Partitioning a graph into isomorphic subgraphs

Marthe Bonamy, Natasha Morrison, Alex Scott

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Necessary conditions for G to admit a perfect H-matching?

1. $|V(H)|$ divides $|V(G)|$
2. Every vertex of G belongs to a copy of H

Perfect matchings

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Theorem (Godsil, Royle 2001)

If G is vertex-transitive, then G admits a perfect matching.

(Vertex-transitive$= \forall u, v,\ \exists$ automorphism f s.t. $f(u) = v$)

Cartesian products and Hypercubes

$G_1 \square G_2$

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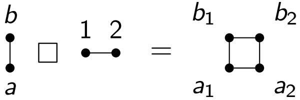

Cartesian products and Hypercubes

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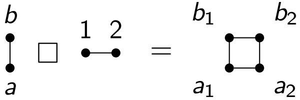
$$V(G_1 \Box G_2) = \{(u_1, u_2) | u_1 \in V(G_1), u_2 \in V(G_2)\}$$



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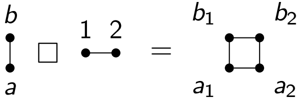

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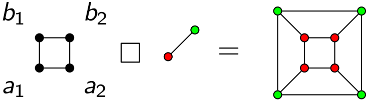

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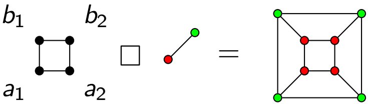

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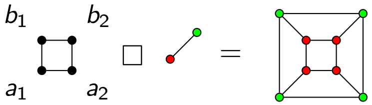

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Hypercube of dimension k: $Q_k = (\bullet\!\!-\!\!\bullet)^k$

Higher dimension

For every $H \subseteq G$ satisfying conditions (1) and (2), does there exist $p \in \mathbb{N}$ such that G^p admits a perfect H-packing?

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Yes.

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Conjecture (Gruslys 2016)

Works for any vertex-transitive G.

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What about if $G = (C_k)^p$ for some k and p?

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Yes for *even k*.

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What about $k = a^{\cdots}$ where a is an odd prime?

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Theorem (Gruslys, Leader, Tan 2015)

$T \subseteq \mathbb{Z}^k$, where T is finite and $\neq \emptyset$. There is n s.t. $\mathbb{Z}^n$ can be partitioned into isometric copies of T.

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Let P be a poset of size 2^k with a greatest and least element. There is n s.t. the Boolean lattice $2^{[n]}$ can be partitioned into copies of P.

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No. For every k, there is $Q_k \subseteq H \subseteq Q_{k+1}$ s.t. no Q_n can be edge-partitioned into copies of H.

The proof

For every even k, for every p, for every $H \subseteq (C_k)^p$ with $k^p \equiv 0 \mod |V(H)|$, there exists $n \in \mathbb{N}$ such that $(C_k)^n$ admits a perfect H-packing.

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If for some $r \geq 1$, the graph $(C_k)^p$ admits both an r-cover and a $(1 \mod r)$-cover, then it admits a perfect H-packing.

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Conjecture (Gruslys, Leader, Tomon 2016)

Let P be a finite poset. Is there a constant $c(P)$ such that, for any n, it is possible to cover all but at most $c(P)$ elements of $2^{[n]}$ with disjoint copies of P?

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For any $t \in \mathbb{N}^*$, $\exists ?d$ s.t. for any $T \subset \mathbb{Z}$ with $|T| = t$, we have that $\mathbb{Z}^d$ can be partitioned into isometric copies of T?

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Thanks!