Dynamique euclidienne : une approche symbolique

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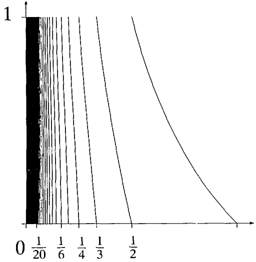
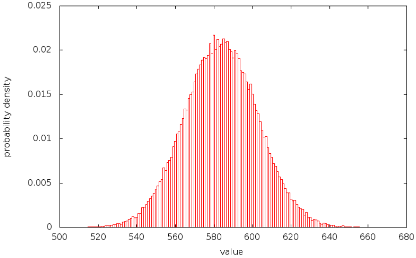
Séminaire de combinatoire Philippe Flajolet

Euclid algorithm...

and...

- continued fractions
- dynamical analysis, costs
- symbolic dynamics : the Sturmian case
- higher-dimensional generalizations





Analysis of algorithms

An algorithm

Euclid algorithm

According to Knuth

'the granddaddy of all algorithms, because it is the oldest nontrivial algorithm that has survived to the present day'

J. Shallit-Origins of the Analysis of the Euclidean Algorithm-
Historia Mathematica (1994)

Euclidean dynamics

An algorithm

Euclid algorithm

together with a dynamical system

Gauss map

$$T\colon [0,1] \to [0,1],\; x \mapsto \{1/x\}$$

Euclid algorithm

We start with two nonnegative integers u_0 and u_1

$$u_0 = u_1 \left\lfloor \frac{u_0}{u_1} \right\rfloor + u_2$$

$$u_1 = u_2 \left\lfloor \frac{u_1}{u_2} \right\rfloor + u_3$$

$$\vdots$$

$$u_{m-1} = u_m \left\lfloor \frac{u_{m-1}}{u_m} \right\rfloor + u_{m+1}$$

$$u_{m+1} = \gcd(u_0, u_1)$$

$$u_{m+2} = 0$$

Euclid algorithm and continued fractions

We start with two coprime integers u_0 and u_1

$$u_0 = u_1 a_1 + u_2$$

$$\vdots$$

$$u_{m-1} = u_m a_m + u_{m+1}$$

$$u_m = u_{m+1} a_{m+1} + 0$$

$$u_{m+1} = 1 = \gcd(u_0, u_1)$$

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$$u_{m+1} = 1 = \gcd(u_0, u_1)$$

Euclid's algorithm yields the digits
for the continued fraction expansion of $\frac{u_1}{u_0}$

Euclid algorithm and continued fractions

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$$\vdots$$
$$u_{m-1} = u_m a_m + u_{m+1}$$
$$u_m = u_{m+1} a_{m+1} + 0$$
$$u_{m+1} = 1 = \gcd(u_0, u_1)$$

$$\frac{u_1}{u_0} = \frac{1}{a_1 + \frac{u_2}{u_1}} \quad \rightsquigarrow \quad \frac{u_1}{u_0} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_m + \cfrac{1}{a_{m+1}}}}}$$

Continued fractions and dynamical systems

Consider the Gauss map

$$T\colon [0,1] \to [0,1],\ x \mapsto \{1/x\}$$

$$x_1 = T(x) = \{1/x\} = \frac{1}{x} - \left[\frac{1}{x}\right] = \frac{1}{x} - a_1$$

$$x = \frac{1}{a_1 + x_1} \qquad\qquad a_n = \left[\frac{1}{T^{n-1}x}\right]$$

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

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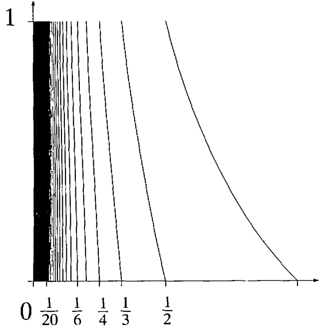


$$T(x) = \{1/x\} = \frac{1}{x} - \left[\frac{1}{x}\right] = \frac{1}{x} - a_1$$

$$\frac{1}{k+1} < x \le \frac{1}{k} \rightsquigarrow a_1 = k$$

Discrete dynamical system

We are given a dynamical system

$$T \colon X \to X$$

Discrete stands for discrete time

We consider orbits/trajectories of points of X under the action of the map T

$$\{T^n x \mid n \in \mathbb{N}\}$$

How well are the orbits distributed ?

According to which measure ?

Continued fractions and ergodicity

Ergodicity has to do with the long term statistical behaviour of orbits

Continued fractions and ergodicity

Ergodicity has to do with the long term statistical behaviour of orbits

The Gauss map is ergodic with respect to the Gauss measure

$$\mu(B) = \frac{1}{\log 2}\int_B \frac{1}{1+x}\mathrm{dx}$$

$$\mu(B) = \mu(T^{-1}B) \quad T\text{-invariance}$$

$$T^{-1}B = B \implies \mu(B) = 0 \text{ or } 1 \quad \text{ergodicity}$$

$$\frac{1}{n}\sum_{j=0}^{n-1} f(T^j x) = \int f d\mu \quad \text{ergodic theorem}$$

The mean behaviour along an orbit=
the mean value of f with respect to μ

Measure-theoretic results

- Gauss measure

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{dx}{1+x}$$

- Convergents

$$\text{For a.e. } x, \quad \lim \frac{\log q_n}{n} = \frac{\pi^2}{12 \log 2}$$

- Densities of partial quotients

  For a.e. x and $a \geq 1$

$$\lim_{N\to\infty} \frac{1}{N}\{k \leq N;\ a_k = a\} = \frac{1}{\log 2} \log \frac{(a+1)^2}{a(a+2)}$$

Rational vs. irrational parameters

Euclid algorithm $\rightsquigarrow$ gcd $\rightsquigarrow$ rational parameters

Continued fractions $\rightsquigarrow$ irrational parameters

Rational vs. irrational parameters

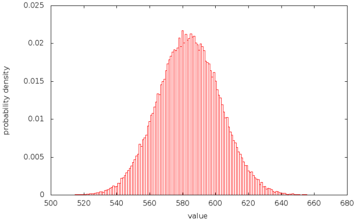
Euclid algorithm $\rightsquigarrow$ gcd $\rightsquigarrow$ rational parameters

Continued fractions $\rightsquigarrow$ irrational parameters

- When computing a gcd, we work with integer/rational parameters
- This set has zero measure
- Ergodic methods produce results that hold only almost everywhere

Is it relevant to compare generic orbits
and orbits for integer parameters ?

Dynamical analysis of Euclid algorithm



Number of steps $\ell(u, v)$

$\ell(u, v)$: number of steps in Euclid algorithm $0 < v < u$

- Worst case

$$\ell(u, v) = O(\log v) \qquad (\leq 5 \log_{10} v, \text{ Lamé } 1844)$$

Reynaud 1821 [$\ell(u, v) < v/2$], see Shallit's survey

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- Mean case $\qquad 0 < v < u \leq N \qquad \gcd(u, v) = 1$

$$\mathbb{E}_N(\ell) \sim \frac{12 \log 2}{\pi^2} \cdot \log N + \eta$$

[see Knuth, Vol. 2]

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- Mean case $0 < v < u \leq N$ $\qquad \gcd(u, v) = 1$

$$\frac{12 \log 2}{\pi^2} \cdot \log N + \eta + O(N^{-\gamma})$$

asymptotically normal distribution

[Heilbronn'69,Dixon'70,Porter'75,Hensley'94,Baladi-Vallée'05...]

Distributional dynamical analysis

$\gcd(u_0, u_1) = 1 \quad N \geq u_0 > u_1 > \cdots \quad u_{k-1} = a_k u_k + u_{k+1}$

Cost of moderate growth $c(a) = O(\log a)$

- Number of steps in Euclid algorithm $c \equiv 1$
- Number of occurrences of a quotient $c = \mathbf{1}_a$
- Binary length of a quotient $c(a) = \log_2(a)$

Distributional dynamical analysis

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Theorem [Baladi-Vallée'05]

$$\mathbb{E}_N[\text{Cost}] = \frac{12 \log 2}{\pi^2} \cdot \hat{\mu}(\text{Cost}) \cdot \log N + O(1)$$

The distribution is asymptotically Gaussian (CLT)

Discrete framework-Euclid algorithm

Ergodic theorem

Theorem [Baladi-Vallée'05]

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$$\mathbb{E}_N[\text{Cost}] = \frac{12\log 2}{\pi^2} \cdot \hat{\mu}(\text{Cost}) \cdot \log N + O(1)$$

$$\mathbb{E}_N[c] = \frac{\text{dimension}}{\text{entropy}} \cdot \hat{\mu}(c) \cdot \log N + O(1)$$

$$\hat{\mu}(c) = \int_0^1 c([1/x]) \frac{1}{\log 2} \frac{1}{1+x} dx$$

Continuous framework-truncated trajectories

Cost of truncated trajectories

Cost of moderate growth

$$c(a_i) = O(\log a_i) \text{ for } a_i \text{ partial quotient}$$

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

Cost of truncated trajectories

Cost of moderate growth

$$c(a_i) = O(\log a_i) \text{ for } a_i \text{ partial quotient}$$

Cost of a truncated trajectory

$$C_n(x) = \sum_{i=1}^{n} c(a_i(x)) \qquad a_i = \left[\frac{1}{T^{i-1}(x)}\right]$$

According to the ergodic theorem, for a.e. $x \in [0, 1]$

$$C_n(x)/n \to \hat{\mu}(x)$$

$$\hat{\mu}(C) = \int_0^1 c(\left[\frac{1}{x}\right]) \cdot \frac{1}{\log 2} \frac{1}{1+x} \cdot dx$$

$$\mathbb{E}_N[C] = \frac{2}{\pi^2/(6\log 2)} \cdot \hat{\mu}(C) \cdot \log N$$

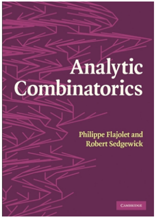
Dynamical analysis of algorithms [Vallée]

It belongs to the area of

- Analysis of algorithms [Knuth'63]

  probabilistic, combinatorial, and analytic methods

- Analytic combinatorics [Flajolet-Sedgewick]



generating functions and complex analysis,
analytic functions, analysis of the singularities

Dynamical analysis of algorithms [Vallée]

It mixes tools from

- dynamical systems (transfer operators, density transformers, Ruelle-Perron-Frobenius operators)
- analytic combinatorics (generating functions of Dirichlet type)

the singularities of (Dirichlet) generating functions
are expressed in terms of transfer operators

Euclidean dynamics [Vallée]

One starts with a discrete algorithm

- This algorithm is extended into a continuous one in terms of a dynamical system

  Orbits/trajectories = executions

- Main parameters of the algorithm are studied in the continuous framework

  rational trajectories $\leftrightarrow$ generic trajectories

- One comes back to the discrete algorithm

A transfer from continuous to discrete

'The probabilistic behaviour of gcd algorithms is quite similar to the behaviour of their continuous counterparts'

Rational vs. irrational parameters

Euclid algorithm $\rightsquigarrow$ gcd $\rightsquigarrow$ rational parameters

Continued fractions $\rightsquigarrow$ irrational parameters

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Average-case analysis vs. a.e. results

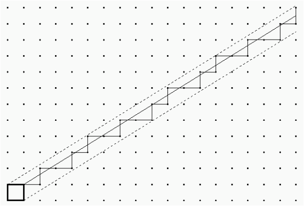
Fact Orbits of rational points tend to behave like generic orbits

And their probabilistic bevaviour can be captured thanks to the methods of dynamical analysis of algorithms

Gauss map

&

symbolic dynamics



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Discrete dynamical system

We are given a dynamical system

$$T\colon X \to X$$

We partition X in to a finite number of subsets $X = \cup_{i=1}^{d} X_i$

We code the trajectory of a point x with respect to (X_i)

$$\{T^n x \mid n \in \mathbb{N}\} \rightsquigarrow (u_n)_{n\in\mathbb{N}} \in \{1, 2, \cdots, d\}^{\mathbb{N}}$$

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$$\{T^n x \mid n \in \mathbb{N}\} \rightsquigarrow (u_n)_{n\in\mathbb{N}} \in \{1, 2, \cdots, d\}^{\mathbb{N}}$$

The map acting on $\{1, 2, \cdots, d\}^{\mathbb{N}}$ is the shift S

$$S((u_n)_n) = (u_{n+1})_n$$

$$(X, T) \rightsquigarrow (Y, S) \text{ with } Y \subset \{1, 2, \cdots, d\}^{\mathbb{N}}$$

From geometric dynamical systems to
symbolic dynamical systems and backwards

Arithmetic dynamics

Arithmetic dynamics [Sidorov-Vershik] arithmetic codings of dynamical systems that preserve their arithmetic structure

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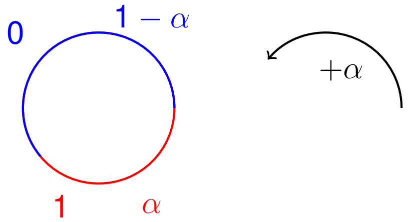
Example Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z},\ x \mapsto x + \alpha$ mod 1
One codes trajectories according to the finite partition

$$\{I_0 = [0, 1 - \alpha[,\ I_1 = [1 - \alpha, 1[\}$$



Sturmian dynamical systems

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This yields a measure-theoretic isomorphism

$$(R_\alpha, \mathbb{R}/\mathbb{Z}) \sim (X_\alpha, S)$$

where S is the shift and $X_\alpha \subset \{0, 1\}^{\mathbb{N}}$

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$$\begin{array}{ccc} \mathbb{R}/\mathbb{Z} & \xrightarrow{R_\alpha} & \mathbb{R}/\mathbb{Z} \\ \downarrow & & \downarrow \\ X_\alpha & \xrightarrow[S]{} & X_\alpha \end{array}$$

Sturmian dynamical systems

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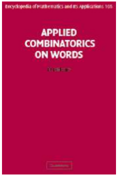
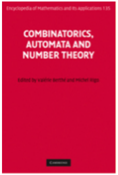




[Lothaire, Algebraic combinatorics on words,
N. Pytheas Fogg, Substitutions in dynamics, arithmetics and combinatorics
CANT Combinatorics, Automata and Number theory]

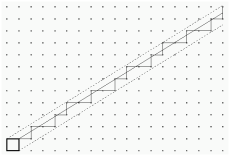
Sturmian words and continued fractions

0110110101101101

Sturmian words and continued fractions

0110110101101101

11 and 00 cannot occur simultaneously



Sturmian words and continued fractions

$$0110110101101101$$

One considers the substitutions

$$\sigma_0 : 0 \mapsto 0,\ \sigma_0 : 1 \mapsto 10$$

$$\sigma_1 : 0 \mapsto 01,\ \sigma_1 : 1 \mapsto 1$$

One has

$$01\,1\,01\,1\,01\,01\,1\,01\,1\,01 = \sigma_1(0101001010)$$

$$0\,10\,10\,0\,10\,10 = \sigma_0(011011)$$

$$01\,1\,01\,1 = \sigma_1(0101)$$

$$01\,01 = \sigma_1(00)$$

Sturmian words and continued fractions

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One considers the substitutions

$$\sigma_0 \colon 0 \mapsto 0, \ \sigma_0 \colon 1 \mapsto 10$$

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The Sturmian words of slope α are provided by an infinite composition of substitutions

$$\lim_{n \to +\infty} \sigma_0^{a_1} \sigma_1^{a_2} \cdots \sigma_{2n}^{a_{2n}} \sigma_{2n+1}^{a_{2n+1}}(0)$$

where the a_i are produced by the continued fraction expansion of α

Sturmian words and continued fractions

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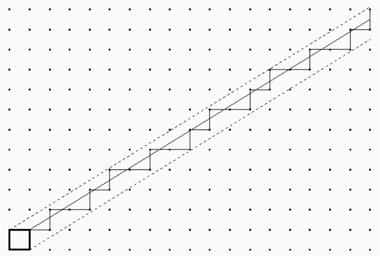

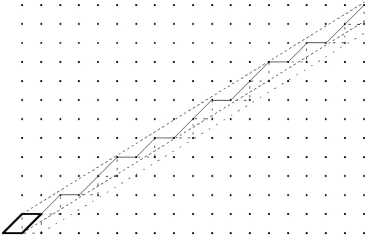

Euclid algorithm and discrete segments

$$\begin{array}{|ccc|}\hline 11 & = & 2\cdot 4+3 \\ 4 & = & 1\cdot 3+1 \\ 3 & = & 3\cdot 1+0 \\ \hline\end{array}$$

$$\frac{4}{11} = \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3}}}$$

$$(11,4) \xleftarrow{\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^2} (3,4) \xleftarrow{\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}} (3,1) \xleftarrow{\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^3} (0,1)$$

$$\mathbf{w} = \mathbf{w}_0 \xleftarrow[\substack{a \mapsto a \\ b \mapsto aab}]{} \mathbf{w}_1 \xleftarrow[\substack{a \mapsto ab \\ b \mapsto b}]{} \mathbf{w}_2 \xleftarrow[\substack{a \mapsto a \\ b \mapsto aaab}]{} \mathbf{w}_3 = b$$

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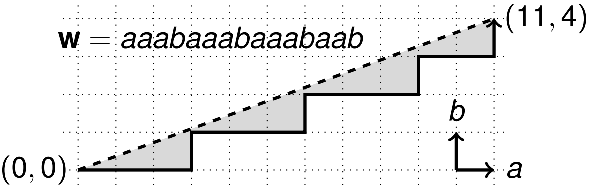

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Higher-dimensional framework

- How to discretize a line in the space ?
- How to compute the gcd of three or more numbers ?
- How to compare gcd/cf algorithms ?
- Integer parameters vs. rational parameters
- Can we generalize the Sturmian framework to translations on $\mathbb{T}^d$?

The Tribonacci fractal

The Tribonacci substitution $\sigma: 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$

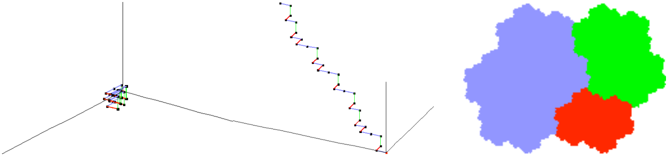
$$\sigma^\infty(1) = 121312112\cdots$$

One represents $\sigma^\infty(1)$ as a broken line

$$1 \mapsto \vec{e}_1,\ 2 \mapsto \vec{e}_2,\ 3 \mapsto \vec{e}_3,$$

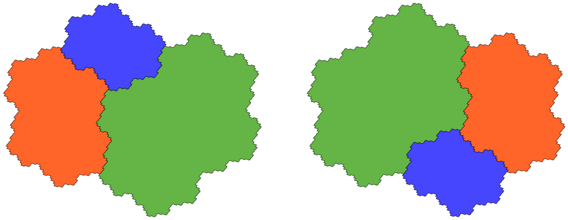
that we will be projected according to the eigenspaces of

$$M_\sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



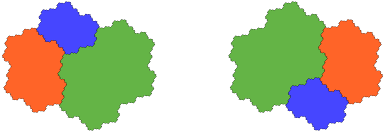
Rauzy fractal and dynamics

One first defines an exchange of pieces acting on the Rauzy fractal

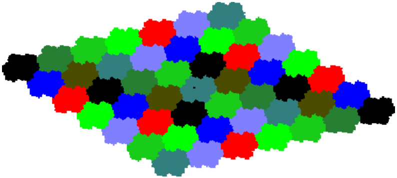


Rauzy fractal and dynamics

One first defines an exchange of pieces acting on the Rauzy fractal.
This due to the fact that the subtiles are disjoint in measure



This exchange of pieces factorizes into a translation of $\mathbb{T}^2$
This due to the fact that the Rauzy fractal tiles periodically the plane



Rauzy fractal and codings

$$\sigma\colon 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$$

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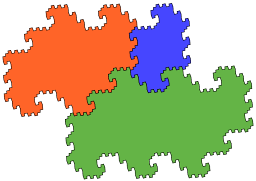

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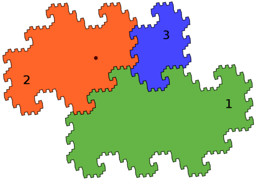
Trajectories are coded according to the partition

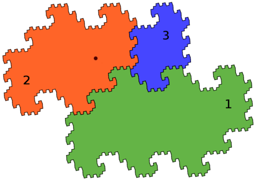


Rauzy fractal and codings

$$\sigma\colon 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$$

$$\sigma^\infty(1) = 12131212112\ldots$$

Trajectory : 2

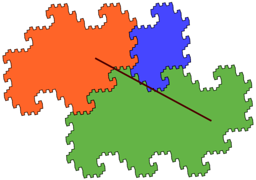


Rauzy fractal and codings

$$\sigma\colon 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$$

$$\sigma^\infty(1) = 12131212112\ldots$$

Trajectory : 21

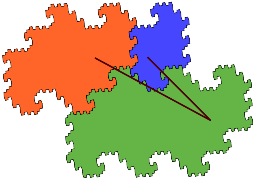


Rauzy fractal and codings

$$\sigma\colon 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$$

$$\sigma^{\infty}(1) = 12131212112\ldots$$

Trajectory : 213

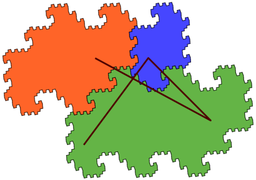


Rauzy fractal and codings

$$\sigma\colon 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$$

$$\sigma^\infty(1) = 12131212112\ldots$$

Trajectory : 2131

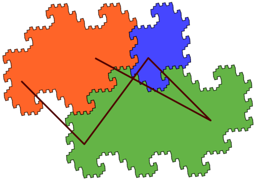


Rauzy fractal and codings

$$\sigma\colon 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$$

$$\sigma^\infty(1) = 12131212112\ldots$$

Trajectory : 21312

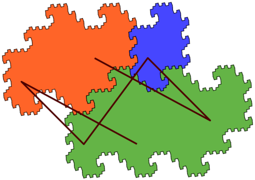


Rauzy fractal and codings

$$\sigma\colon 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$$

$$\sigma^{\infty}(1) = 12131212112\ldots$$

Trajectory 2 : 213121

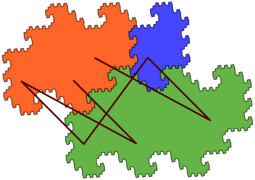


Rauzy fractal and codings

$$\sigma\colon 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$$

$$\sigma^{\infty}(1) = 12131212112\ldots$$

Trajectory : 2131212

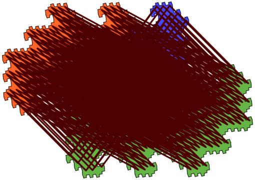


Rauzy fractal and codings

$$\sigma\colon 1 \mapsto 12,\ 2 \mapsto 1312,\ 3 \mapsto 112$$

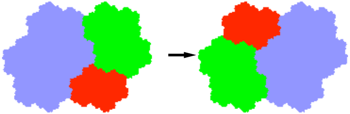
$$\sigma^\infty(1) = 12131212112\ldots$$

Density and even equidistribution of orbits



Tribonacci rotation $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$

Theorem [Rauzy, Chekhovaya-Hubert-Messaoudi]

- (X_σ, S) is measure-theoretically isomorphic with a two-dimensional translation and is equal to the codings of the orbits under the action of the translation 

$$R_\beta : \mathbb{T}^2 \to \mathbb{T}^2,\ x \mapsto x + (1/\beta, 1/\beta^2)$$

with respect to the pieces of the Rauzy fractal

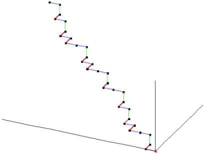
Tribonacci rotation $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$

Theorem [Rauzy, Chekhovaya-Hubert-Messaoudi]

- (X_σ, S) is measure-theoretically isomorphic with a two-dimensional translation and is equal to the codings of the orbits under the action of the translation

$$R_\beta : \mathbb{T}^2 \to \mathbb{T}^2,\ x \mapsto x + (1/\beta, 1/\beta^2)$$

  with respect to the pieces of the Rauzy fractal
- The points of the broken line corresponding to $\sigma^n(1)$, $n \in \mathbb{N}$, produce the sequence of best approximations for the vector $(\frac{1}{\beta}, \frac{1}{\beta^2})$ for a given norm associated with the incidence matrix M_σ



S-adic Rauzy fractals

We want to find

- 'good' symbolic codings for d-dimensional translations
$$R_{(\alpha_1,\cdots,\alpha_d)}: \mathbb{T}^d \to \mathbb{T}^d$$
- 'good' partitions of the torus $\mathbb{T}^d$

Take a multidimensional continued fraction algorithm and transform it into substitutions

[B.-Steiner-Thuswaldner,
B.-Jolivet-Siegel,Arnoux-B.-Labbé]

Comparing Euclid/cf algorithms

- Number of steps and costs functions for algorithms defined on rational entries

  worst-case, mean behavior, average-case analysis

- Convergence properties
- Ergodic properties

  ergodic invariant measure, natural extension

- Arithmetic properties

  cubic numbers and periodic expansions, Diophantine approximation

Multidimensional Euclid's algorithms

- Jacobi-Perron We subtract the first one to the two other ones with $0 \leq u_1, u_2 \leq u_3$

$$(u_1, u_2, u_3) \mapsto (u_2 - [\frac{u_2}{u_1}]u_1, u_3 - [\frac{u_3}{u_1}]u_1, u_1)$$

- Brun We subtract the second largest entry and we reorder. If $u_1 \leq u_2 \leq u_3$

$$(u_1, u_2, u_3) \mapsto (u_1, u_2, u_3 - u_2)$$

- Poincaré We subtract the previous entry and we reorder

$$(u_1, u_2, u_3) \mapsto (u_1, u_2 - u_1, u_3 - u_2)$$

- Selmer We subtract the smallest to the largest and we reorder

$$(u_1, u_2, u_3) \mapsto (u_1, u_2, u_3 - u_1)$$

- Fully subtractive We subtract the smallest one to the other ones and we reorder

$$(u_1, u_2, u_3) \mapsto (u_1, u_2 - u_1, u_3 - u_1)$$

Number of steps

Consider parameters $(u_1, \cdots, u_d)$ with $0 \leq u_1, \cdots, u_d \leq N$

Thm Expectation of the number of steps $= \dfrac{\text{dimension}}{\text{Entropy}} \times \log N$

Dimension

- d= Number of parameters

Number of steps

Consider parameters $(u_1, \cdots, u_d)$ with $0 \leq u_1, \cdots, u_d \leq N$

Thm Expectation of the number of steps $= \dfrac{\text{dimension}}{\text{Entropy}} \times \log N$

- Euclid algorithm

$$\frac{2}{\pi^2/(6\log 2)} \log N$$

[Heilbronn'69,Dixon'70,Hensley'94,Baladi-Vallée'03,Lhote-Vallée'08,...]

Number of steps

Consider parameters $(u_1, \cdots, u_d)$ with $0 \leq u_1, \cdots, u_d \leq N$

Thm Expectation of the number of steps $= \dfrac{\text{dimension}}{\text{Entropy}} \times \log N$

- Jacobi-Perron

  [Fischer-Schweiger'75]

- Brun

  [B.-Lhote-Vallée, work in progress]

Number of steps

Consider parameters $(u_1, \cdots, u_d)$ with $0 \le u_1, \cdots, u_d \le N$

Thm Expectation of the number of steps $= \dfrac{\text{dimension}}{\text{Entropy}} \times \log N$

- Formal power series with coefficients in a finite field and polynomials with degree less than m

$$\frac{2}{2\frac{q}{q-1}} m = \frac{q-1}{q} m$$

[Knopfmacher-Knopfmacher'88, Friesen-Hensley'96, Lhote-Vallée'06'08, B.-Nakada-Natsui-Vallée'12]

Formal power series

Let q be a power of a prime number p

We have the correspondence

- $\mathbb{Z} \sim \mathbb{F}_q[X]$
- $\mathbb{Q} \sim \mathbb{F}_q(X)$
- $\mathbb{R} \sim \mathbb{F}_q((X^{-1}))$

$$f = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0 + a_{-1} X^{-1} + \cdots$$

Laurent formal power series

Formal power series

Let $f \in \mathbb{F}_q((X^{-1}))$ $\qquad f \neq 0$

$$f = a_n X^n + a_{n-1} X^{n-1} + \cdots \qquad a_n \neq 0$$

- Degree $\qquad \deg f = n$
- Distance $\qquad |f| = q^{\deg f}$

Ultrametric space

$$|f + g| \leq \max(|f|, |gl)$$

No carry propagation !

Continued fractions

One can expand series f into continued fractions

$$f = a_0(X) + \cfrac{1}{a_1(X) + \cfrac{1}{a_2(X) + \ddots}} := [a_0(X); a_1(X), a_2(X), \cdots],$$

The digits $a_i(X)$ are polynomials of positive degree

$$a_k \geq 1 \leadsto \deg a_k(X) \geq 1$$

- Unique expansion even if f does not belong to $\mathbb{F}_q(X)$
- Finite expansion iff $f \in \mathbb{F}_q(X)$
- But there exist explicit examples of algebraic series with bounded partial quotients [Baum-Sweet]
- Roth's theorem does not hold for algebraic series (see e.g. [Lasjaunias-de Mathan])

[B.-Nakada, Expositiones Mathematicae]

Why is everything simpler ?

Ultrametric space !

- Digits are equidistributed : the Haar measure is invariant
- Hence, understanding the 'polynomial case' can help the understanding of the 'integer case'

And now..

- Numeration dynamics $T_\beta : x \mapsto \{\beta x\}$
- Discrete lines and planes
- Invariant measures