321-avoiding affine permutations, heaps, and periodic parallelogram polyominoes

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joint work with

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Séminaire Philippe Flajolet
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Introduction and motivations

321-avoiding permutations

A permutation $\sigma \in S_n$ is 321-avoiding if no integers $i < j < k$ are such that $\sigma(i) > \sigma(j) > \sigma(k)$

In S_6, $\sigma = 513624$ is not 321-avoiding while $\sigma = 231564$ is.

They are counted by Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$.

The inversion number $inv(\sigma)$ is the number of inversions of the permutation σ i.e.

$$inv(\sigma) = |\{(i,j) \in [n]^2 \mid i < j \text{ and } \sigma(i) > \sigma(j)\}|.$$

For example $\sigma = 513624$ has 4+0+1+2+0=6 inversions.

Affine permutations

Definition (Affine permutations)

The group $\widetilde{S}_n$ is the set of permutations σ of $\mathbb{Z}$ satisfying

$$\sigma(i+n) = \sigma(i) + n \text{ and } \sum_{i=1}^{n} \sigma(i) = \sum_{i=1}^{n} i.$$

Note that $\sigma(i) \equiv \sigma(j) \pmod{n}$ if and only if $i \equiv j \pmod{n}$.

An element of $\widetilde{S}_4$ is

$$\ldots \mid 2, -7, -5, 4, \mid \mathbf{6}, \mathbf{-3}, \mathbf{-1}, \mathbf{8}, \mid 10, 1, 3, 12, \mid 14, 5, 7, 16 \mid \ldots$$

denoted simply by $[6, -3, -1, 8] = [\sigma(1), \sigma(2), \sigma(3), \sigma(4)]$.

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321-avoiding affine permutations

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An affine permutation is 321-avoiding if there are not $i < j < k$ in $\mathbb{Z}$ such that $\sigma(i) > \sigma(j) > \sigma(k)$. We write $\sigma \in \widetilde{S}_n(321)$

For example

$$\ldots, |\, 2, -7, -5, 4, \,|\, \mathbf{6}, -\mathbf{3}, -\mathbf{1}, \mathbf{8}, \,|\, 10, 1, 3, 12, \,|\, 14, 5, 7, 16 \,|\, \ldots \in \widetilde{S}_4(321)$$

$$\ldots \,|\, -3, -5, 5 \,|\, \mathbf{0}, -\mathbf{2}, \mathbf{8} \,|\, 3, 1, 11 \,|\, 6, 4, 14 \,|\, 9, 7, 17 \,|\, \ldots \notin \widetilde{S}_3(321)$$

Definition (Affine inversions)

$$inv(\sigma) = |\{(i,j) \in [n] \times \mathbb{P} \mid i < j \text{ and } \sigma(i) > \sigma(j)\}|.$$

For example $inv([6, -3, -1, 8]) = 9$.

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Generating function for permutations in $S_n(321)$

$$A_{n-1}(q) := \sum_{\sigma \in S_n(321)} q^{inv(\sigma)} \text{ and } A(x,q) = \sum_{n \geq 0} A_n(q) x^n.$$

Theorem (Barcucci, Del Lungo, Pergola, Pinzani, 2001)

We have $A(x,q) = \frac{1}{1-xq} \times \frac{J(xq)}{J(x)}$, *where*

$$J(x) := \sum_{n \geq 0} \frac{(-x)^n q^{\binom{n}{2}}}{(q)_n (xq)_n}$$

Here $(a)_0 := 1$

and $(a)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}),\ n \geq 1$

Generating function for affine permutations in $\tilde{S}_n(321)$

$$\widetilde{A}_{n-1}(q) := \sum_{\sigma \in \widetilde{S}_n(321)} q^{inv(\sigma)} \text{ and } \widetilde{A}(x,q) := \sum_{n \geq 1} \widetilde{A}_{n-1}(q) x^n$$

Theorem (B., Bousquet-Mélou, Jouhet, Nadeau, 2016)

$$\widetilde{A}(x,q) = -x \frac{J'(x)}{J(x)} - \sum_{n \geq 1} \frac{x^n q^n}{1-q^n}$$

- Hanusa and Jones [2009] found a complicated expression for $\widetilde{A}(x,q)$ and showed that the coefficients of the series $\widetilde{A}_{n-1}(q)$ are ultimately periodic of period dividing n.
- BJN [2013] characterized the series $\widetilde{A}_{n-1}(q)$ by a systems of non-linear q-equations.
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Generating function for affine permutations in $\tilde{S}_n(321)$

This computational approach does not explain the simplicity of $\widetilde{A}(x,q)$ and $A(x,q)$.
In this talk, we provide two bijective explanations of them.

Today's combinatorial methods.

Encode 321-avoiding (affine) permutations by :

(affine) alternating diagrams, then by

- (periodic) parallelogram polyominoes, and
- (marked) heaps of segments;

or by

- Motzkin type paths, and
- (marked) pyramids of monomers and dimers.

Fully commutative elements

The original motivation was the computation of the series

$$\sum_{w \in W^{FC}} q^{\ell(w)}$$

where W^{FC} denotes the set of **fully commutative elements** in the Coxeter group W, and ℓ the Coxeter length.

Coxeter group

(W, S) Coxeter group W given by Coxeter matrix $(m_{st})_{s,t \in S}$.

Relations: $\begin{cases} s^2 = 1 \\ \underbrace{sts\cdots}_{m_{st}} = \underbrace{tst\cdots}_{m_{st}} \end{cases}$ (called Braid relations)

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Reduced decompositions

Definition (Length)

$\ell(w) =$ minimal l such that $w = s_1 s_2 \cdots s_l$ with $s_i \in S$

Such a minimal word is a reduced decomposition of w.

Proposition (Matsumoto-Tits property)

Given two reduced decompositions of w, there is a sequence of braid or commutation relations which can be applied to transform one into the other.

Definition

An element w is fully commutative if given two reduced decompositions of w, there is a sequence of commutation relations which can be applied to transform one into the other.

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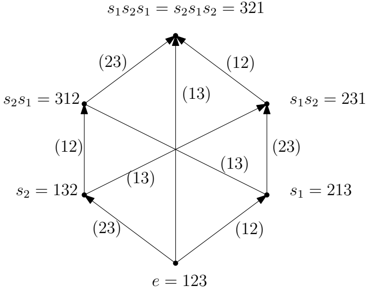
The symmetric group

Example

The symmetric group S_n is generated as a Coxeter group by the set S of simple transpositions $s_i = (i, i+1)$ with

Relations: $\begin{cases} s^2 = 1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ (braid relations)} \\ s_i s_j = s_j s_i \text{ if } j \neq i \pm 1 \text{ (commutation relations)} \end{cases}$



All elements of S_3 are FC except $321 = s_1s_2s_1 = s_2s_1s_2$.

Note that $\ell(\sigma) = inv(\sigma)$.

Fully commutative elements

Theorem (Billey-Jockush-Stanley, 1993)

A permutation in S_n is fully commutative if and only if it is 321-avoiding.

Theorem (Green, 2001)

An affine permutation in $\widetilde{S}_n$ is fully commutative if and only if it is 321-avoiding.

Theorem (Lusztig, 1983)

- $\widetilde{S}_n$ is a Coxeter group of type $\widetilde{A}_{n-1}$;
- $s_0 = \cdots(-1-n,-n)(-1,0)(-1+n,n)\cdots$;
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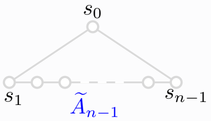

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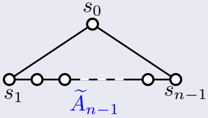
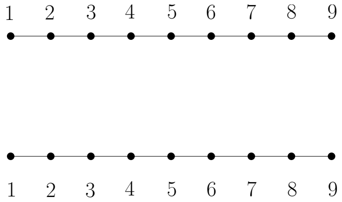


Affine alternating diagrams

From line diagrams to alternating diagrams

Take the 321-avoiding permutation $\sigma = 461279358 \in S_9$.



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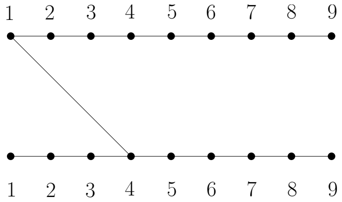

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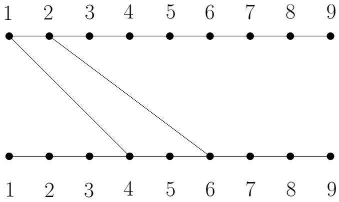

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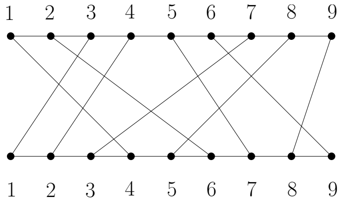

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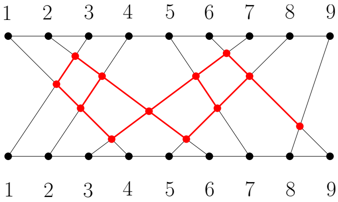
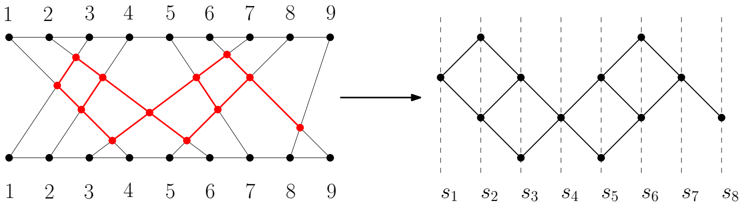


From line diagrams to alternating diagrams

The alternating diagram of $\sigma = 461279358 \in S_9$: $inv(\sigma) = 12$.



Definition (Alternating diagram)

An alternating diagram of rank n is a poset :

- its elements are labeled by the generators $\{s_1, \ldots, s_{n-1}\}$ of S_n
- $\forall i$, elements with labels s_i, s_{i+1} form an alternating chain ;
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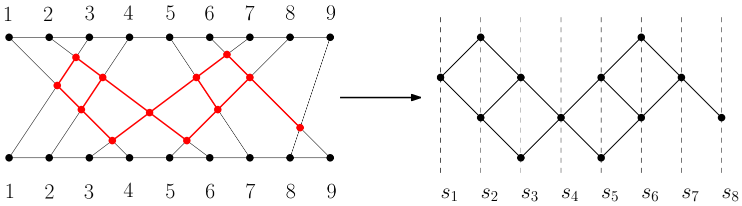

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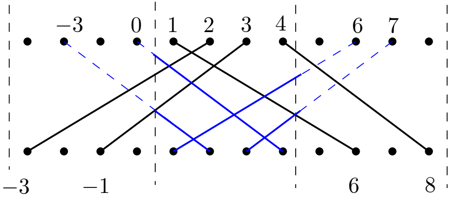
From line diagrams to affine alternating diagrams

$\sigma = \ldots \mid 2, -7, -5, 4, \mid \mathbf{6}, -\mathbf{3}, -\mathbf{1}, \mathbf{8}, \mid 10, 1, 3, 12, \mid 14, 5, 7, 16 \mid \ldots$

Here $\sigma \in \widetilde{S}_4(321)$ and $inv(\sigma) = 9$.



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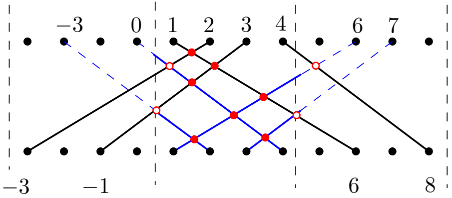

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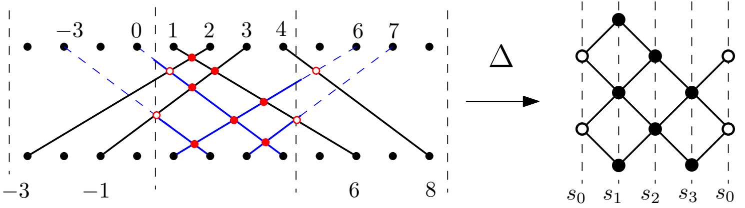

Proposition (Characterization of affine alternating diagrams)

- **Same** *number of occurrences of* s_0 *in the first and last column.*

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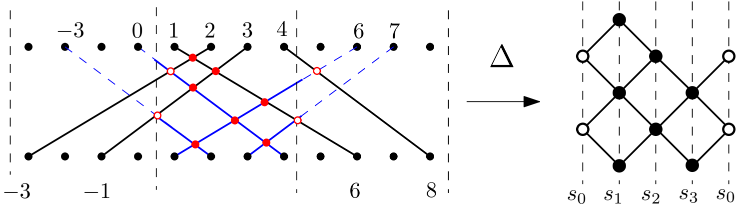

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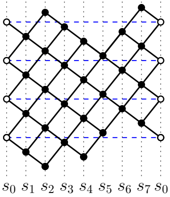
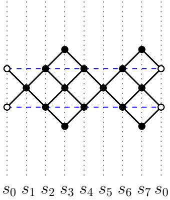
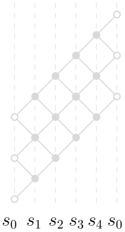
Representations of alternating diagrams on a cylinder

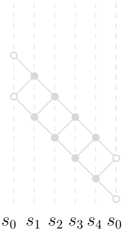




Two affine alternating diagrams of $\widetilde{S}_8$.

The second is self-dual.





Two excluded diagrams !
(rectangular shape)

They do not represent posets.

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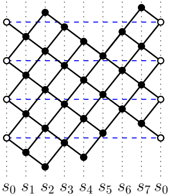

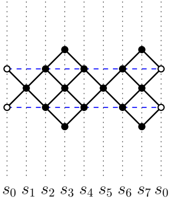

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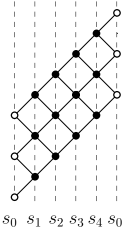

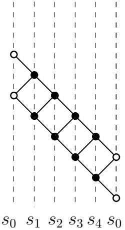

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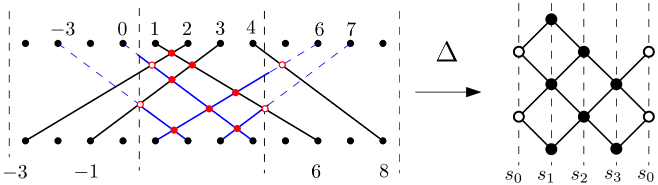
Bijection between $\widetilde{S}_n(321)$ and affine alternating diagrams

Theorem (B, Jouhet, Nadeau, 2015)

The map Δ between $\widetilde{S}_n(321)$ and affine alternating diagrams is a bijection such that:

- $\sigma \in S_n(321)$ *if and only if* $\Delta(\sigma)$ *do not contain any* s_0
- σ *is an involution if and only if* $\Delta(\sigma)$ *is self-dual.*



The inverse bijection

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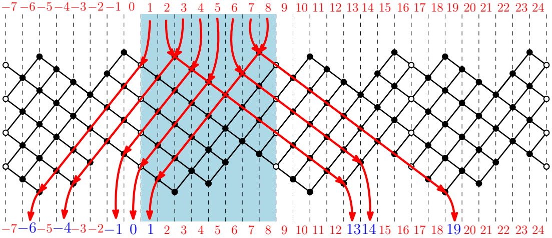
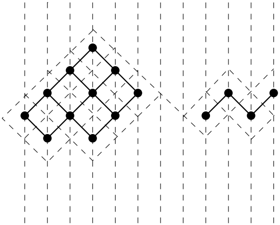


Periodic parellologram polyominoes

From alternating diagrams to parallelogram polyominoes



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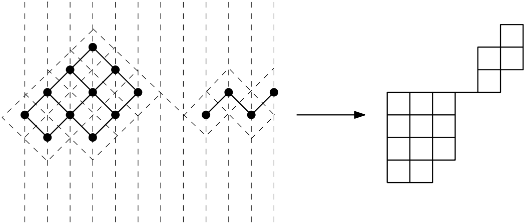

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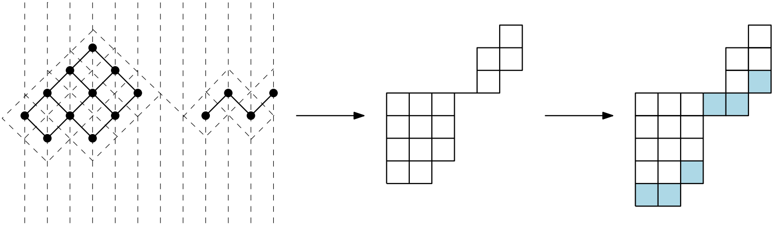
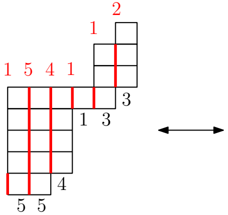


Theorem (Viennot, < 1992)

There is a bijection between alternating diagrams and parallelogram polyominoes (PP).

A *PP* is a convex polyomino enclosed by two paths consisting of unit horizontal and vertical steps, both starting in the same point, ending in the same point, and non-intersecting elsewhere.

Bousquet-Mélou, Viennot encoding



(a_1, b_1) (a_n, b_n)

$(1, 5), (5, 5), (4, 4), (1, 1), (1, 3), (2, 3)$

Convention: $a_1 = 1$

Parallelogram polyominoes are coded by sequences $(a_i, b_i)_{1 \leq i \leq n}$ with $a_1 = 1$, where:

- b_i is the height of the column C_i;
- a_i is the number of commun rows between C_{i-1} and C_i.

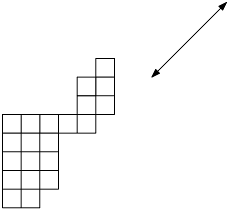
Any of such finite sequences $(a_i, b_i)_{1 \leq i \leq n}$ satisfies :

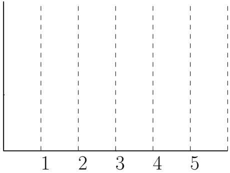
$$1 \leq b_1 \geq a_2 \leq b_2 \geq \ldots \leq b_{n-1} \geq a_n \leq b_n.$$

From parallelogram polyominoes to heaps of segments

Classical case (Bousquet-Mélou–Viennot, 1992)

$(1,5),(5,5),(4,4),(1,1),(1,3),(2,3)$

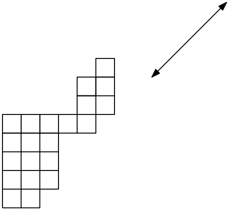


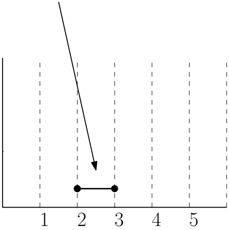


From parallelogram polyominoes to heaps of segments

Classical case (Bousquet-Mélou–Viennot, 1992)

$$(1,5),(5,5),(4,4),(1,1),(1,3),(2,3)$$





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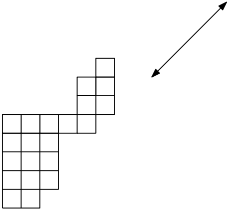

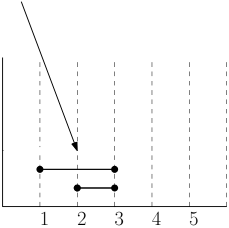

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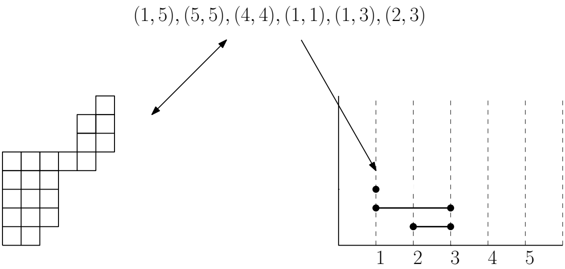

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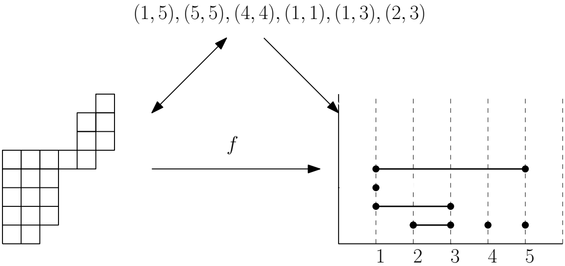


Half pyramid: a unique max $[1, 5]$

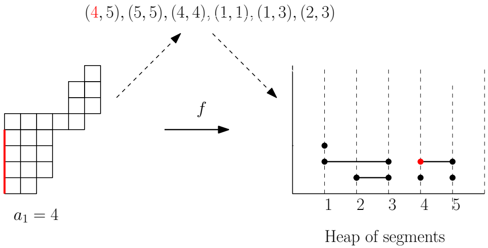
Theorem (Bousquet-Mélou–Viennot, 1992)

The map f is a bijection between the set of parallelogram polyominoes and the set of half pyramids of segments.

From $\mathcal{S}$ to heaps of segments

Let $\mathcal{S}$ be the set of finite sequences $(a_i, b_i)_{1 \leq i \leq n}$ satisfying

$$a_1 \leq b_1 \geq a_2 \leq b_2 \geq \ldots \leq b_{n-1} \geq a_n \leq b_n.$$



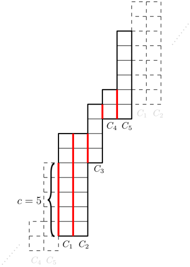
Proposition

The map f extends to a bijection between the set of (PP) marked in their first column and the set of heaps of segments $\mathcal{H}$.

Periodic parallelogram polyominoes (PPP)

Definition (PPP)

A **periodic parallelogram polyomino** is a couple (P, c), where P is a marked PP and c is an integer between 1 and the height of the last column of P.



Periodic parellelogram polyominoes as sequences

These naturally correspond to sequences $(a_i, b_i)_{1 \leq i \leq n} \in \mathcal{S}$ such that $b_n \geq a_1$, i.e.

$$b_n \geq a_1 \leq b_1 \geq a_2 \leq b_2 \geq \ldots \leq b_{n-1} \geq a_n \leq b_n.$$

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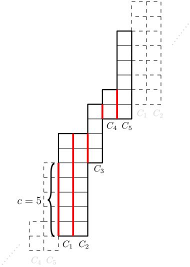

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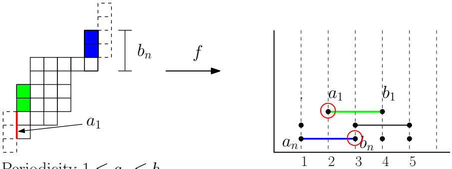
From PPP to heaps of segments

$(2,4),(3,5),(5,5),(4,4),(1,1),(1,3)$



Periodicity $1 \le a_1 \le b_n$

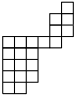
Condition $(\sim)$ on the heaps
read on left min et right max elements

Let $\widetilde{\mathcal{H}}$ be the set of heaps satisfying **condition $(\sim)$** i.e. the beginning of the rightmost maximal segment should be on the left of the end of the leftmost minimal segment.

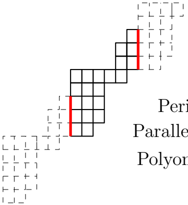
Proposition

The map f induces a bijection between the set PPP and $\widetilde{\mathcal{H}}$.

Summary

$S_n(321)$ $\longleftrightarrow$  $\longleftrightarrow$ $(\mathcal{S}, a_1 = 1)$ $\longleftrightarrow$ Half pyramids

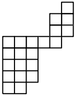
Parallelogram Polyominoes

 $\longleftrightarrow$ $(\mathcal{S}, a_1 \leq b_n)$ $\longleftrightarrow$ Heaps of segments in $\widetilde{\mathcal{H}}$

Periodic Parallelogram Polyominoes

Recent papers (2016) on *PPP* also by Aval, Boussicault, Laborde–Zubieta, and Pétréolle.

Summary

$S_n(321)$ $\longleftrightarrow$  $\longleftrightarrow$ $(\mathcal{S}, a_1 = 1)$ $\longleftrightarrow$ Half pyramids

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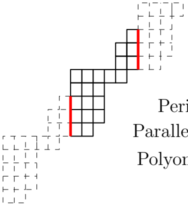 $\longleftrightarrow$ $(\mathcal{S}, a_1 \leq b_n)$ $\longleftrightarrow$ Heaps of segments in $\widetilde{\mathcal{H}}$

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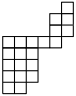
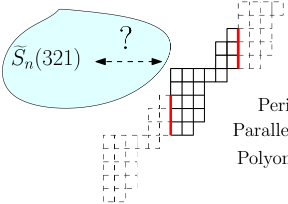


$\longleftrightarrow$ $(\mathcal{S}, a_1 = 1)$ $\longleftrightarrow$ Half pyramids

Parallelogram Polyominoes

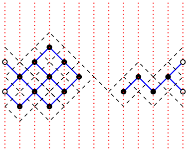


$\longleftrightarrow$ $(\mathcal{S}, a_1 \leq b_n)$ $\longleftrightarrow$ Heaps of segments in $\widetilde{\mathcal{H}}$

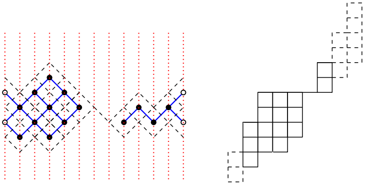
Periodic Parallelogram Polyominoes

Connection between $\widetilde{S}_n(321)$ and PPP

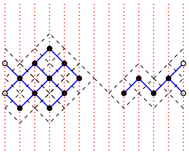
Back to 321-avoiding affine permutations

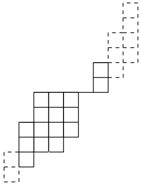


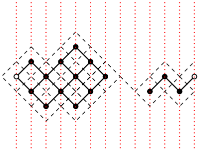
From 321-avoiding affine permutations to PPP^*



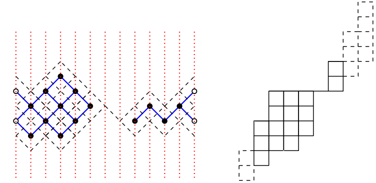
From 321-avoiding affine permutations to PPP^*

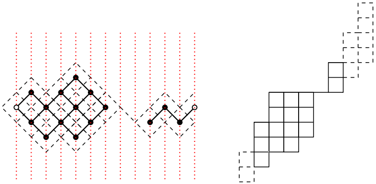






From 321-avoiding affine permutations to PPP^*





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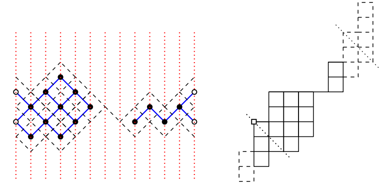
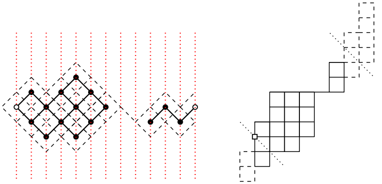


A mark in $[a_1, b_1]$
to recover s_0



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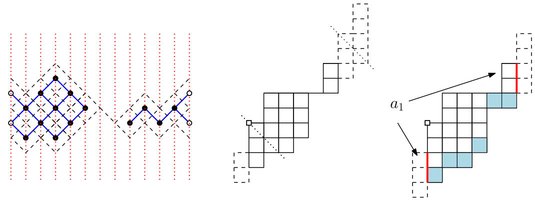
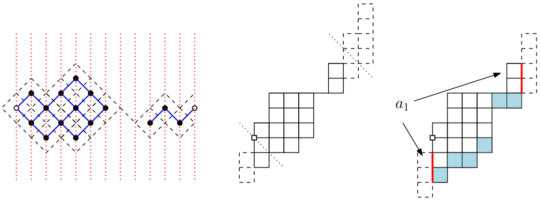


A mark in $[a_1, b_1]$ to recover s_0

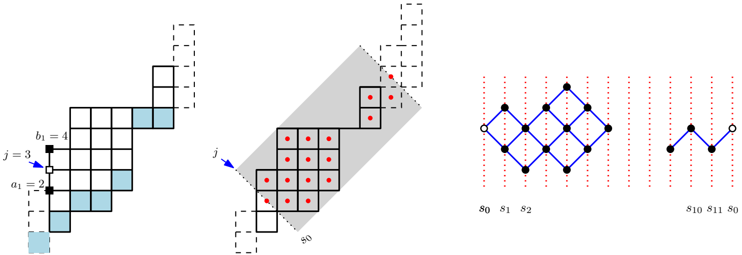
The periodicity identifies a mark a_1 in $[1, b_n]$



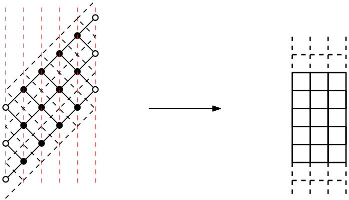
From PPP^* to 321-avoiding affine permutations

Theorem (B, Jouhet, Nadeau, (2016))

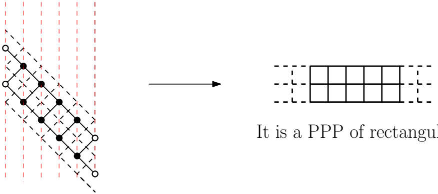
The previous application is a bijection between 321-avoiding affine permutations and marked PPP of non-rectangular shape.



Why the rectangular shape ?



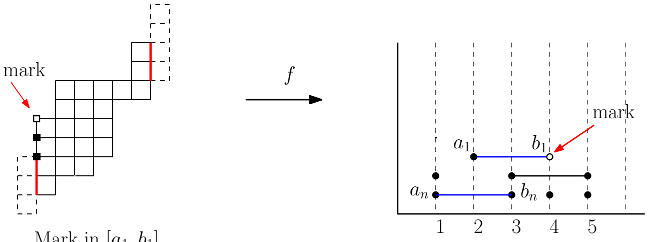
It is not a PPP



It is a PPP of rectangular shape.

Marked PPP

$(2,4),(3,5),(5,5),(4,4),(1,1),(1,3)$



Mark in $[a_1, b_1]$

Definition

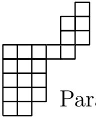
$PPP^* = PPP$ with a mark between a_1 and b_1 ;
$\widetilde{\mathcal{H}}^* =$ heaps in $\widetilde{\mathcal{H}}$ with a mark in their rightmost maximal segment.

Corollary

The map f induces a bijection between PPP^* and $\widetilde{\mathcal{H}}^*$.

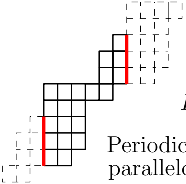
Summary

$S_n(321)$ $\longleftrightarrow$



PP $\longleftrightarrow$ Half pyramids

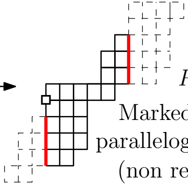
Parallelogram polyominoes



PPP $\longleftrightarrow$ Heaps of segments in $\widetilde{\mathcal{H}}$

Periodic parallelogram polyominoes

$\widetilde{S}_n(321)$ $\longleftrightarrow$

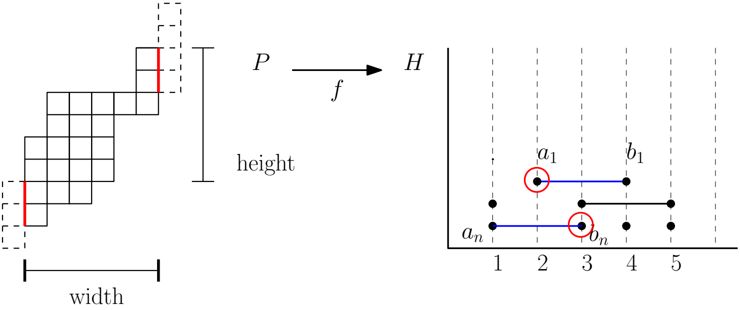


PPP^* $\longleftrightarrow$ Marked heaps of segments in $\widetilde{\mathcal{H}}^*$

Marked periodic parallelogram polyominoes (non rectangular)

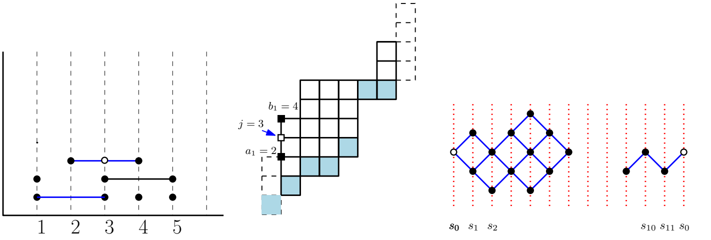
Generating functions

Statistics on *PPP*



- width$(P) = |H|$ the number of segments in H
- height$(P) = \ell(H)$ the sum of the lengths of the segments of H
- area$(P) = e(H)$ sum of right endpoints of the segments of H

Statistics on *PPP*



$e(H) - |H|$ ⟷ area(P)-width(P) ⟷ inv(σ)

$\ell(H) + |H|$ half perimeter(P) n = number of vertical columns : $\sigma \in \widetilde{S}_n(3$

Hence we have that

$$\widetilde{A}(x,q) = \sum_{n\geq 1}\left(\sum_{\sigma\in\widetilde{S}_{n+1}(321)} q^{inv(\sigma)}\right)x^n = \sum_{H\in\widetilde{\mathcal{H}}^*} x^{\ell(H)+|H|}q^{e(H)-|H|} - \sum_{n\geq 1}\frac{x^n q^n}{1-q^n}.$$

Generating functions for *PPP*

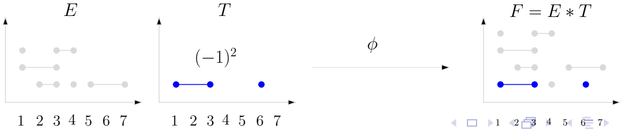
Goal: to compute the series

$$\widetilde{\mathcal{H}}^*(x,y,q) = \sum_{H \in \widetilde{\mathcal{H}}^*} x^{\ell(H)} y^{|H|} q^{e(H)}.$$

Theorem (Inversion Lemma - Viennot, 1985)

$$\mathcal{H}(x,y,q) = \frac{1}{\mathcal{T}(x,y,q)} \text{ and } \mathcal{HP}(x,y,q) = \frac{\mathcal{T}^c(x,y,q)}{\mathcal{T}(x,y,q)}$$

where $\mathcal{T}$ (resp. $\mathcal{T}^c$) is the signed GF for trivial heaps (resp. not touching abscissa 1), and $\mathcal{HP}$ denotes the half pyramids.



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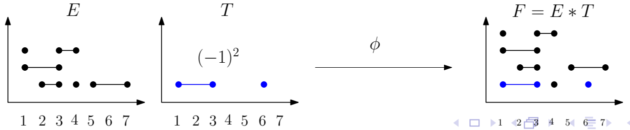
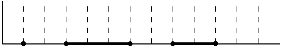


Trivial heaps

Trivial heaps $\mathcal{T}$

A trivial heap $T \in \mathcal{T}$ has no two pieces in concurrence.

$T =$ 

$v(T) = x^5 y^3 q^{17}$

Signed generating function for trivial heaps

$$\mathcal{T} = \mathcal{T}(x, y, q) = \sum_{T \in \mathcal{T}} (-1)^{|T|} x^{\ell(T)} y^{|T|} q^{e(T)}.$$

Generating functions for heaps of segments

Theorem (Bousquet-Mélou, Viennot, 1992)

$$\mathcal{T} = \sum_{n\geq 0} \frac{(-y)^n q^{\binom{n+1}{2}}}{(q)_n (xq)_n} \text{ and } \mathcal{T}^c = \sum_{n\geq 1} \frac{(-y)^n q^{\binom{n+1}{2}}}{(q)_{n-1} (xq)_n}.$$

Since 321-avoiding permutations are in bijection with half pyramids, we obtain back the result of Barcucci et al, by setting $y \to y/q$ (recall that we added a box in each column), and then $y \to x$, in Viennot formula. Note that $inv(\sigma) = e(H)$.

Theorem (Barcucci et al.)

$$A(x,q) = \frac{1}{1-xq} \times \frac{J(xq)}{J(x)} \text{ where } J(x) := \sum_{n\geq 0} \frac{(-x)^n q^{\binom{n}{2}}}{(q)_n (xq)_n}$$

Adaptation to our special heaps of segments in $\widetilde{\mathcal{H}}^*$

We can adapt the Viennot's technique to $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}}^*$, but conditiion ($\sim$) is very complicated to handle.

Theorem (B, Jouhet, Nadeau, 2016)

$$PPP(x,y,q) = -y\frac{\partial_y \mathcal{T}}{\mathcal{T}} \qquad PPP^*(x,y,q) = -x\frac{\partial_x \mathcal{T}}{\mathcal{T}}.$$

Since marked PPP^* (minus those of rectangular shape) of half-perimeter n are in bijection with 321-avoiding affine permutations of size n, we obtain (after taking care about the weight, $y \to y/q$, and $y \to x$) that

Theorem (B., Bousquet-Mélou, Jouhet, Nadeau)

$$\widetilde{A}(x) = -x\frac{J'(x)}{J(x)} - \sum_{n\geq 1}\frac{x^n q^n}{1-q^n}$$

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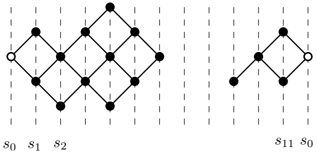
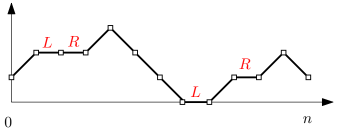
A different encoding

A different bijection

H



$\varphi'(H)$

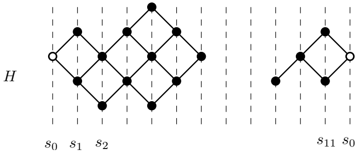


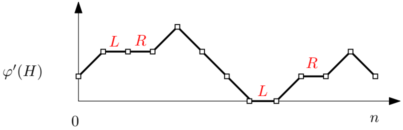
Theorem (BJN, 2013)

The map φ' is a bijection between :

1. $\widetilde{S}_n(321)$ *and*
2. $\mathcal{O}_n^* \setminus \{$*paths at constant height* $h > 0$ *with all steps having the same label L or R*$\}$, *where* $\mathcal{O}_n^*$ *is the set of length n paths with starting and ending point at the same height, with steps in* $(1,1)$, $(1,-1)$ *and* $(1,0)$ *satisfying condition* $(*)$.

A different bijection





Theorem (BJN, 2013)

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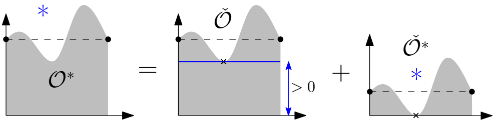
GF and beginning of periodicity

Corollary

$$\tilde{A}^{FC}_{n-1}(q) = \mathcal{O}^*_n(q) - \frac{2q^n}{1-q^n} = \frac{q^n(\check{\mathcal{O}}_n(q)-2)}{1-q^n} + \check{\mathcal{O}}^*_n(q),$$

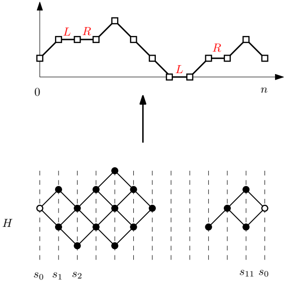
from which the periodicity follows.



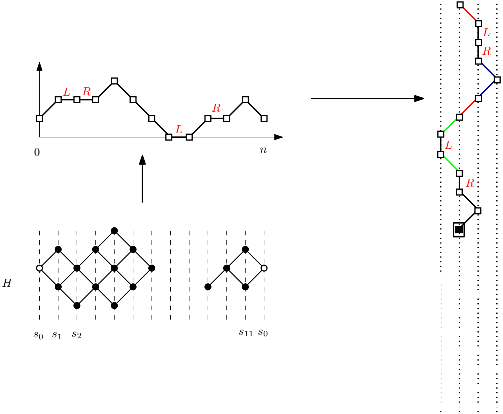
Corollary (Hanusa and Jones, 2010)

The coefficients of $\tilde{A}^{FC}_{n-1}(q)$ are ultimately periodic of period dividing n.

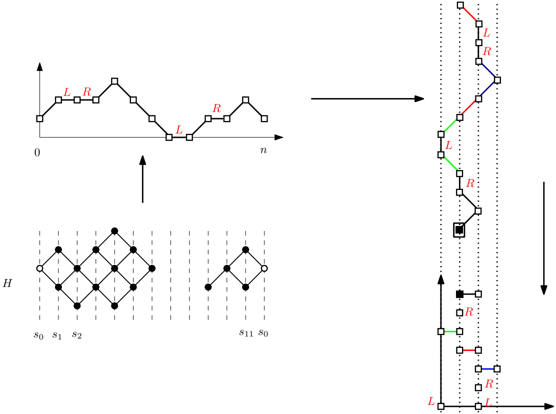
Encoding by heaps of monomers and dimers



Encoding by heaps of monomers and dimers

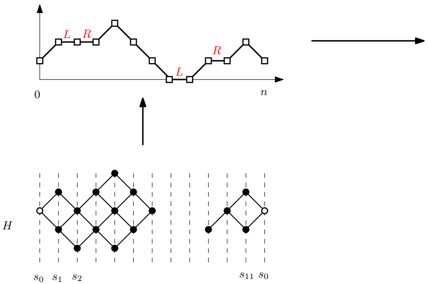


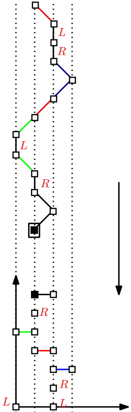
Encoding by heaps of monomers and dimers



A marked pyramid (P_m) of dimers and monomers L, R with condition (*)
(*) only in column 0

Encoding by heaps of monomers and dimers





Monomers : weight xq^i — i

Dimers : weight x^2q^{2i+1} — i $i+1$

A marked pyramid (P_m) of dimers and monomers L, R with condition (*)
(*) only in column 0

GF of heaps of monomers and dimers

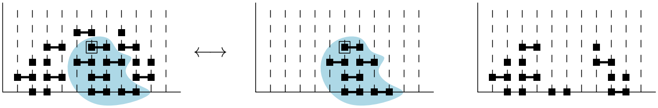
For a marked heap $\mathcal{E}$, the weight is

$$v(\mathcal{E}) := \prod_{\text{monomers } [i]} xq^i \prod_{\text{dimers } [i,i+1]} x^2 q^{2i+1}.$$

We need to compute the GF of marked pyramids $\Pi_m(x)$.
If the GF of heaps is $E(x)$, the GF for marked heaps is $xE'(x)$.

Proposition (Viennot)

$$xE'(x) = \Pi_m(X) \times E(x).$$



GF of heaps of monomers and dimers

Once again we conclude using the Inversion Lemma.

Theorem (Inversion Lemma - Viennot, 1985)

$$E(x) = \frac{1}{\mathcal{T}^*(x)},$$

where $\mathcal{T}^*$ is the signed GF for trivial heaps satisfying condition $(*)$.

A computation shows that $\mathcal{T}^*(x) = (xq; q)_\infty J(x)$ from which we obtain the previous result

$$\widetilde{A}(x) = -x\frac{J'(x)}{J(x)} - \sum_{n\geq 1} \frac{x^n q^n}{1-q^n}$$

.

321-avoiding involutions in S_n and $\widetilde{S}_n$

$$\mathcal{A} = \sum_{n\geq 0} \mathcal{A}_n^{Invo}(q)x^n \qquad \text{and} \qquad \tilde{\mathcal{A}} = \sum_{n\geq 1} \tilde{\mathcal{A}}_{n-1}^{Invo}(q)x^n.$$

Theorem (B., Bousquet-Mélou, Jouhet, Nadeau, 2016)

We have

$$\mathcal{A} = \frac{\mathcal{J}(-xq)}{\mathcal{J}(x)} \qquad \textit{and} \qquad \tilde{\mathcal{A}} = -x\frac{\mathcal{J}'(x)}{\mathcal{J}(x)}, \qquad \textit{where}$$

$$\mathcal{J}(x) = \sum_{n\geq 0} \frac{(-1)^{\lceil n/2\rceil} x^n q^{\binom{n}{2}}}{((q^2))_{\lfloor n/2\rfloor}}.$$

Give a proof of this results using PPP^*.

Open problem : pyramids

Denote by Π the set of pyramids (heaps with a **unique maximal element**).

By using the bijection ϕ we find

$$\sum_{H \in \Pi} x^{\ell(H)} y^{|H|} q^{e(H)} = -y \frac{\partial_y T}{T} = \sum_{H \in \widetilde{\mathcal{H}}} x^{\ell(H)} y^{|H|} q^{e(H)}$$

A bijection between the set $\widetilde{\mathcal{H}}$ and the set of pyramids Π would be nice, as would be a direct way of encoding periodic parallelogram polyominoes as pyramids.

End of the talk

The end