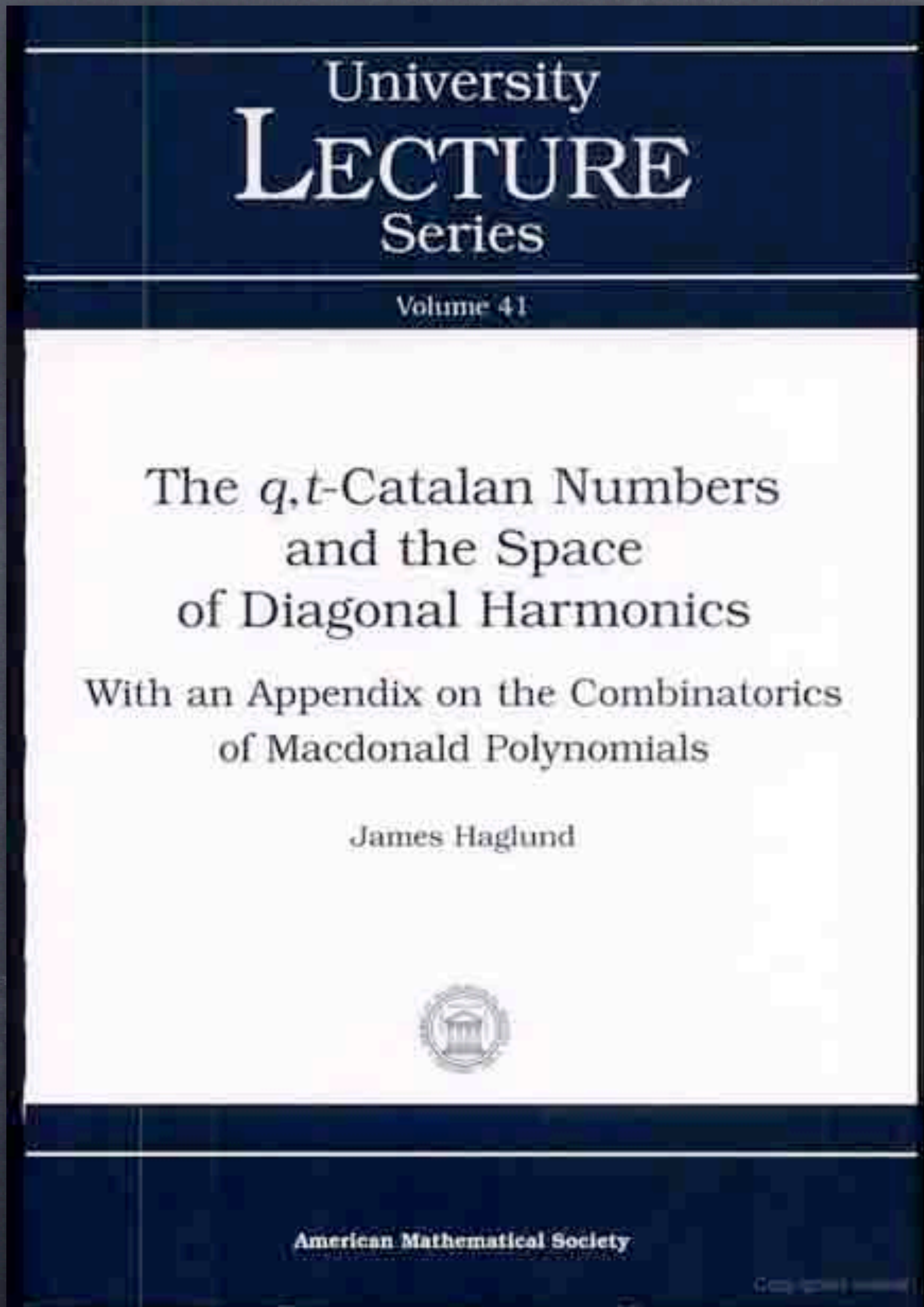
Combinatoire de Catalan et Polynômes Harmoniques Diagonaux

François Bergeron, LaCIM



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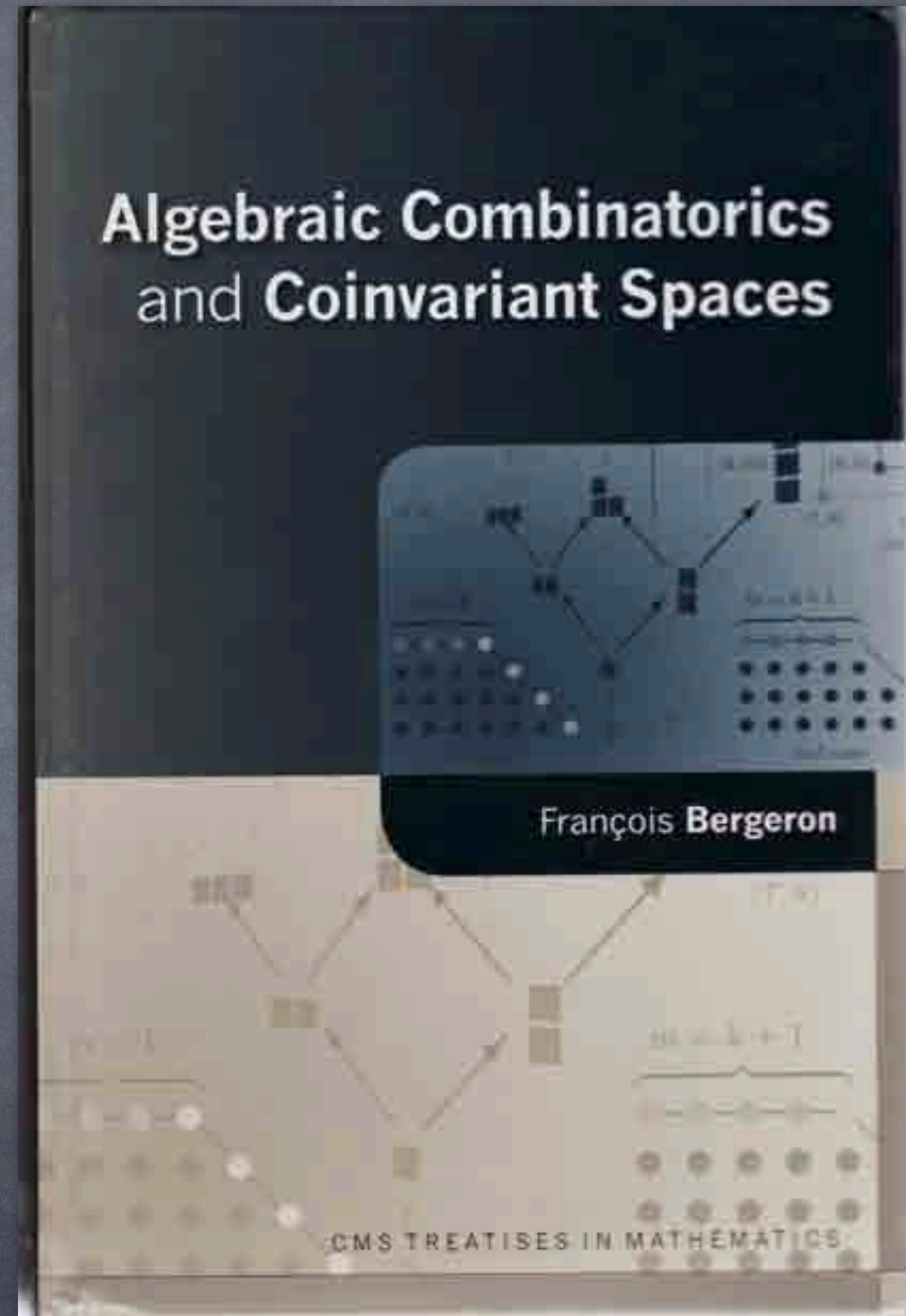

2009

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L'ANNEAU DES POLYNÔMES

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ℓ ensembles de n variables

$$X = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & \cdots & z_n \end{pmatrix}$$

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ℓ ensembles de n variables

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

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ℓ ensembles de n variables

$$X^A := x_1^{a_1} \cdots x_n^{a_n} \, y_1^{b_1} \cdots y_n^{b_n} \cdots$$

$$\cdots z_1^{c_1} \cdots z_n^{c_n}$$

DEGRÉ

$$\mathrm{DEG}(x^A) := (a_1 + \dots + a_m, b_1 + \dots + b_m, \dots$$

$$\dots, c_1 + \dots + c_m)$$

$f(x)$ HOMOGÈNE DE DEGRÉ

$$d = (d_1, d_2, \dots, d_\ell)$$

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DEGRÉ

$$\mathrm{DEG}(x^A) := (a_1 + \dots + a_n, b_1 + \dots + b_n, \dots, c_1 + \dots + c_n)$$

$$f\left(\begin{pmatrix} g_1 & & 0 \\ & g_2 & \\ 0 & & \ddots \; g^{\ell} \end{pmatrix} X\right) =$$

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DEGRÉ

$$\mathrm{DEG}(x^A) := (a_1 + \ldots + a_m, b_1 + \ldots + b_m, \ldots$$
$$\ldots, c_1 + \ldots + c_m)$$

$$f\left(\begin{pmatrix} q_1 & & 0 \\ & q_2 & \\ 0 & & \ddots & q_\ell \end{pmatrix} X\right) = q^d f(X)$$

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DEGRÉ

$$\mathrm{DEG}(f) := d \qquad q^{d} := q_1^{d_1}\, q_2^{d_2} \cdots q_\ell^{d_\ell}$$

$$f\left(\begin{pmatrix} q_1 & & 0 \\ & q_2 & \\ 0 & & \ddots \; q^{\ell} \end{pmatrix} X\right) = q^{d}\, f(x)$$

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SÉRIE DE HILBERT

$\mathcal{V}$ SOUS-ESPACE DE $\mathbb{Q}[\mathbf{x}]$

$\mathcal{B}$: BASE HOMOGÈNE DE $\mathcal{V}$

$$\mathcal{V}(q) := \sum_{f \in \mathcal{B}} q^{\deg(f)}$$

POLYNÔMES DIAGONALEMENT SYMÉTRIQUES

$$X\sigma = \begin{pmatrix} x_{\sigma(1)} & \cdots & x_{\sigma(n)} \\ \vdots & \ddots & \vdots \\ z_{\sigma(1)} & \cdots & z_{\sigma(n)} \end{pmatrix}$$

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POLYNÔMES DIAGONALEMENT SYMÉTRIQUES

$$\forall \sigma \in S_n \qquad f(x\sigma) = f(x)$$

EXEMPLE

$$x_1^a y_1^b z_1^c + x_2^a y_2^b z_2^c + \dots$$

$$\dots + x_n^a y_n^b z_n^c$$

PETIT RAPPEL

$$h_k(q_1, q_2, \ldots, q_\ell) := \sum_{a+b+\ldots+c=k} q_1^a q_2^b \cdots q_\ell^c$$

EXEMPLE

$$h_3(q_1, q_2) = q_1^3 + q_1^2 q_2 + q_1 q_2^2 + q_2^3$$

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Petit rappel

$$h_k(q_1, q_2, \ldots, q_\ell) := \sum_{a+b+\cdots+c=k} q_1^a q_2^b \cdots q_\ell^c$$

$$h_k(\underbrace{1, 1, \ldots, 1}_{\ell}) = \binom{\ell+k-1}{k}$$

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Partages de n

$$\mu = \mu_1 \mu_2 \cdots \mu_k$$

$$\mu_i \in \mathbb{N}$$

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$$

$$n = \mu_1 + \mu_2 + \cdots + \mu_k$$

$$\mu \vdash n$$

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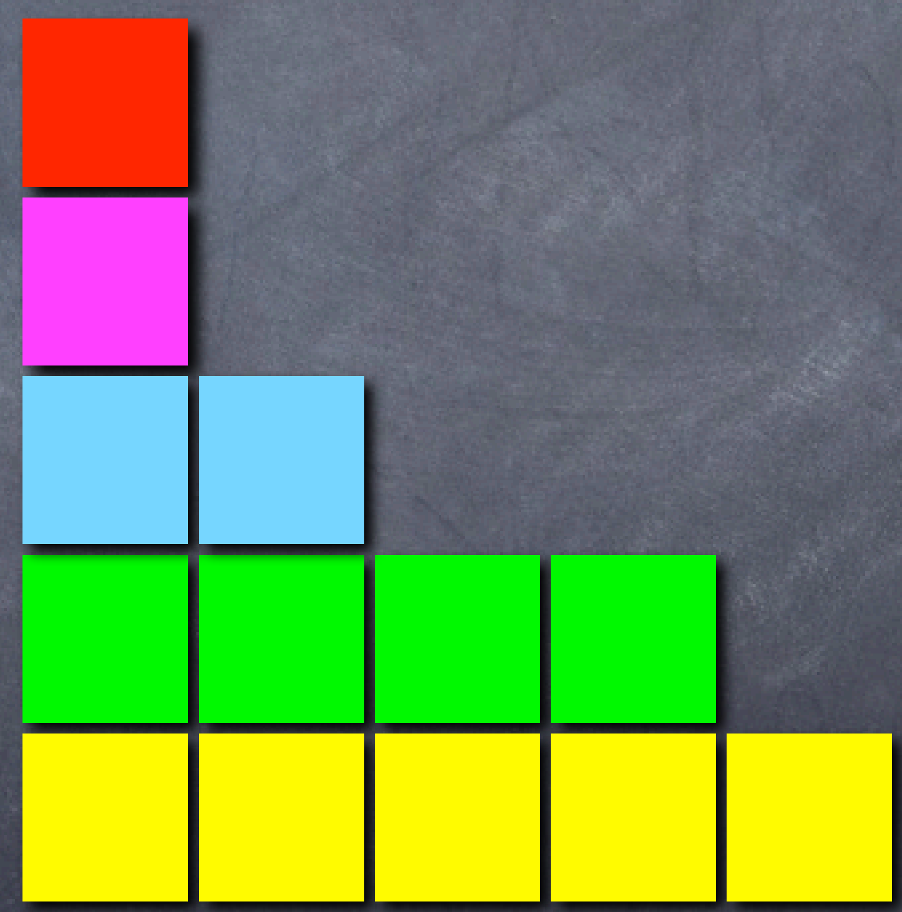
Partages de n

$$\mu = 53211$$

$$\mu =$$



$$
\begin{aligned}
m_5 &= x_1^5 + x_2^5 + x_3^5 + x_4^5 \\
m_{41} &= x_1^4 x_2 + x_1^4 x_3 + x_1^4 x_4 + x_2^4 x_3 + x_2^4 x_4 + x_3^4 x_4 \\
&\quad + x_1 x_2^4 + x_1 x_3^4 + x_1 x_4^4 + x_2 x_3^4 + x_2 x_4^4 + x_3 x_4^4 \\
m_{32} &= x_1^3 x_2^2 + x_1^3 x_3^2 + x_1^3 x_4^2 + x_2^3 x_3^2 + x_2^3 x_4^2 + x_3^3 x_4^2 \\
&\quad + x_1^2 x_2^3 + x_1^2 x_3^3 + x_1^2 x_4^3 + x_2^2 x_3^3 + x_2^2 x_4^3 + x_3^2 x_4^3 \\
m_{311} &= x_1^3 x_2 x_3 + x_1^3 x_2 x_4 + x_1^3 x_3 x_4 + x_2^3 x_3 x_4 \\
&\quad + x_1 x_2^3 x_3 + x_1 x_2^3 x_4 + x_1 x_3^3 x_4 + x_2 x_3^3 x_4 \\
&\quad + x_1 x_2 {x_3}^3 + x_1 x_2 x_4^3 + x_1 x_3 x_4^3 + x_2 x_3 x_4^3 \\
m_{221} &= x_1^2 x_2^2 x_3 + x_1^2 x_2^2 x_4 + x_1^2 x_3^2 x_4 + x_2^2 x_3^2 x_4 \\
&\quad + x_1^2 x_2 x_3^2 + x_1^2 x_2 x_4^2 + x_1^2 x_3 x_4^2 + x_2^2 x_3 x_4^2 \\
&\quad + x_1 x_2^2 x_3^2 + x_1 x_2^2 x_4^2 + x_1 x_3^2 x_4^2 + x_2 x_3^2 x_4^2 \\
m_{2111} &= x_1^2 x_2 x_3 x_4 + x_1 x_2^2 x_3 x_4 + x_1 x_2 x_3^2 x_4 + x_1 x_2 x_3 x_4^2 \\
m_{11111} &= 0
\end{aligned}
$$

$$h_\mu := h_{\mu_1} h_{\mu_2} \cdots h_{\mu_k}$$

$$\begin{aligned}
h_{11111} &= m_5 + 5\,m_{41} + 10\,m_{32} + 20\,m_{311} + 30\,m_{221} + 60\,m_{2111} + 120\,m_{11111}\\
h_{2111} &= m_5 + 4\,m_{41} + 7\,m_{32} + 13\,m_{31,1} + 18\,m_{221} + 33\,m_{2111} + 60\,m_{11111}\\
h_{221} &= m_5 + 3\,m_{41} + 5\,m_{32} + 8\,m_{311} + 11\,m_{221} + 18\,m_{2111} + 30\,m_{11111}\\
h_{311} &= m_5 + 3\,m_{41} + 4\,m_{32} + 7\,m_{311} + 8\,m_{221} + 13\,m_{2111} + 20\,m_{11111}\\
h_{32} &= m_5 + 2\,m_{41} + 3\,m_{32} + 4\,m_{311} + 5\,m_{221} + 7\,m_{2111} + 10\,m_{11111}\\
h_{41} &= m_5 + 2\,m_{41} + 2\,m_{32} + 3\,m_{311} + 3\,m_{221} + 4\,m_{2111} + 5\,m_{11111}\\
h_{5} &= m_5 + m_{41} + m_{32} + m_{311} + m_{221} + m_{2111} + m_{11111}
\end{aligned}$$

$$e_k := m_{11\dots1} \qquad p_k := m_k$$

$$e_\mu := e_{\mu_1} e_{\mu_1} \cdots e_{\mu_k} \qquad p_\mu := p_{\mu_1} p_{\mu_1} \cdots p_{\mu_k}$$

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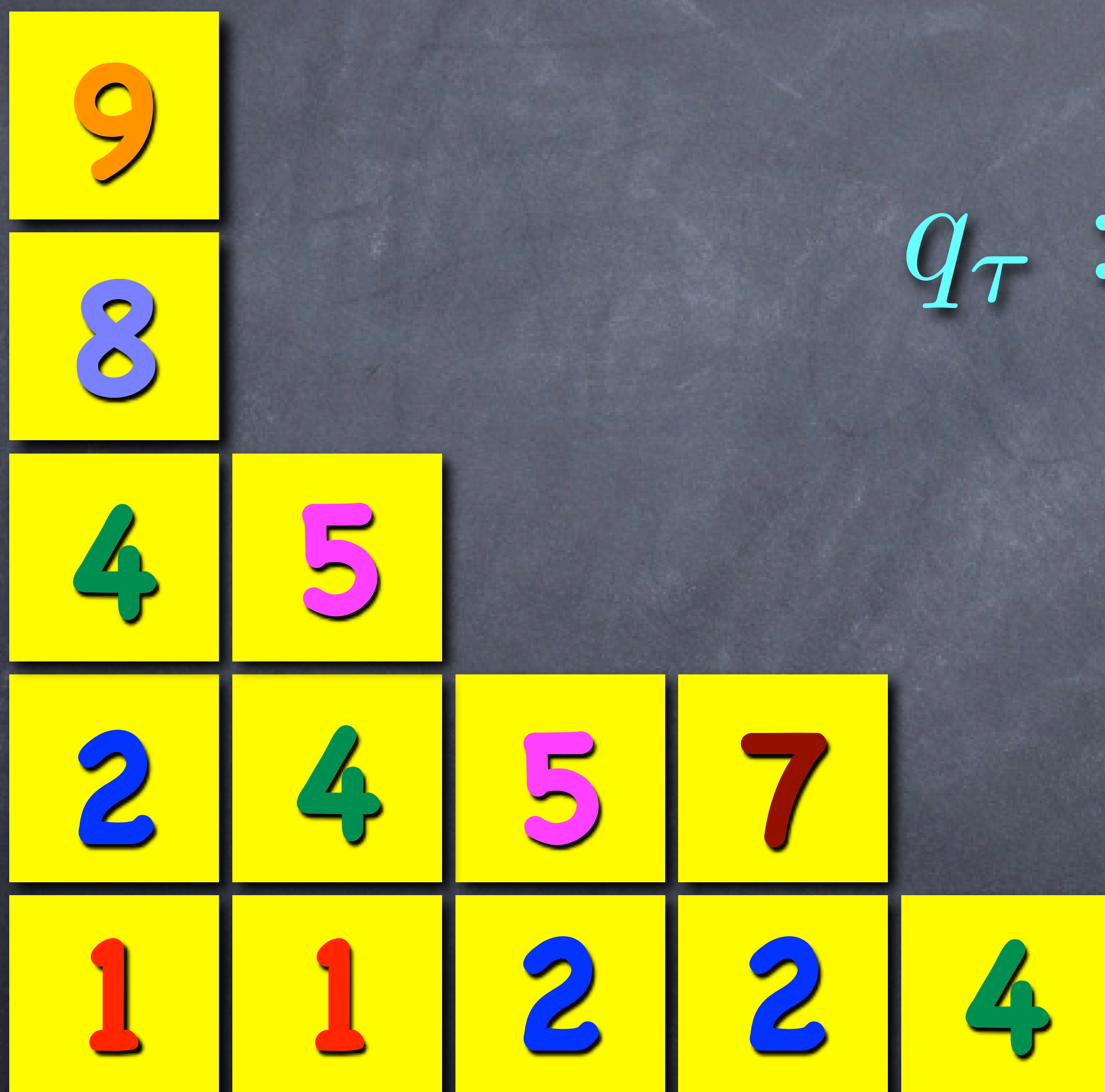
Tableaux semi-standards



$$q_\tau := q_1^2\, q_2^3\, q_4^3\, q_5^2\, q_7\, q_8\, q_9$$

Polynômes de Schur s_μ

$$s_\mu := \sum_{\tau \text{ de forme } \mu} q_\tau$$

Polynômes de Schur s_μ

$$s_{\begin{smallmatrix}\square & \\ \square & \square\end{smallmatrix}} = \cdots + \begin{array}{|c|c} \hline b & \\ \hline a & \multicolumn{1}{c|}{a} \\ \hline \end{array} + \begin{array}{|c|c} \hline b & \\ \hline a & \multicolumn{1}{c|}{b} \\ \hline \end{array} +$$

$$\cdots + \begin{array}{|c|c} \hline b & \\ \hline a & \multicolumn{1}{c|}{c} \\ \hline \end{array} + \begin{array}{|c|c} \hline c & \\ \hline a & \multicolumn{1}{c|}{b} \\ \hline \end{array} + \cdots$$

$$s_{\begin{smallmatrix}\square & \\ \square & \square\end{smallmatrix}} = m_{\begin{smallmatrix}\square & \\ \square & \square\end{smallmatrix}} + 2\, m_{\square\square\square}$$

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Polynômes de Schur s_μ

- Charactères des représentations irréductibles de GL_ℓ

- Codage des représentations irréductibles de $\mathbb{S}_n$

Série de Hilbert

$$\mathcal{V}(q) := \sum_{f \in \mathcal{B}} q^{\deg(f)}$$

$$\mathcal{V}(q) := \sum_{\mu} c_\mu S_\mu$$

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Polynômes Harmoniques Diagonaux

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POLYNÔMES HARMONIQUES DIAGONAUX

$$\mathcal{D}_n := \{\, g(x) \mid f(\partial x)\, g(x) = 0,\ f(0)=0 \,\}$$

$$\downarrow$$

POUR TOUT $f(x)$ DIAGONALEMENT SYMÉTRIQUE.

Polynômes Harmoniques Diagonaux

$$\mathcal{D}_n := \{\, g(x) \mid f(\partial x)\, g(x) = 0,\ f(0)=0 \,\}$$

$$\downarrow$$

$$f(\partial x) = \sum_{A} f_A \cdot \partial x^{A}$$

POLYNÔMES HARMONIQUES DIAGONAUX

$$\mathcal{D}_n := \{ g(x) \mid f(\partial x)\, g(x) = 0,\ f(\alpha) = 0 \}$$

$$\downarrow$$

$$\partial x^A := \partial x_1^{a_1} \cdots \partial x_n^{a_n}\, \partial y_1^{b_1} \cdots \partial y_n^{b_n} \cdots$$

$$\cdots \partial z_1^{c_1} \cdots \partial z_n^{c_n}$$

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POLYNÔMES HARMONIQUES DIAGONAUX

$\mathcal{D}_m$ ADMET UNE BASE HOMOGÈNE (FINIE)

$$\dim(\mathcal{D}_m) = \mathcal{D}_m(\underbrace{1, 1, \ldots, 1}_{\ell})$$

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ALTERNANTS

$$\mathcal{A}_n := \left\{ g(x) \in \mathcal{D}_n \;\middle|\; g(x\sigma) = (-1)^{\mathrm{inv}(\sigma)}\, g(x), \text{ pour tout } \sigma \in \mathbb{S}_n \right\}$$

CAS $\ell = 1$

$$\mathcal{D}_3 = \mathbb{Q}[\partial x^\alpha \Delta_3 \mid \alpha \in \mathbb{N}^3]$$

$$\Delta_3 = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$

$$\mathcal{D}_3 = \mathbb{Q}\{\Delta_3, \partial x_1 \Delta_3, \partial x_2 \Delta_3, \partial x_1^2 \Delta_3, \partial x_1 \partial x_2 \Delta_3, 1\}$$

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SÉRIE DE HILBERT

$$\mathcal{D}_n(q) = \prod_{i=1}^{n} \frac{1-q^i}{1-q}$$

$$\mathcal{D}_1(q) = 1$$

$$\mathcal{D}_2(q) = 1 + q$$

$$\mathcal{D}_3(q) = 1 + 2q + 2q^2 + q^3$$

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CAS $\ell = 2$

$$\dim(\mathcal{D}_n) = (n+1)^{n-1}$$

$$\dim(A_n) = \frac{1}{n+1}\binom{2n}{n}$$

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CAS $n = 2$

$$\mathcal{D}_2 = \mathbb{Q}\{1, x_{11} - x_{12}, \ldots, x_{\ell 1} - x_{\ell 2}\}$$

SÉRIE DE HILBERT

$$\mathcal{D}_2(q_1, q_2, \ldots, q_\ell) = 1 + h_1(q)$$

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Série de Hilbert générique

Théorème: La série de Hilbert de l'espace des polynômes harmoniques diagonaux admet une description indépendante de ℓ.

$$\mathcal{D}_n = \sum_{\mu} d_\mu^{(n)} s_\mu$$

$\mu \vdash d, \qquad 0 \le d \le \binom{n}{2}.$

nombre de parts(μ) ≤ $\min(n-1, \binom{n}{2} - d)$

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$$\mathcal{D}_n(1) = n!$$

$$\mathcal{D}_n(1,1) = (n+1)^{n-1}$$

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CAS $\ell = 3$

$$\mathcal{D}_1(q_1, q_2, q_3) = 1$$

$$\mathcal{D}_2(q_1, q_2, q_3) = 1 + (q_1 + q_2 + q_3)$$

$$\begin{aligned}\mathcal{D}_3(q_1, q_2, q_3) = 1 &+ 2(q_1 + q_2 + q_3) + \\ &(q_1 + q_2 + q_3)^2 + \\ &(q_1^2 + q_2^2 + q_3^2 + q_1 q_2 + \\ &\quad q_1 q_3 + q_2 q_3) + \\ &(q_1^3 + \cdots + q_1 q_2 q_3)\end{aligned}$$

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Dimensions

$1^\ell := \underbrace{11\cdots1}_{\ell\text{-copies}}$

$$\mathcal{D}_1(1^\ell) = 1$$

$$\mathcal{D}_2(1^\ell) = 1 + \binom{\ell}{1}$$

$$\mathcal{D}_3(1^\ell) = 1 + 2\binom{\ell}{1} + \binom{\ell}{1}^2 + \binom{\ell+1}{2} + \binom{\ell+2}{3}$$

$$\mathcal{D}_4(1^\ell) = 1 + 3\binom{\ell}{1} + 3\binom{\ell}{1}^2 + 2\binom{\ell+1}{2} + \binom{\ell}{1}^3 + 3\binom{\ell}{1}\binom{\ell+1}{2} + 2\binom{\ell+2}{3} + 4\binom{\ell}{1}\binom{\ell+2}{3} + \binom{\ell+3}{4} + \binom{\ell}{1}\binom{\ell+3}{4} + 2\binom{\ell+4}{5} + \binom{\ell+5}{6}$$

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Conjecture: $\mathfrak{D}_n$ est h-positif

$$\Longrightarrow \mathfrak{D}_n = \sum_{\sigma \in S_n} h_{\mu(\sigma)}$$

où

$\mu(\sigma)$ PARTAGE DE $inv(\sigma)$

$n = 3$

0, $\underbrace{1, \quad 1,}$ $\underbrace{11, \quad 2,}$ 3,

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$$\mathcal{D}_1 = 1$$

$$\mathcal{D}_2 = 1 + h_1$$

$$\mathcal{D}_3 = 1 + 2h_1 + h_1^2 + h_2 + h_3$$

$$\mathcal{D}_4 = 1 + 3h_1 + 3h_1^2 + 2h_2 + h_1^3 + 3h_1h_2 + 2h_3 + 4h_1h_3 + h_4 + h_1h_4 + 2h_5 + h_6$$

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Série de Hilbert générique

Théorème: La série de Hilbert de l'espace des polynômes harmoniques diagonaux alternants admet une description indépendante de ℓ.

$$A_n = \sum_{\mu} a_{\mu}^{(n)} s_{\mu}$$

$a_{\mu}^{(n)} \in \mathbb{N}$ indépendants de ℓ

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$$\mathcal{A}_1 = 1$$

$$\mathcal{A}_2 = s_1$$

$$\mathcal{A}_3 = s_{11} + s_3$$

$$\mathcal{A}_4 = s_{111} + s_{31} + s_{41} + s_6$$

$$\mathcal{A}_5 = s_{1111} + s_{311} + s_{411} + s_{42} + s_{43}$$

$$+ s_{511} + s_{61} + s_{62} + s_{71} + s_{81} + s_{10}$$

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$$a_n(1,1) = \frac{1}{n+1}\binom{2n}{n}$$

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q,t-Catalan

$$A_1(q,t) = 1$$
$$A_2(q,t) = q+t$$
$$A_3(q,t) = q^3+q^2t+qt^2+t^3+qt$$
$$A_4(q,t) = q^6+q^5t+q^4t^2+q^3t^3+q^2t^4+qt^5+t^6$$
$$+q^4t+q^3t^2+q^2t^3+qt^4+q^3t+q^2t^2+qt^3$$
$$A_5(q,t) = q^{10}+q^9t+q^8t^2+q^7t^3+q^6t^4+q^5t^5+q^4t^6+q^3t^7+q^2t^8+qt^9+t^{10}$$
$$+q^8t+q^7t^2+q^6t^3+q^5t^4+q^4t^5+q^3t^6+q^2t^7+qt^8$$
$$+q^7t+2\,q^6t^2+2\,q^5t^3+2\,q^4t^4+2\,q^3t^5+2\,q^2t^6+qt^7$$
$$+q^6t+q^5t^2+2\,q^4t^3+2\,q^3t^4+q^2t^5+qt^6+q^4t^2+q^3t^3+q^2t^4$$

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Dimensions Alternants

$$A_1(1^\ell) = 1$$

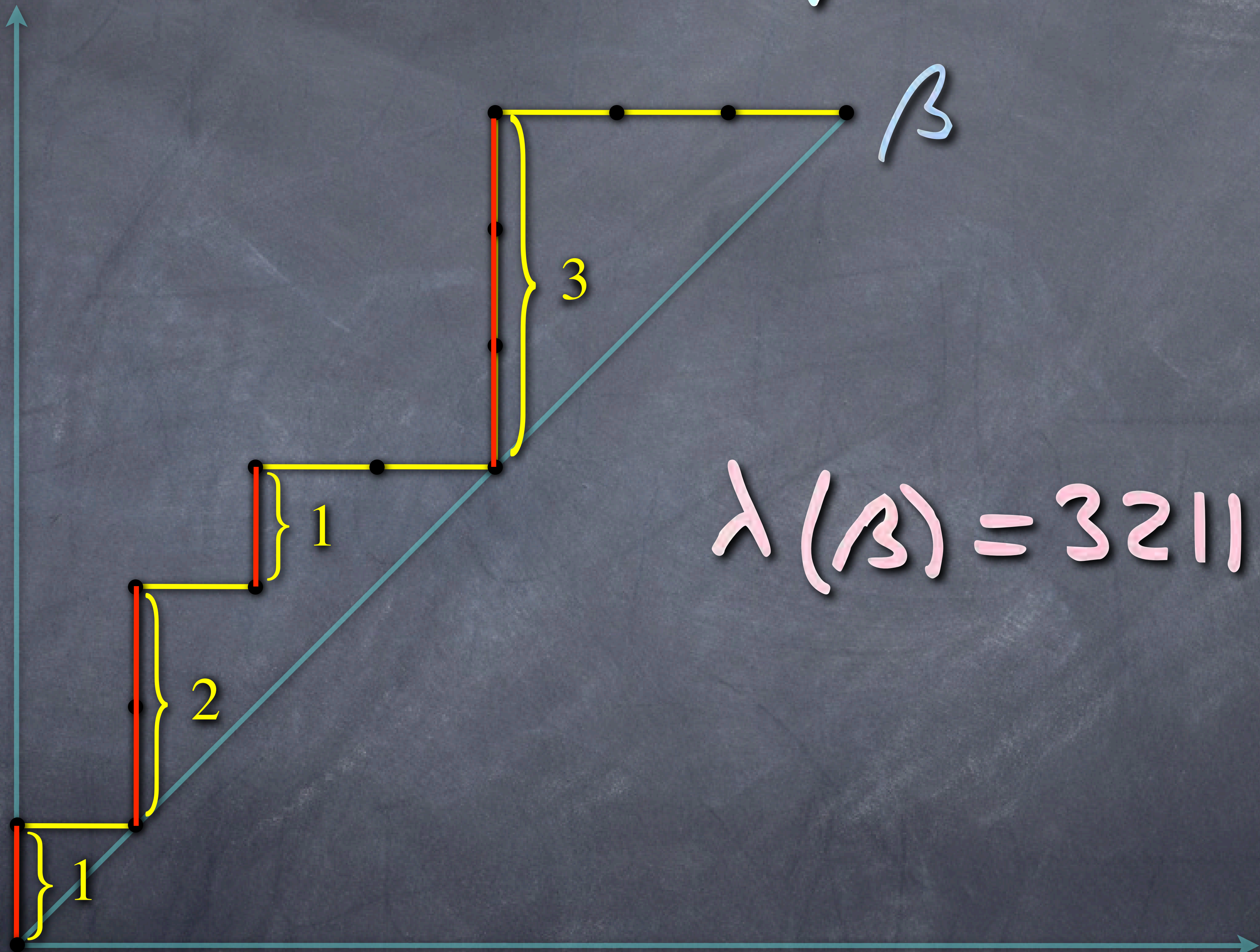
$$A_2(1^\ell) = 1 + \binom{\ell-1}{1}$$

$$A_3(1^\ell) = 1 + 2\binom{\ell-1}{1} + \binom{\ell-1}{1}^2 + \binom{\ell+1}{3}$$

$$A_4(1^\ell) = 1 + 3\binom{\ell-1}{1} + 3\binom{\ell-1}{1}^2 + \binom{\ell-1}{1}^3 + 2\binom{\ell+1}{3}$$
$$+2\binom{\ell-1}{1}\binom{\ell+1}{3} + \binom{\ell-1}{1}\binom{\ell+2}{4} + \binom{\ell+4}{6}$$

CAS $\ell = 3$

λ(β): Forme de β



$\lambda(\beta) = 3211$

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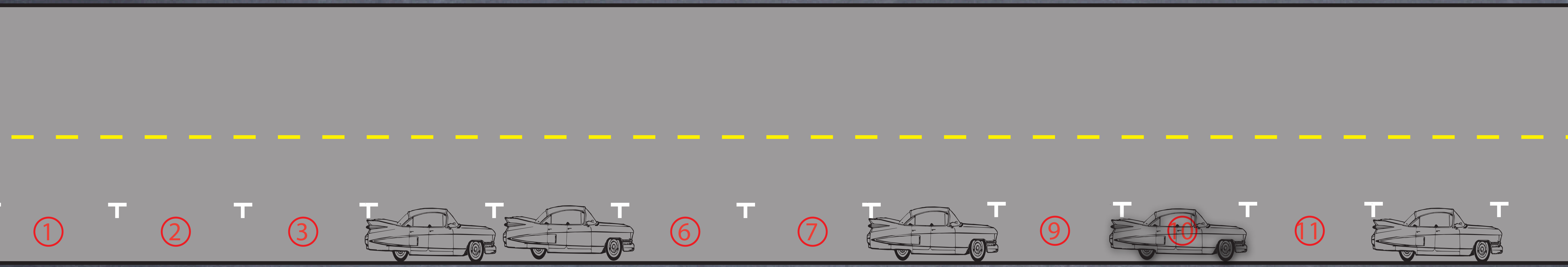
Fonctions de stationnement.



$$p : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$$

$$\# p^{-1}(\{1, \dots, k\}) \geq k$$

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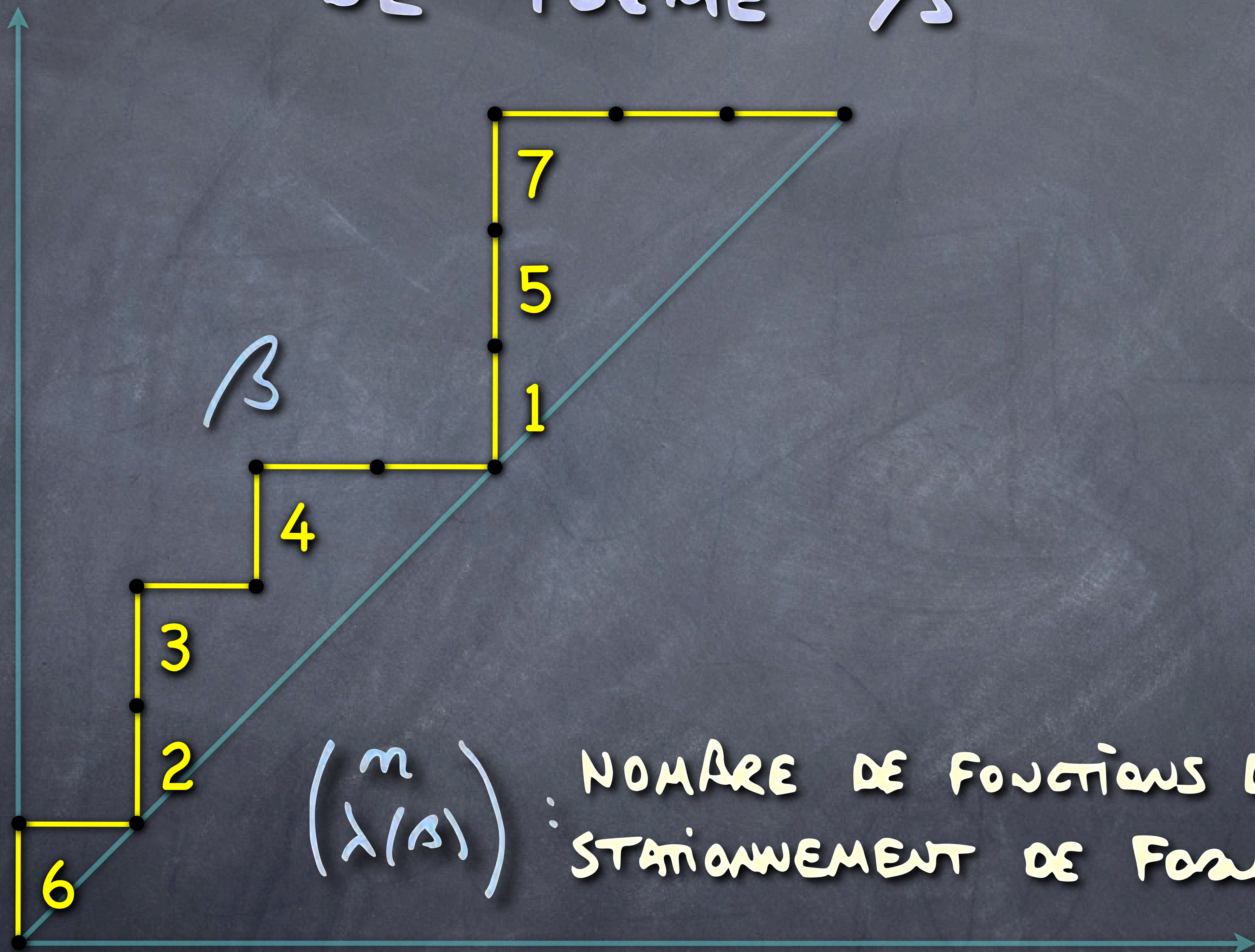
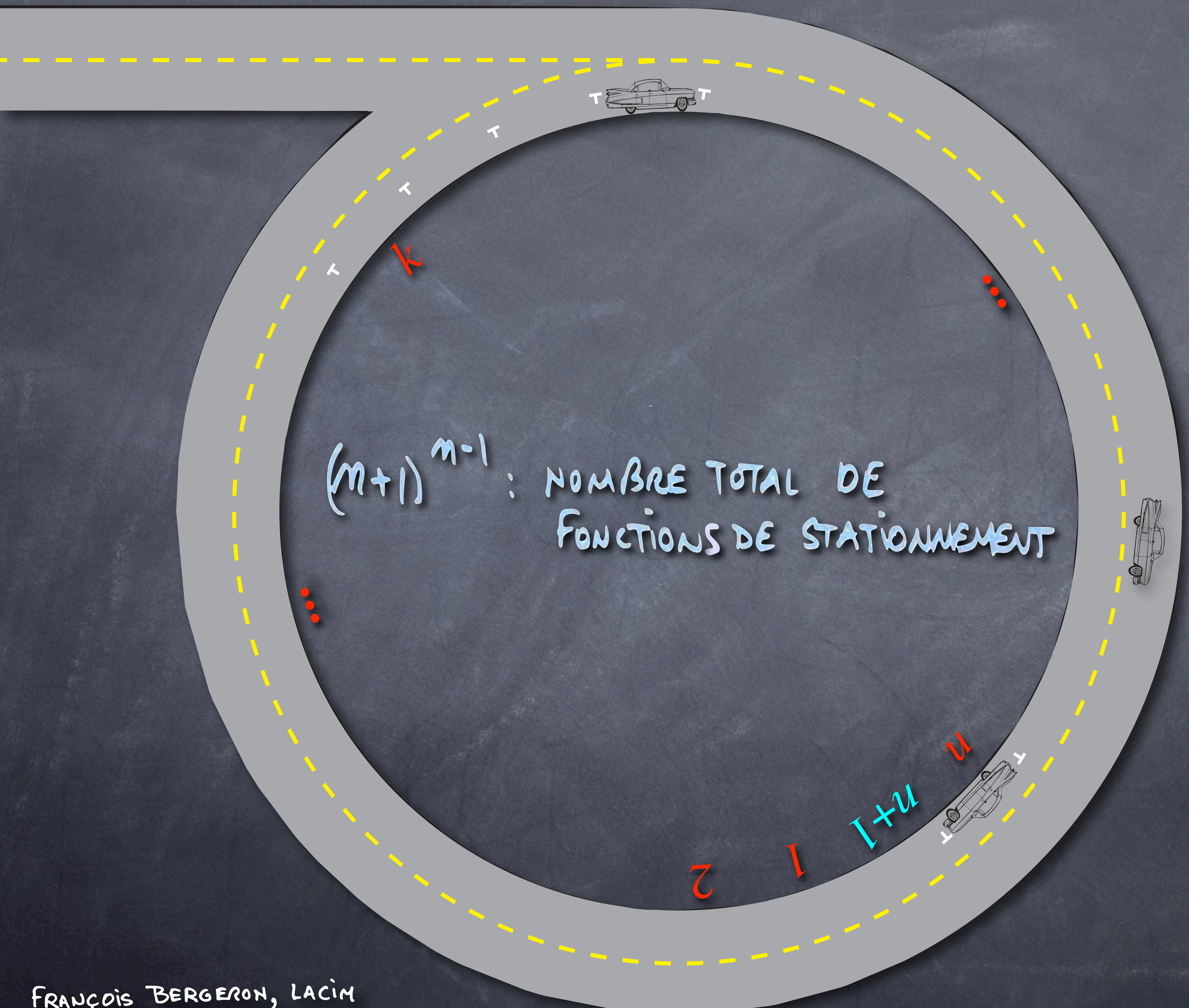
Fonction de stationnement de forme β



$\binom{n}{\lambda(\beta)}$: Nombre de fonctions de stationnement de forme β



$(n+1)^{n-1}$: nombre total de fonctions de stationnement

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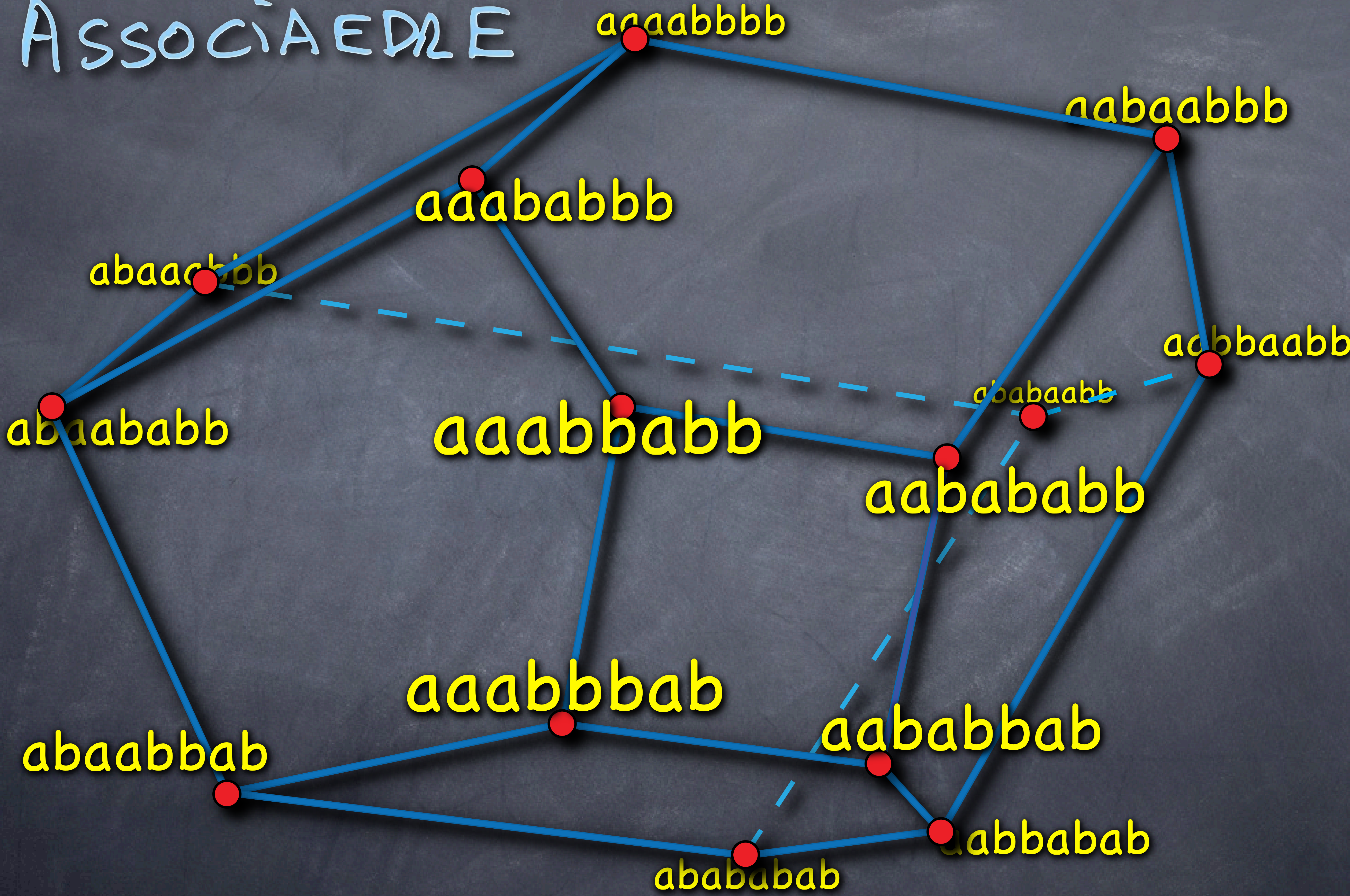
Fonctions de Stationnement.

$$(n+1)^{n-1} = \sum_{\beta} \binom{n}{\lambda(\beta)}$$

Associaèdre



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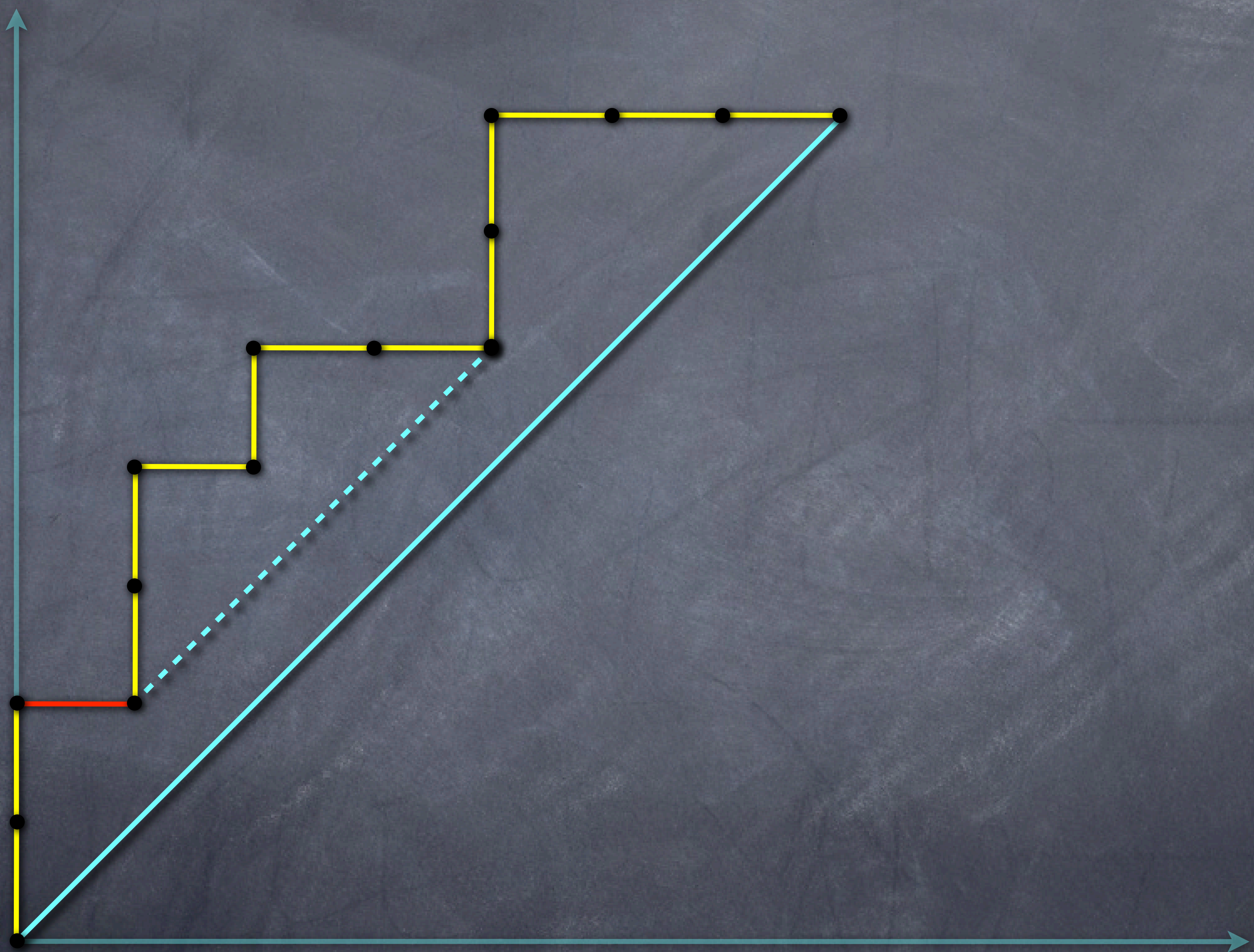

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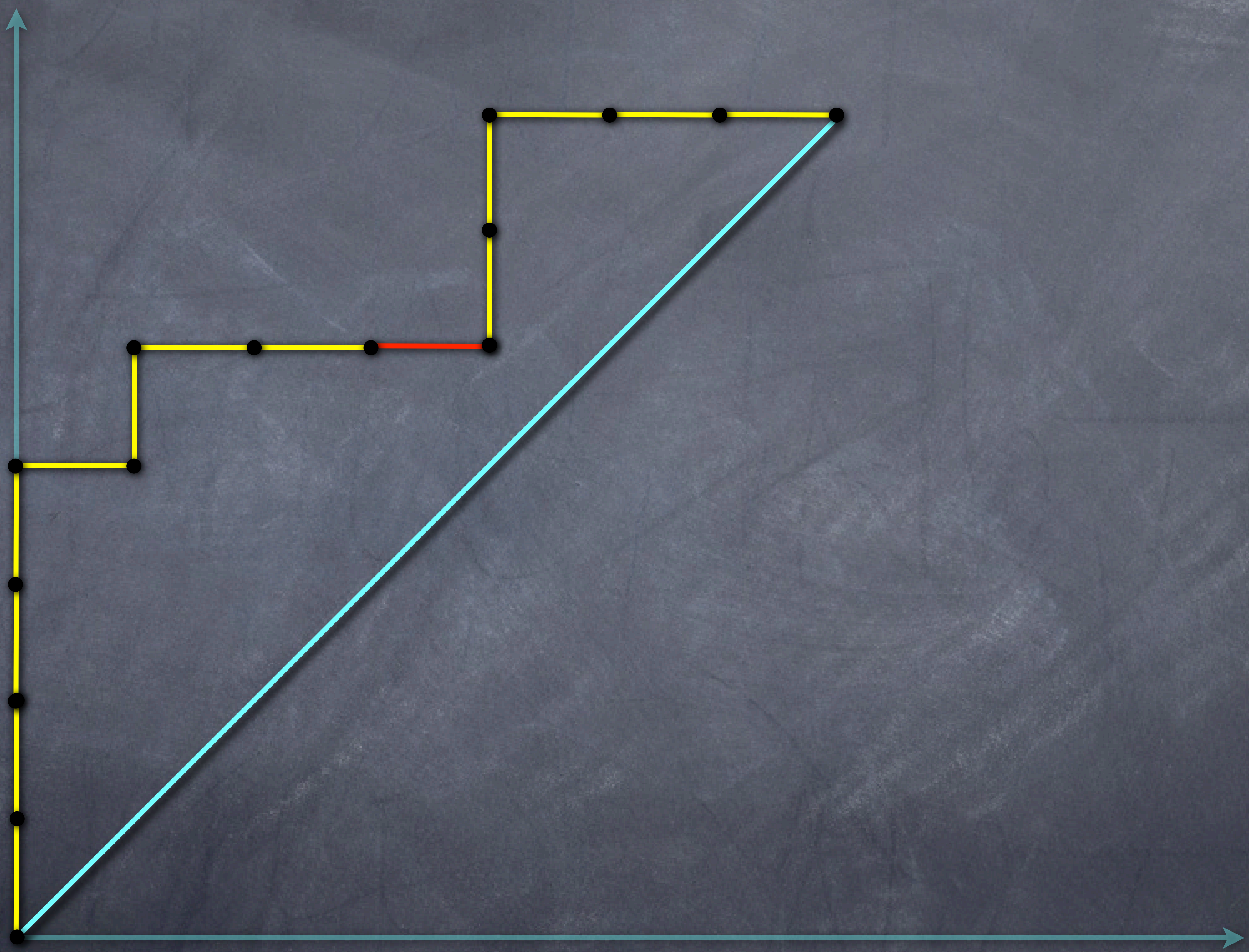
Treillis de Tamari



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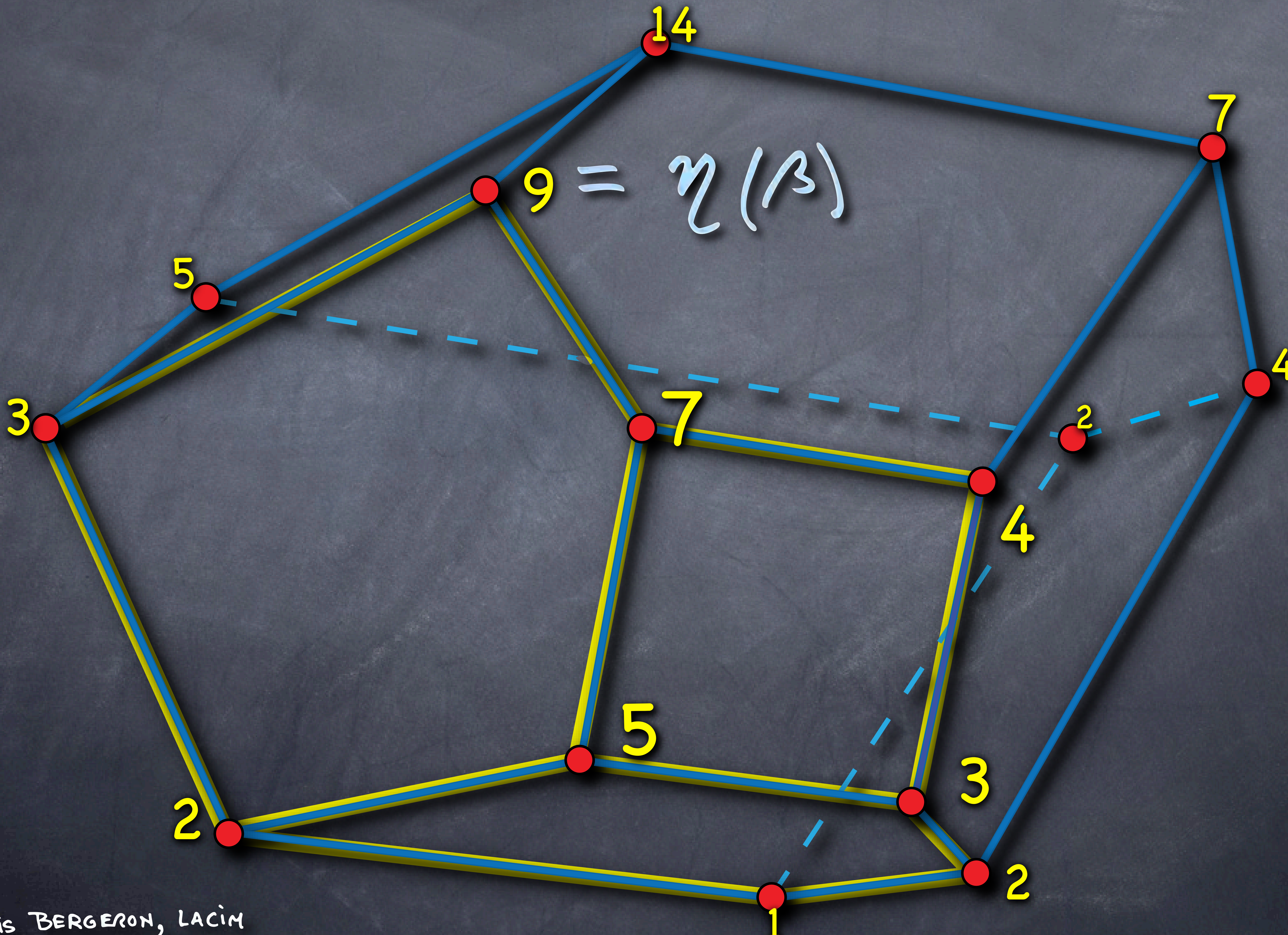
$\eta(\beta)$: NOMBRE D'ÉLÉMENTS $\leq \beta$



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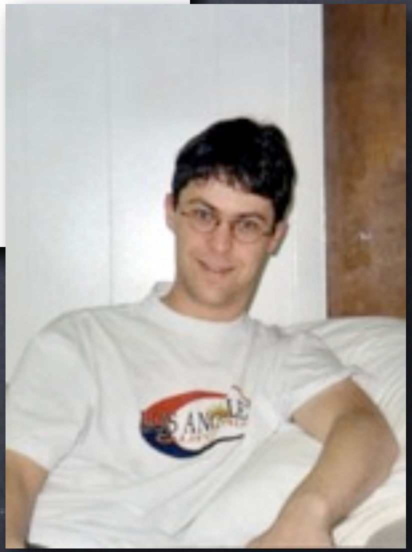
Séminaire Lotharingien de Combinatoire 55 (2006), Article B55f

SUR LE NOMBRE D'INTERVALLES DANS LES TREILLIS DE TAMARI

F. CHAPOTON

Résumé. On compte le nombre d'intervalles dans les treillis de Tamari. On utilise pour cela une description récursive de l'ensemble des intervalles. On introduit ensuite une notion d'intervalle nouveau dans les treillis de Tamari et on compte les intervalles nouveaux. On obtient aussi l'inverse de deux séries particulières dans un groupe de séries formelles en arbres.

Abstract. We enumerate the intervals in the Tamari lattices. For this, we introduce an inductive description of the intervals. Then a notion of "new interval" is defined and these are also enumerated. As a side result, the inverse of two special series is computed in a group of tree-indexed series.



$$\frac{2}{n(n+1)}\binom{4n+1}{n-1} = \sum_{\substack{\beta \\ \text{DYCK}}} \eta(\beta)$$

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QUESTION ALGÉBRIQUE OUVERTE

$$\mathcal{D}_n(1,1,1) \stackrel{?}{=} 2^n (n+1)^{n-2}$$

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Question Algébrique Ouverte

$$A_n(1,1,1) \stackrel{?}{=} \frac{2}{n(n+1)} \binom{4n+1}{n-1}$$

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QUESTION COMBINATOIRE OUVERTE

$$2^n (n+1)^{n-2} \stackrel{?}{=} \sum_{\substack{\beta \\ \text{Dyck}}} \eta(\beta) \binom{n}{\lambda(\beta)}$$

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Question combinatoire ouverte

$$2^n (n+1)^{n-2} \overset{?}{=} \sum_{\substack{\beta \\ \text{Dyck}}} \eta(\beta) \binom{n}{\lambda(\beta)}$$

Louis-François
Préville-Ratelle

m-DYCK

m-TAMARI

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m-DYCK

← PENTE $1/m$

ORDRE DE m-TAMARI

m-FONCTIONS DE STATIONNEMENT

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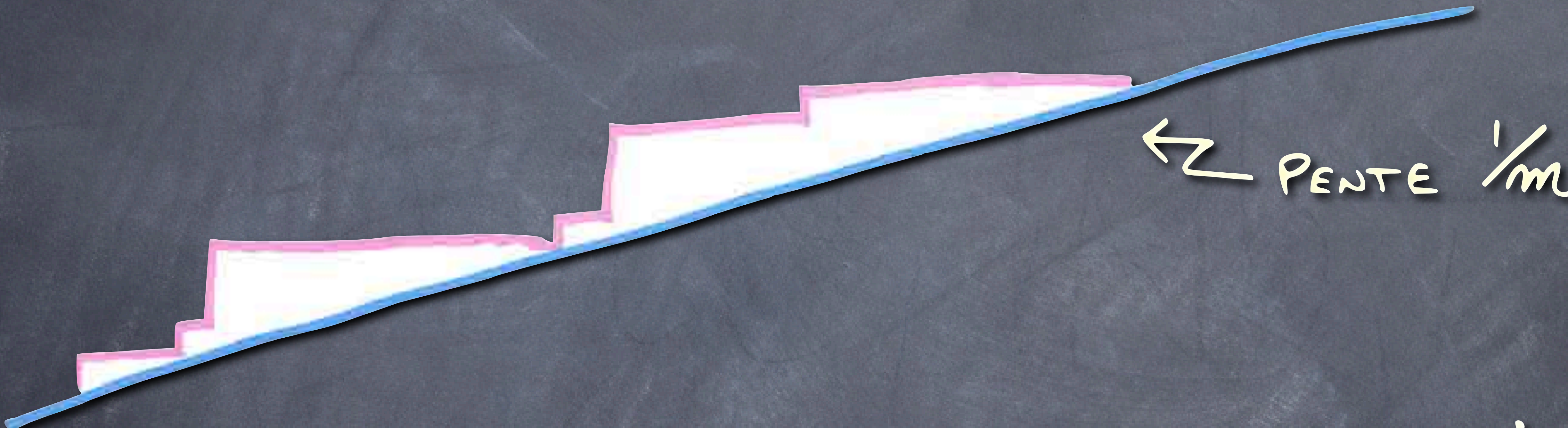
m-DYCK



ORDRE DE m-TAMARI

m-FONCTIONS DE STATIONNEMENT

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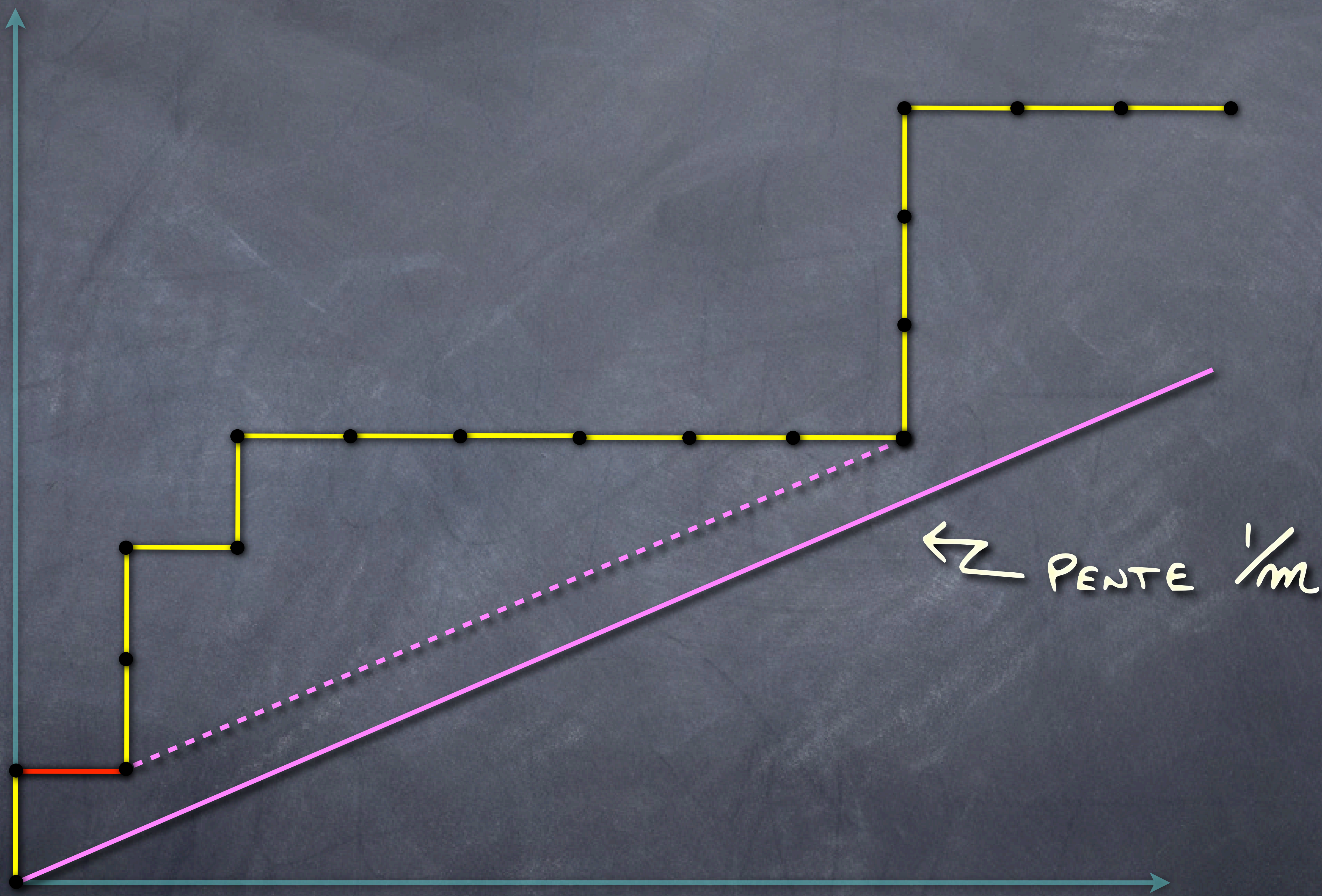
ORDRE DE m-TAMARI



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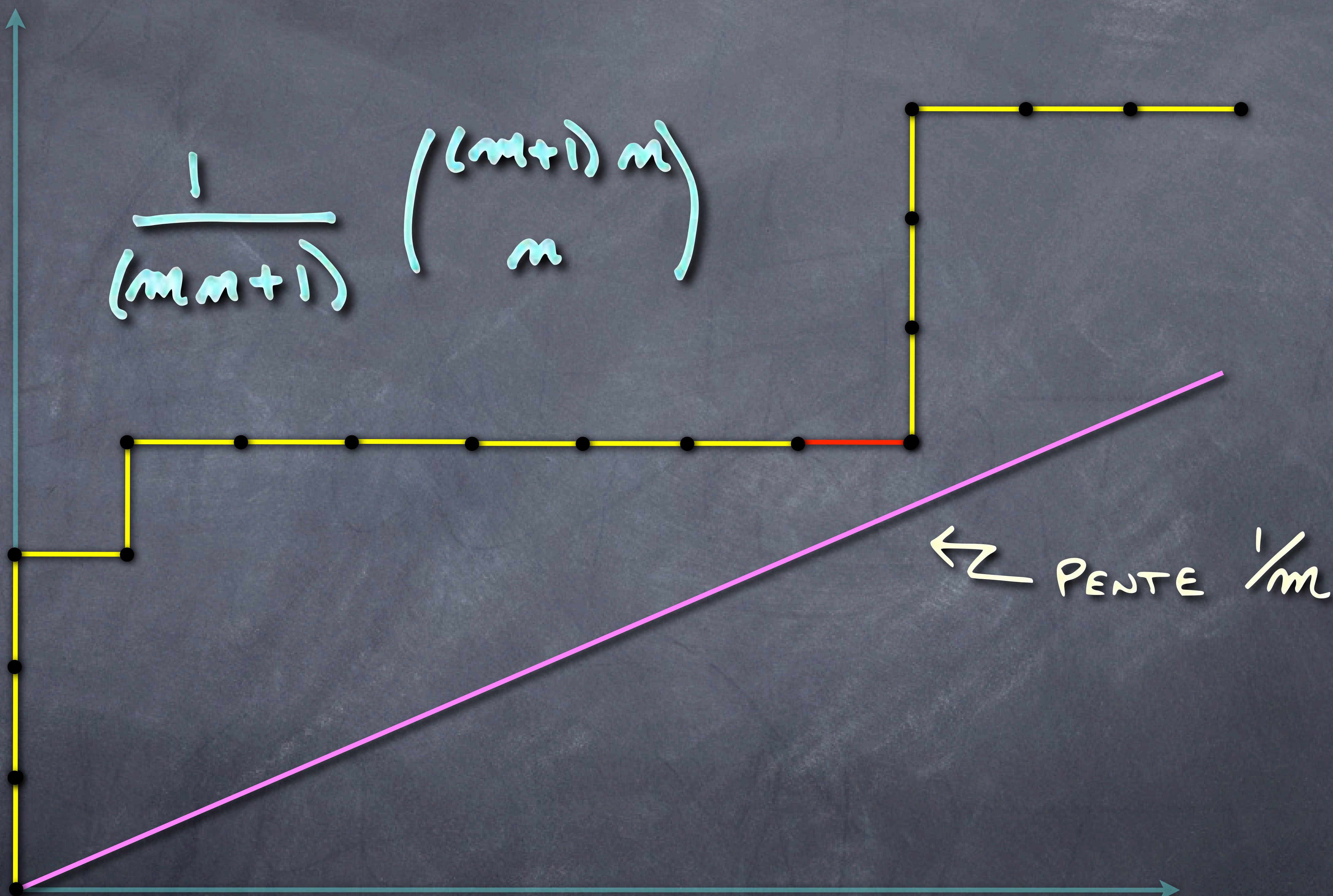
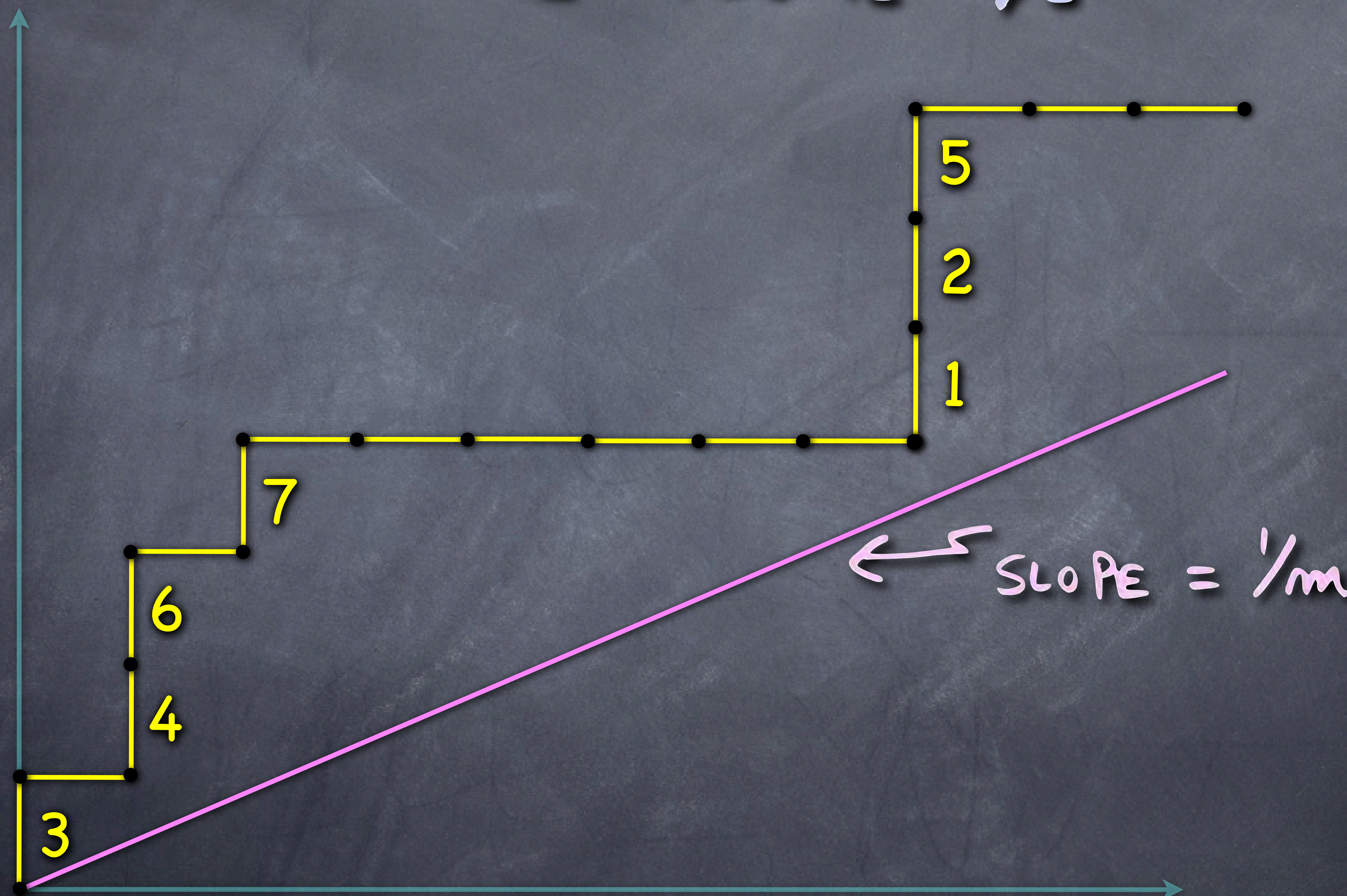
ORDRE DE m-TAMARI

$$\frac{1}{(mn+1)}\binom{(m+1)n}{n}$$



m-FONCTION DE STATIONNEMENT DE FORME β



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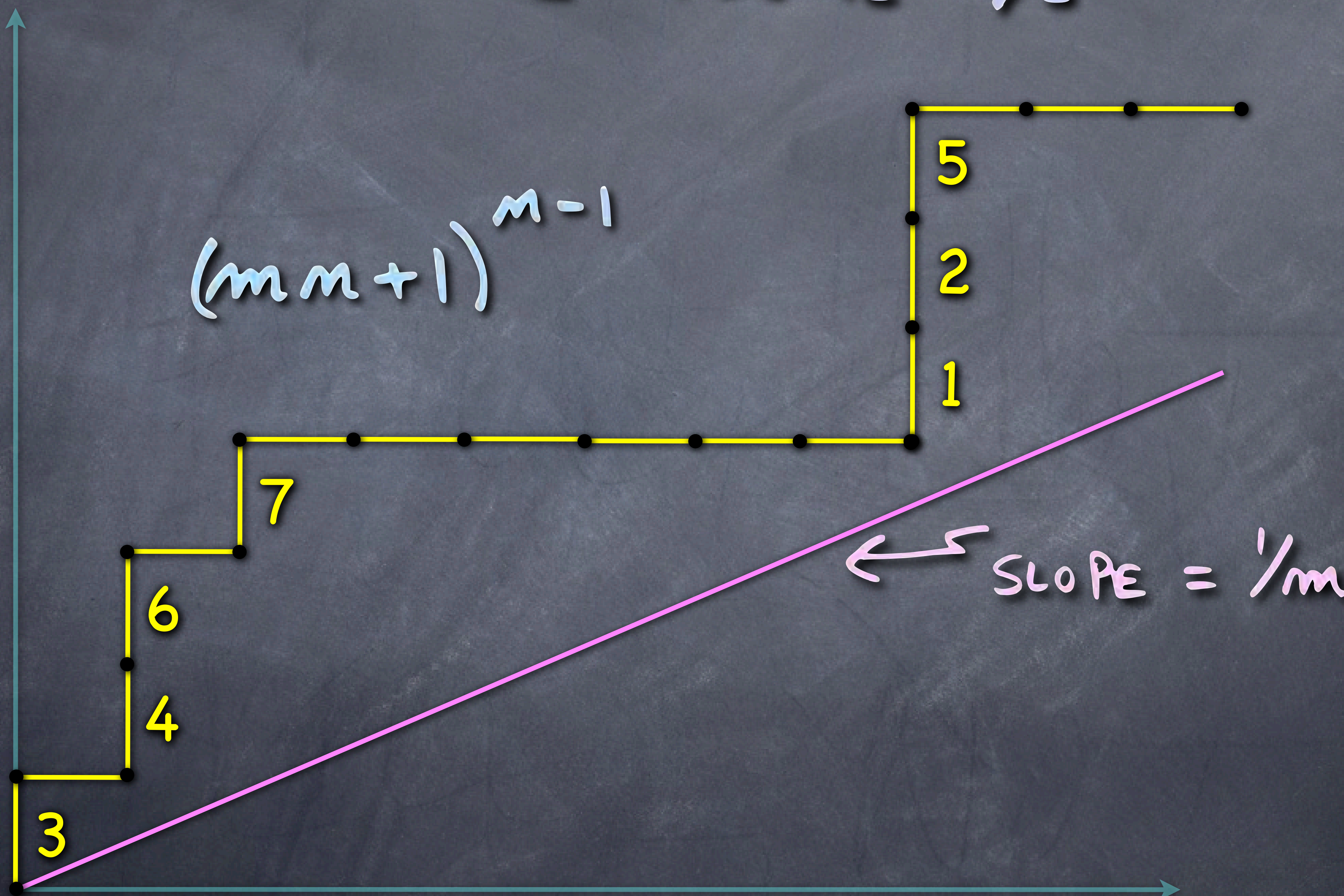
m-Fonction de stationnement de forme β



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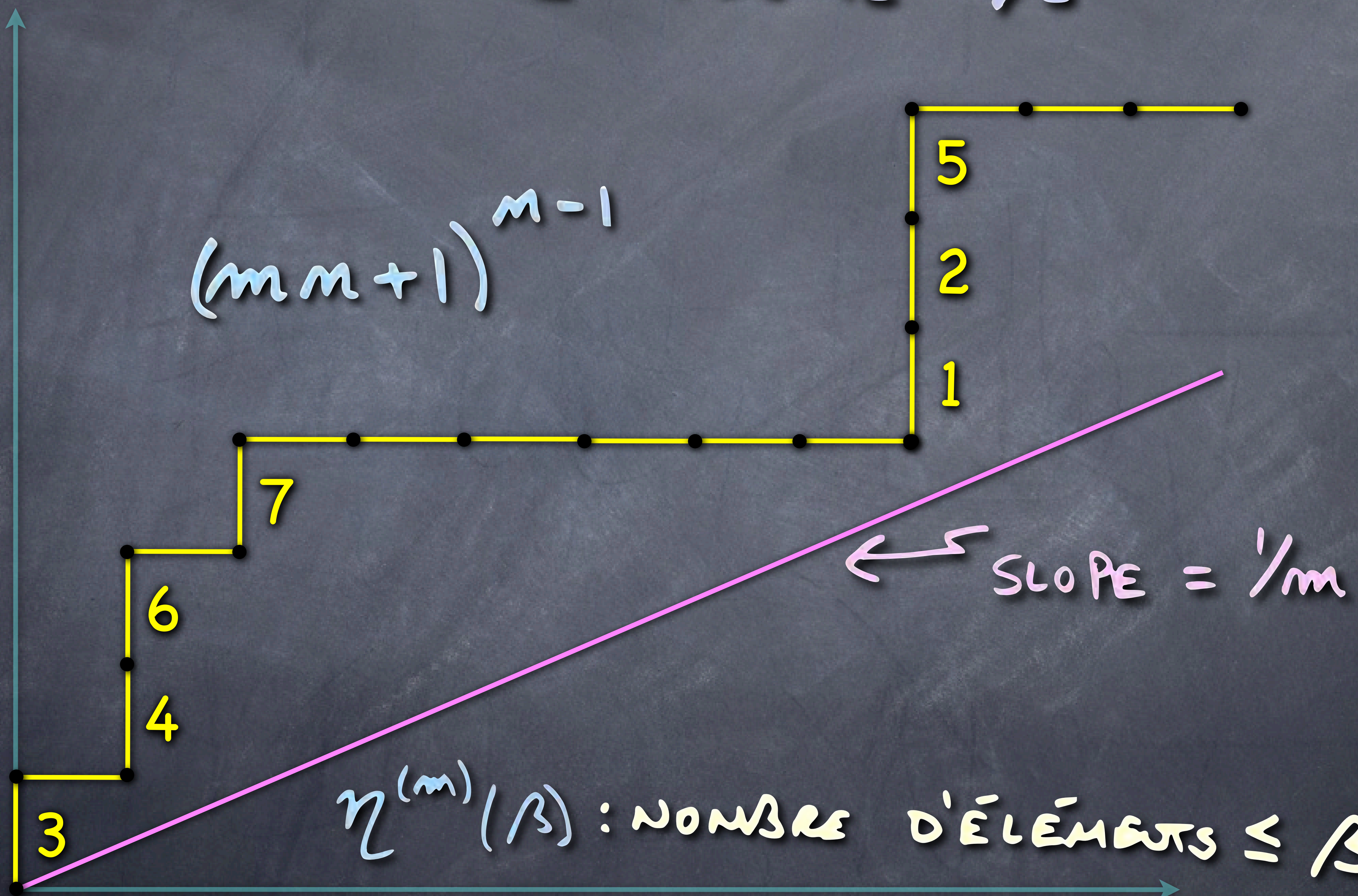
m-FONCTION DE STATIONNEMENT DE FORME β



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m-fonction de stationnement

$$p : \{1, \ldots, n\} \longrightarrow \{1, \ldots, nm\}$$

$$\# p^{-1}(\{1, \ldots, km\}) \geq k$$

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m-FONCTION DE STATIONNEMENT

$$(mn+1)^{n-1} = \sum_{\substack{\beta \\ m\text{-DYCK}}} \binom{n}{\lambda(\beta)}$$

$$\mathcal{D}_n^{(m)} := \mathbb{A}^{m-1} / I\mathbb{A}^{m-1}$$

$\mathbb{A}$: IDÉAL ENGENDRÉ PAR LES ALTERNANTS DIAGONAUX

I : IDÉAL ENGENDRÉ PAR LES POLYNÔMES DIAGONALEMENT SYMÉTRIQUES

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$$\mathcal{D}_n^{(m)} := \mathbb{A}^{m-1} \big/ \mathrm{I}\,\mathbb{A}^{m-1}$$

$$\mathcal{D}_n^{(m)}(1,1) = (mn+1)^{n-1}$$

$$A_n^{(m)}(1,1) = \frac{1}{(mn+1)} \binom{(m+1)\,n}{n}$$

François Bergeron, LACIM

Question algébrique ouverte

$$\mathcal{D}_n^{(m)}(1,1,1) \overset{?}{=} (m+1)^n\,(mn+1)^{n-2}$$

$$A_n^{(m)}(1,1,1) \overset{?}{=} \frac{(m+1)}{n\,(mn+1)} \binom{(m+1)^2 n + m}{n-1}$$

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QUESTION COMBINATOIRE OUVERTE

$$\frac{(m+1)}{n(mn+1)}\binom{(m+1)^2 n+m}{n-1} \overset{?}{=} \sum_{\substack{\beta \\ m\text{-DYCK}}} \eta^{(m)}(\beta)$$

$$\overset{?}{=} \sum_{\mu \vdash n} \frac{1}{z_\mu}(mn+1)^{\ell(\mu)-2} \prod_{k\in\mu} \binom{k(m+1)}{k}$$

$$z_\mu := 1^{d_1} d_1! 2^{d_2} d_2! \cdots n^{d_n} d_n!$$

QUESTION COMBINATOIRE OUVERTE

$$(m+1)^n (mn+1)^{n-2} \stackrel{?}{=} \sum_{\substack{\beta \\ m\text{-DYCK}}} \eta^{(m)}(\beta) \binom{n}{\lambda(\beta)}$$

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2 COMMUTING ACTIONS
(GL_ℓ AND $\mathfrak{S}_n$)

$\mathcal{V}$ INVARIANT FOR BOTH ACTIONS

$\forall f \in \mathcal{V}$ $\quad \sigma \cdot f \in \mathcal{V}$ AND $f \cdot \tau \in \mathcal{V}$

Frobenius Transform of the Graded Character of $\mathcal{D}$

$$\mathcal{D}_n(w;q) := \sum_{d\in\mathbb{N}^\ell} q^d \frac{1}{n!}\sum_{\sigma\in S_n} x^{\gamma_d(\sigma)} p_{\lambda(\sigma)}$$

Frobenius Transform of the Graded Character of $\mathcal{D}$

$$\mathcal{D}_2(w;q) = m_2(w) + (1+h_1(q))\, m_{11}(w)$$

$$\mathcal{D}_3(w;q) = m_3 + (1+h_1+h_2)\, m_{21}$$

$$+ (1+2h_1+h_1^2+h_2+h_3)\, m_{111}$$

QUESTION ALGÉBRIQUE OUVERTE

$$\mathcal{D}_n(w;q) \overset{?}{=} \sum_{\lambda \vdash n} m_\lambda \sum_{\mathrm{Desc}(\sigma) \subseteq S(\lambda)} h_{\mu(\sigma)}$$

$$S(\lambda) = \{\lambda_1, \lambda_1 + \lambda_2, \ldots\}$$

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QUESTION ALGÉBRIQUE OUVERTE

$$\mathcal{D}_n(w;q) \stackrel{?}{=} \sum_{\lambda \vdash n} m_\lambda \sum_{\mathrm{Desc}(\sigma) \subseteq S(\lambda)} h_{\mu_\lambda(\sigma)}$$

$$S(\lambda) = \{\lambda_1, \lambda_1 + \lambda_2, \ldots\}$$

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Théorèmes

$$\mathcal{D}_n(w;1) = h_1^n$$

$$\mathcal{D}_n(w;1,1) = \sum_{\beta} e_{\lambda(\beta)}$$

$$\mathcal{D}_n^{(m)}(w;1,1) = \sum_{\substack{\beta \\ m\text{-DYCK}}} e_{\lambda(\beta)}$$

QUESTION ALGÉBRIQUE OUVERTE

$$\mathcal{D}_n(w; 1, 1, 1) \stackrel{?}{=} \sum_{\beta} \eta(\beta)\, e_{\lambda(\beta)}$$

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QUESTION ALGÉBRIQUE OUVERTE

$$\mathcal{D}_n(w;1,1,1) \overset{?}{=} \sum_{\beta} \eta(\beta)\, e_{\lambda(\beta)}$$

$$\mathcal{D}_n^{(m)}(w;1,1,1) = \sum_{\substack{\beta \\ m\text{-DYCK}}} \eta^{(m)}(\beta)\, e_{\lambda(\beta)}$$

Question algébrique ouverte

$$\mathcal{D}_n(w;1,1,1) \stackrel{?}{=} \sum_{\beta} \eta(\beta)\, e_{\lambda(\beta)}$$

$$\mathcal{D}_n^{(m)}(w;1,1,1) \stackrel{?}{=} \sum_{\substack{\beta \\ m\text{-Dyck}}} \eta^{(m)}(\beta)\, e_{\lambda(\beta)}$$

François Bergeron, LaCIM/LaBRI

QUESTION ALGÉBRIQUE OUVERTE

$$\mathcal{D}_n^{(m)}(w; 1,1,1) \overset{?}{=}$$

$$\sum_{\mu \vdash n} \frac{(-1)^{n-\ell(\mu)}}{z_\mu} p_\mu(w)$$

$$(nm+1)^{\ell(\mu)-2} \prod_{k \in \mu} \binom{k(m+1)}{k}$$

FRANÇOIS BERGERON, LACIM/LABRI

Fin