The Z-invariant Ising model on isoradial graphs

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Outline

The Ising model

The Ising model via dimers

Z-invariance

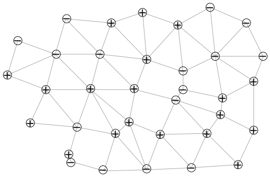
Z-invariant Ising model out of criticality

The Ising model

The Ising model

- (planar) graph G
- spin configurations: $\sigma : G \to \pm 1$
- parameters: coupling constants $(J_e)_{e \in E(G)} > 0$
- Energy of a configuration:

$$\mathcal{H}(\sigma) = -\sum_{e=xy} J_e \sigma_x \sigma_y$$

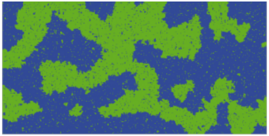


- Probability of a configuration:

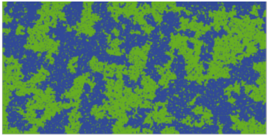
$$\mathbb{P}(\sigma) = \frac{1}{Z(G,(J_e))} \times \exp\left(-\mathcal{H}(\sigma)\right)$$

The Ising model on the square lattice

- A single parameter to study possible phase transitions: $\beta \mapsto J(e, \beta)$ increasing
- On a regular graph: $J(e, \beta) = \beta J \qquad (\beta = 1/T)$



$\beta > \beta_c$ (low T)



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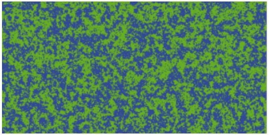


$\beta < \beta_c$ (high T)

Simulation pictures: Raphaël Cerf

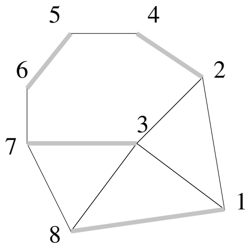
The Ising model via dimers

The Ising model is free fermionic

Physics folklore: the Ising model is a model of free fermions

Kasteleyn: the partition of the Ising model on any planar graph can be written as a Pfaffian, in connection with dimers

dimer configurations = perfect matchings = 1-factors



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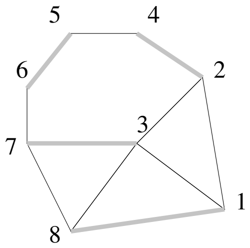
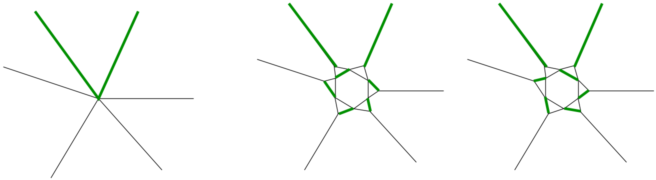


Fisher: another explicit correspondence with dimers on a decorated graph

Fisher's bijection

Ising spins$\leftrightarrow$ contours (separating spins) $\leftrightarrow$ dimers



This version (Dubédat) is not a bijection: 2 choices for each decoration of a vertex

Kasteleyn's theory of dimer models

Let $\mathcal{G}$ a finite planar graph.

- weights (ν_e) on edges of $\mathcal{G}$
- probability of a dimer conf. $\mathcal{C} \propto \prod_{e \in \mathcal{C}} \nu_e$

Theorem (Kasteleyn)

Let K be the weighted oriented adjacency matrix of G for an admissible *orientation. Then:*

- *The partition function $Z_{dimers} := \sum_{\mathcal{C}} \prod_{e \in \mathcal{C}} \nu_e$ is $\pm\,\mathrm{Pfaff}\, K$,*
- *The probability that $e_1 = (v_{i_1}, v_{i_2}), \ldots, e_k = (v_{i_{2k-1}}, v_{i_{2k}})$ occur in a random dimer configuration is*

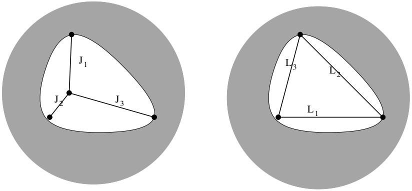
$$\left(\prod_j K(v_{i_{2j-1}}, v_{i_{2j}})\right) \mathrm{Pfaff}_{1 \le p,q \le 2k} K^{-1}(v_{i_p}, v_{i_q})^T$$

Pfaffian process

Z-invariance

Star-triangle transformation

G and G': planar graphs differing by a $Y - \nabla$ transformation



Coupling constants so that the Ising models are equivalent?

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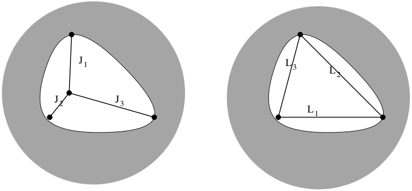

Coupling constants so that the Ising models are equivalent?

$\sigma_1\sigma_2\sigma_3$	G	G'
$\pm\pm\pm$	$2\cosh(J_1+J_2+J_3)$	$e^{L_1+L_2+L_3}$
$\pm\pm\mp$	$2\cosh(-J_1-J_2+J_3)$	$e^{-L_1-L_2+L_3}$
$\pm\mp\pm$	$2\cosh(-J_1+J_2-J_3)$	$e^{-L_1+L_2-L_3}$
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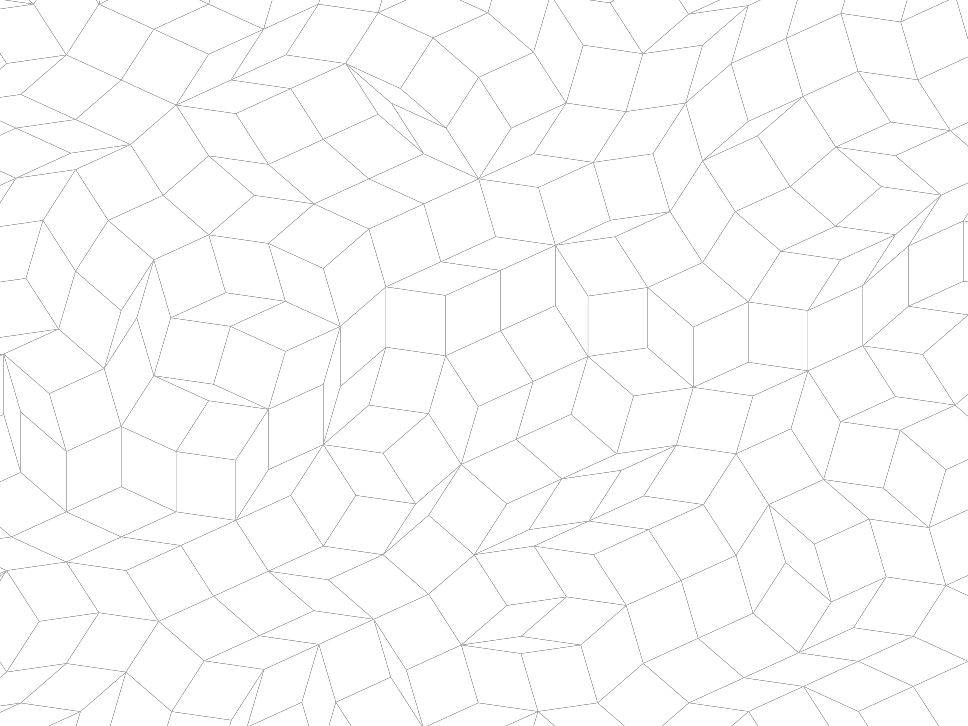
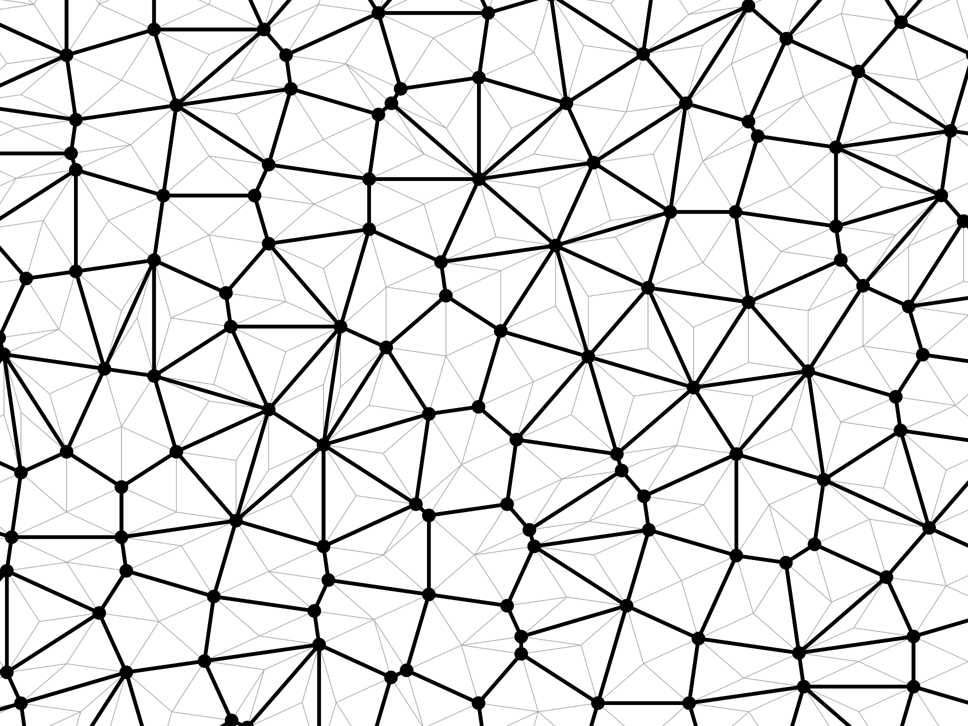
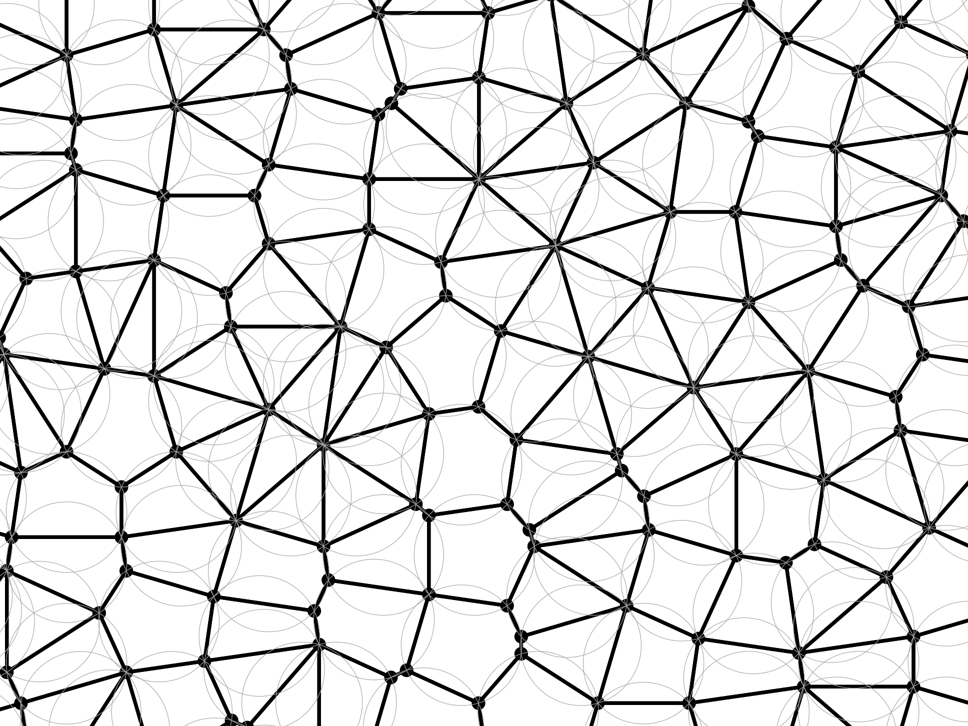
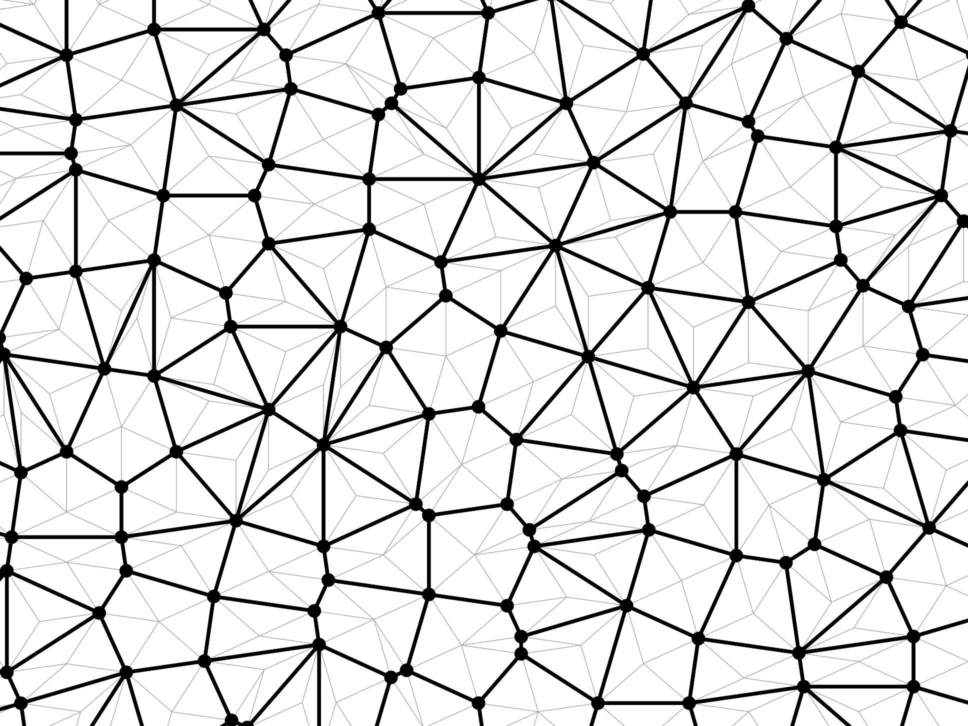


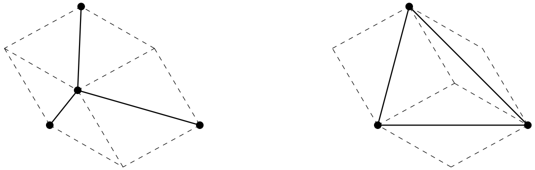




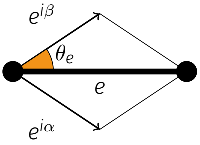


Isoradial graphs

- quad graph : projection of a surface in $\mathbb{Z}^d$
- star-triangle transformation: natural flip operation



- Each edge e has a natural parameter $\theta_e = \frac{\beta-\alpha}{2}$



Parametrization of coupling constants with angles

If we require that for isoradial graphs:

- for any edge e, $J(e) = J(\theta_e)$
- invariance under star-triangle transformations

1-parameter family of coupling constants:

$$\sinh(2J(\theta|k)) = \mathrm{sc}(\theta \frac{2K(k)}{\pi}|k) = \frac{\mathrm{sn}(\theta \frac{2K(k)}{\pi}|k)}{\mathrm{cn}(\theta \frac{2K(k)}{\pi}|k)} \qquad \text{[Baxter]}$$

The Ising model is then said to be Z-invariant

Z-invariant coupling constants

$$\sinh(2J(\theta|k)) = \mathrm{sc}(\theta\frac{2K(k)}{\pi}|k) = \frac{\mathrm{sn}(\theta\frac{2K(k)}{\pi}|k)}{\mathrm{cn}(\theta\frac{2K(k)}{\pi}|k)}$$

k: elliptic modulus $\qquad k' = \sqrt{1-k^2} \in (0,\infty) \leftrightarrow$ temperature

$K(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\cos^2(t)}}$ elliptic integral of 1st kind

$\mathrm{sn}(\cdot|k), \mathrm{cn}(\cdot|k), \mathrm{sc}(\cdot|k)$ Jacobi elliptic functions functions: generalization of $\sin$, $\cos$, $\tan$ respectively.

Bonus: Kramers-Wannier duality built-in

$$\sinh(2J(\theta|k)) \times \sinh(2J(\frac{\pi}{2} - \theta|k^*)) = 1 \qquad \text{with } k' \times (k^*)' = 1$$

Critical Z-invariant Ising model ($k = 0$)

Self-duality: $k^* = k \Leftrightarrow k = 0$

- Elliptic functions $\curvearrowright$ trigonometric: $\mathrm{sinh}(2J(\theta|0)) = \mathrm{tan}(\theta)$
- really critical [Li, Cimasoni–Duminil-Copin]
- discrete harmonic fermionic observable
- conformally invariant scaling limit [Mercat, Chelkak-Smirnov...]

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- construction of probability measure in infinite volume on isoradial graphs (dimers, Fisher correspondence) [B.–de Tilière]
- locality of dimers (and thus spin) correlations
- related to local expr. for Green function on isoradial graphs for conductances $\mathrm{tan}(\theta)$ [Kenyon]

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Question: does locality still holds out of criticality?

Z-invariant Ising model out of criticality

Inverse Kasteleyn operator

Consider the dimer model on the Fisher graph $\mathcal{G}$ coming from a Z-invariant Ising model on an isoradial graph G:

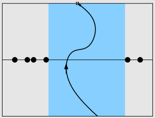
$$\nu_e = \begin{cases} \frac{\mathrm{sn}(\frac{2K\theta}{\pi}|k)}{1+\mathrm{cn}(\frac{2K\theta}{\pi}|k)} & \text{if } e \text{ is an edge coming from } G \\ 1 & \text{otherwise} \end{cases}$$

Let K the corresponding (infinite) Kasteleyn matrix on $\mathcal{G}$

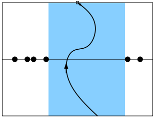
Theorem (B.–de Tilière – Raschel)

- *For $k \neq 0$, the Kasteleyn operator on the Fisher graph has a unique inverse with bounded coefficients $K^{-1}_{x,y}$.*
- *These coefficients have a local expression*

$$K^{-1}_{x,y} = \frac{k'}{8\pi} \int_{\Gamma_{x,y}} f_x(u+2K) f_y(u)\, \mathrm{Exp}_{\mathbf{x},\mathbf{y}}(u|k) du$$

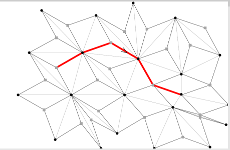


$$K_{x,y}^{-1} = \frac{k'}{8\pi} \int_{\Gamma_{x,y}} f_x(u+2K) f_y(u)\, \mathrm{Exp}_{\mathbf{x},\mathbf{y}}(u|k) du$$



Definition (massive exponential functions)

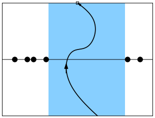
$$\mathrm{Exp}_{x,y}(u|k) = \prod_j i\sqrt{k'}\, \mathrm{sc}\left(\frac{u-\alpha_j}{2}|k\right),\ u \in T_k$$



Definition (function f)

- If x internal to a decoration $f_x(u) = \pm\, \mathrm{cn}(\frac{u-\alpha}{2}|k)^{-1}$, where $e^{i\alpha}$ edge of the quad-graph
- If x connected to an edge of G, f_x is the sum of two such terms

$$K^{-1}_{x,y} = \frac{k'}{8\pi} \int_{\Gamma_{x,y}} f_x(u+2K) f_y(u)\, \mathsf{Exp}_{\mathbf{x},\mathbf{y}}(u|k)\, du$$



- This expression is local: $K^{-1}_{x,y}$ depends on the geometry of the graph only along a path from x to y
- It can be used to define a Gibbs measure on dimer configurations of the Fisher graph, and thus on Ising contours (without assumption on periodicity of the graph)
- Dimer statistics are local

On periodic isoradial graphs: spectral curve

- If G is periodic, the Kasteleyn operator K is also periodic
- $K(z,w)$ Fourier transform of K: matrix with rows/columns indexed by vertices in a fund. domain with extra $z^{\pm 1}$ or $w^{\pm 1}$ weight for edges crossing its boundary
- $P(z,w) = \det K(z,w)$ characteristic polynomial
- Fourier formula for K^{-1}:

$$K^{-1}_{x,y+(m,n)} = \iint_{|z|=|w|=1} z^{-m} w^{-n} \frac{Q_{x,y}(z,w)}{P(z,w)} \frac{dz}{2i\pi z} \frac{dw}{2i\pi w}$$

  where $Q_{x,y}$ cofactor of $K(z,w)$.

Asymptotics depends on the zeros of P.

$C = \{(z,w) : P(z,w) = 0\}$ is called the spectral curve

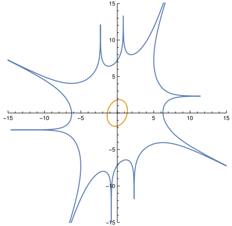
Theorem (B–de Tilière–Raschel)

spectral curve of a Z-invariant
Ising model on isoradial graph

$$\Updownarrow$$

Harnack curve of genus 1 with symmetry

$$(z, w) \leftrightarrow \left(\tfrac{1}{z}, \tfrac{1}{w}\right)$$



- Parametrization: $u \mapsto (\mathrm{Exp}_{x,x+(1,0)}(u|k), \mathrm{Exp}_{x,x+(0,1)}(u|k))$
- Area of the hole as a function of k and the local geom. of G
- Same curve for the Ising model with param. k and k^*

On periodic isoradial graphs: free energy

free energy F_{Ising}: normalized log of the partition function

Theorem

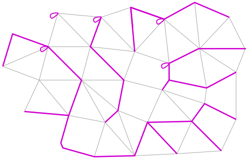
$$F_{Ising}(k) = -\frac{\log 2}{2}|V_1| - |V_1| \int_0^K 2H'(\theta) \log \mathrm{sc}(\theta) d\theta + \sum_{e \in E_1} \left(-H(2\theta) \log \mathrm{sc}(\theta) + \int_0^{\theta_e} 2H'(\theta) \log \mathrm{sc}(\theta) d\theta \right).$$

As k goes to 0,

$$F_{Ising}(k) = F_{Ising}(0) - \frac{|V_1|}{2} k^2 \log k^{-1} + O(k^2)$$

Z-invariant Ising model and rooted spanning forests

- This free energy is half the free energy of rooted spanning forests, "counted" by the determinant of a massive Laplacian on isoradial graphs, with conductances $\mathrm{sc}(2K\theta/\pi|k)$ we introduced.



- Same phase transition in Ising as from spanning forests to spanning trees
- Massive exponential functions: harmonic for this massive Laplacian (elliptic generalisation of Mercat's harmonic exponential functions)

Phase transition in the Ising model