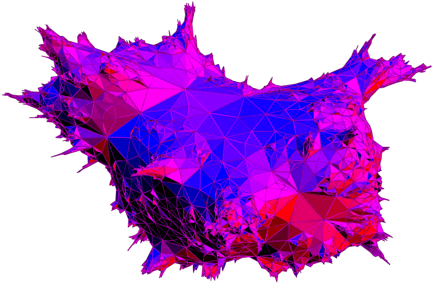
Bipolar orientations of maps and quadrant walks

Mireille Bousquet-Mélou, CNRS, LaBRI, Université de Bordeaux

joint work with
Éric Fusy & Kilian Raschel

Outline

I. Bipolar orientations of planar maps

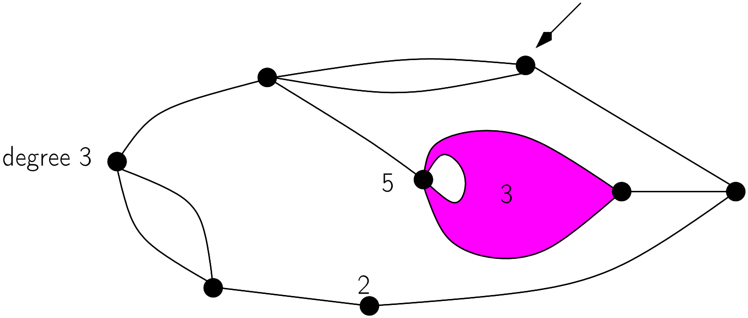
II. From bipolar orientations to quadrant walks
[Kenyon, Miller, Sheffield, Wilson 15(a)]

III. Enumeration of quadrant walks

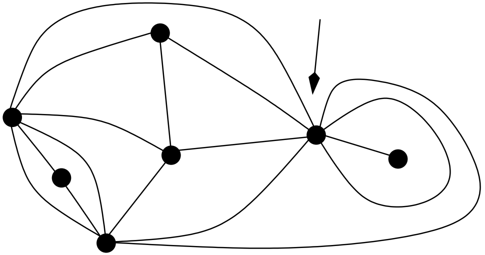
IV. A bijective proof

V. Asymptotics

Rooted planar maps

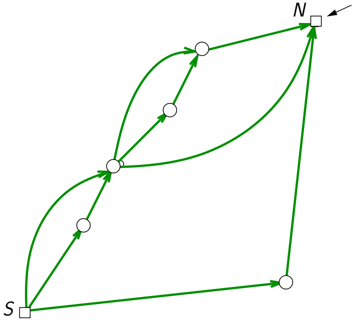


With degree constraints: rooted triangulations



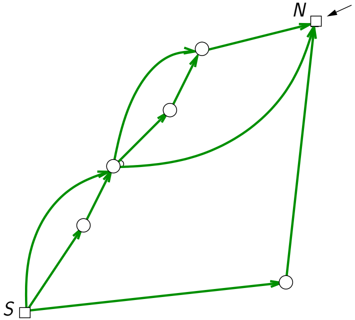
Bipolar orientations of maps

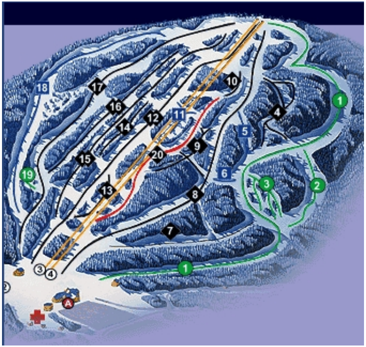
- a rooted planar map, with root vertex N (the north pole)
- another marked vertex S (the south pole) in the outer face
- an acyclic orientation
- S is the only source and N the only sink



Bipolar orientations of maps

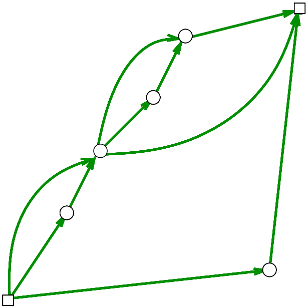
- a rooted planar map, with root vertex N (the north pole)
- another marked vertex S (the south pole) in the outer face
- an acyclic orientation
- S is the only source and N the only sink

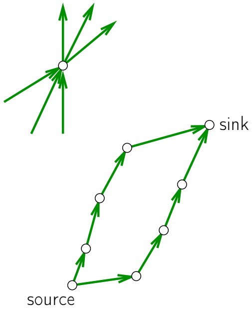




Bipolar maps: basic facts

- simple orientations around a vertex/face

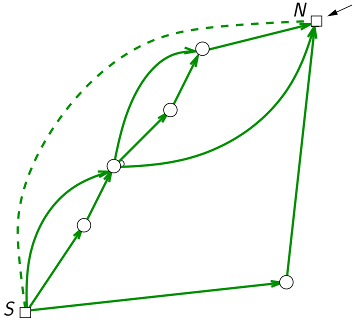




Bipolar maps: basic facts

- simple orientations around a vertex/face
- M admits a bipolar orientation from S to N iff $M \cup \{S, N\}$ is 2-connected

[De Fraysseix, Ossona de Mendez, Rosenstiehl 95]



Bipolar maps: basic facts

- simple orientations around a vertex/face
- M admits a bipolar orientation from S to N iff $M \cup \{S, N\}$ is 2-connected
- the number $\mathrm{bip}(M)$ of bipolar orientations of M from N to S can be computed from the chromatic polynomial of $M \cup \{S, N\}$

[De Fraysseix, Ossona de Mendez, Rosenstiehl 95]

[Greene & Zaslavsky 83], [Lass 01]

Bipolar maps: basic facts

- simple orientations around a vertex/face
- M admits a bipolar orientation from S to N iff $M \cup \{S, N\}$ is 2-connected
- the number $\mathrm{bip}(M)$ of bipolar orientations of M from N to S can be computed from the chromatic polynomial of $M \cup \{S, N\}$

[De Fraysseix, Ossona de Mendez, Rosenstiehl 95]

[Greene & Zaslavsky 83], [Lass 01]

• Aim: compute, or characterize, the generating function

$$\sum_{M \text{ map}} \mathrm{bip}(M) t^{\mathrm{e}(M)} = \sum_{O \text{ bip. orient.}} t^{\mathrm{e}(O)},$$

where the sum runs over a given family of planar maps M (or the corresponding bipolar orientations), and $\mathrm{e}(M)$ is the edge number.

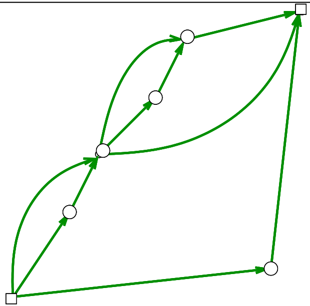
Maps equipped with an additional structure

In combinatorics, and in theoretical physics

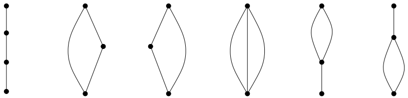
- Spanning trees [Mullin 67, Bernardi 07]
- Spanning forests [Bouttier et al., Sportiello et al., mbm-Courtiel 13]
- Proper colourings [Tutte 68-84]
- Self-avoiding walks [Duplantier-Kostov 88]
- Hard particles [Bouttier et al. 02, mbm, Schaeffer, Jehanne]
- The q-state Potts model (equivalent to the Tutte polynomial) [Eynard-Bonnet 99, Baxter 01, Bernardi-mbm 09, Borot et al. 12, Guionnet et al.]
- Loop models [Borot et al., Eynard 99, Kristjansen, Zinn-Justin 00]
- Eulerian orientations [Kostov 00, Zinn-Justin 00, Bonichon et al. 17, Elvey Price & Guttmann 18, mbm & Elvey Price 18(a)]

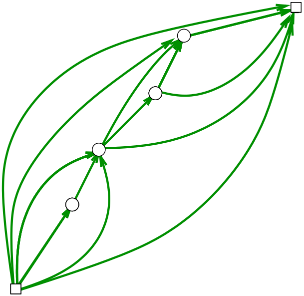
Bipolar orientations with n edges: Two main questions

No degree restriction on faces	Triangulations, quadrangulations, etc.

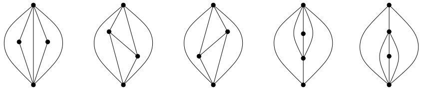


$b(n) = 1, 2, 6, 22, 92, 422, 2074$





$a(3k+1) = 1, 1, 5, 42, 462, 6006, 87516$



The number of bipolar orientations with n edges

Proposition [R. Baxter 01]

The number of bipolar orientations with n edges is

$$b(n) = \frac{2}{n(n+1)^2} \sum_{k=1}^{n} \binom{n+1}{k-1} \binom{n+1}{k} \binom{n+1}{k+1} \sim \frac{32}{\sqrt{3}\pi} 8^n n^{-4}.$$

This sequence is P-recursive (the associated generating function $\sum b(n) t^n$ is D-finite):

$$(n+6)(n+5)b(n+2) = (7n^2+49n+82)b(n+1) + 8(n+2)(n+1)b(n)$$

The number of bipolar orientations with n edges

Proposition [R. Baxter 01]

The number of bipolar orientations with n edges is

$$b(n) = \frac{2}{n(n+1)^2} \sum_{k=1}^{n} \binom{n+1}{k-1} \binom{n+1}{k} \binom{n+1}{k+1} \sim \frac{32}{\sqrt{3}\pi} 8^n n^{-4}.$$

This sequence is P-recursive (the associated generating function $\sum b(n) t^n$ is D-finite):

$$(n+6)(n+5)b(n+2) = (7n^2+49n+82)b(n+1) + 8(n+2)(n+1)b(n)$$

But but but... these numbers count Baxter permutations!
[G. Baxter 64] [Chung, Graham, Hoggatt & Kleiman 78]

The number of bipolar orientations with n edges

Proposition [R. Baxter 01]

The number of bipolar orientations with n edges is

$$b(n) = \frac{2}{n(n+1)^2} \sum_{k=1}^{n} \binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1} \sim \frac{32}{\sqrt{3}\pi} 8^n n^{-4}.$$

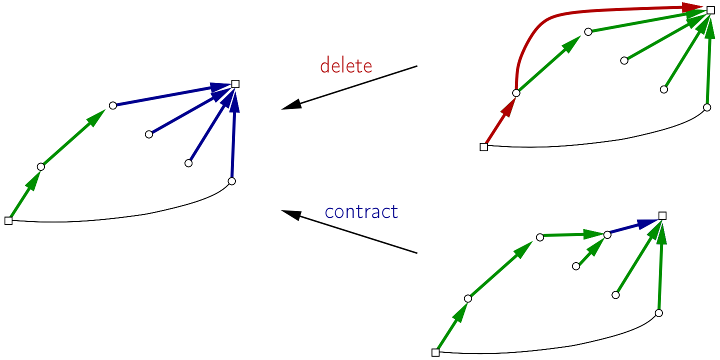
This sequence is P-recursive (the associated generating function $\sum b(n)t^n$ is D-finite):

$$(n+6)(n+5)b(n+2) = (7n^2 + 49n + 82)b(n+1) + 8(n+2)(n+1)b(n)$$

But but but... these numbers count Baxter permutations!
[G. Baxter 64] [Chung, Graham, Hoggatt & Kleiman 78]

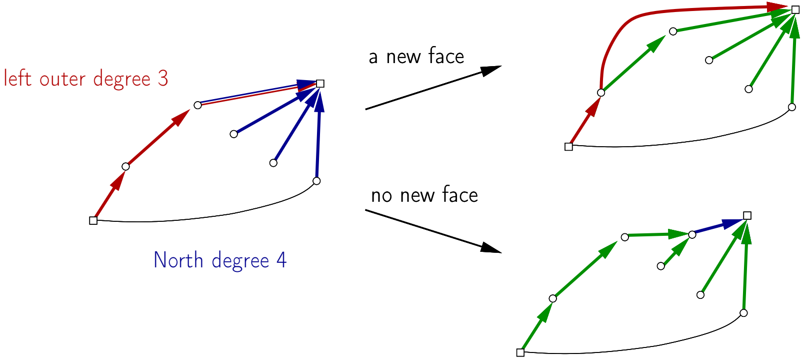
$\Rightarrow$ Bijections with Baxter permutations, non-intersecting 3-tuples of paths [Bonichon, mbm & Fusy 09, Felsner, Fusy, Noy & Orden 11, Fusy, Poulalhon & Schaeffer 09]

Bipolar orientations: a simple recursive structure



Bipolar orientations: a simple recursive structure

Two ways of adding an edge:



- Every bipolar map is obtained exactly once
- The left outer degree and the North degree can be described recursively

Prescribing face degrees

- Due to edge contractions, the above recursive construction behaves badly (apart from triangulations)

Prescribing face degrees

- Due to edge contractions, the above recursive construction behaves badly (apart from triangulations)

Proposition [Tutte 73]

The number of bipolar orientations of triangulations of a digon having $n = 3k + 1$ edges is

$$a(k) = \frac{2(3k)!}{k!(k+1)!(k+2)!} \sim \frac{\sqrt{3}}{\pi} 27^k k^{-4}.$$

The sequence is P-recursive (hypergeometric):

$$(k+1)(k+2)a(k) = 3(3k-1)(3k-2)a(k-1).$$

This is also the number of rectangular Young tableaux of height 3 and width k.

Prescribing face degrees

- Due to edge contractions, the above recursive construction behaves badly (apart from triangulations)
- A new construction: a bijection with lattice paths [Kenyon, Miller, Sheffield, Wilson, 15(a)]

Bipolar orientations with prescribed face degrees

Denote $\bar{x} := 1/x$, $\bar{y} := 1/y$, and let

$$S(x,y) := x\bar{y} + \sum_{i,j\geq 0} z_{i+j}\bar{x}^i y^j.$$

Enumeration by face degrees [mbm, Fusy & Raschel 18(a)]

The generating function of bipolar orientations of a digon, with each edge weighted by t and each (inner) face of degree $k+2$ weighted by z_k, is

$$B = -[x^0y^0]\frac{t\,y^2}{x}\frac{S_2'(x,y)}{1-t\,S(x,y)}\left(1-\frac{\bar{x}^2}{t}+\sum_{k\geq 0} z_k(k+1)\bar{x}^{k+2}\right)$$

When degrees are bounded, the RHS is a rational series and B is a D-finite series.

Bipolar orientations with prescribed face degrees

Denote $\bar{x} := 1/x$, $\bar{y} := 1/y$, and let

$$S(x,y) := x\bar{y} + \sum_{i,j\geq 0} z_{i+j}\bar{x}^i y^j.$$

Let $Y_1 = tx + O(t^2)$ is the unique power series in t (with coefficients that are Laurent polynomials in x) satisfying $1 = t\,S(x, Y_1)$.

Enumeration by face degrees [mbm, Fusy & Raschel 18(a)]

The generating function of bipolar orientations of a digon, with each edge weighted by t and each (inner) face of degree $k+2$ weighted by z_k, is

$$B = -[x^0y^0]\frac{t\,y^2}{x}\frac{S_2'(x,y)}{1-t\,S(x,y)}\left(1 - \frac{\bar{x}^2}{t} + \sum_{k\geq 0} z_k(k+1)\bar{x}^{k+2}\right)$$

When degrees are bounded, the RHS is a rational series and B is a D-finite series. Equivalently,

$$B = [x^0]\frac{Y_1}{x}\left(1 - \frac{\bar{x}^2}{t} + \sum_{k\geq 0} z_k(k+1)\bar{x}^{k+2}\right)$$

Recurrence relations for $(k+2)$-angulations by edges

Bipolar orientations: a D-finite series

$$B = -[x^0 y^0] \frac{t\,y^2}{x} \frac{S_2'(x,y)}{1 - t\,S(x,y)} \left(1 - \frac{\bar{x}^2}{t} + \sum_{k \geq 0} z_k (k+1) \bar{x}^{k+2}\right)$$

Recurrence relations for $(k+2)$-angulations by edges

Bipolar orientations: a D-finite series

$$B = -[x^0y^0]\frac{t\,y^2}{x}\frac{S_2'(x,y)}{1-t\,S(x,y)}\left(1-\frac{\bar{x}^2}{t}+\sum_{k\geq 0} z_k(k+1)\bar{x}^{k+2}\right)$$

- $k=1$ (triangulations)

$$(n+3)(n+2)a(n+1) = 3(3n+2)(3n+1)a(n)$$

- $k=2$ (quadrangulations)

$$(n+4)(n+3)^2 a(n+2) = 4(2n+3)(n+3)(n+1)a(n+1)+12(2n+3)(2n+1)(n+1)a(n)$$

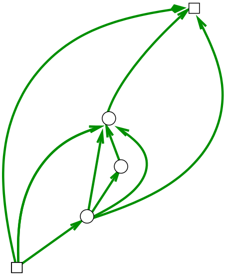
- $k=3$ (pentagulations)

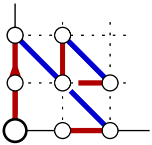
$$\begin{aligned}&27(3n+8)(3n+4)(5n+3)(3n+5)^2(3n+7)^2(n+2)^2 a(n+2) =\\ &60(5n+7)(3n+5)(5n+9)(5n+6)(3n+4)(8+5n)(145n^3+532n^2+626n+233)a(n+1)\\ &\quad -800(5n+6)(5n+1)(5n+7)(5n+2)(5n+3)(5n+9)(5n+4)(8+5n)^2 a(n)\end{aligned}$$

Software: [Bostan, Lairez, Salvy 13]

II. From bipolar orientations to quadrant walks

[Kenyon, Miller, Sheffield, Wilson, 15(a)]

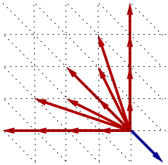




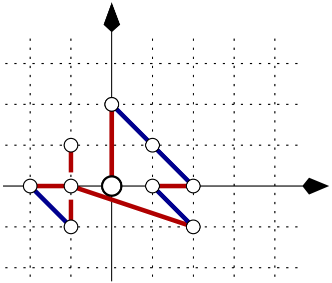
The KMSW construction

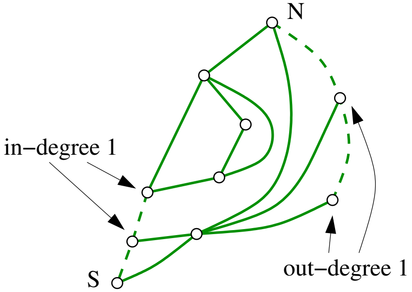
Take a lattice walk with two kinds of steps:

- SE steps $(1, -1)$
- NW steps $(-i, j)$ with $i, j \geq 0$



The construction starts from a walk and a bipolar orientation reduced to an edge, and yields an incomplete bipolar orientation.

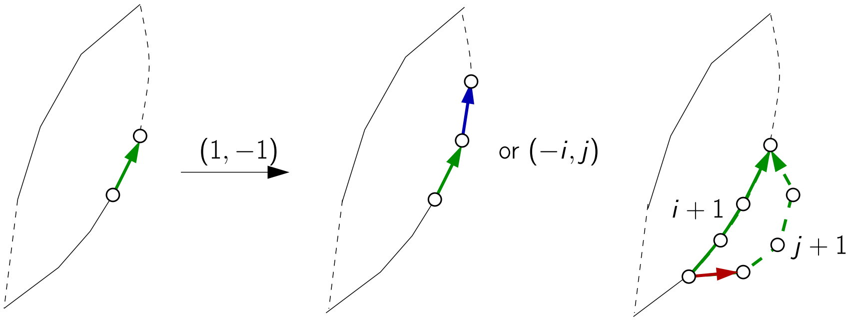




The KMSW construction

The construction starts from a walk and a bipolar orientation reduced to an edge, and yields an incomplete bipolar orientation.

- every SE step $(1, -1)$ creates an edge.
- every NW step $(-i, j)$ creates a face of degree $i + j + 2$ and an edge.

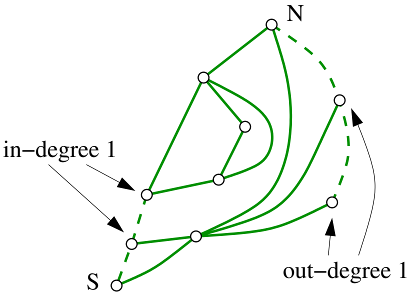


Example: walk
$(0, 2)(1, -1)(1, -1)(-1, 0)(1, -1)(-3, 1)(-1, 0)(1, -1)(0, 1)(0, 1)$

The KMSW construction

Proposition [Kenyon et al. 15(a)]

This construction is a bijection from lattice paths to incomplete bipolar orientations.

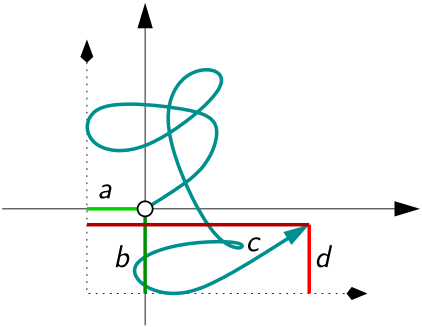


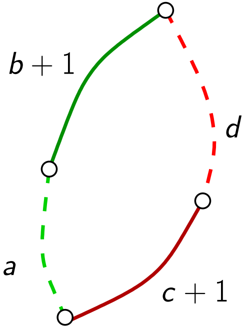
The KMSW construction

Proposition [Kenyon et al. 15(a)]

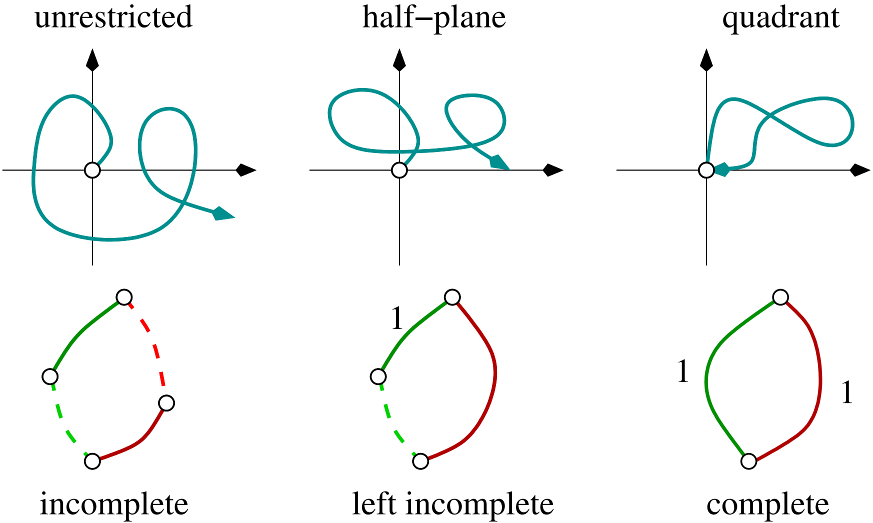
This construction is a bijection from lattice paths to incomplete bipolar orientations.

- steps $\Leftrightarrow$ (solid) edges in the orientation (minus 1)
- steps $(-i, j)$ $\Leftrightarrow$ inner faces of oriented degree $(i+1, j+1)$
- coordinates of the endpoints $\Leftrightarrow$ left and right boundaries of the map.

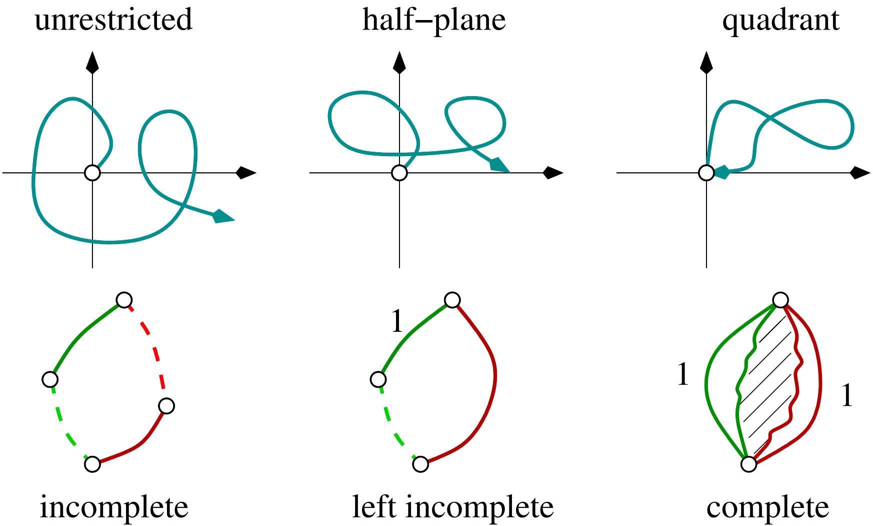




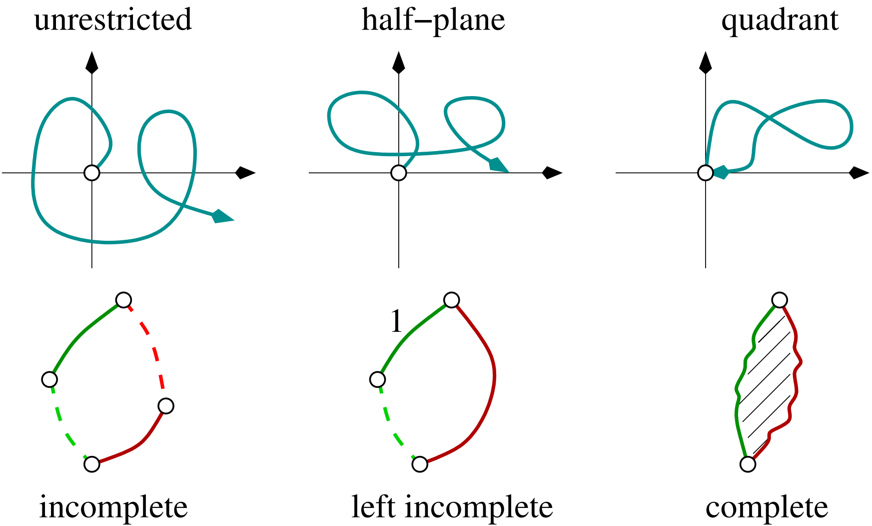
The KMSW construction: Some specializations



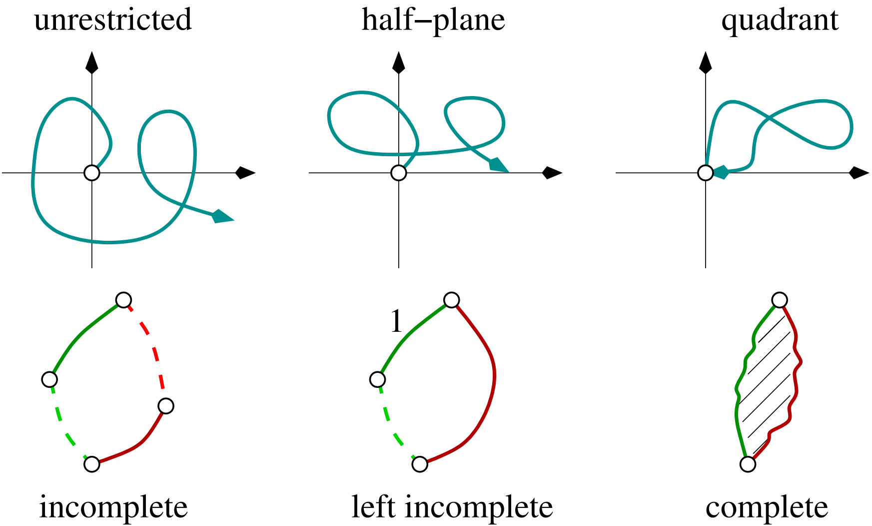
The KMSW construction: Some specializations



The KMSW construction: Some specializations

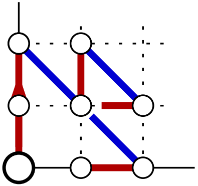


The KMSW construction: Some specializations

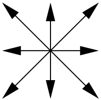


Enumeration of walks confined to the quadrant

III. Counting quadrant walks: a very active topic

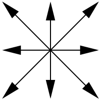


Counting quadrant walks



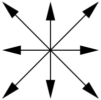
- With small steps (included in $\{-1,0,1\}^2$)
  - sporadic cases [Gessel, Gouyou-Beauchamps, Kreweras, Krattenthaler, Niederhausen, Sagan...]
  - uniform approach [Mishna, mbm-Mishna 10]
  - D-finite and algebraic cases [Bostan & Kauers 10, mbm-Mishna 10, Zeilberger]
  - non-D-finite cases [Kurkova & Raschel 12, Bostan, Raschel, Salvy 14]
  - D-algebraic cases [Bernardi, mbm & Raschel 18(a)]
  - non-D-algebraic cases [Dreyfus, Hardouin, Roques & Singer 17(a)]
  - an attractive mixture of methods: power series algebra, bijections, complex analysis, computer algebra, differential Galois theory...

Counting quadrant walks



- With small steps (included in $\{-1,0,1\}^2$)
  - sporadic cases [Gessel, Gouyou-Beauchamps, Kreweras, Krattenthaler, Niederhausen, Sagan...]
  - uniform approach [Mishna, mbm-Mishna 10]
  - D-finite and algebraic cases [Bostan & Kauers 10, mbm-Mishna 10, Zeilberger]
  - non-D-finite cases [Kurkova & Raschel 12, Bostan, Raschel, Salvy 14]
  - D-algebraic cases [Bernardi, mbm & Raschel 18(a)]
  - non-D-algebraic cases [Dreyfus, Hardouin, Roques & Singer 17(a)]
  - an attractive mixture of methods: power series algebra, bijections, complex analysis, computer algebra, differential Galois theory...
- With arbitrary steps
  - an approach that solves (some) D-finite cases [Bostan, mbm & Melczer 18(a)]

Counting quadrant walks



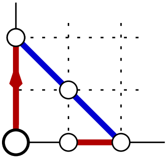
- With small steps (included in $\{-1,0,1\}^2$)
  - sporadic cases [Gessel, Gouyou-Beauchamps, Kreweras, Krattenthaler, Niederhausen, Sagan...]
  - uniform approach [Mishna, mbm-Mishna 10]
  - D-finite and algebraic cases [Bostan & Kauers 10, mbm-Mishna 10, Zeilberger]
  - non-D-finite cases [Kurkova & Raschel 12, Bostan, Raschel, Salvy 14]
  - D-algebraic cases [Bernardi, mbm & Raschel 18(a)]
  - non-D-algebraic cases [Dreyfus, Hardouin, Roques & Singer 17(a)]
  - an attractive mixture of methods: power series algebra, bijections, complex analysis, computer algebra, differential Galois theory...
- With arbitrary steps
  - an approach that solves (some) D-finite cases
    [Bostan, mbm & Melczer 18(a)]
    including those corresponding to bipolar orientations
    [mbm, Fusy & Raschel 18(a)]

Walk enumeration for bipolar orientations

Parameters and variables:

- steps/edges: variable t
- steps $(-i, j)$ (faces): variable z_{i+j} (degree selection)
- coordinates of the endpoint: variables x, y

Example:



$$\mathrm{weight}(w) = t^4 z_2 z_1 x^1 y^0$$

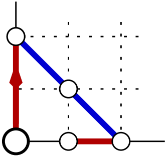
The step polynomial (generating function of the steps)

$$S(x, y) := x\bar{y} + \sum_{i,j \geq 0} z_{i+j} \bar{x}^i y^j$$

Walk enumeration for bipolar orientations

Parameters and variables:

- steps/edges: variable t
- steps $(-i, j)$ (faces): variable z_{i+j} (degree selection)
- coordinates of the endpoint: variables x, y



Example:

$$\mathrm{weight}(w) = t^4 z_2 z_1 x^1 y^0$$

The step polynomial (generating function of the steps)

$$S(x,y) := x\bar{y} + \sum_{i,j\geq 0} z_{i+j}\bar{x}^i y^j$$

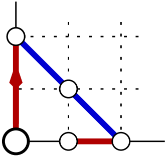
Unrestricted walks: a rational series

$$U(x,y) = \frac{1}{1 - tS(x,y)}$$

Walk enumeration for bipolar orientations

Parameters and variables:

- steps/edges: variable t
- steps $(-i, j)$ (faces): variable z_{i+j} (degree selection)
- coordinates of the endpoint: variables x, y



Example:

$$\mathrm{weight}(w) = t^4 z_2 z_1 x^1 y^0$$

The step polynomial (generating function of the steps)

$$S(x,y) := x\bar{y} + \sum_{i,j\geq 0} z_{i+j}\bar{x}^i y^j$$

Bipolar orientations

$$B = -[x^0 y^0]\frac{t\,y^2}{x}\frac{S_2'(x,y)}{1 - t\,S(x,y)}\left(1 - \frac{\bar{x}^2}{t} + \sum_{k\geq 0} z_k (k+1)\bar{x}^{k+2}\right)$$

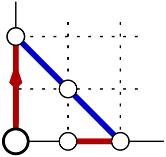
Walk enumeration for bipolar orientations

Parameters and variables:

- steps/edges: variable t
- steps $(-i, j)$ (faces): variable z_{i+j} (degree selection)
- coordinates of the endpoint: variables x, y

Example:

$$\mathrm{weight}(w) = t^4 z_2 z_1 x^1 y^0$$



The step polynomial (generating function of the steps)

$$S(x, y) := x\bar{y} + \sum_{i,j \geq 0} z_{i+j} \bar{x}^i y^j$$

Bipolar orientations: an alternative formula

$$B = [x^0] \frac{Y_1(x)}{x} \left(1 - \frac{\bar{x}^2}{t} + \sum_{k \geq 0} z_k (k+1) \bar{x}^{k+2} \right)$$

Walks in a half-plane

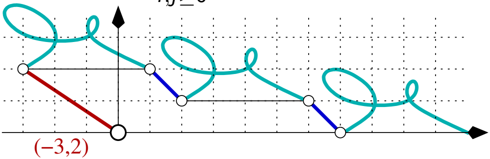
Half-plane walks: an algebraic series

$$H(x) = \frac{Y_1(x)}{tx},$$

where $Y_1(x)$ is the unique series in t satisfying $1 = t\,S\,(x, Y_1(x))$.

Proof. First return decomposition (largest down move $= -1$)

$$H(x) = 1 + \sum_{i,j \geq 0} z_{i+j}(t\bar{x}^i) H(x)^{j+1} (tx)^j.$$



This gives for $Y = txH(x)$ the equation $tS(x, Y) = 1$, with

$$S(x, y) := x\bar{y} + \sum_{i,j \geq 0} z_{i+j} \bar{x}^i y^j.$$

Walk enumeration: the quadrant case

Quadrant walks: a D-finite series

$$Q(0,0) = [x^0]\frac{Y_1(x)}{tx}\left(1 - \frac{\bar{x}^2}{t} + \sum_k (k+1) z_k \bar{x}^{k+2}\right).$$

Walk enumeration: the quadrant case

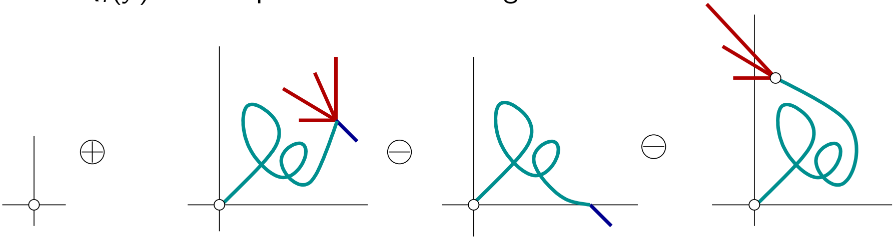
Quadrant walks: a D-finite series

$$Q(0,0) = [x^0]\frac{Y_1(x)}{tx}\left(1 - \frac{\bar{x}^2}{t} + \sum_k (k+1) z_k \bar{x}^{k+2}\right).$$

A functional equation:

$$Q(x,y) = 1 + tQ(x,y)S(x,y) - tx\bar{y}Q(x,0)$$
$$- t \sum_{i>0, j\geq 0} z_{i+j}\bar{x}^i y^j \left(Q_0(y) + xQ_1(y) + \cdots + x^{i-1}Q_{i-1}(y)\right)$$

where $Q_i(y)$ counts quadrant walks ending at abscissa i.



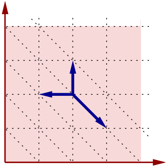
Walk enumeration: the quadrant case

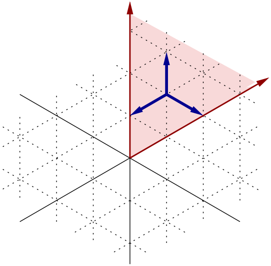
Quadrant walks: a D-finite series

$$Q(0,0) = [x^0]\frac{Y_1(x)}{tx}\left(1 - \frac{\bar{x}^2}{t} + \sum_k (k+1) z_k \bar{x}^{k+2}\right).$$

A simple case: triangulations. Take $z_1 = 1$ and $z_i = 0$ if $i \neq 1$

$$Q(x,y) = 1 + Q(x,y)S(x,y) - tx\bar{y}Q(x,0) - t\bar{x}Q(0,y)$$





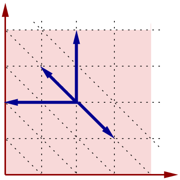
Walks confined to a Weyl chamber, solvable using the reflection principle [Gessel-Zeilberger 92]

Walk enumeration: the quadrant case

Quadrant walks: a D-finite series

$$Q(0,0) = [x^0]\frac{Y_1(x)}{tx}\left(1 - \frac{\bar{x}^2}{t} + \sum_k (k+1) z_k \bar{x}^{k+2}\right).$$

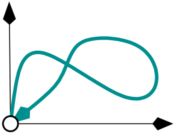
Quadrangulations. Take $z_2 = 1$ and $z_i = 0$ if $i \neq 2$



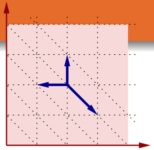
$Q(x,y) = 1 + S(x,y)Q(x,y) - tx\bar{y}Q(x,0) - t\bar{x}^2\left(Q_0(y) + xQ_1(y)\right) - t\bar{x}yQ_0(y$

where $Q_i(y)$ counts quadrant walks ending at abscissa i.

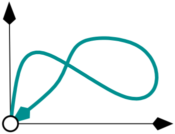
IV. Enumeration of quadrant walks: a bijective proof

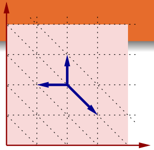


$$Q = [x^0]\frac{Y_1(x)}{tx}\left(1-\frac{1}{tx^2}+\frac{2}{x^3}\right)$$



IV. Enumeration of quadrant walks: a bijective proof

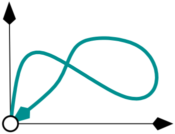


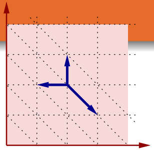


$$
\begin{aligned}
Q &= [x^0]\frac{Y_1(x)}{tx}\left(1-\frac{1}{tx^2}+\frac{2}{x^3}\right)\\
&= [x^0]H(x)\left(1-\frac{1}{tx^2}+\frac{2}{x^3}\right)
\end{aligned}
$$

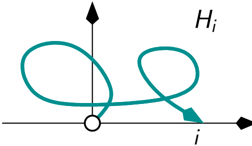
half-plane walks

IV. Enumeration of quadrant walks: a bijective proof

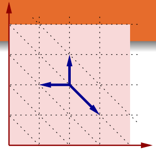


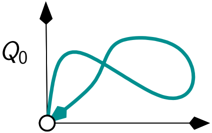


$$
\begin{aligned}
Q &= [x^0]\frac{Y_1(x)}{tx}\left(1-\frac{1}{tx^2}+\frac{2}{x^3}\right)\\
&= [x^0]H(x)\left(1-\frac{1}{tx^2}+\frac{2}{x^3}\right)\\
&= H_0-\frac{1}{t}H_2+2H_3
\end{aligned}
$$

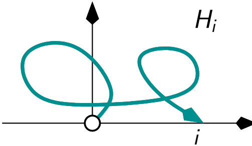


IV. Enumeration of quadrant walks: a bijective proof

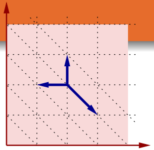


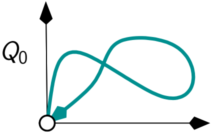


$$Q_0 = [x^0]\frac{Y_1(x)}{tx}\left(1 - \frac{1}{tx^2} + \frac{2}{x^3}\right)$$
$$= [x^0]H(x)\left(1 - \frac{1}{tx^2} + \frac{2}{x^3}\right)$$
$$= H_0 - \frac{1}{t}H_2 + 2H_3$$

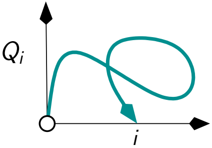


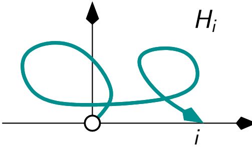
IV. Enumeration of quadrant walks: a bijective proof





$$Q_0 = [x^0]\frac{Y_1(x)}{tx}\left(1 - \frac{1}{tx^2} + \frac{2}{x^3}\right)$$
$$= [x^0]H(x)\left(1 - \frac{1}{tx^2} + \frac{2}{x^3}\right)$$
$$= H_0 - \frac{1}{t}H_2 + 2H_3$$



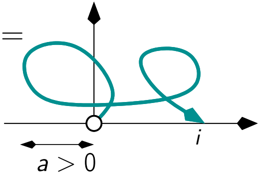


In fact,

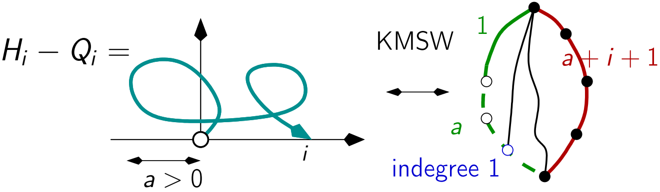
$$\boxed{Q_i = H_i - \frac{1}{t}H_{i+2} + 2H_{i+3}}$$

A bijective proof: $H_i - Q_i = H_{i+2}/t - 2H_{i+3}$

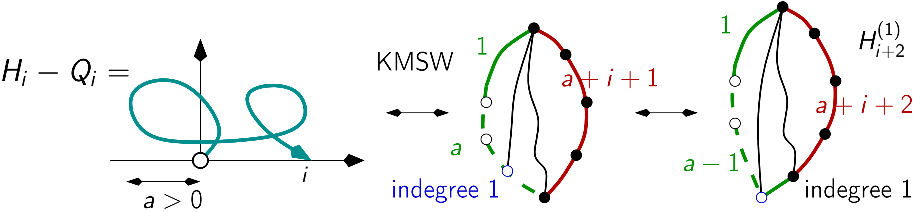
$H_i - Q_i =$



A bijective proof: $H_i - Q_i = H_{i+2}/t - 2H_{i+3}$

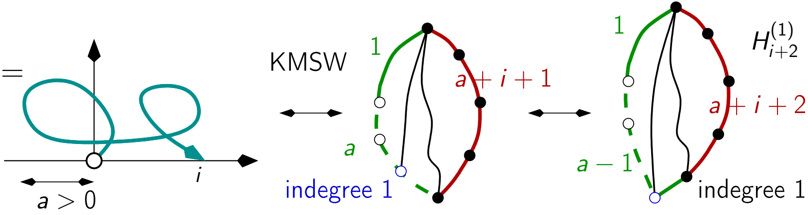


A bijective proof: $H_i - Q_i = H_{i+2}/t - 2H_{i+3}$



A bijective proof: $H_i - Q_i = H_{i+2}/t - 2H_{i+3}$

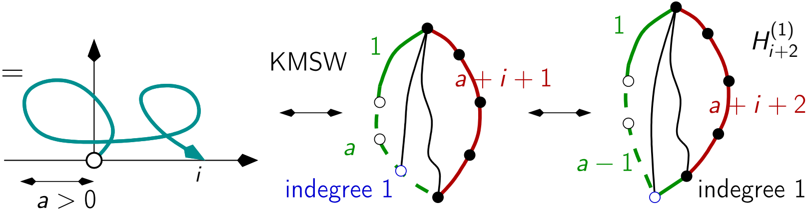
$H_i - Q_i =$



$= \frac{1}{t} H^{(1)}_{i+2}$

A bijective proof: $H_i - Q_i = H_{i+2}/t - 2H_{i+3}$

$H_i - Q_i =$

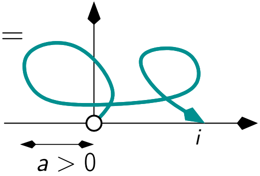


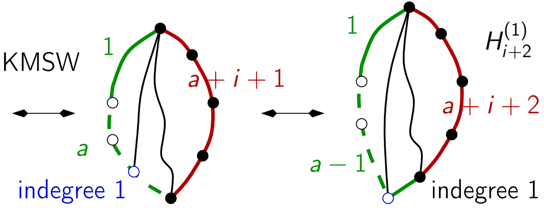
$$= \frac{1}{t} H_{i+2}^{(1)}$$

$$= \frac{1}{t} \left(H_{i+2} - H_{i+2}^{(2)} \right)$$

A bijective proof: $H_i - Q_i = H_{i+2}/t - 2H_{i+3}$

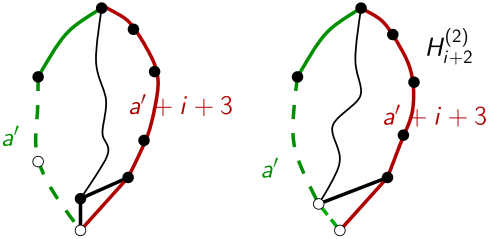
$H_i - Q_i =$





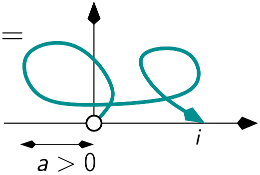
$= \frac{1}{t} H^{(1)}_{i+2}$

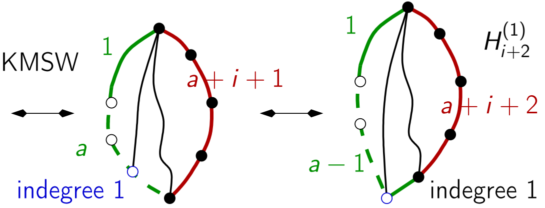
$= \frac{1}{t} \left(H_{i+2} - H^{(2)}_{i+2} \right)$



A bijective proof: $H_i - Q_i = H_{i+2}/t - 2H_{i+3}$

$H_i - Q_i =$

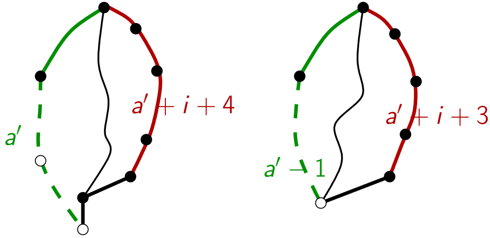




$$= \frac{1}{t} H_{i+2}^{(1)}$$

$$= \frac{1}{t} \left(H_{i+2} - H_{i+2}^{(2)} \right)$$

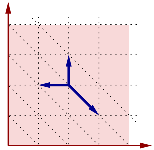
$$= \frac{1}{t} H_{i+2} - 2H_{i+3}$$



An algebraicity phenomenon

• Known: Young tableaux of height at most three are counted by Motzkin numbers [Regev 81]

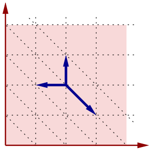
They correspond to quadrant walks ending anywhere.



An algebraicity phenomenon

• Known: Young tableaux of height at most three are counted by Motzkin numbers [Regev 81]

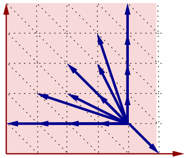
They correspond to quadrant walks ending anywhere.



• Generalization: the generating function of quadrant walks ending anywhere is Y/t, where $Y = Y_1(1)$ is the only power series solution of $tS(1, Y) = 1$. Equivalently,

$$Y = t + t \sum_{i,j \geq 0} z_{i+j} Y^{j+1}$$

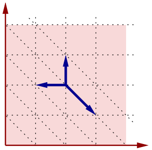
[mbm, Fusy, Raschel 18(a)]:
an algebraic and a bijective proof



An algebraicity phenomenon

• Known: Young tableaux of height at most three are counted by Motzkin numbers [Regev 81]

They correspond to quadrant walks ending anywhere.

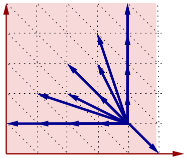


Bijections [Gouyou-Beauchamps 89, Eu 10, Chyzak & Yeats 18(a)]

• Generalization: the generating function of quadrant walks ending anywhere is Y/t, where $Y = Y_1(1)$ is the only power series solution of $tS(1, Y) = 1$. Equivalently,

$$Y = t + t \sum_{i,j \geq 0} z_{i+j} Y^{j+1}$$

[mbm, Fusy, Raschel 18(a)]:
an algebraic and a bijective proof



VI. Asymptotics

A universal asymptotic behaviour

Given a finite set Ω of degrees, define α by

$$1 = \sum_{s \in \Omega} \binom{s-1}{2} \alpha^{-s},$$

and let

$$\gamma = \sum_{s \in \Omega} \binom{s}{2} \alpha^{-s+2}.$$

A universal behaviour [mbm, Fusy, Raschel 18(a)]

The number of bipolar orientations of a digon with n edges, in which all inner faces have degree in Ω, satisfies (with periodicity contraints)

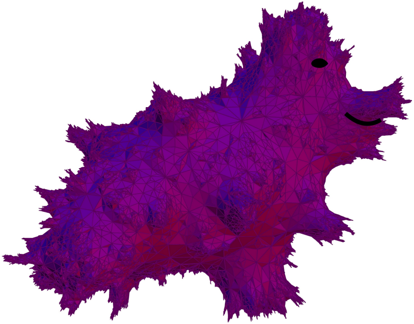
$$b^{\Omega}(n) \sim \kappa \gamma^n n^{-4}$$

where the constant κ is also explicit.

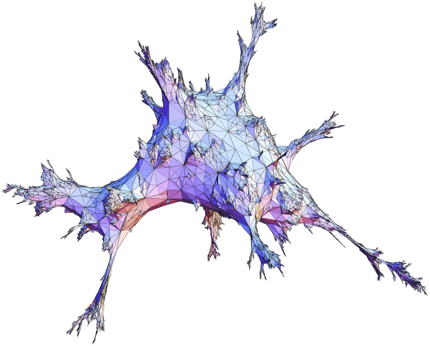
builds on enumerative results + the approach of [Denisov & Wachtel 15]

In conclusion

- Very rich combinatorics
- Connection with quadrant walks, with the longest increasing sequence in (Baxter) permutations...
- Enumerative results
- What about large random bipolar maps?



© Jérémie Bettinelli



© Nicolas Curien