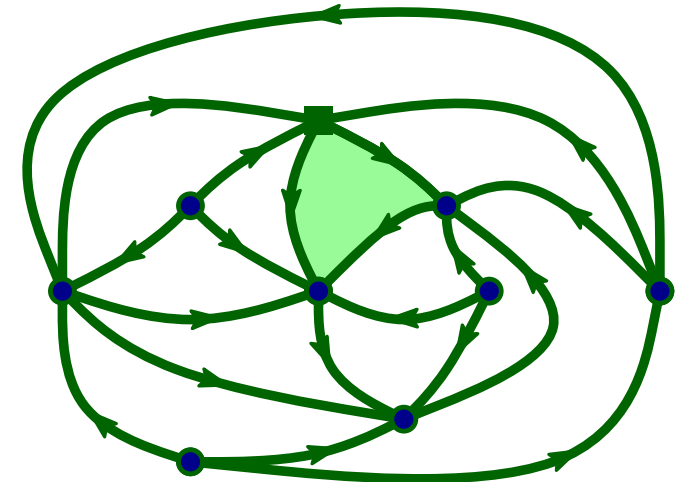


3 outgoing edges / non-root vertex
1 outgoing edge / root vertex

**Many families admit a caracterization via orientations**
(description of the orientation = outdegree for each vertex is prescribed)

# Orientations

## 4-regular maps
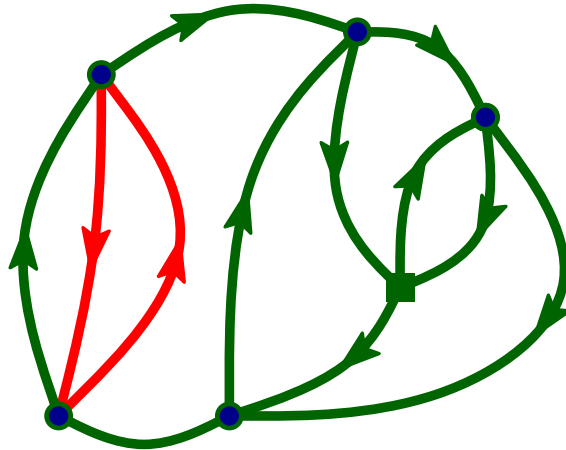


2 outgoing edges/vertex
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## Simple triangulations
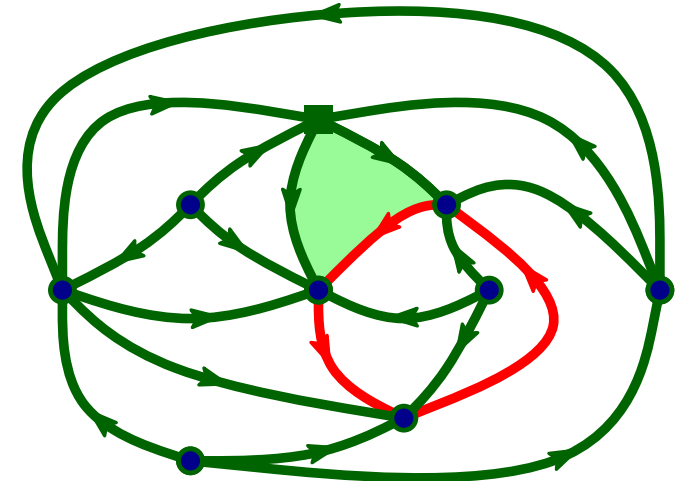


3 outgoing edges / non-root vertex
1 outgoing edge / root vertex

Theorem requires accessible orientation without ccw cycles:
Too much too ask ?

# Orientations

## 4-regular maps
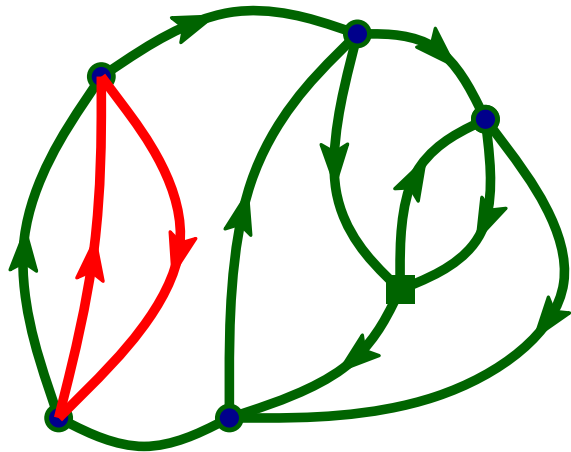


2 outgoing edges/vertex
2 ingoing edges/vertex

## Simple triangulations



3 outgoing edges / non-root vertex
1 outgoing edge / root vertex

Theorem requires accessible orientation without ccw cycles:   **NO !**
Too much too ask ?

**Proposition:** [Felsner '04]
For a given map and orientation, there exists a unique orientation with the same outdegrees and without ccw cycles.
If there exists one accessible such orientation, all of them are accessible.

# Summary

- Take a family of maps,
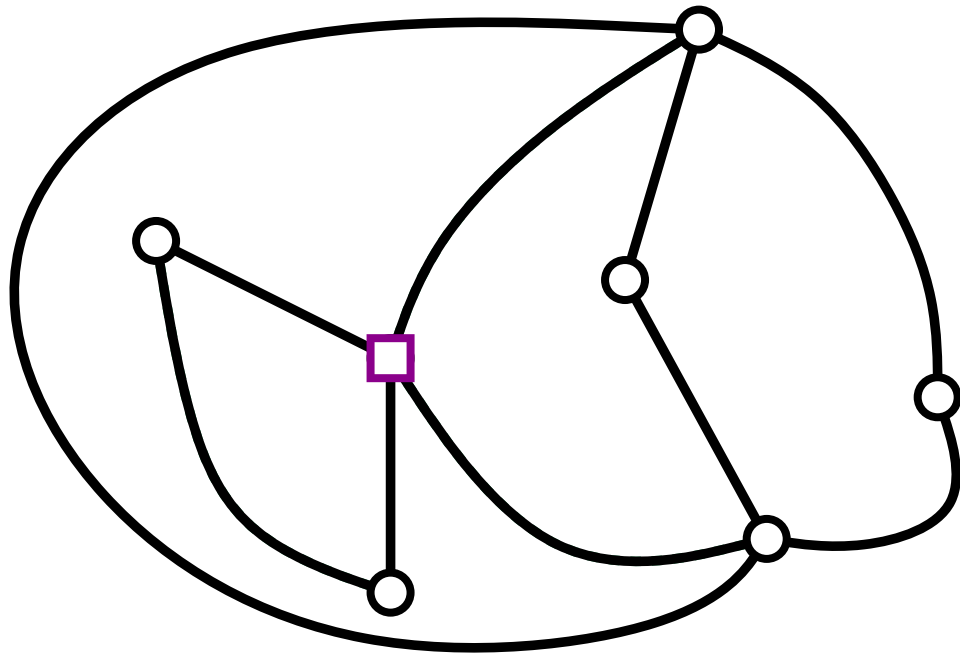


Maps with even degrees.

# Summary

- Take a family of maps,
- Try to find a caracterization of the family by an orientation,



Maps with even degrees.
Orientations with same out/in degrees

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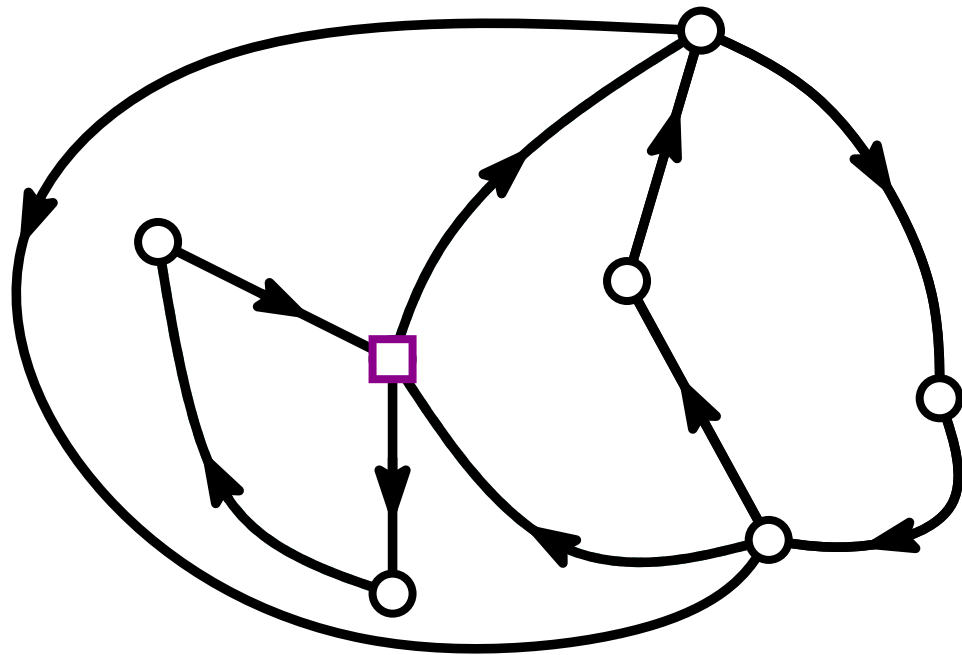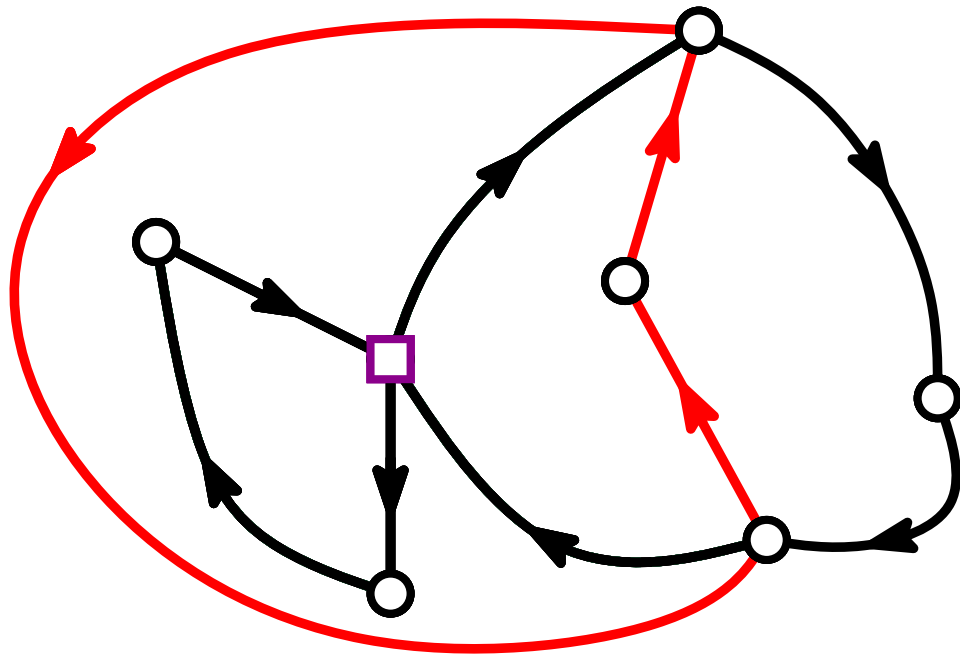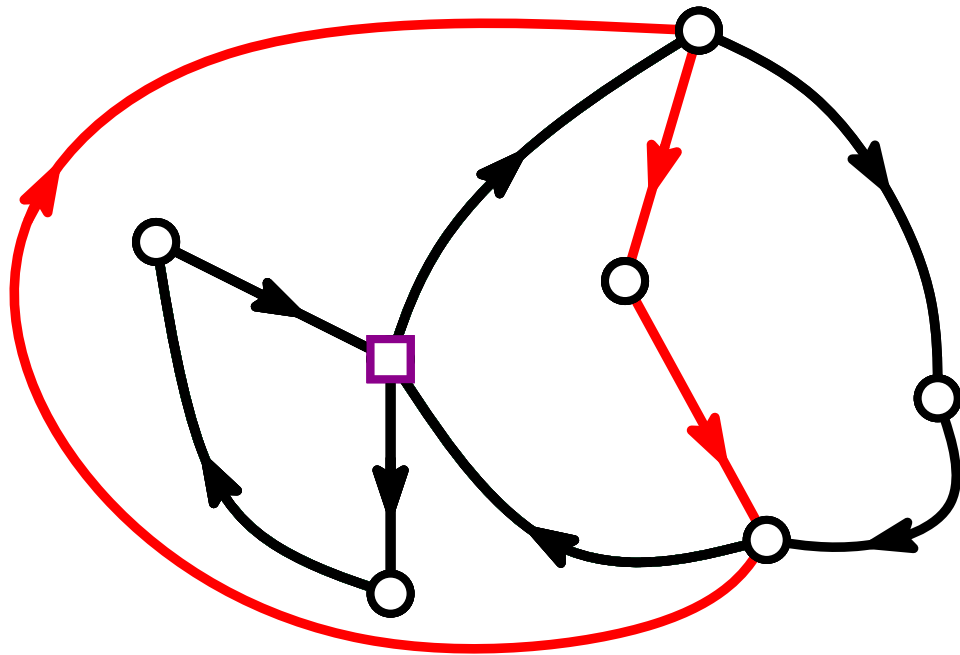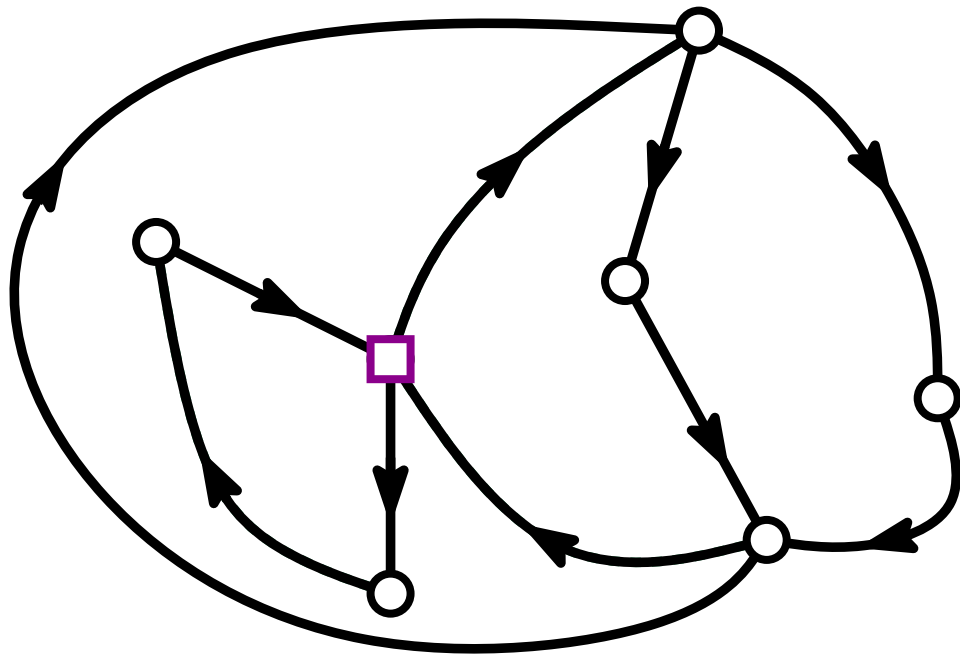- Consider the unique orientation without counterclockwise cycles,

Maps with even degrees.
Orientations with same out/in degrees

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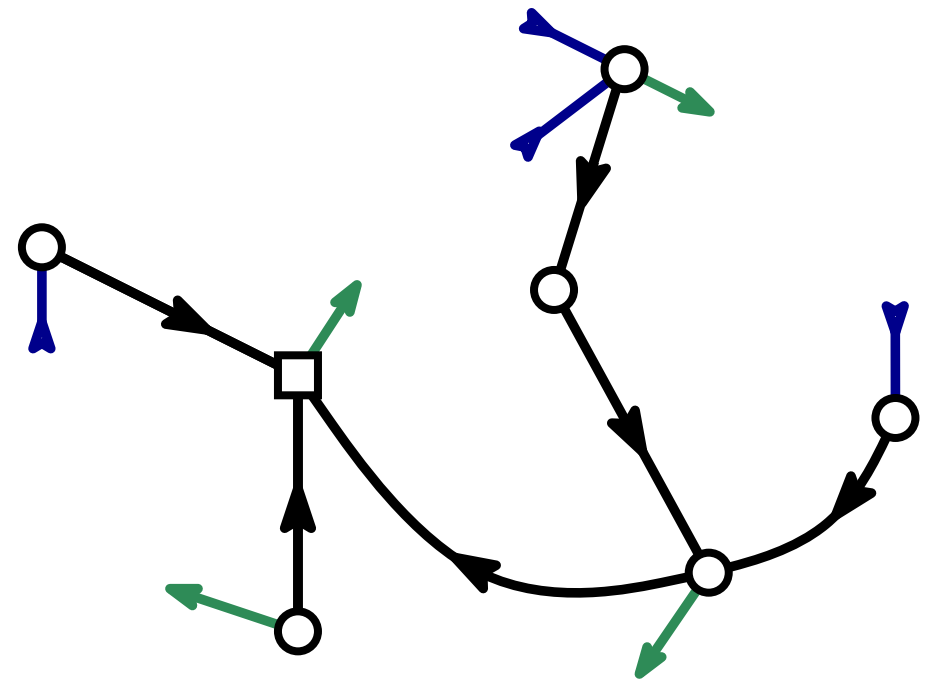- Consider the unique orientation without counterclockwise cycles,



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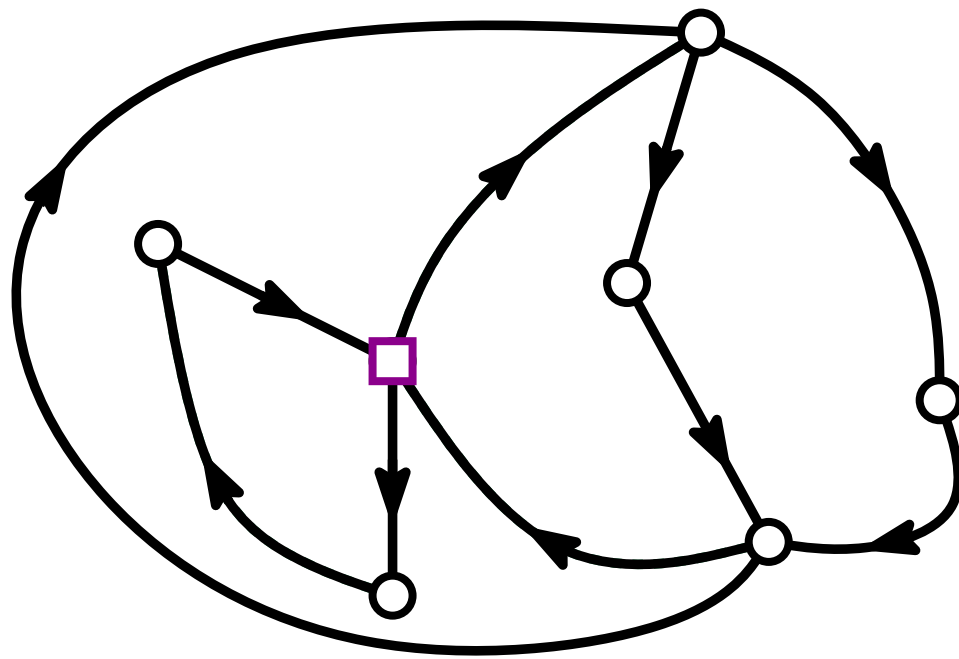- Consider the unique orientation without counterclockwise cycles,
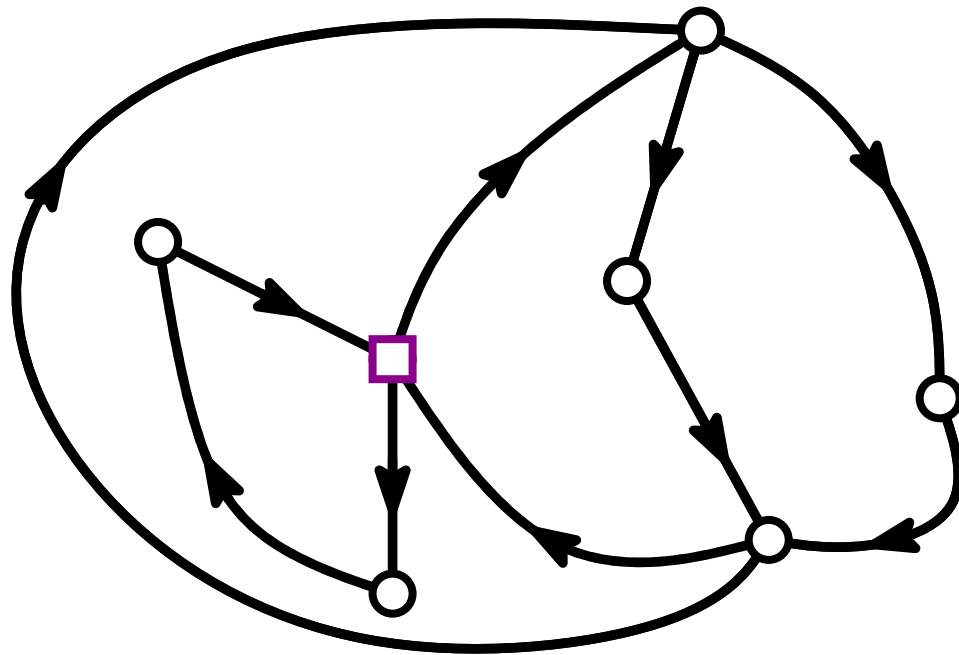


Maps with even degrees.
Orientations with same out/in degrees

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# Summary

- Take a family of maps,
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- Apply the bijection,



Maps with even degrees.
Orientations with same out/in degrees

# Summary

- Take a family of maps,
- Try to find a caracterization of the family by an orientation,
- Consider the unique orientation without counterclockwise cycles,
- Apply the bijection,
- Study the family of blossoming trees.
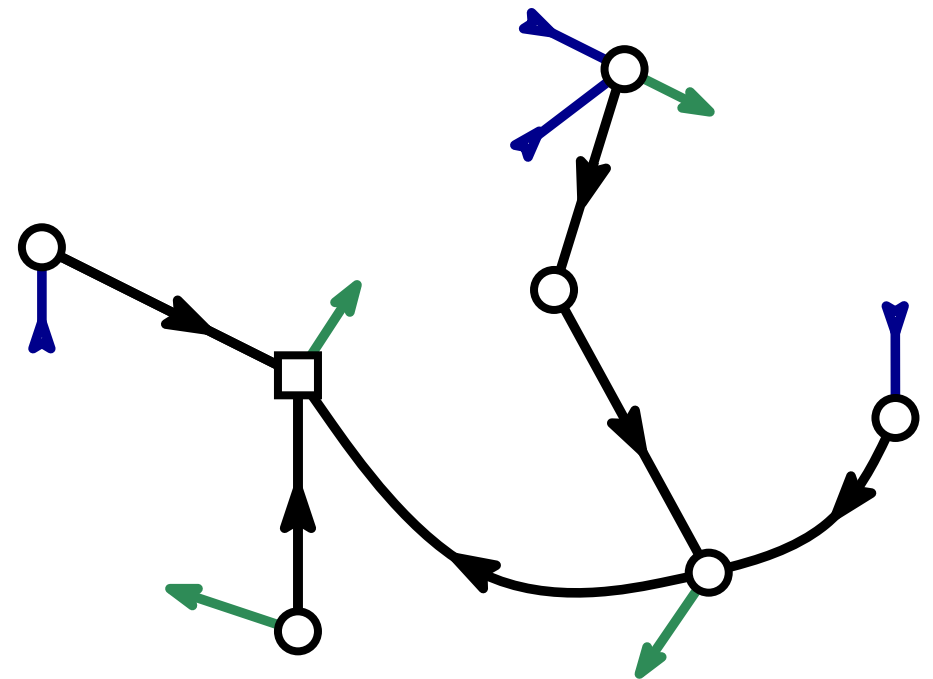


Maps with even degrees.
Orientations with same out/in degrees
⟷
Trees with same out/in degrees

# Distances in blossoming trees: simple triangulations
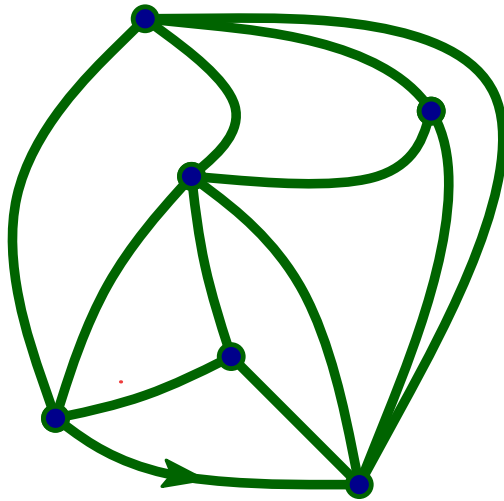


Simple Triangulation :
no multiple edges
no loops

Euler Formula : $v + f = 2 + e$
Triangulation : $2e = 3f$

$\mathcal{M}_n = \{$Simple triangulations of size $n\}$
$= n + 2$ vertices, $2n$ faces, $3n$ edges

$M_n$ = Random element of $\mathcal{M}_n$

**What is the behavior of $M_n$ when $n$ goes to infinity ?
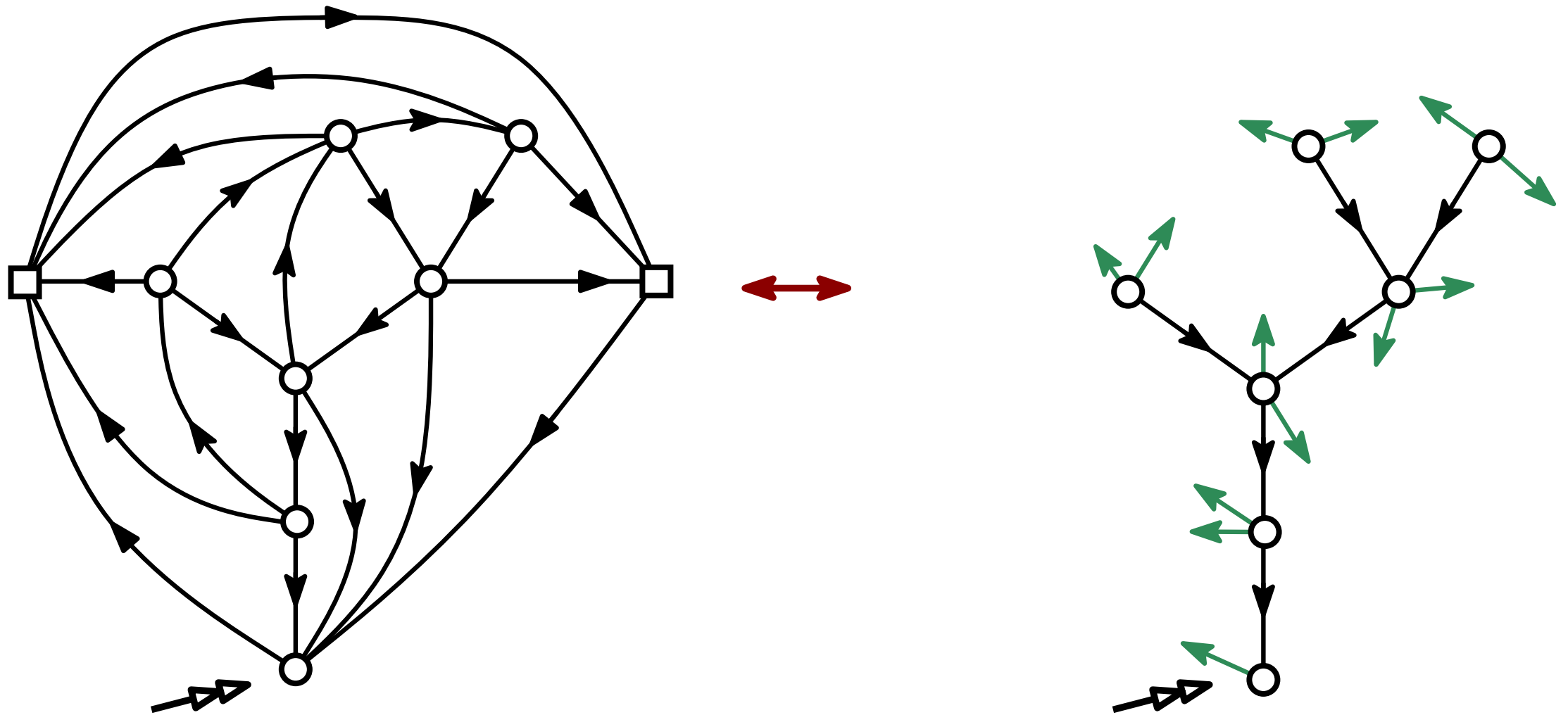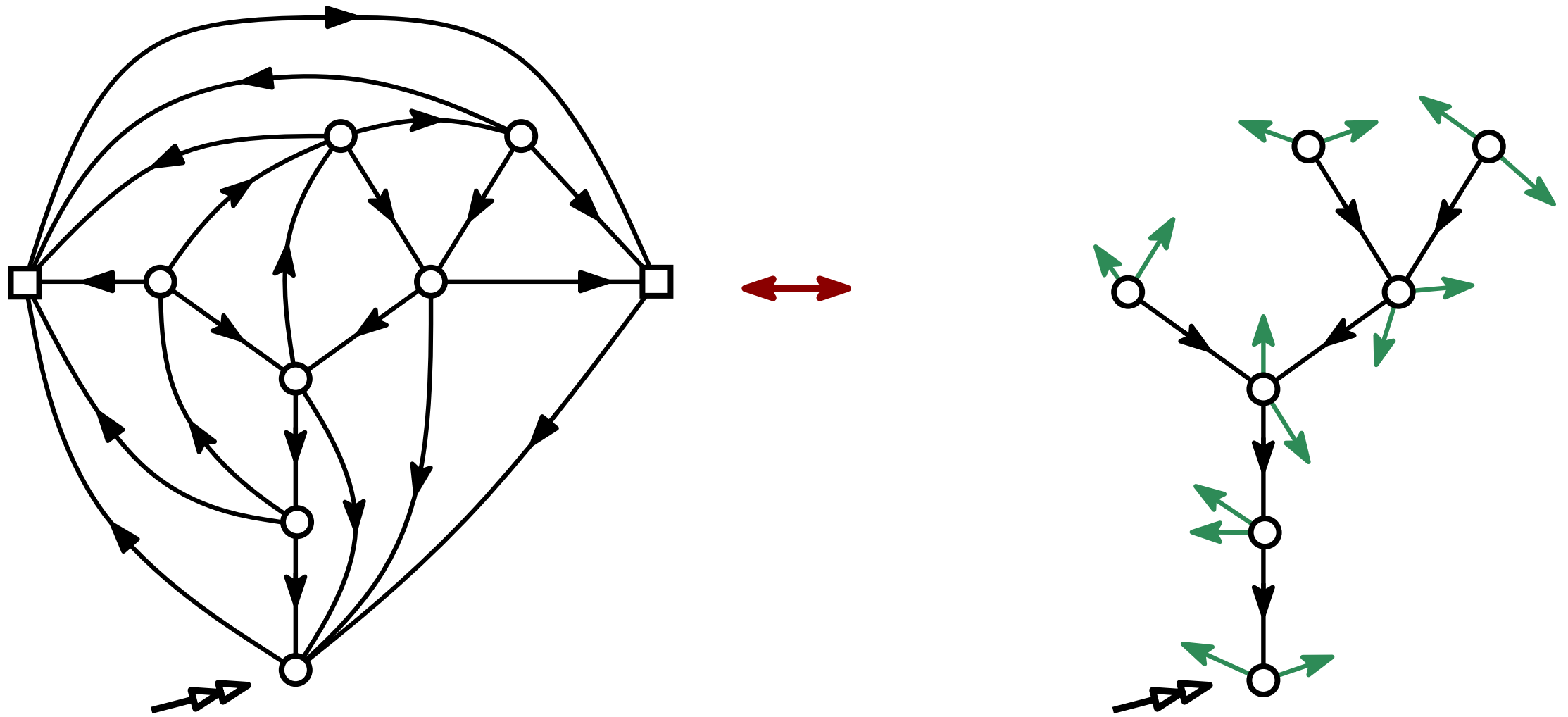typical distances ? Scaling limit of $M_n$ ?**

# From simple triangulations to blossoming trees



Simple triangulation endowed with
its unique orientation such that :
- no counterclockwise cycle
- $\mathsf{out}(v) = 3$ for $v$ an inner vertex
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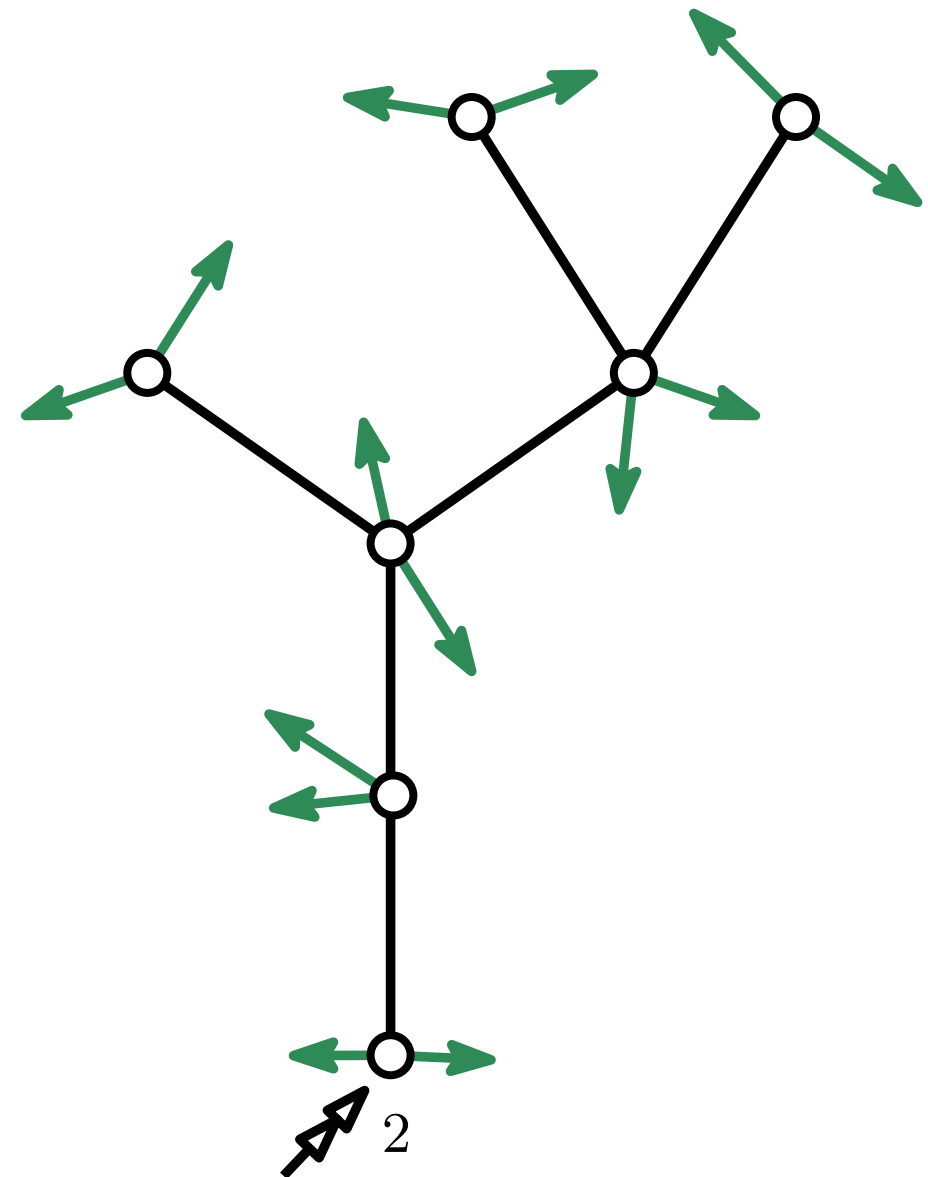
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# From simple triangulations to blossoming trees



**Theorem:** [Poulalhon, Schaeffer '05]

The closure operation is a bijection between balanced 2-blossoming trees and simple triangulations.

# Same bijection with corner labels

- Start with a planted 2-blossoming tree.
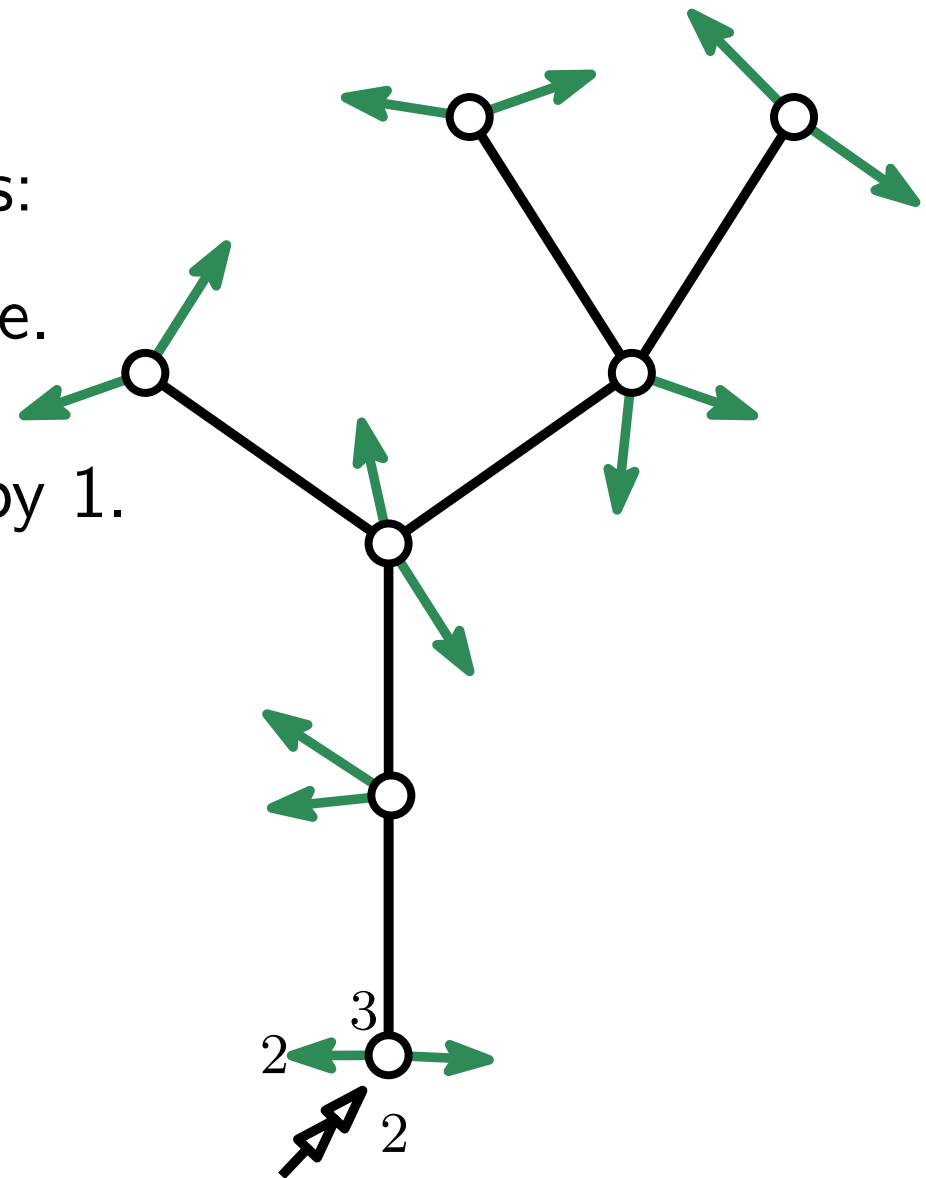- Give the root corner label 2.

# Same bijection with corner labels

- Start with a planted $2$-blossoming tree.
- Give the root corner label $2$.

In contour order, apply the following rules:

- Non-leaf to leaf, label does not change.
- Leaf to non-leaf, label increases by $1$.
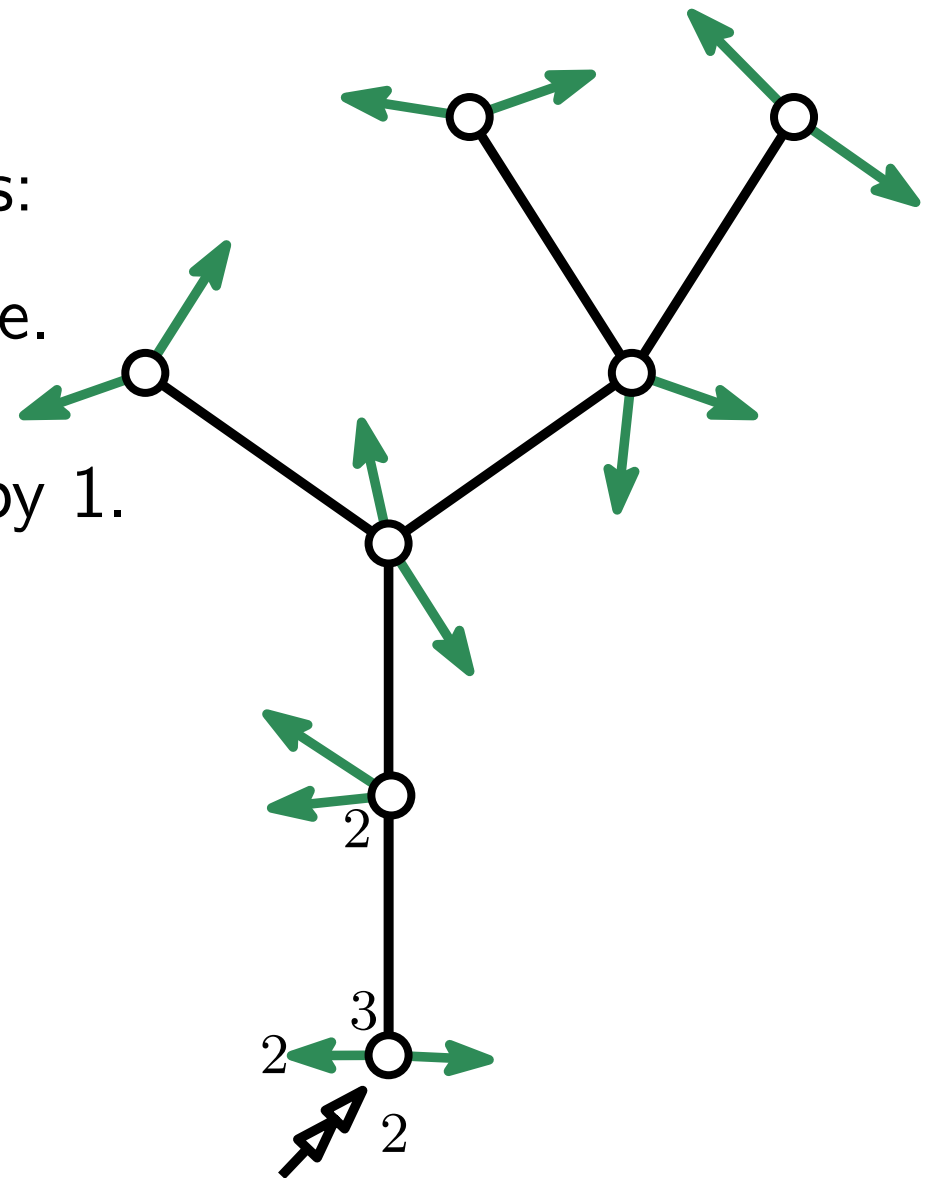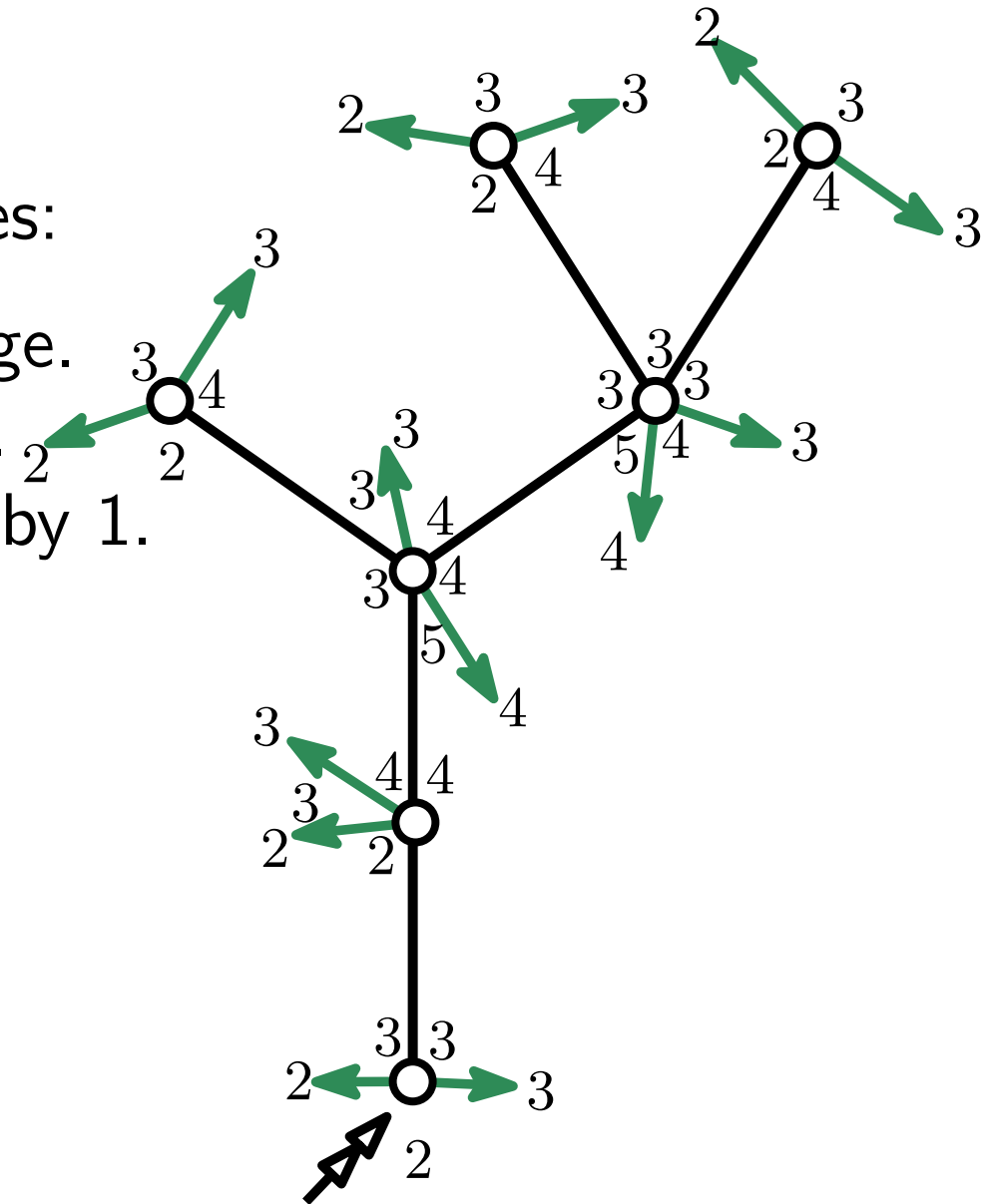- Non-leaf to non-leaf, label decreases by $1$.

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Aside: Tree is balanced $\Leftrightarrow$
  all labels $\geq 2$
  +root corner incident to two stems
Closure: Merge each leaf with the first subsequent corner with a smaller label.
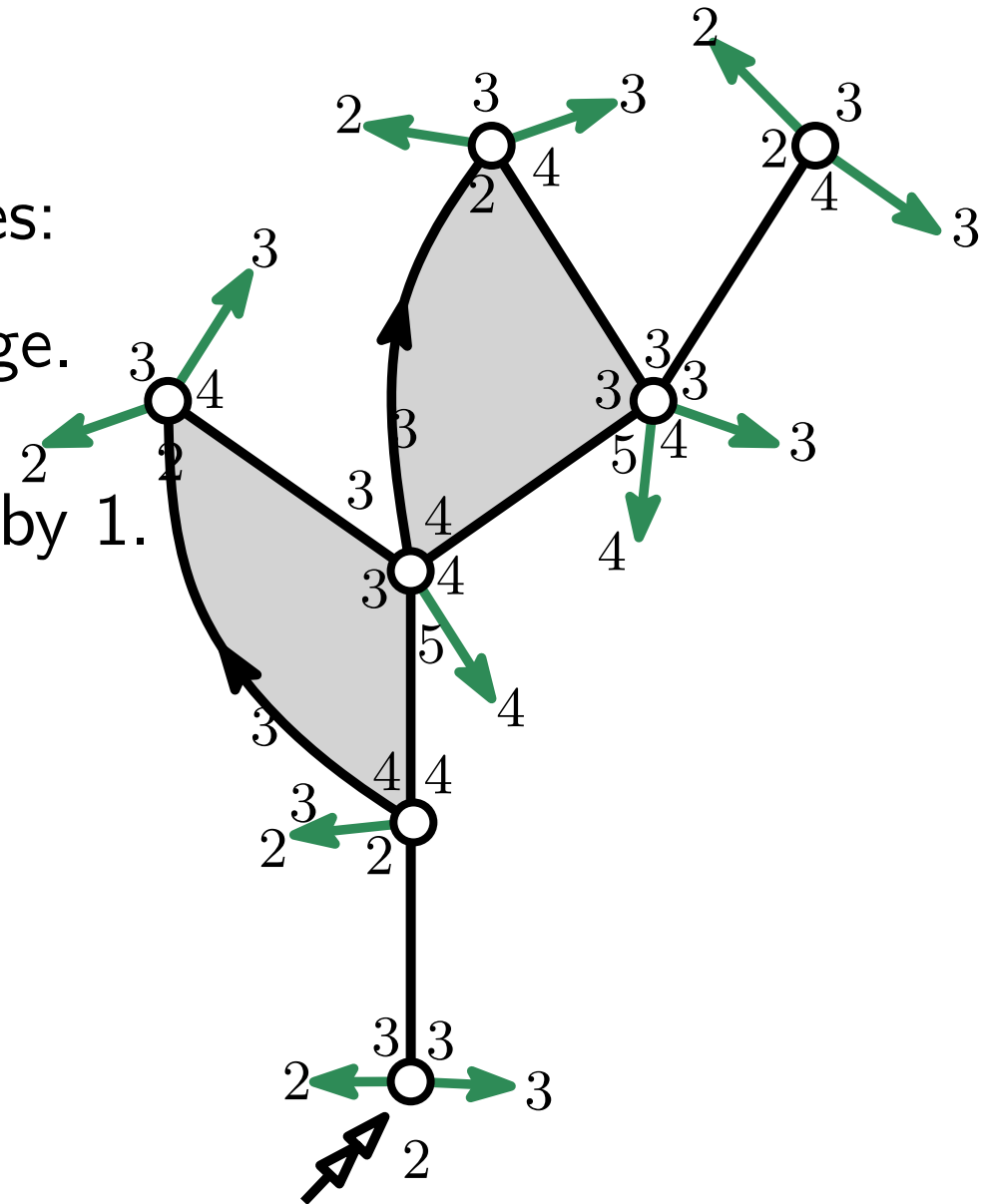
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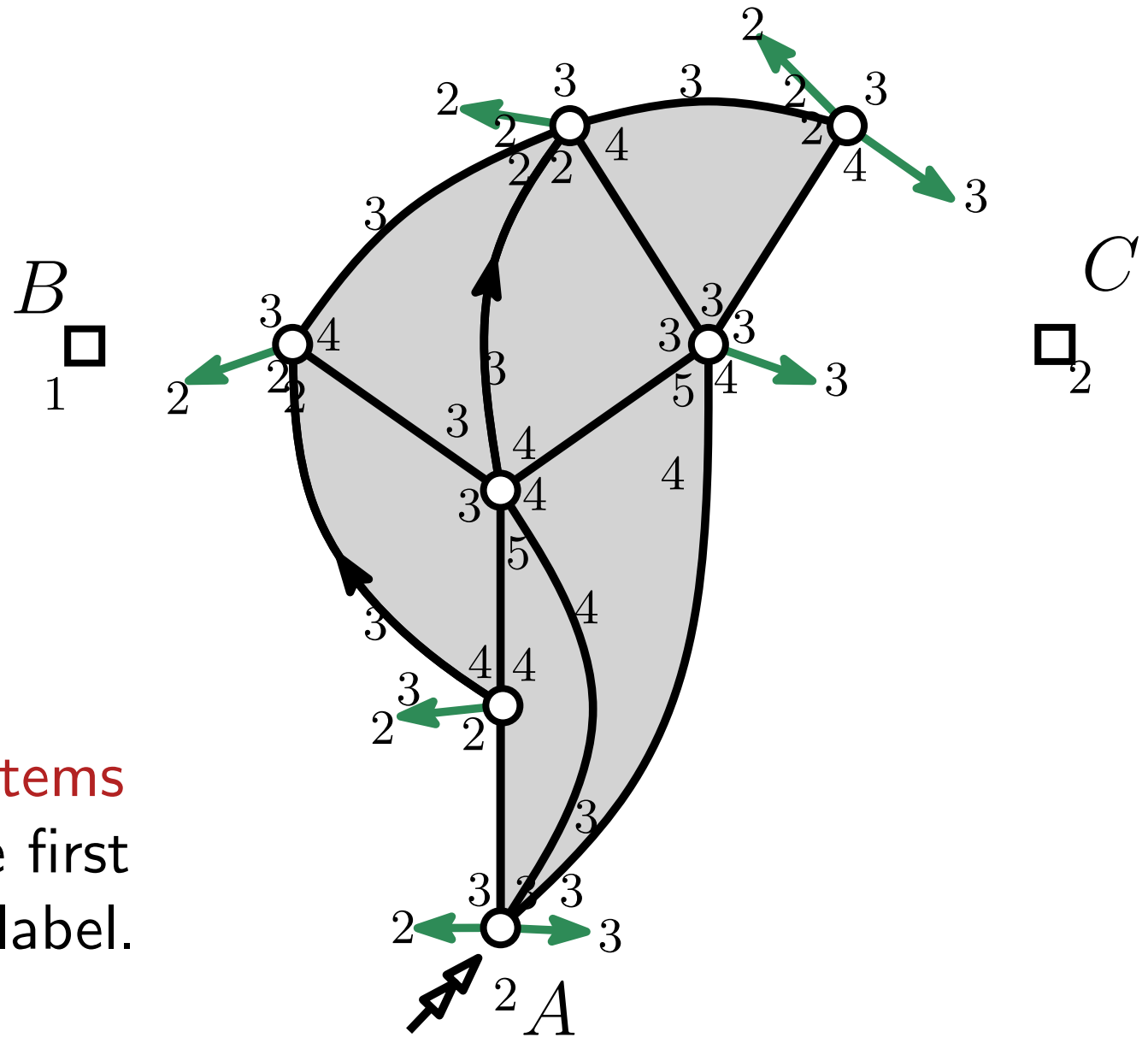


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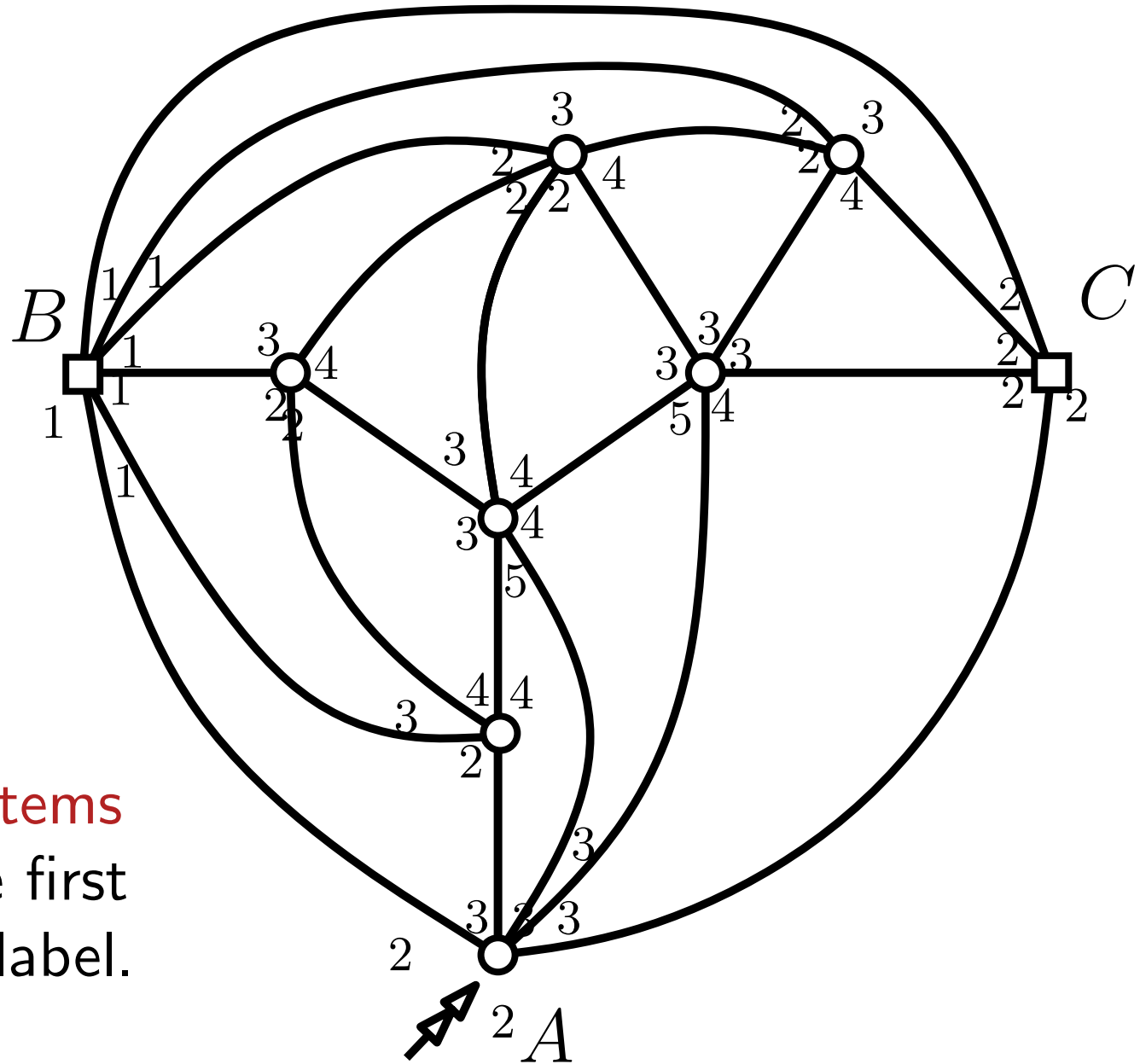


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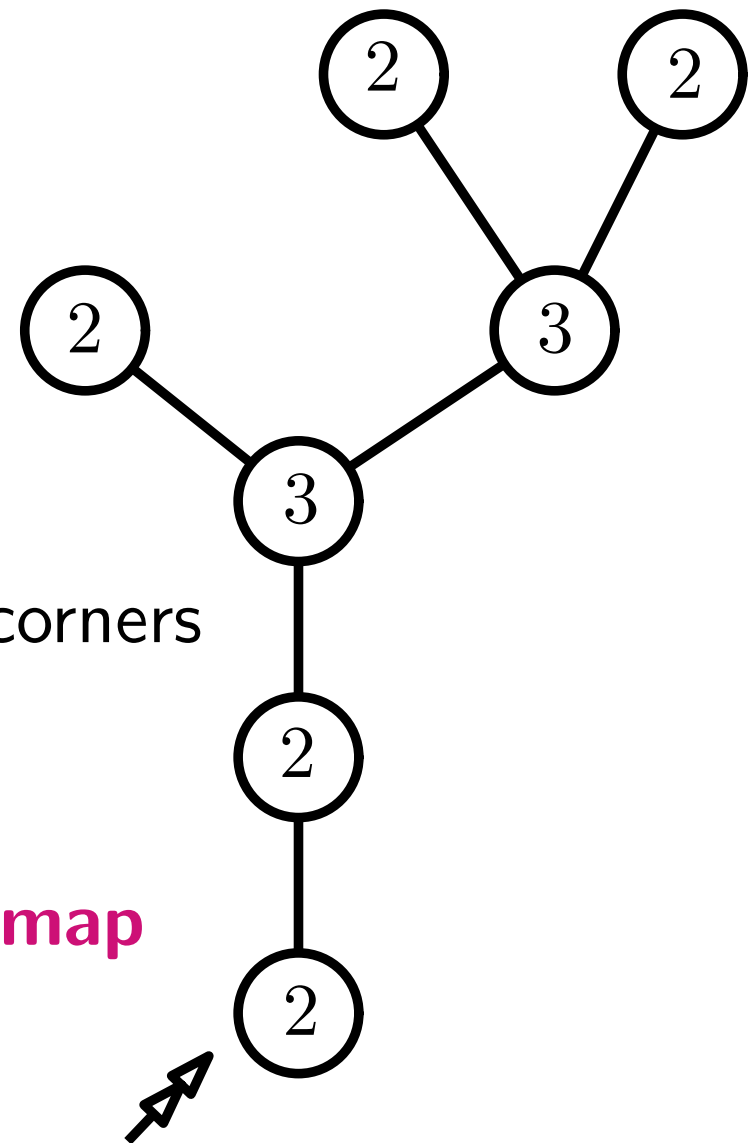  all labels ≥ 2
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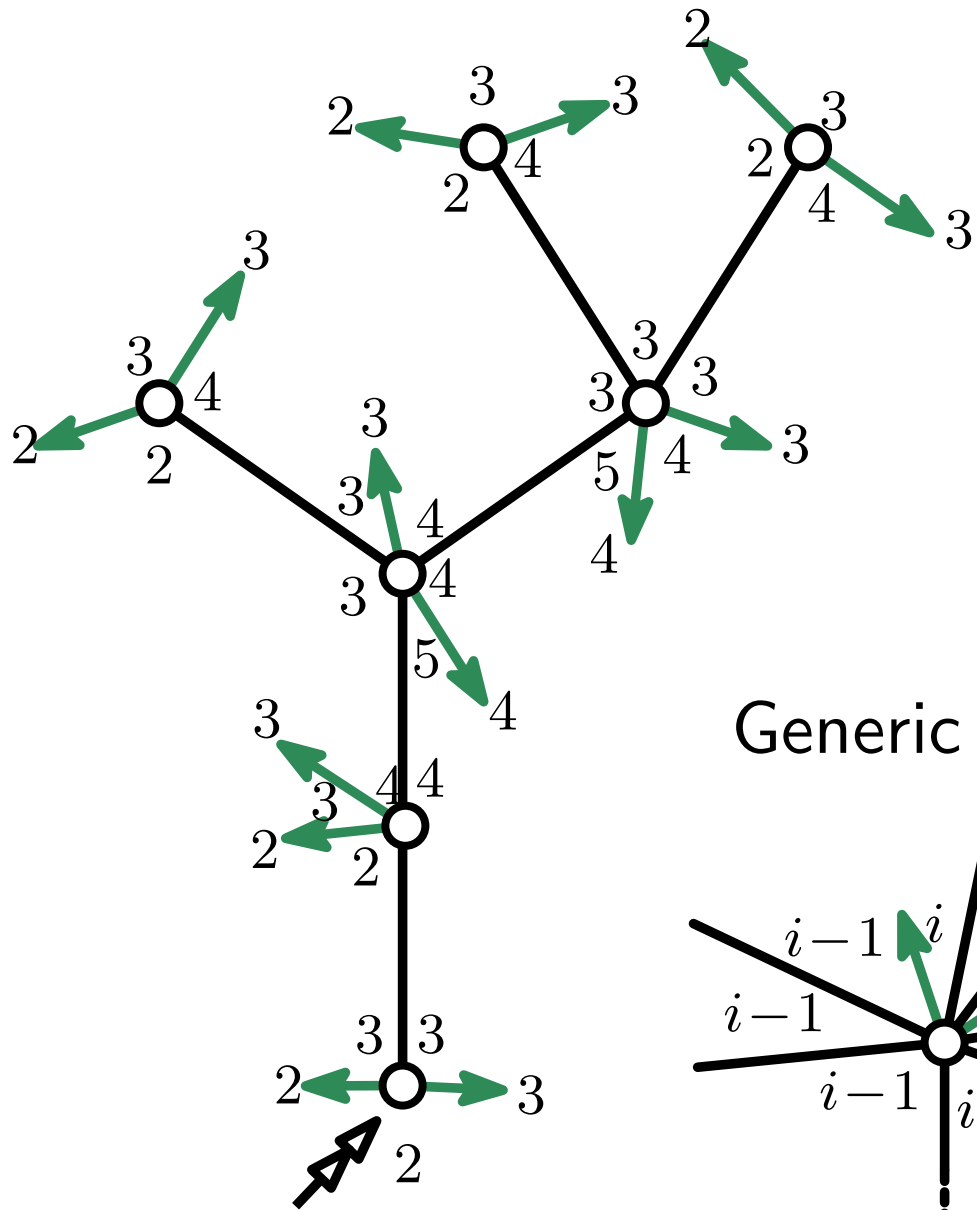
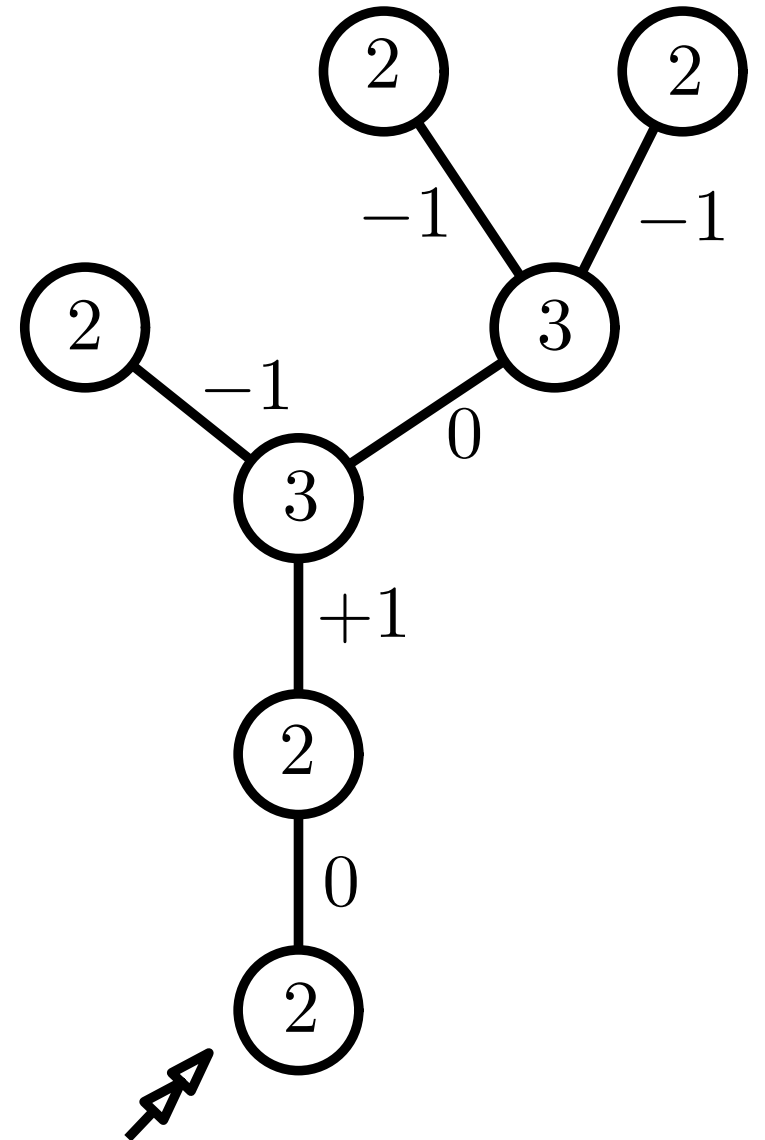# From blossoming trees to labeled trees
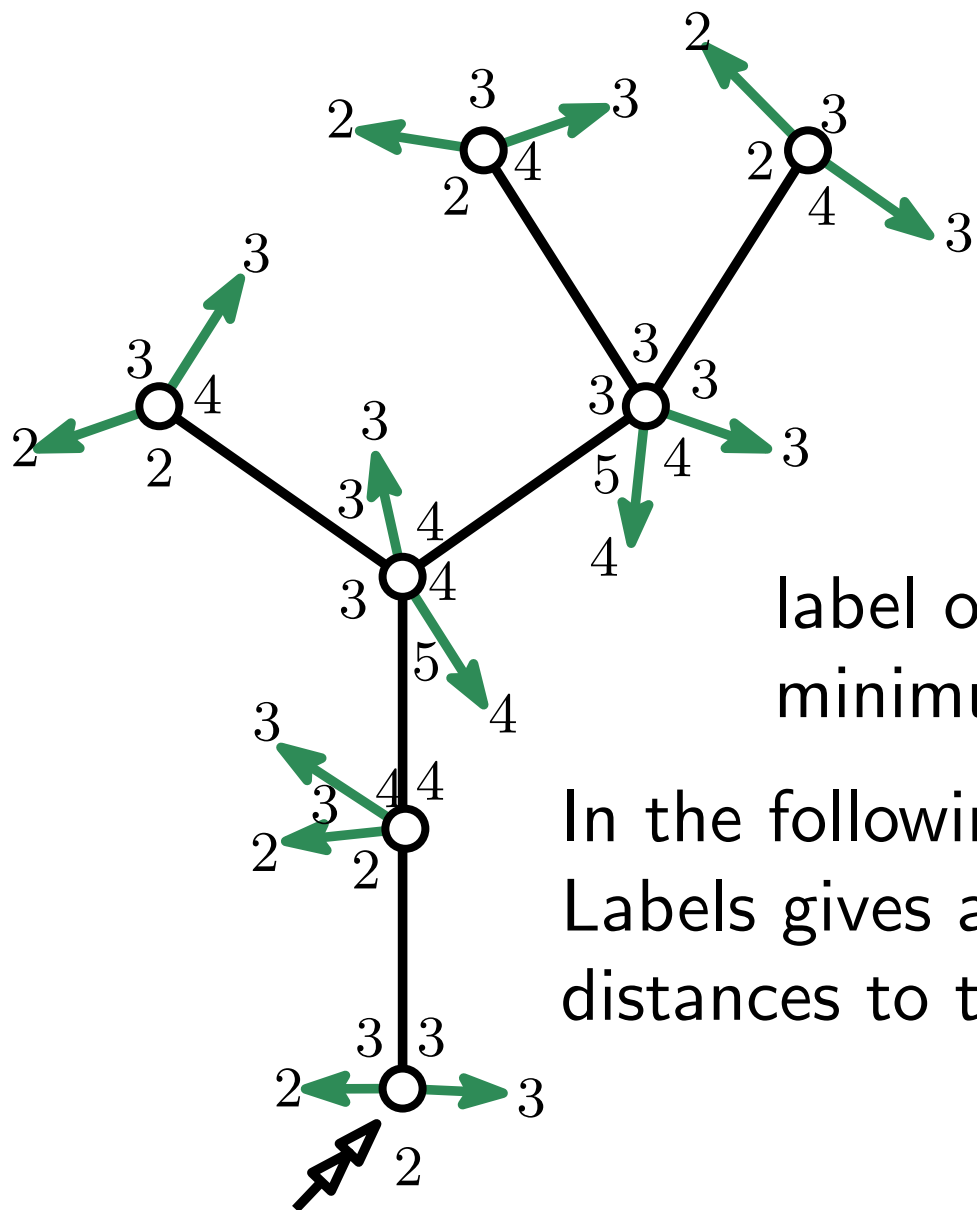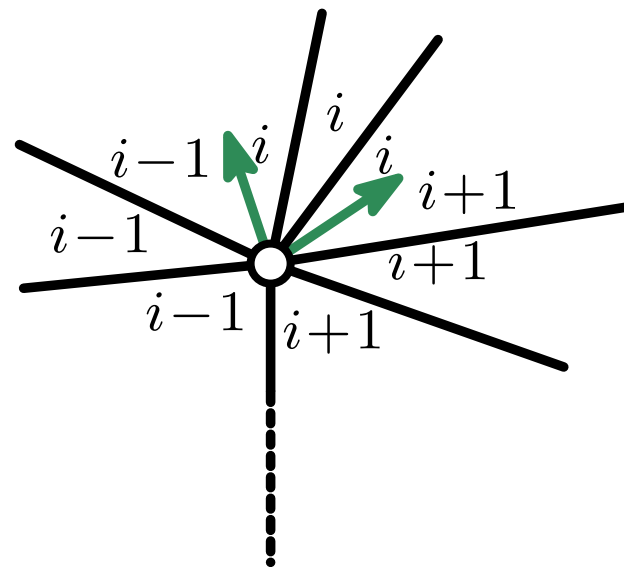


label of a vertex =
minimum label of its corners

In the following:
Labels gives approximate
distances to the root **in the map**

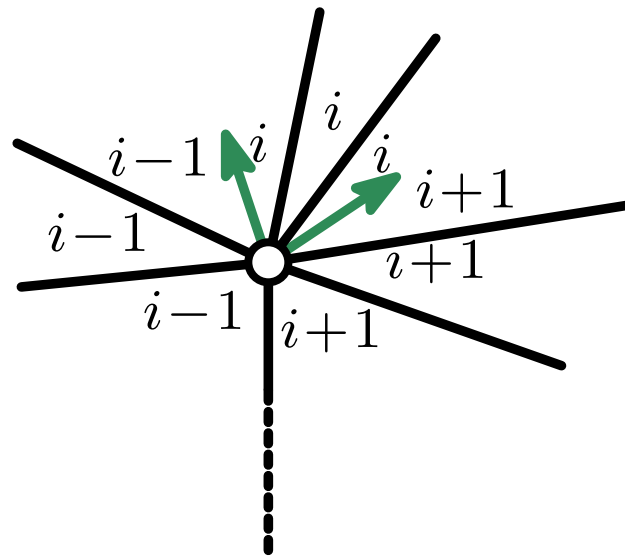# From blossoming trees to labeled trees
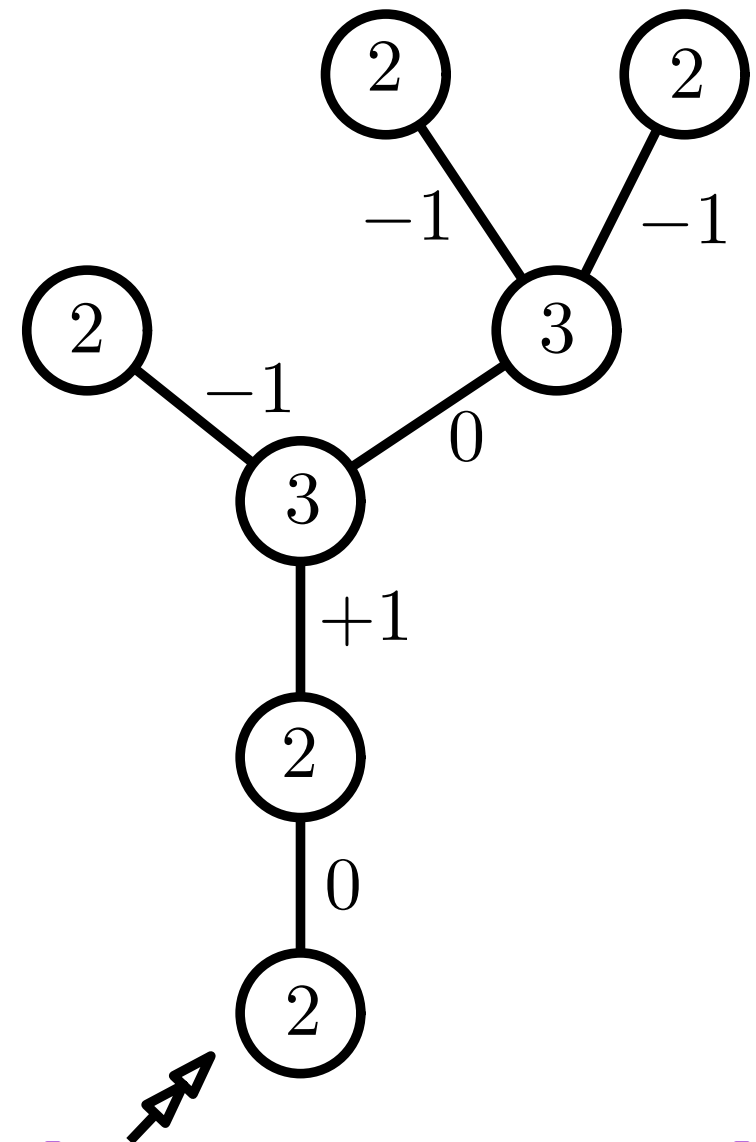


Generic vertex :

# From blossoming trees to labeled trees

Generic vertex :



- Can retrieve the blossoming tree from the labeled tree.

- Labeled tree = GW trees + random displacements on edges uniform on

$$\{(-1, -1, \ldots, -1, 0, 0, \ldots, 0, 1, 1 \ldots, 1)\}.$$

**almost the setting of [Janson-Marckert] and [Marckert-Miermont] but r.v are not "locally centered" $\Rightarrow$ symmetrization required**
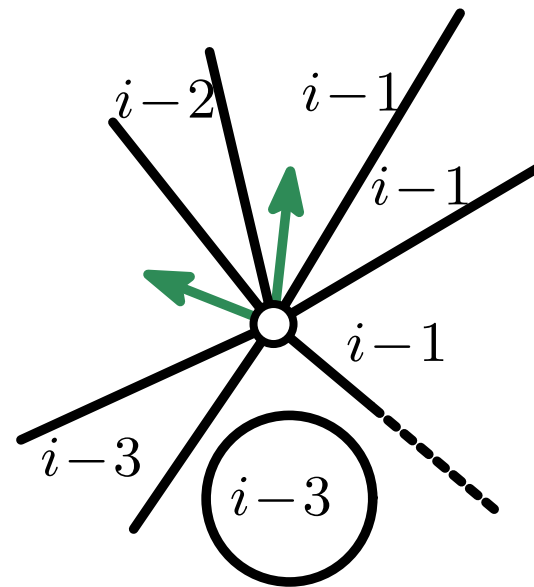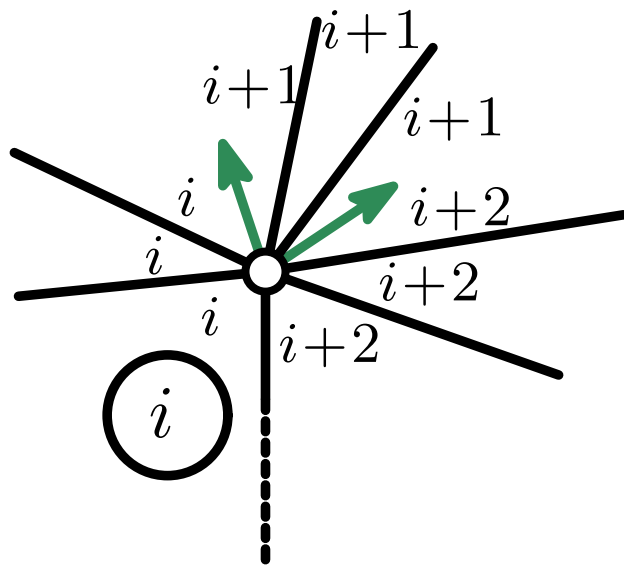
# Distances in simple triangulations

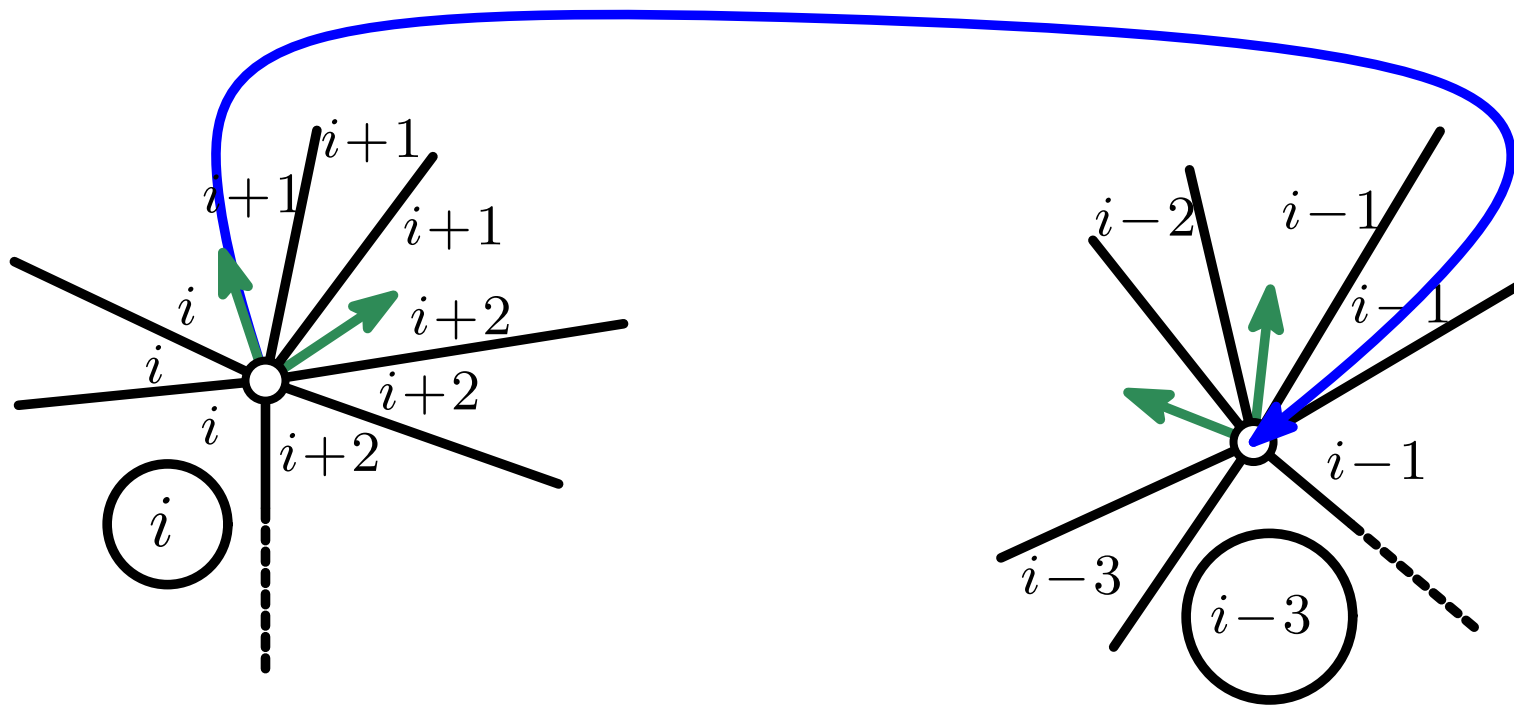**Claim 1:** $3d_M(root, u) \geq$ Label of $u$

First observation : In the tree, the labels of two adjacent vertices differ by at most 1. **What can go wrong with closures ?**

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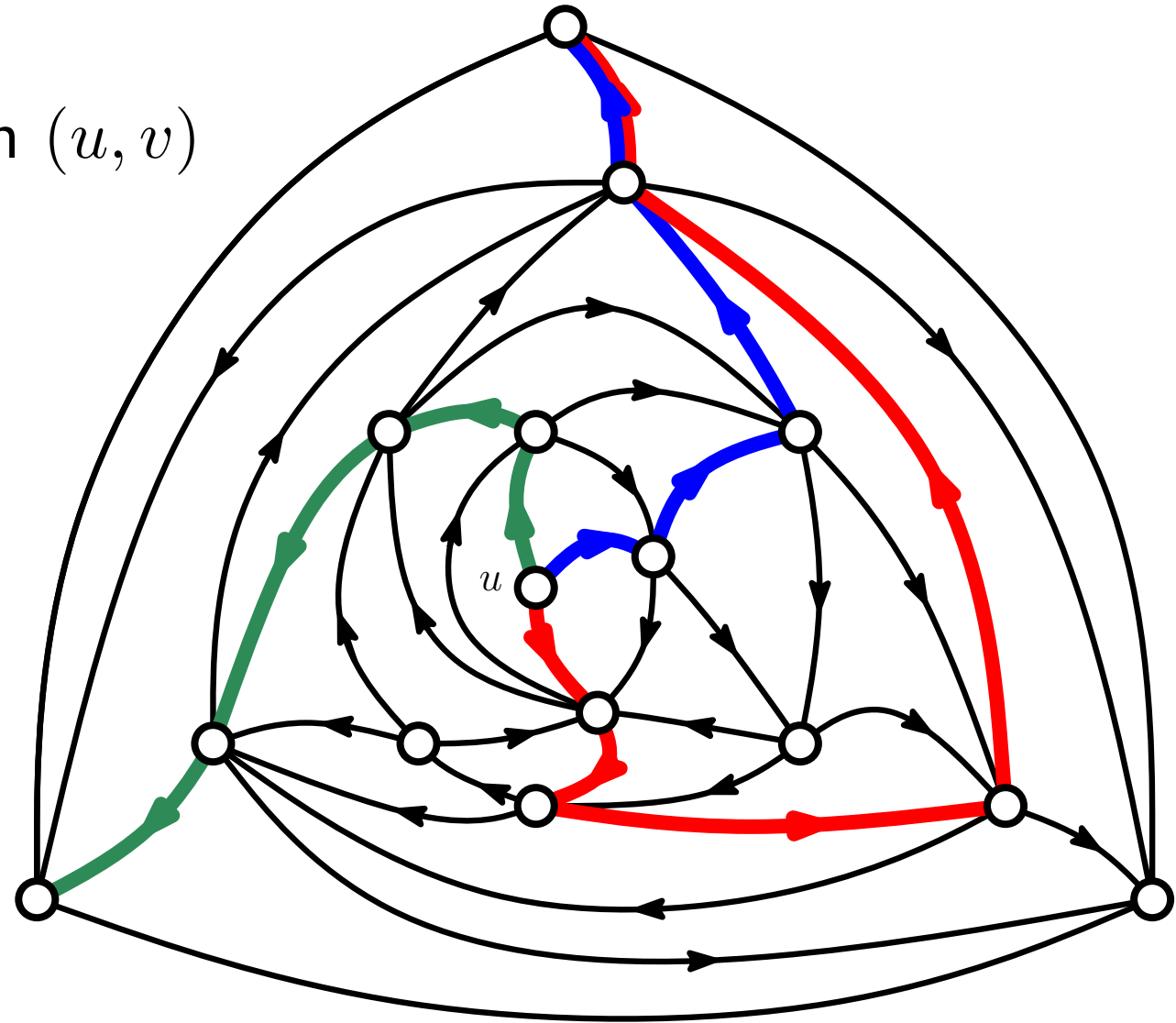
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- For each inner vertex : 3 LMP

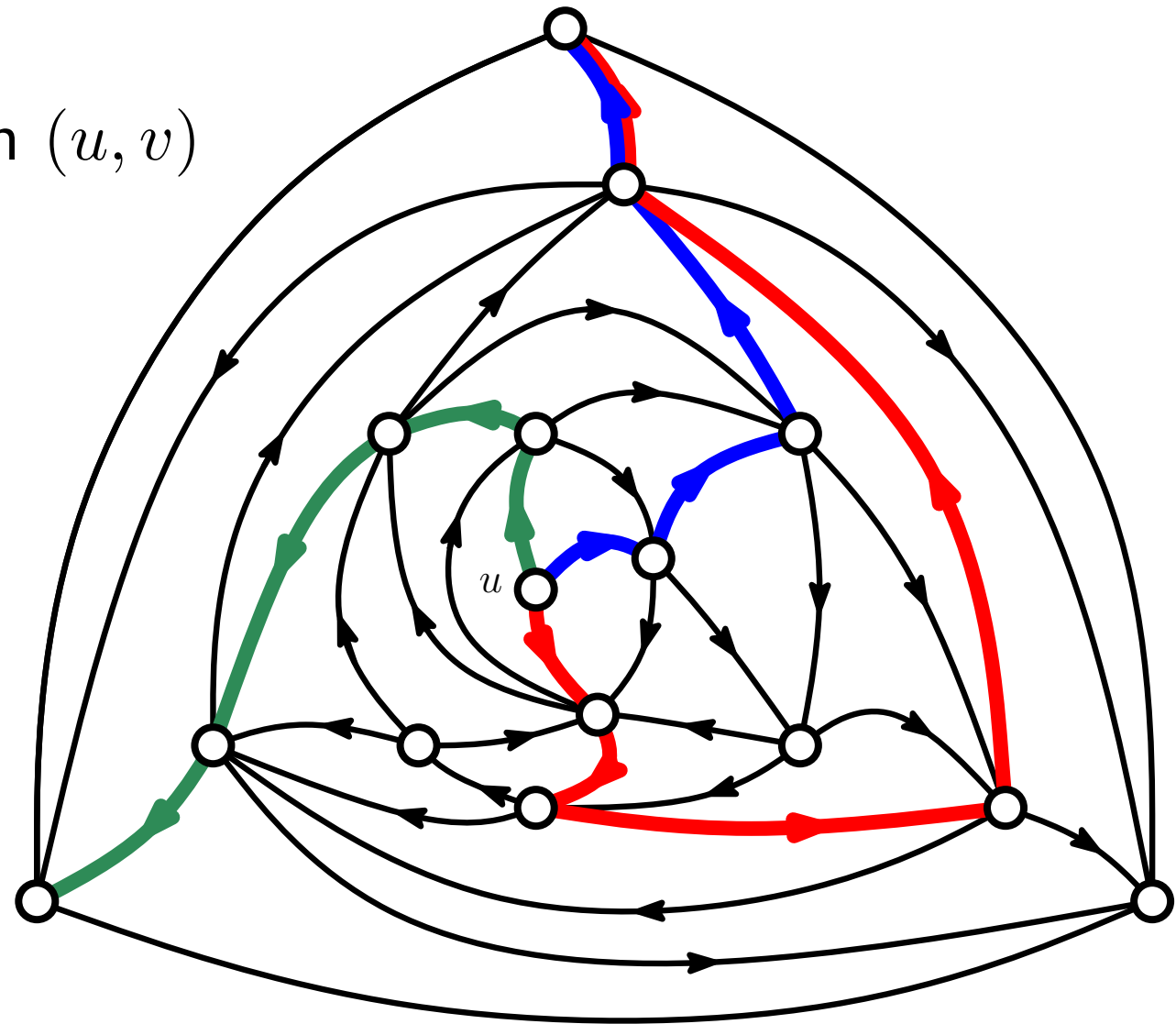# Distances in simple triangulations

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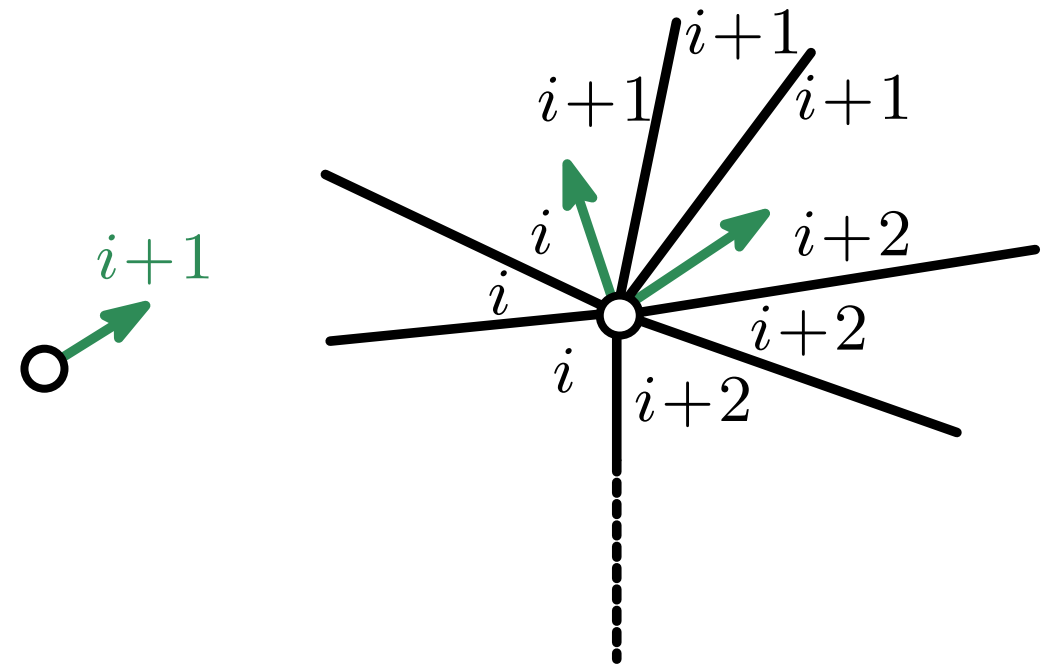- LMP are not self-intersecting $\Rightarrow$ they reach the outer face

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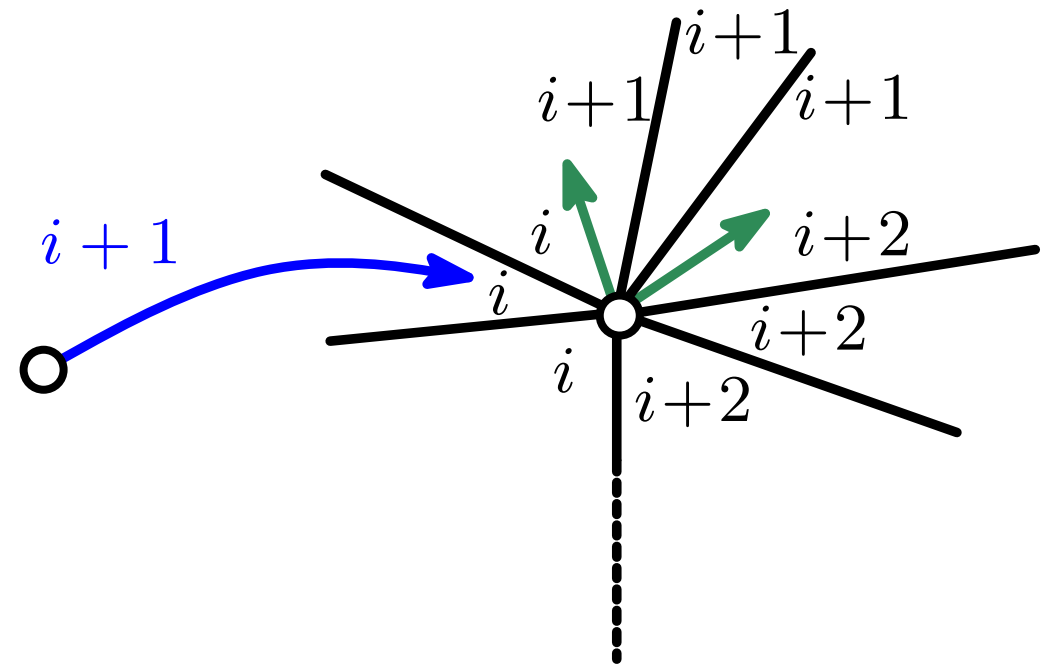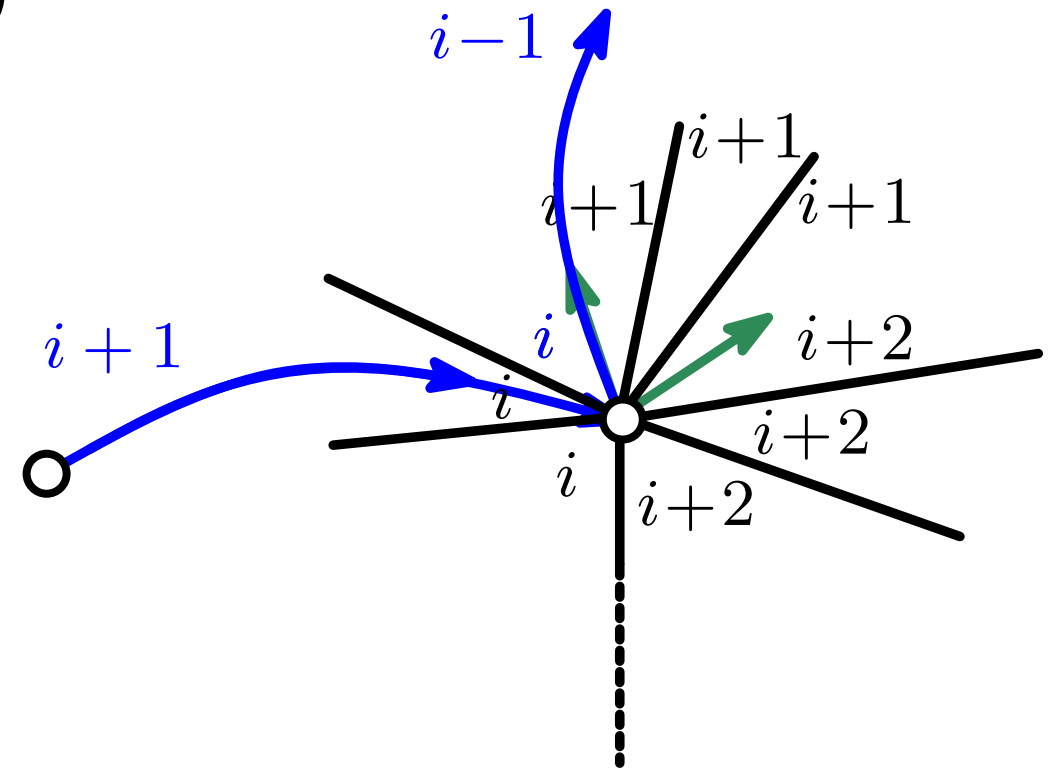- On the left of a LMP, corner labels decrease exactly by one.

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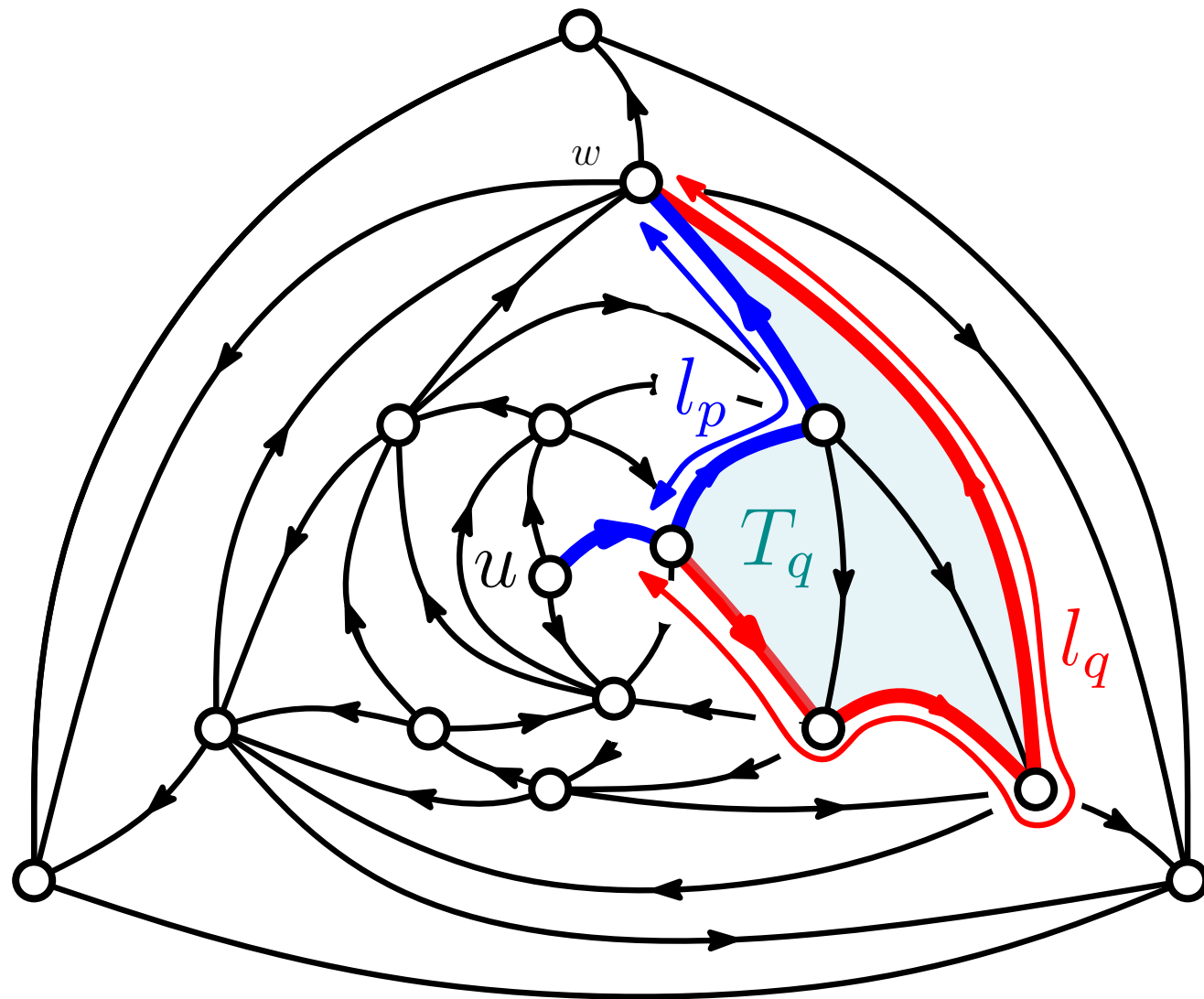
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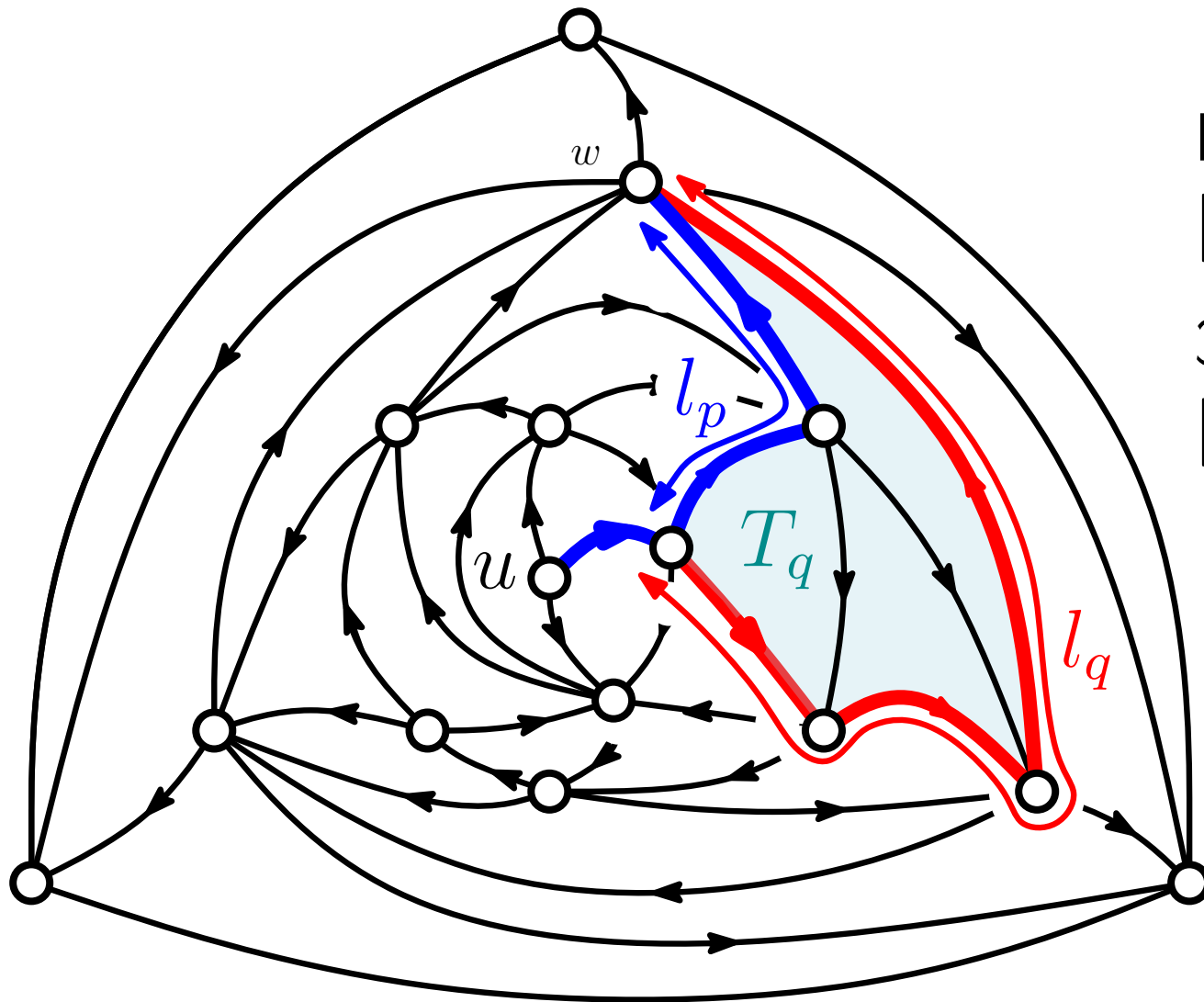- On the left of a LMP, corner labels decrease exactly by one.

# LMP are almost geodesic



Leftmost path

Another path: can it be shorter ?

# LMP are almost geodesic



Euler Formula :
$$|E(T_q)| = 3|V(T_q)| - 3 - (\ell_p + \ell_q)$$

3-orientation + LMP :
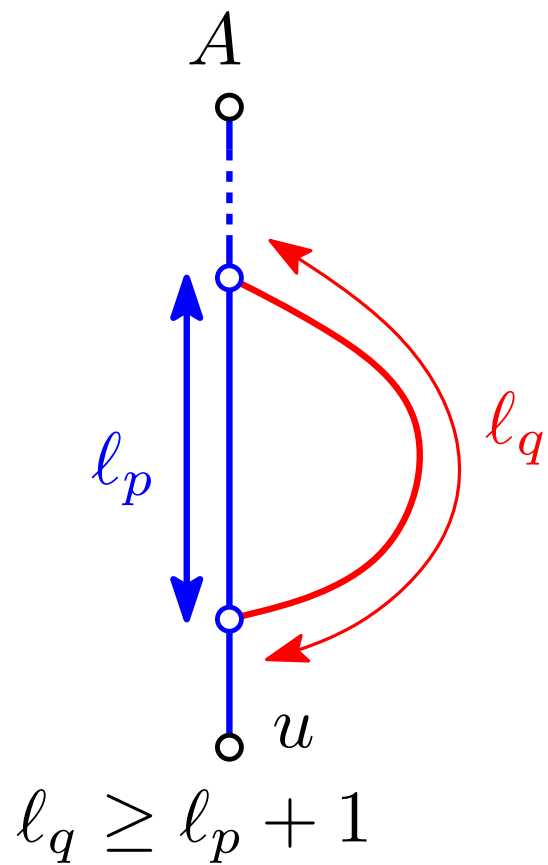$$|E(T_q)| \geq 3|V(T_q)| - 2\ell_q - 2$$
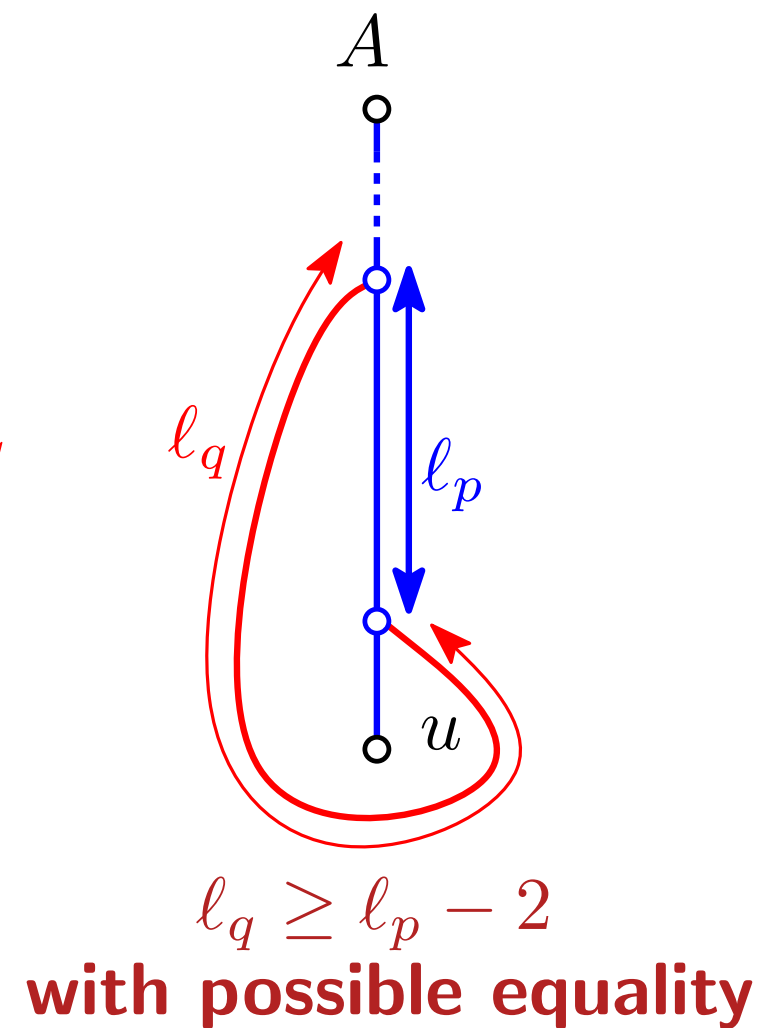
$$\implies \ell_q \geq \ell_p + 1$$

Leftmost path
Another path: can it be shorter ?
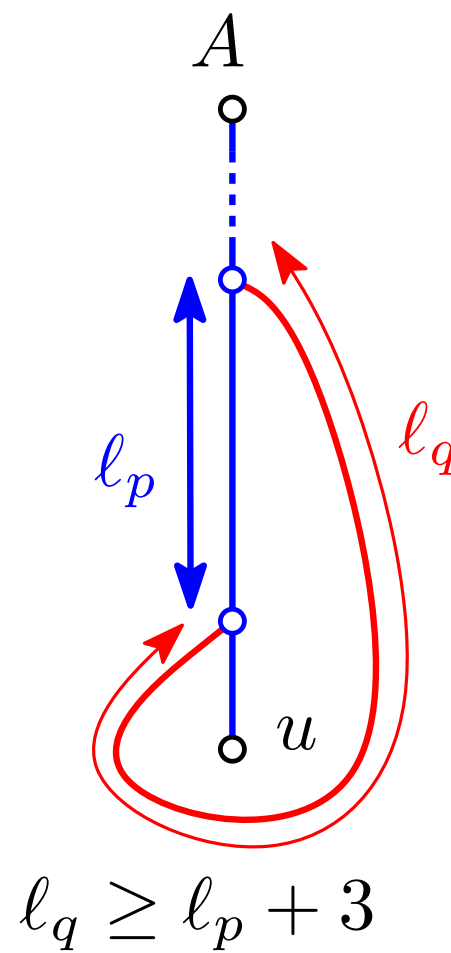
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$A$

$\ell_p$

$\ell_q$

$u$

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# LMP are almost geodesic

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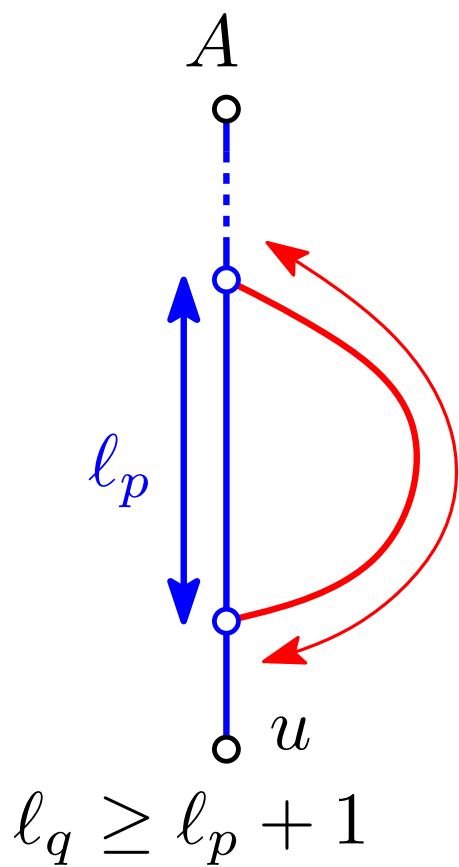Another path: can it be shorter ?   YES



$\ell_q \geq \ell_p + 1$      $\ell_q \geq \ell_p$      $\ell_q \geq \ell_p + 3$      $\ell_q \geq \ell_p - 2$

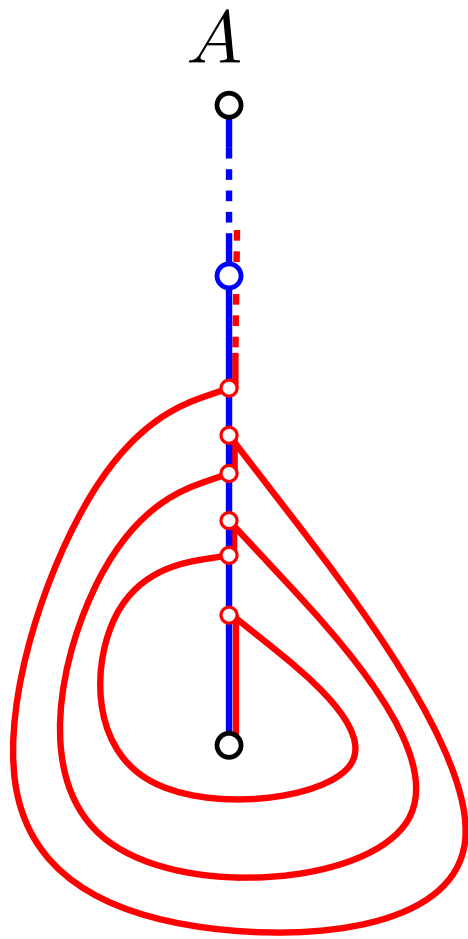**with possible equality**

# LMP are almost geodesic

Leftmost path

Another path: can it be shorter ?   YES ... but not too often

$A$
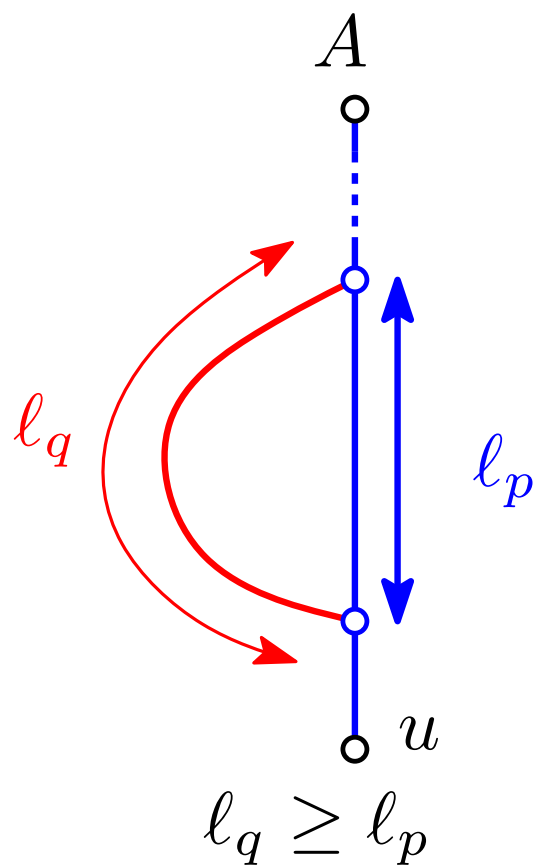
Bad configuration $=$
too many **windings** around the LMP

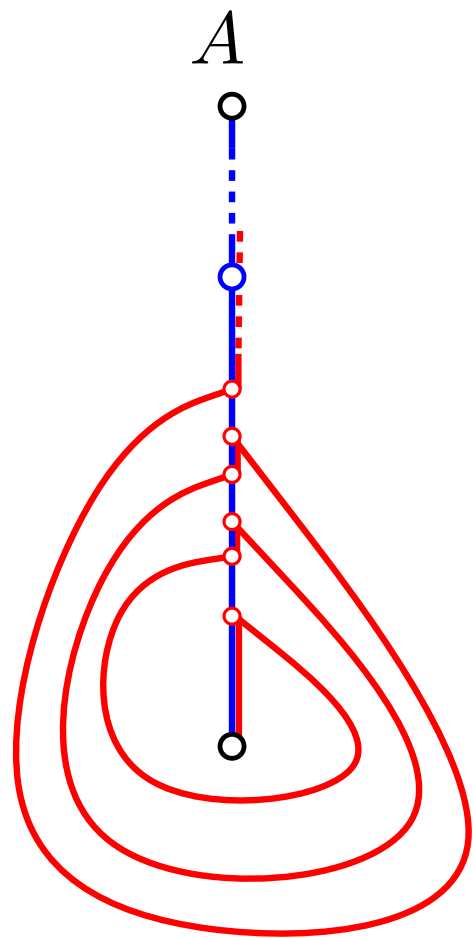But w.h.p a winding cannot be too short.

$\implies$ w.h.p the number of windings is $o(n^{1/4})$.

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Bad configuration =
too many **windings** around the LMP

But w.h.p a winding cannot be too short.

$\implies$ w.h.p the number of windings is $o(n^{1/4})$.

**Proposition:**
For $\varepsilon > 0$, let $A_{n,\varepsilon}$ be the event that there exists
$u \in M_n$ such that
Label of $u \geq d_{M_n}(u, root) + \varepsilon n^{1/4}$.
Then under the uniform law on $\mathcal{M}_n$, for all $\varepsilon > 0$:

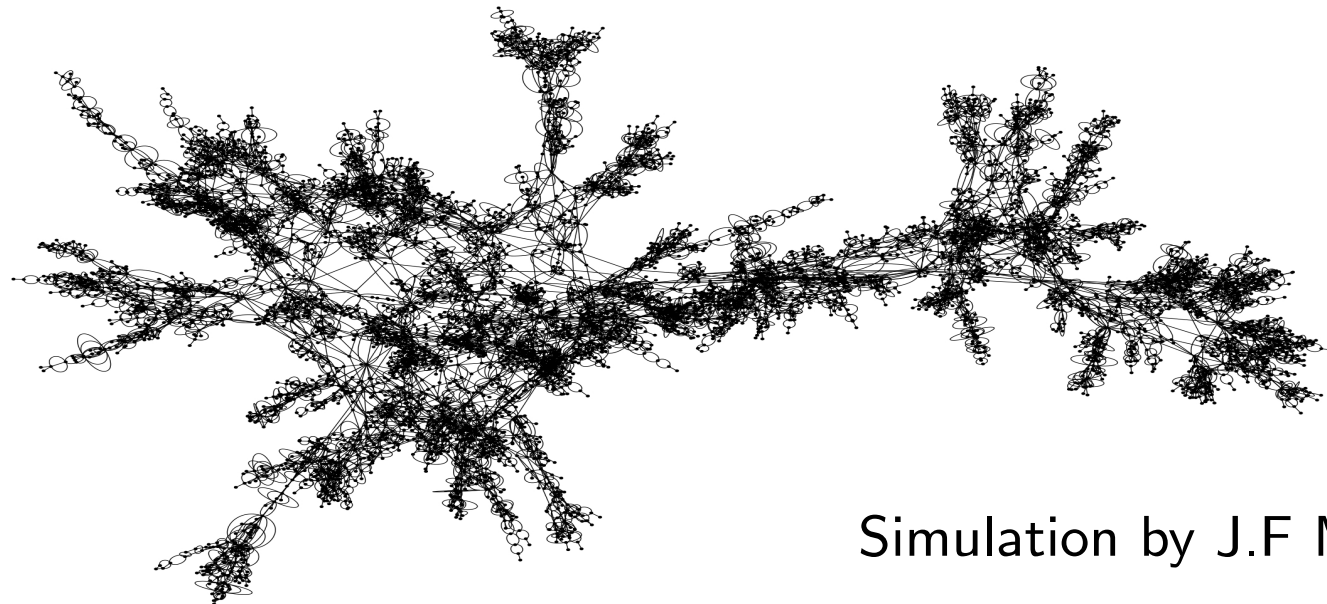$$\mathbb{P}\left(A_{n,\varepsilon}\right) \to 0.$$

# The result

**Theorem :** [Addario-Berry, A.]

$(M_n) = $ sequence of random **simple** triangulations, then:

$$\left( M_n, \left( \frac{3}{4n} \right)^{1/4} d_{M_n} \right) \xrightarrow{(d)} \text{Brownian map}$$

for the distance of Gromov-Hausdorff on the isometry classes of compact metric spaces.



Simulation by J.F Marckert

# Beyond the universality

Simple triangulations converge to the Brownian map
$\Rightarrow$ properties of the Brownian map from the simple triangulations ?

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One motivation : Circle-packing theorem

Each simple triangulation $M$ has a unique (up to
Möbius transformations and reflections) circle
packing whose tangency graph is $M$.
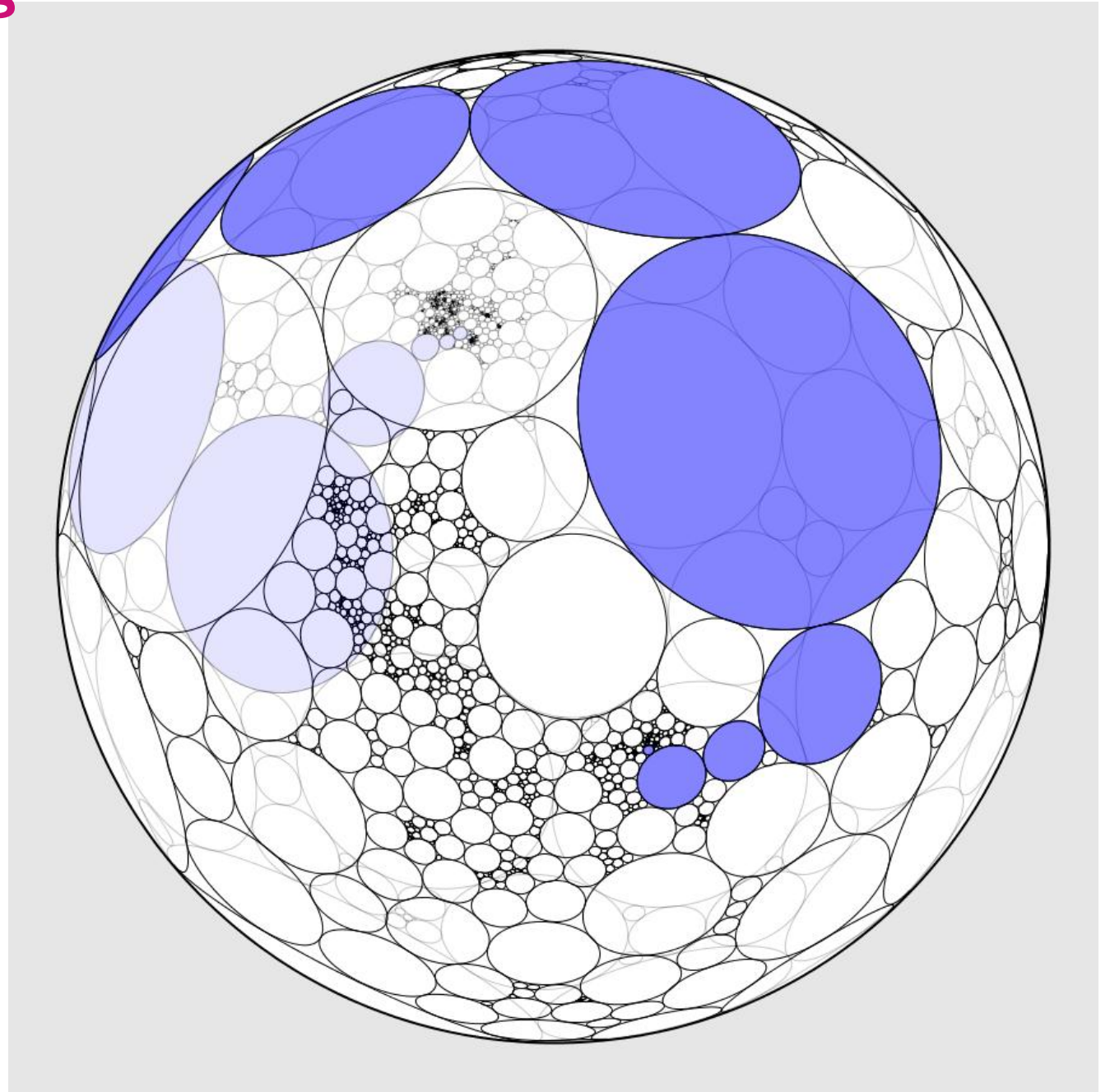[Koebe-Andreev-Thurston]



Gives a canonical embedding of simple
triangulations in the sphere and possibly of their
limit.
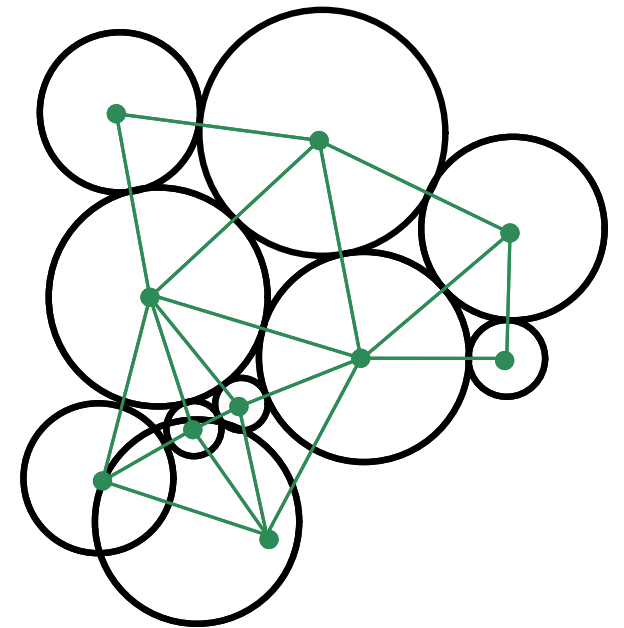
# Random circle packing

Random circle packing = canonical embedding of random simple triangulation in the sphere.

Gives a way to define a canonical embedding of their limit ?



Team effort : code by Kenneth Stephenson, Eric Fusy and our own.

# Perspectives

Same approach works also for simple quadrangulations.

Can we make this approach work for the general setting of bijections developped in [A.,Poulalhon] and in [Bernardi, Fusy] ?

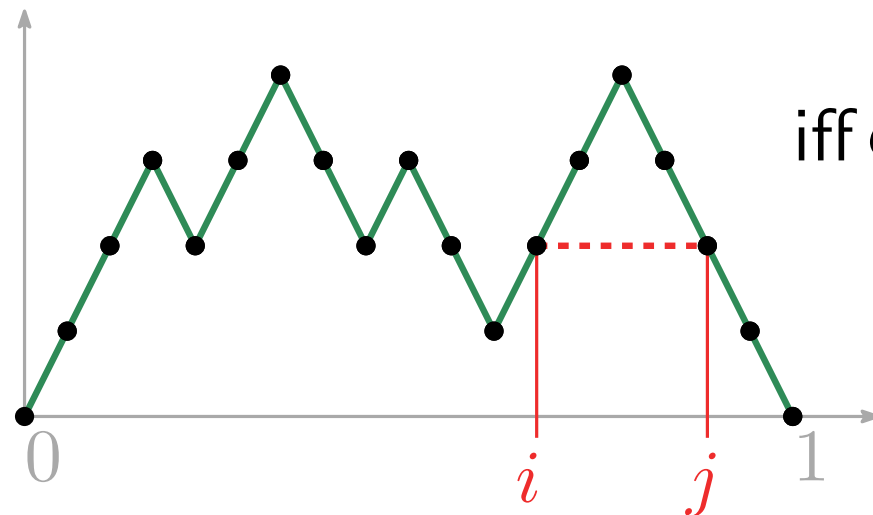Can we say something about a random circle packing ?

## Perspectives

Same approach works also for simple quadrangulations.

Can we make this approach work for the general setting of bijections developped in [A.,Poulalhon] and in [Bernardi, Fusy] ?

Can we say something about a random circle packing ?

# Thank you !

# Brownian snake $(e_t, Z_t)_{0 \le t \le 1}$
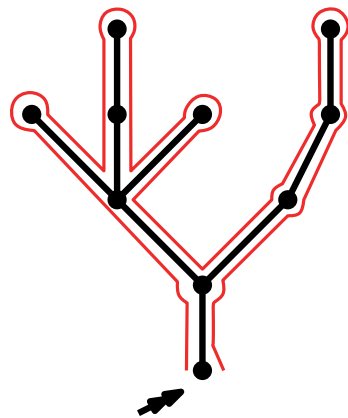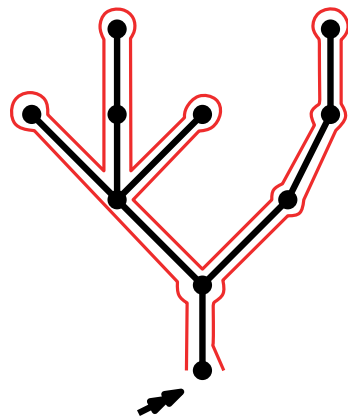
## 1st step : the Brownian tree [Aldous]



$i$ and $j =$ same vertex of $T$

iff $C_n(i) = C_n(j) = \min\limits_{i \le k \le j} C_n(k)$

$T \longrightarrow$ $C_n^T$ (or $C_n$) = contour process

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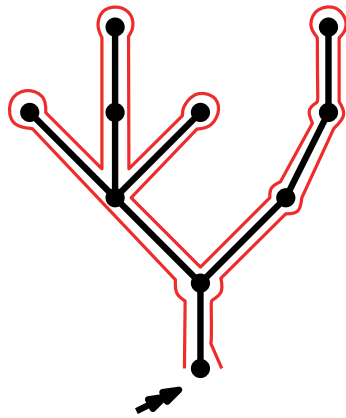
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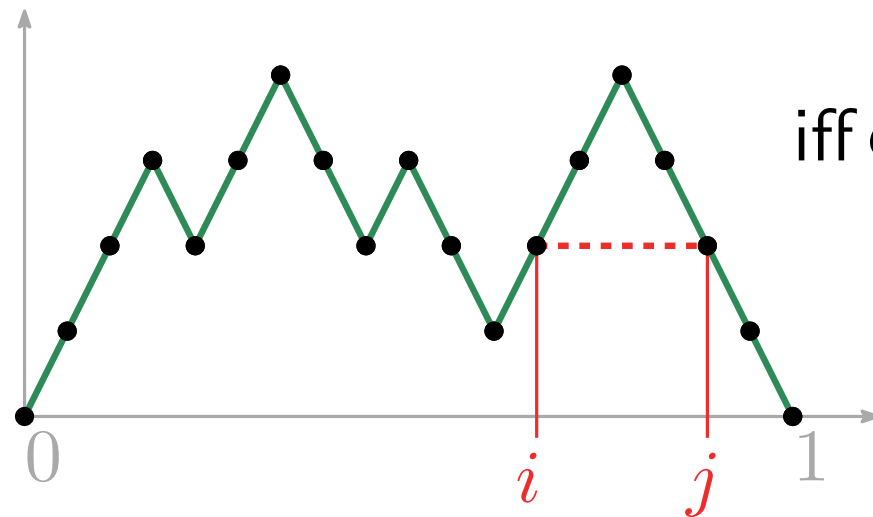
$C_n^T$ (or $C_n$) = contour process

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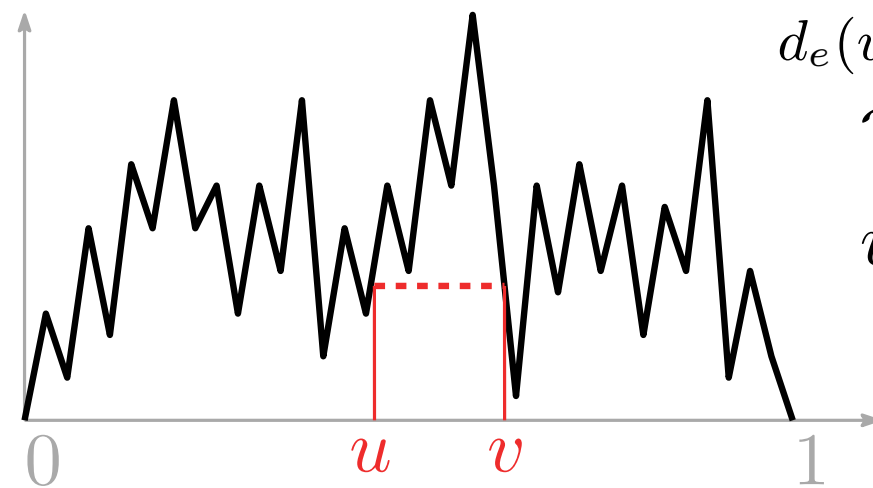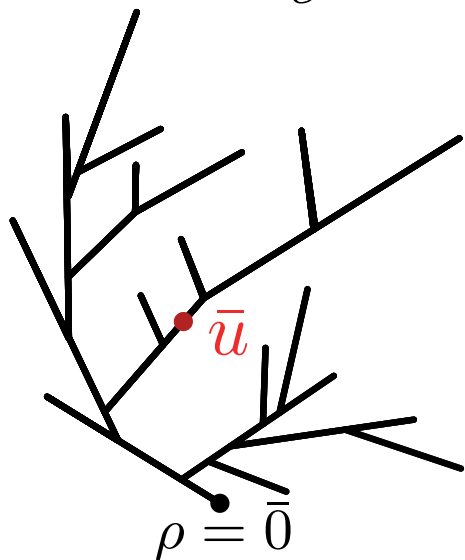## 1st step : the Brownian tree [Aldous]



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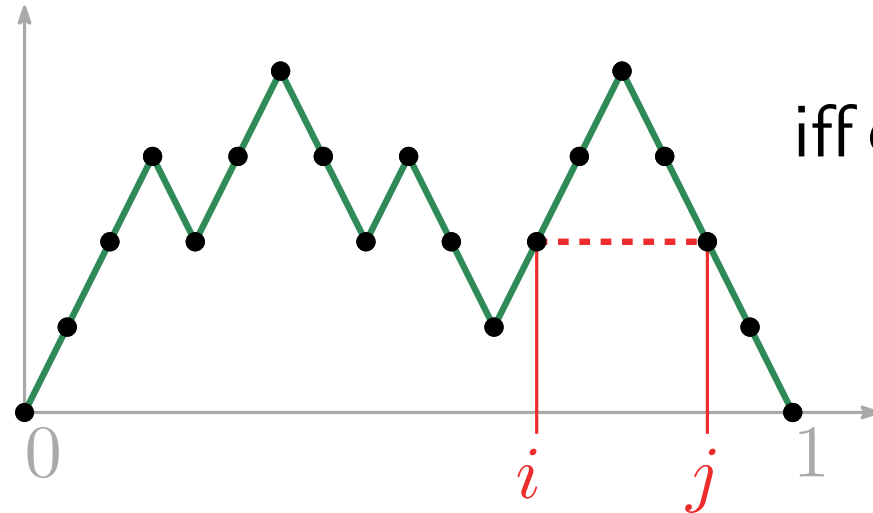
$C_n^T$ (or $C_n$) = contour process

$\mathcal{T}_e \longleftarrow$

$(e_t)_{0 \leq t \leq 1} =$ Brownian excursion

$d_e(u, v) = e_u + e_v - 2 \min_{u \leq s \leq v} e_s$

$\mathcal{T}_e = [0, 1]/ \sim_e$

$u \sim_e v$ iff $d_e(u, v) = 0$

# Brownian snake $(e_t, Z_t)_{0 \leq t \leq 1}$
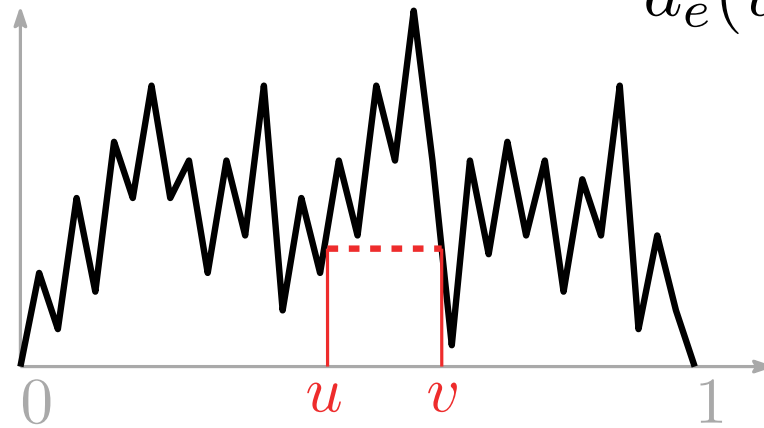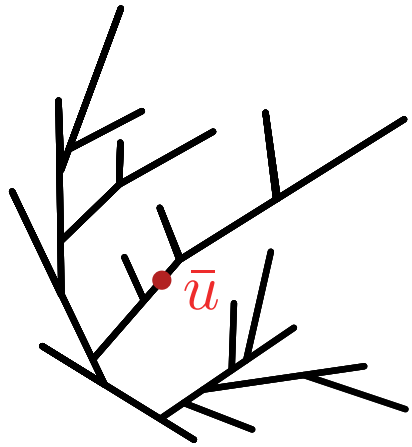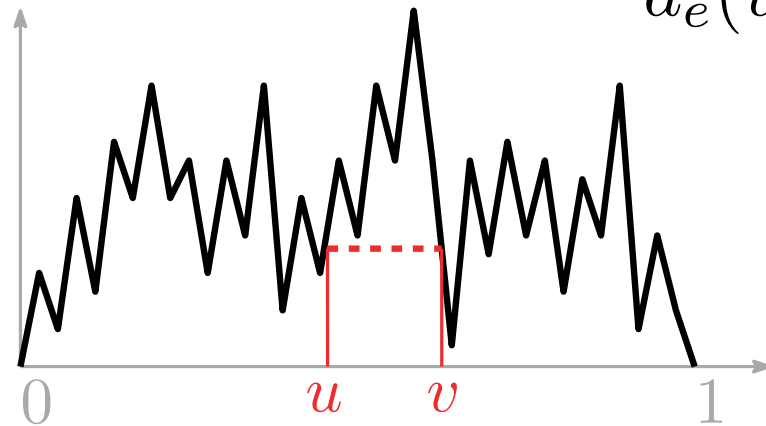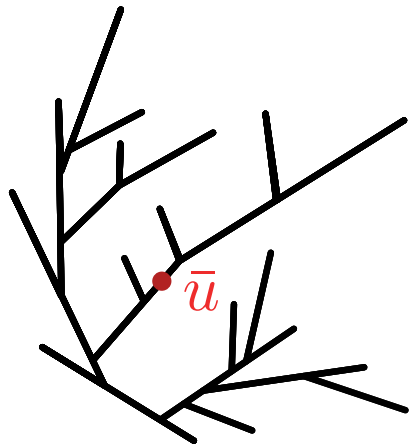
## 1st step : the Brownian tree [Aldous]



$$d_e(u, v) = e_u + e_v - 2\min_{u \leq s \leq v} e_s$$

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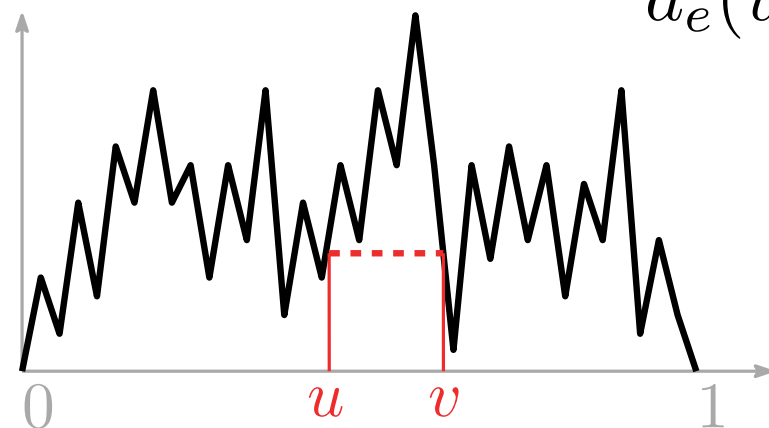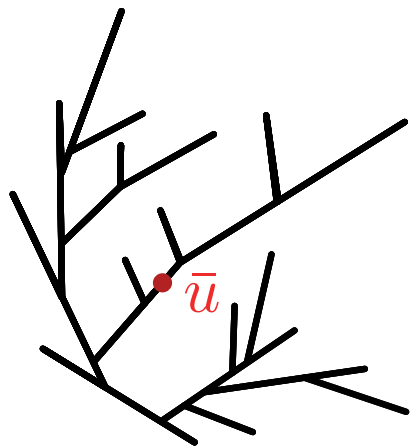$$\mathcal{T}_e = [0, 1] / \sim_e$$
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## 2nd step : Brownian labels

Conditional on $\mathcal{T}_e$, $Z$ a centered Gaussian process with $Z_\rho = 0$ and
$E[(Z_s - Z_t)^2] = d_e(s, t)$

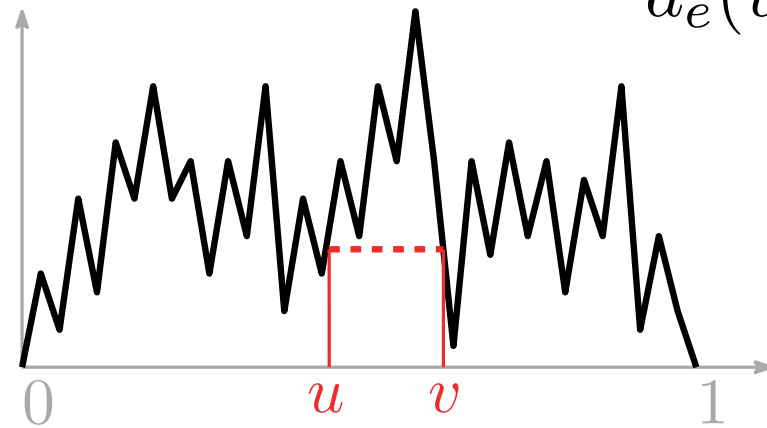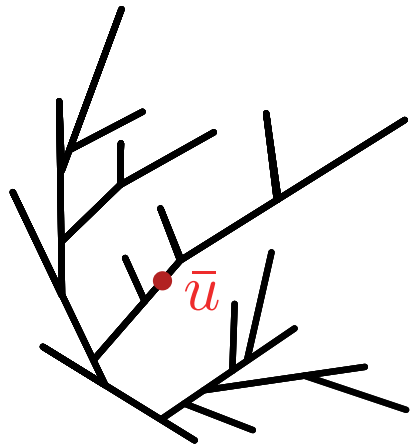$Z \sim$ **Brownian motion on the tree**

# The Brownian map



$$d_e(u, v) = e_u + e_v - 2\min_{u \leq s \leq v} e_s$$
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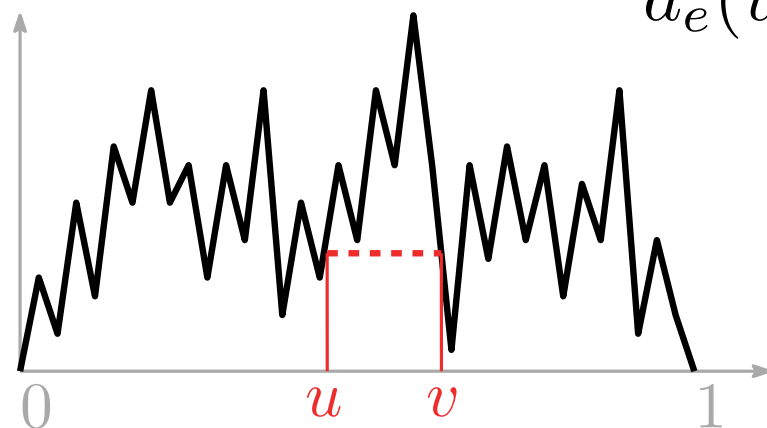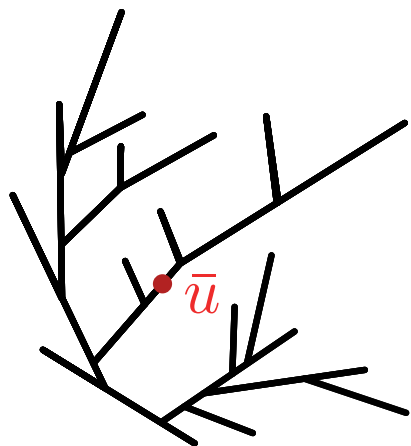


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$$D^\circ(s,t) = Z_s + Z_t - 2\max\left(\inf_{s \le u \le t} Z_u, \inf_{t \le u \le s} Z_u\right), \quad s,t \in [0,1].$$

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$\bar{u}$

$0 \qquad u \quad v \qquad 1$

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$$D^*(a,b) = \inf\left\{ \sum_{i=1}^{k-1} D^\circ(a_i, a_{i+1}) : k \ge 1, a = a_1, a_2, \ldots, a_{k-1}, a_k = b \right\},$$

# The Brownian map



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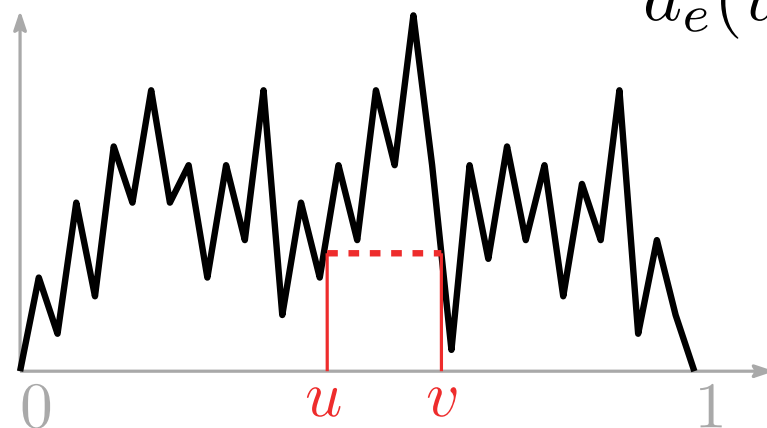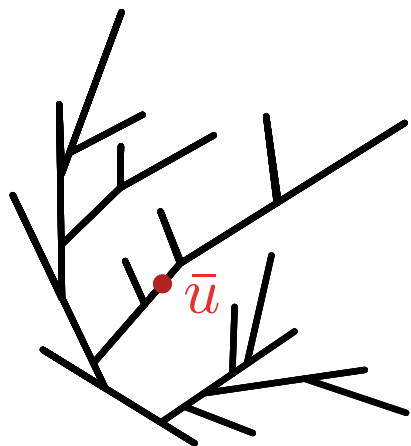$$\mathcal{T}_e = [0,1]/\sim_e$$
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Then $M = (\mathcal{T}_e/\sim_{D^\star}, D^*)$ is the **Brownian map**.