De la mesure uniforme à la mesure de Plancherel sur les partitions d'entiers

Jérémie Bouttier
Travail en commun avec Dan Betea
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Institut de Physique Théorique, CEA Saclay
Laboratoire de Physique, ENS de Lyon

Séminaire Flajolet, IHP
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Contexte

Le contexte général de notre travail est celui des processus de Schur [Okounkov '01, O.-Reshetikhin '03-'06, Johansson '0x, Borodin-Rains '05...].

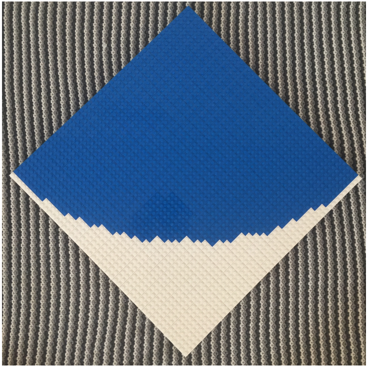
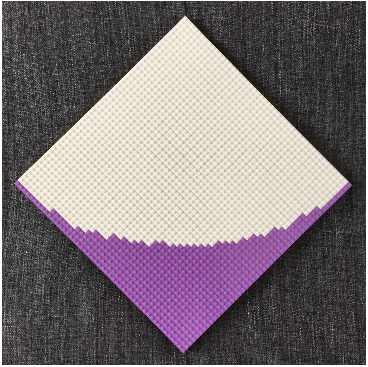
Nous nous intéressons particulièrement au cas des conditions aux limites périodiques [Borodin '07] et libres [Betea-Bouttier-Nejjar-Vuletić '18].

Avec Dan Betea, nous avons compris comment reformuler le processus de Schur périodique en termes de fermions libres, qui ont récemment fait l'objet de plusieurs travaux en lien avec les matrices aléatoires [Dean-Le Doussal-Majumdar-Schehr '15-'18, Cunden-Mezzadri-O'Connell '17, Liechty-Wang '17, Stéphan '19...].

Dans cet exposé je présenterai certains aspects de notre travail dans le cadre combinatoire le plus simple, qui se formule en termes de partitions aléatoires.

Outline

Random partitions

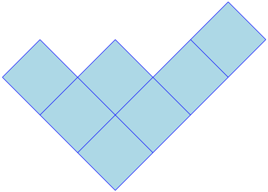
Pictures by courtesy of Dan Betea

Integer partitions

An integer partition λ is a nonincreasing sequence of integers

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots$$

that vanishes eventually. Its size is $|\lambda| := \sum \lambda_i$. It is commonly represented by a Young diagram.



The Young diagram of $\lambda = (4, 2, 1)$ in "Russian" convention.

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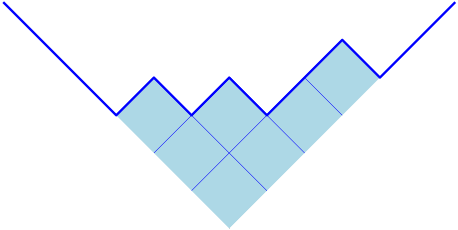

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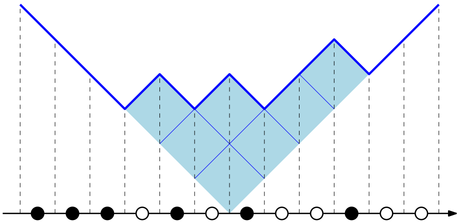
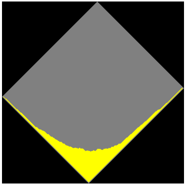


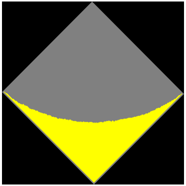
The Young diagram of $\lambda = (4, 2, 1)$ in "Russian" convention.
The boundary may be viewed as a piecewise linear curve with slope ± 1.
It may also be viewed as a 1D particle configuration ("fermions").

Random partitions

Let us fix a size N. There are two contenders for the title of the "most natural" probability distribution on the set of partitions of size N.

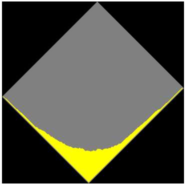
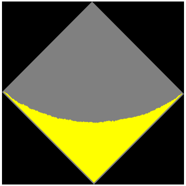
- The uniform measure: $\mathrm{Prob}(\lambda) = 1/p(N)$ with $p(N)$ "the" partition function.
- The Plancherel measure: $\mathrm{Prob}(\lambda) = \dim(\lambda)^2/N!$ with $\dim(\lambda)$ the number of Standard Young Tableaux of shape λ (dimension of irrep of $\mathfrak{S}_N$, hook-length formula...).





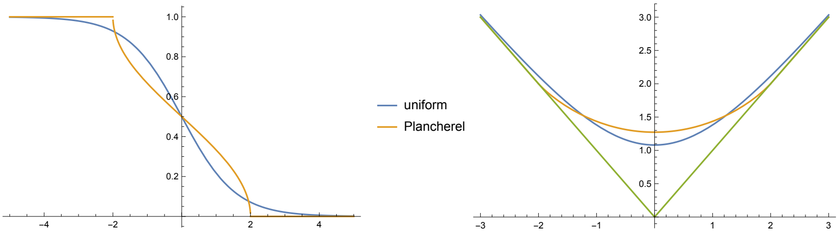
Pictures by courtesy of Dan Betea, $N = 10000$. Note the scales are different!

Limit shape

In both cases there is a limit shape phenomenon: the suitably rescaled boundary of the Young diagram converges to a deterministic curve as $N \to \infty$.
(The natural scale factor is $N^{-1/2}$ so that the rescaled area of the Young diagram is 1.)

Limit shape



The analytical expression for the limit shape is most easily expressed through the local density ρ of particles, giving the local slope $(1-2\rho)$:

- uniform: $\rho(x) = \frac{1}{1+e^{\pi x/\sqrt{6}}}$ (Fermi-Dirac distribution) [Vershik 1996]
- Plancherel: $\rho(x) = \begin{cases} \frac{\arccos(x/2)}{\pi} & \text{if } x \in (-2,2), \\ 1 & \text{if } x \leqslant -2, \\ 0 & \text{if } x \geqslant 2. \end{cases}$

[Logan-Shepp, Vershik-Kerov 1977]

Beyond the limit shape

But, at a “microscopic” or “mesoscopic” level, uniform and Plancherel random partitions look quite different:

- bulk limit: locally around a point of density ρ, particles form a discrete point process

- edge limit: we look at the position of the rightmost particle(s), which corresponds to the largest part(s) λ_1 ($\lambda_2, \ldots$) of the partition.

Beyond the limit shape

But, at a "microscopic" or "mesoscopic" level, uniform and Plancherel random partitions look quite different:

- bulk limit: locally around a point of density ρ, particles form a discrete point process:
  - in the uniform case, particle occupation numbers are i.i.d. Bernoulli(ρ) ("the partition is locally a random walk") [Okounkov 2001]
  - in the Plancherel case, we observe a determinantal point process with the discrete sine kernel $K_{ds}(i,j) := \frac{\sin(\rho(i-j))}{\pi(i-j)}$ [Borodin-Okounkov-Olshanski 2000]
- edge limit: we look at the position of the rightmost particle(s), which corresponds to the largest part(s) λ_1 $(\lambda_2, \ldots)$ of the partition.

Edge limit/extreme value statistics

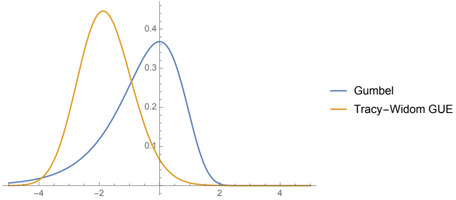
We consider λ_1, the first (largest) part of λ.

- Uniform case: $\lambda_1 = \sqrt{\frac{3N}{2\pi^2}} \ln N + XN^{1/2} + \cdots$
  with X a Gumbel-distributed random variable.

  [Erdős-Lehner 1941]

- Plancherel case: $\lambda_1 = 2\sqrt{N} + YN^{1/6} + \cdots$
  with Y following the Tracy-Widom GUE ($\beta = 2$) distribution.

  [Baik-Deift-Johansson 1999, Borodin-Okounkov-Olshanski 2000]



Canonical ensemble/Poissonization

It is easier to study ensembles of partitions where the size is allowed to fluctuate.

- "Uniform" (canonical) measure:

$$\mathrm{Prob}(\lambda) = \frac{u^{|\lambda|}}{Z(u)}, \qquad u \in (0,1).$$

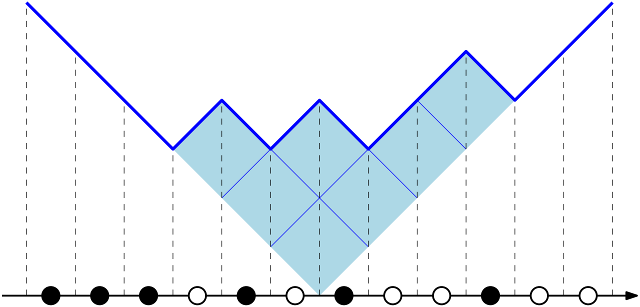
- Poissonized Plancherel measure:

$$\mathrm{Prob}(\lambda) = \left(\frac{\vartheta^{|\lambda|}\,\mathrm{dim}(\lambda)}{|\lambda|!}\right)^2 e^{-\vartheta^2}, \qquad \vartheta \in (0,\infty).$$

Large partitions are obtained by taking $u \to 1$ or $\vartheta \to \infty$. In both cases $|\lambda|$ concentrates around its expected value, so we have (or expect) "equivalence of ensembles".

Outline

Integer partitions and fermionic configurations



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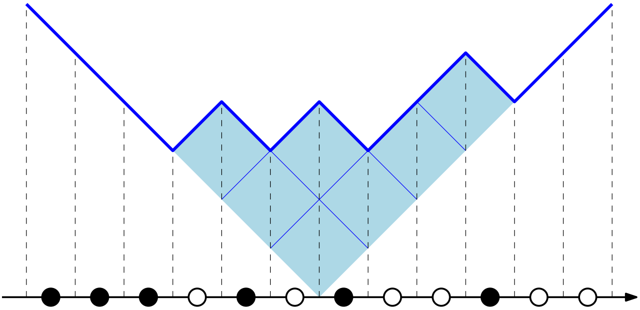

To a partition λ we associate the fermionic configuration

$$S(\lambda) = \left\{ \lambda_1 - \frac{1}{2}, \lambda_2 - \frac{3}{2}, \lambda_3 - \frac{5}{2}, \cdots \right\}$$

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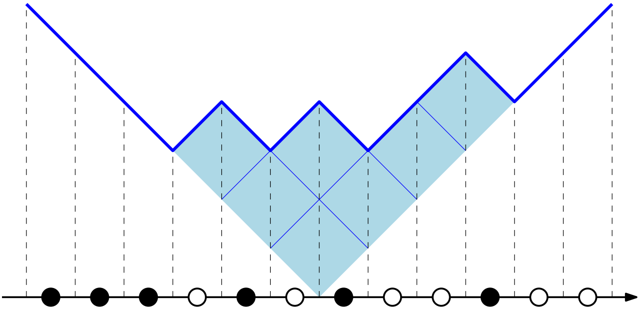

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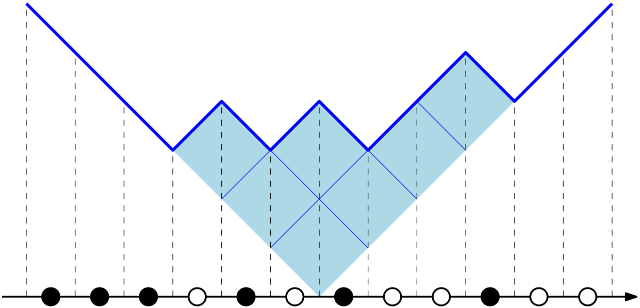
that gives the position of particles ($\bullet$), here $S(\lambda) = \{\frac{7}{2}, \frac{1}{2}, -\frac{3}{2}, \cdots\}$.

Fermionic configurations



A fermionic configuration is a subset of $\mathbb{Z}' := \mathbb{Z} + \frac{1}{2}$ which:

- has a maximal element,
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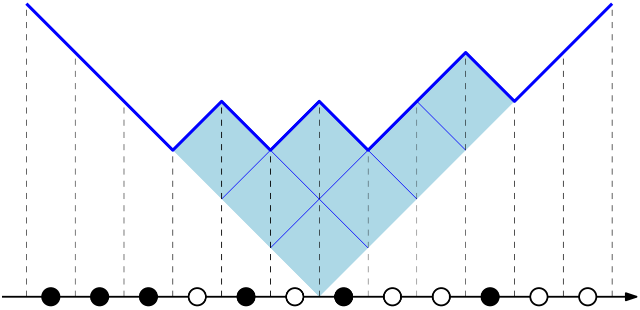

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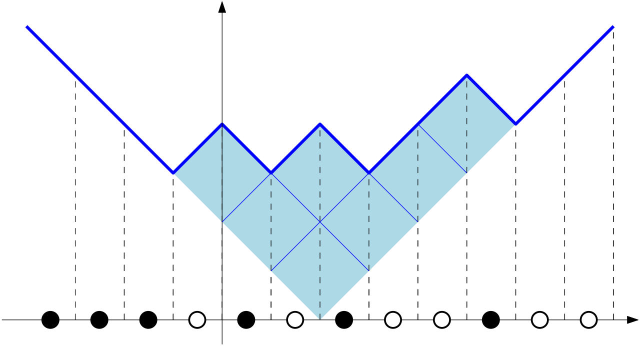
The mapping $\lambda \mapsto S(\lambda)$ is injective but not surjective because the charge of $S(\lambda)$ is zero.

Charged partitions



To fix this, we consider charged partitions, i.e. pairs (λ, c) with λ a partition and c an integer. To such pair we associate

$$S(\lambda) + c = \left\{ \lambda_1 - \frac{1}{2} + c, \lambda_2 - \frac{3}{2} + c, \lambda_3 - \frac{5}{2} + c, \cdots \right\}$$

and the mapping is now bijective.

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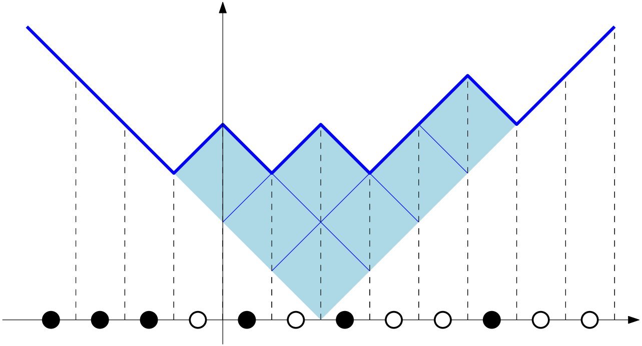

If $(\lambda, c) \mapsto S$, we have

$$|\lambda| + \frac{c^2}{2} = E(S)$$

where the energy $E(S)$ is defined by

$$E(S) := \sum_{\substack{s \in S \\ s > 0}} s - \sum_{\substack{s \notin S \\ s < 0}} s$$

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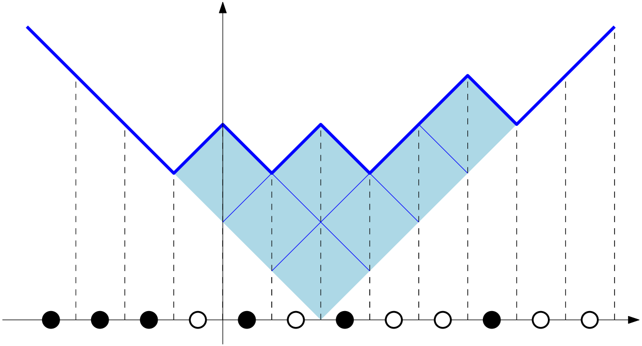

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Jacobi triple product identity

If $(\lambda, c) \mapsto S$, we have

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From this we deduce the identity

$$\sum_{(\lambda,c)} z^c q^{|\lambda| + \frac{c^2}{2}} = \prod_{\substack{s \in \mathbb{Z}' \\ s>0}} (1 + zq^s) \prod_{\substack{s \in \mathbb{Z}' \\ s<0}} (1 + z^{-1} q^{-s}).$$

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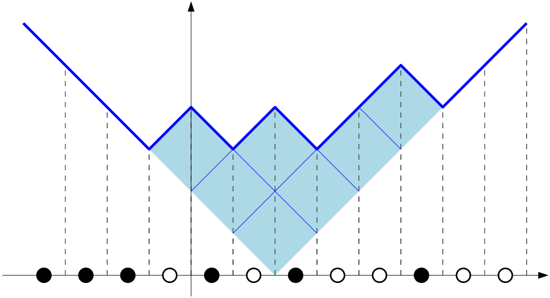
It yields the Jacobi triple product identity

$$\sum_{c \in \mathbb{Z}} z^c q^{\frac{c^2}{2}} = \prod_{n=1}^{\infty} (1 - q^n)(1 + zq^{n-\frac{1}{2}})(1 + z^{-1} q^{n-\frac{1}{2}}).$$

Fermi-Dirac statistics



The probabilistic meaning is the following: consider the probability distribution $\mathrm{Prob}(\lambda, c) = \frac{1}{Z} z^c q^{|\lambda| + \frac{c^2}{2}}$ over charged partitions with $0 \leqslant q < 1$ and $z > 0$.

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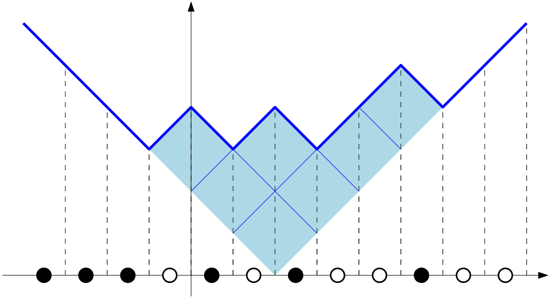

The probabilistic meaning is the following: consider the probability distribution $\mathrm{Prob}(\lambda, c) = \frac{1}{Z} z^c q^{|\lambda| + \frac{c^2}{2}}$ over charged partitions with $0 \leqslant q < 1$ and $z > 0$. Then, in the corresponding fermionic configuration S, there is a particle at position s with probability

$$\mathrm{Prob}(s \in S) = \frac{zq^s}{1 + zq^s}$$

independently of all the other positions.

Asymptotics

Now we let $q = e^{-r} \to 1$ and take $z = 1$ (without loss of generality).
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(Technical details: $cR^{-1/2}$ is asymptotically normal so the charge shift may be neglected, and $|\lambda|$ concentrates around $\frac{\pi^2}{6} R^2$...)

Outline

The cylindric Plancherel measure

The cylindric Plancherel measure (CPM) interpolates between the two measures:

$$\mathrm{Prob}(\lambda) \propto \sum_{\mu \subset \lambda} u^{|\mu|} \left(\frac{\vartheta^{|\lambda/\mu|} \dim(\lambda/\mu)}{|\lambda/\mu|!} \right)^2$$

$\dim(\lambda/\mu)$ is the number of Standard Young Tableaux of skew shape λ/μ.

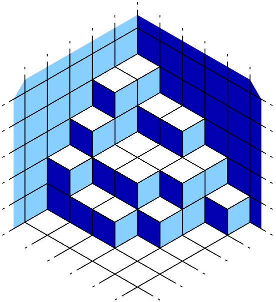
It reduces to uniform for $\vartheta = 0$, and to Plancherel for $u = 0$.

It is the simplest instance of a periodic Schur process. [Borodin 2007]

Schur processes

The Schur process was originally introduced to study plane partitions.

[Okounkov-Reshetikhin 2003]



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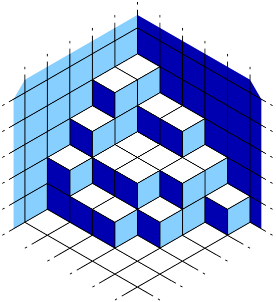
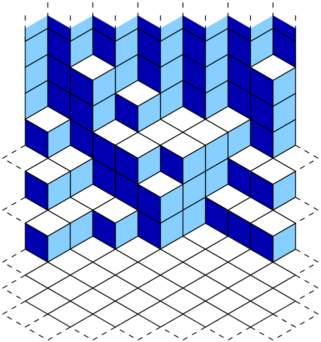


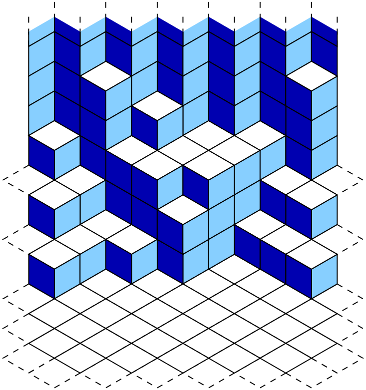


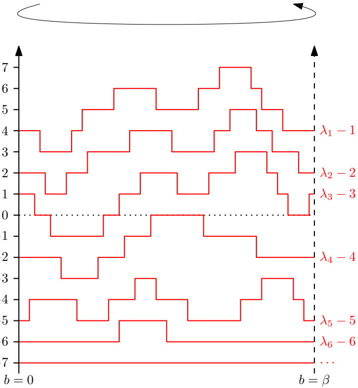
The periodic Schur process is its analogue for studying cylindric partitions.

[Borodin 2007]

The cylindric Plancherel process

By taking a certain "poissonian" limit of a measure on cylindric partitions, we obtain the cylindric Plancherel process (CPP) (to be defined later).





The CPM is the fixed-time marginal of the CPP.

Bulk limit of the CPM

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(We can check that for $\vartheta \to \infty$ and $u < 1$ fixed we obtain the same limits up to a scale factor as for $u = 0$.)

Towards the edge limit

We now analyze the edge behavior (which Borodin did not consider). Note that $0 < \rho(x) < 1$ for all x. For $x \to \infty$ we have

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with I_0 the modified Bessel function.
Naively we expect to find the rightmost particle where $\rho(x) \sim r$, i.e.

$$\lambda_1 \sim r^{-1} \ln \frac{I_0(2\vartheta)}{r}.$$

This turns out to be essentially correct even for $\vartheta \to \infty$, where we have

$$\lambda_1 \sim 2L, \qquad L := r^{-1}\vartheta.$$

But what is the order of magnitude of fluctuations?

Edge limit: thermal vs quantum fluctuations

Intuitively, we expect to have a competition between two types of fluctuations:

- thermal fluctuations of order r^{-1} (as in the uniform case $\vartheta = 0$),
- quantum fluctuations of order $L^{1/3}$ (as in the Plancherel case $u = 0$).

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So, in fact, there are three possible regimes:

- the high-temperature regime $r^{-1} \gg L^{1/3}$ i.e. $\vartheta \ll r^{-2}$: thermal fluctuations win, we expect Gumbel,
- the low-temperature regime $r^{-1} \ll L^{1/3}$ i.e. $\vartheta \gg r^{-2}$: quantum fluctuations win, we expect Tracy-Widom,
- the crossover regime $r^{-1} \propto L^{1/3}$ i.e. $\vartheta \propto r^{-2}$: we expect a new behaviour.

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Note that the edge crossover regime is not the same as that in the bulk! In the intermediate regime $1 \ll \vartheta \ll r^{-2}$ the bulk behaves as at zero temperature, and the edge as at infinite temperature.

Edge limit: main theorem

Theorem [Betea-B. 2018]

Consider the cylindric Plancherel measure with $u = e^{-r}$ and $L := \vartheta/(1-u)$. Then, the largest part λ_1 has the following limiting distributions:

- (High-temperature) For $r \to 0$ and $rL^{1/3} \to 0$, we have

$$\mathbb{P}\left(r\lambda_1 - \ln \frac{I_0(2Lr)}{r} \leqslant s \right) \to e^{-e^{-s}}, \qquad s \in \mathbb{R}$$

- (Crossover and low-temperature) For $L \to \infty$ and $rL^{1/3} \to \alpha \in (0, \infty]$, we have

$$\mathbb{P}\left(\frac{\lambda_1 - 2L}{L^{1/3}} \leqslant s \right) \to F_\alpha(s), \qquad s \in \mathbb{R}$$

  where F_α is Johansson's "interpolating" distribution (which reduces to the Tracy-Widom GUE distribution for $\alpha = \infty$).

Johansson's interpolating distribution

When studying the Moshe-Neuberger-Shapiro random matrix model, Johansson encountered the distribution

$$F_\alpha(s) := \det(I - M_\alpha)_{L^2(s,\infty)}$$

with M_α the "finite-temperature Airy kernel"

$$M_\alpha(x,y) = \int_{-\infty}^{\infty} \frac{e^{\alpha s}}{1+e^{\alpha s}} \operatorname{Ai}(x+s)\operatorname{Ai}(y+s) ds.$$

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It interpolates between the Gumbel and Tracy-Widom GUE distributions:

$$\lim_{\alpha\to\infty} F_\alpha(s) = F_{GUE}(s)$$

$$\lim_{\alpha\to 0} F_\alpha\left(\frac{s-\frac{1}{2}\ln(4\pi\alpha^3)}{\alpha}\right) = e^{-e^{-s}}.$$

[Johansson 2007]
see also [Dean-Le Doussal-Majumdar-Schehr 2015 and Liechty-Wang 2017]

Ideas of the proof

We use the fact that the grand canonical particle configuration is determinantal (i.e. we have free fermions). The correlation kernel for the CPM is the discrete finite-temperature Bessel kernel

$$K_{dftB}(a,b) = \sum_{\ell \in \mathbb{Z}+1/2} \frac{J_{a+\ell}(2L) J_{b+\ell}(2L)}{1+u^\ell}$$

with $J_n(\cdot)$ a Bessel function.
We need to show that, upon suitable rescalings, this kernel converges to:

- the finite-temperature Airy kernel $M_\alpha(x,y)$ in the crossover/low-temperature regime,
- the (degenerate) Poisson kernel $e^{-x}\delta_{x,y}$ in the high-temperature regime.

The theorem follows from standard arguments on Fredholm determinants.

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We may prove such convergences directly from the sum representation for K_{dftB}, or alternatively by analyzing the integral representation

$$K_{dftB}(a,b) = \frac{1}{(2i\pi)^2} \oiint\limits_{\substack{|z|=1^+ \\ |w|=1^-}} \frac{dzdw}{z^{a+1}w^{-b+1}} \frac{e^{L(z-z^{-1})}}{e^{L(w-w^{-1})}} \kappa(z,w)$$

where $\kappa(z,w)$ is the "fermionic finite-temperature propagator"

$$\kappa(z,w) = \langle \psi(z)\psi^*(w) \rangle_u = \sum_{\ell \in \mathbb{Z}+1/2} \frac{(z/w)^\ell}{1+u^{-\ell}}.$$

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This propagator is an elliptic function. For $u = e^{-r}$ and $z/w = e^\zeta$ we have by Poisson summation

$$\kappa(z,w) = \sum_{\ell \in \mathbb{Z}} (-1)^\ell \frac{\pi}{r \sin \pi \frac{\zeta - 2i\pi\ell}{r}}.$$

For $r \to 0$ and $|\arg(\zeta)| \leqslant \pi$ the term $\ell = 0$ is exponentially dominant.

Ideas of the proof

Thus we get

$$K_{dftB}(a,b) \simeq \frac{1}{(2i\pi)^2} \oiint_{\substack{|z|=1^+ \\ |w|=1^-}} \frac{dzdw}{z^{a+1}w^{-b+1}} \frac{e^{L(z-z^{-1})}}{e^{L(w-w^{-1})}} \frac{\pi}{r \sin \pi \frac{\zeta}{r}}.$$

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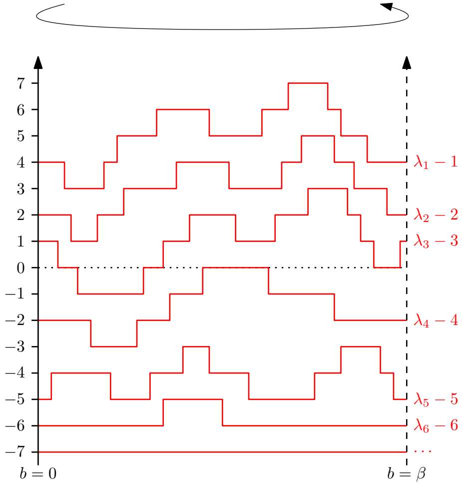
From there the analysis depends on the regime we consider:

- in the crossover/low-temperature regime ($r = O(L^{1/3})$), we perform a saddle-point analysis very similar to that for the usual poissonized Plancherel measure (i.e. zero temperature) [see e.g. Okounkov 2002],
- the high-temperature regime ($r \gg L^{1/3}$) requires a new analysis, we can see that the integral is dominated by the contribution of the pole at $z = w/u$ i.e. $\zeta = r$.

Outline

The cylindric Plancherel process



The cylindric Plancherel process

We now provide a definition for the CPP. For $L, t \geqslant 0$ and λ, μ two partitions, we define a transition kernel by

$$T_L(t)_{\lambda,\mu} := e^{L^2(e^{-t}-1)} \sum_{\nu} e^{-t|\nu|} \frac{(L(1-e^{-t}))^{|\lambda/\nu|+|\mu/\nu|} \dim(\lambda/\nu) \dim(\mu/\nu)}{|\lambda/\nu|!|\mu/\nu|!}.$$

It satisfies the semi-group property

$$T_L(t) T_L(t') = T_L(t+t')$$

and is in fact related to a Markov process introduced by Borodin and Olshanski, which leaves the poissonized Plancherel measure invariant.

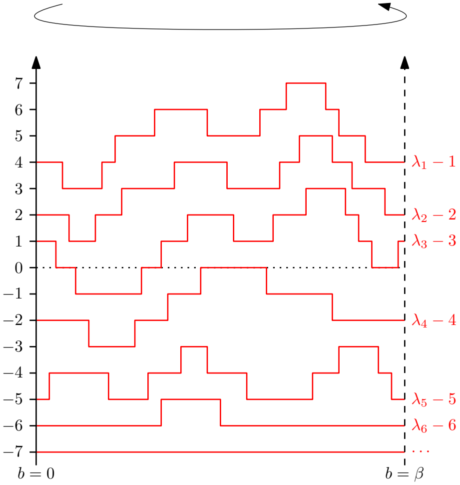
The cylindric Plancherel process

Fix an intensity L and a period β. We define the cylindric Plancherel process (CPP) as the partition-valued continuous-time β-periodic process $\lambda(\cdot)$ whose finite-dimensional marginals are given by

$$\mathrm{Prob}(\lambda(b_1), \lambda(b_2), \ldots, \lambda(b_n)) = \sum_{\lambda(0)} T_L(b_1)_{\lambda(0),\lambda(b_1)} T_L(b_2 - b_1)_{\lambda(b_1),\lambda(b_2)} \cdots T_L(\beta - b_n)_{\lambda(b_n),\lambda(0)}$$

where $0 \leqslant b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n \leqslant \beta$.
At any fixed time b, the law of $\lambda(b)$ is the cylindric Plancherel measure of parameters $u = e^{-\beta}$, $\vartheta = L(1 - e^{-\beta})$.
We believe the process admits a more natural PNG-type definition (work in progress).

The cylindric Plancherel process



Extended discrete finite-temperature Bessel kernel

The associated grand canonical particle configurations is still determinantal, and is described by the extended kernel

$$K_{edftB}(b,k;b',k') = \begin{cases} \sum_\ell J_{k+\ell}(2L) J_{k'+\ell}(2L) \frac{e^{(b-b')\ell}}{1+e^{\beta\ell}} & \text{if } b \leqslant b', \\ -\sum_\ell J_{k+\ell}(2L) J_{k'+\ell}(2L) \frac{e^{(b-b')\ell}}{1+e^{-\beta\ell}} & \text{if } b > b'. \end{cases}$$

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Theorem [BB18]

In the edge crossover or low-temperature regime $L \to \infty$, $\beta \to 0$, $L^{1/3}\beta \to \alpha \in (0,\infty]$, $b = \beta\tau/\alpha$, $k = \lfloor 2L + xL^{1/3} \rfloor$ (and similarly for b', k')

$$L^{1/3} K_{edftB}(b,k;b',k') \to \begin{cases} \int_{-\infty}^{\infty} \frac{e^{(\tau-\tau')v}}{1+e^{-\alpha v}} \operatorname{Ai}(x+v) \operatorname{Ai}(x'+v) dv & \text{if } \tau \leqslant \tau', \\ -\int_{-\infty}^{\infty} \frac{e^{(\tau-\tau')v}}{1+e^{\alpha v}} \operatorname{Ai}(x+v) \operatorname{Ai}(x'+v) dv & \text{if } \tau > \tau' \end{cases}$$

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Work in progress for the high-temperature regime.

Summary and conclusion

We have analyzed a measure on random partitions which interpolates between the uniform and Plancherel partitions.

We have a complete picture of the transition from one case to another in the thermodynamic limit. It is quite nontrivial as the transition takes place in different regimes for the bulk and the edge. (Is this universal?)

Summary and conclusion

We have analyzed a measure on random partitions which interpolates between the uniform and Plancherel partitions.

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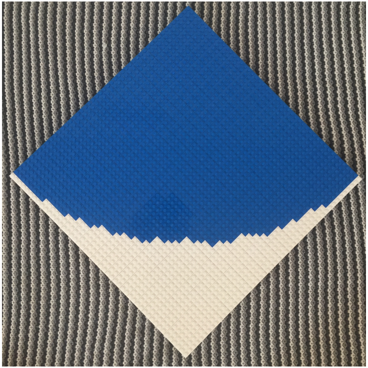
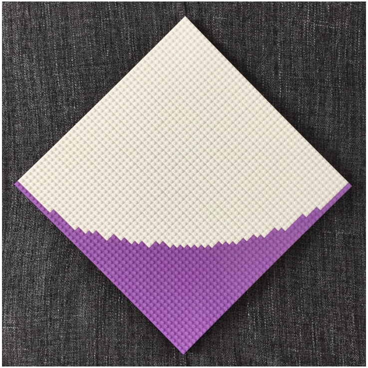
Open questions/future directions:

- finite-temperature analogues of other limiting kernels? (Pearcey, multicritical potentials...)
- applications to last-passage percolation and TASEP?
- connection with finite-time solutions of the KPZ equation?
- analysis of Johansson's interpolating distribution?

  connected with the "lower tail of the KPZ equation" [Corwin-Ghosal 2018]
- what about the free boundary Schur process?

  [Betea-B.-Nejjar-Vuletić 2018, 2019 + work in progress]

Thanks !

Questions ?