Operators of equivalent sorting power and related Wilf-equivalences

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joint work with
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Séminaire de combinatoire Philippe Flajolet
décembre 2013

Outline

1. Definitions and some history
   - Permutation patterns and partial sorting devices/algorithms
   - Permutation classes and Wilf-equivalences
2. Some operators with equivalent sorting power
   - How many permutations can we sort with the operators $\mathbf{S} \circ \alpha \circ \mathbf{S}$, where $\mathbf{S}$ is the *stack sorting operator* of Knuth, and α is any symmetry of the square?
3. Longer operators with equivalent sorting power
   - How many permutations can we sort with longer compositions of stack sorting and symmetries $\mathbf{S} \circ \alpha \circ \mathbf{S} \circ \beta \circ \mathbf{S} \circ \ldots$?
4. Related Wilf-equivalences
   - These are obtained from a (surprisingly little known) bijection between Av(231) and Av(132) which appears in our study.

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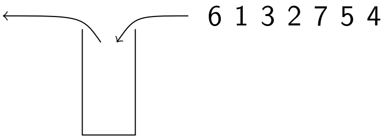

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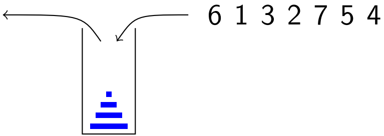

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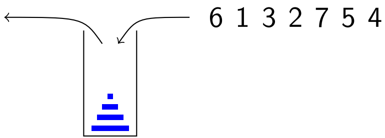

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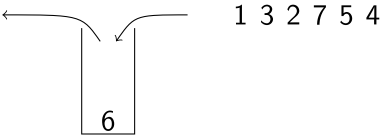

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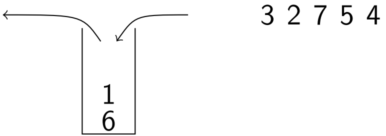

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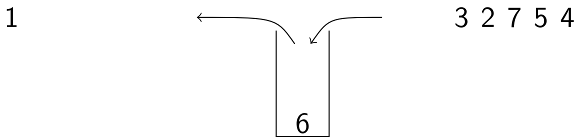

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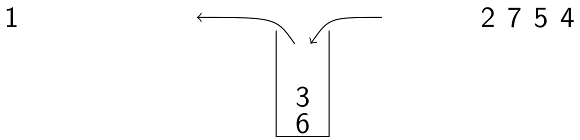

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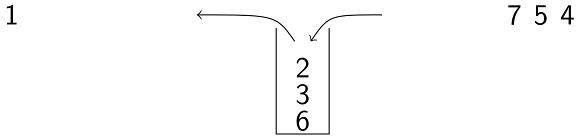

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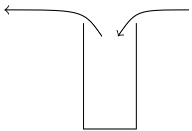

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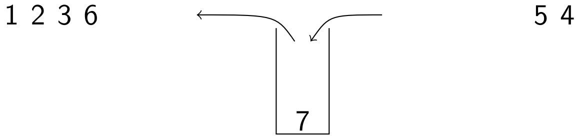

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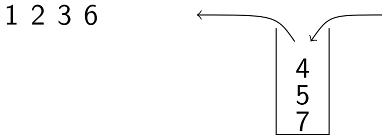

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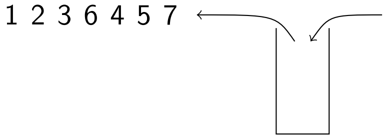

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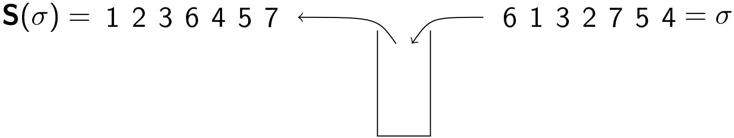

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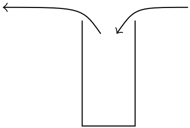

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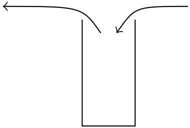
$\mathbf{S}(\sigma) = 1\ 2\ 3\ 6\ 4\ 5\ 7$  $6\ 1\ 3\ 2\ 7\ 5\ 4 = \sigma$

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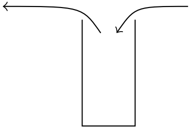
$\mathbf{S}(\sigma) = 1\ 2\ 3\ 6\ 4\ 5\ 7$  $6\ 1\ 3\ 2\ 7\ 5\ 4 = \sigma$

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iff there are no $i < j < k$ such that $\sigma_k < \sigma_i < \sigma_j$
iff σ avoids the pattern 231.

More sorting devices

- several stacks in series
- several stacks in parallel
- networks of stacks
- a single stack used several times
- queue(s)
- double-ended queue (= deque)
- pop-stacks
- ...

Pioneers in the seventies: Knuth, Pratt, Tarjan, ...

From the nineties until today:

Albert, Atkinson, Bousquet-Mélou, Claesson, Linton, Magnusson, Murphy, Pierrot, Rossin, Smith, Ulfarsson, Vatter, West, Zeilberger, ...

Patterns in permutations

- $\pi \in \mathfrak{S}_k$ is a **pattern** of $\sigma \in \mathfrak{S}_n$ when
  $\exists\ i_1 < \ldots < i_k$ such that $\sigma_{i_1} \ldots \sigma_{i_k}$ is order isomorphic ($\equiv$) to π.
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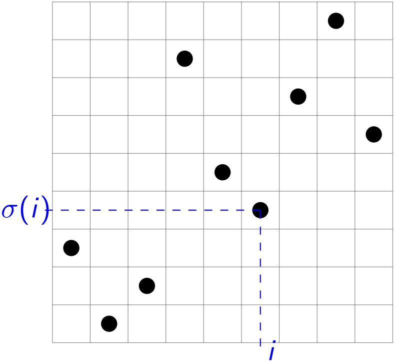

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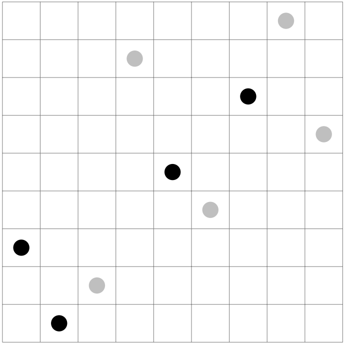

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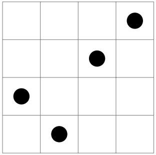

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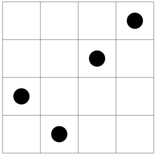

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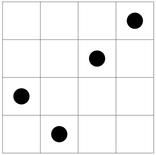

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Permutation classes are sets $\text{Av}(B)$ (with B finite or infinite).

Some early enumeration results about permutation classes

- Av(231) is enumerated by the Catalan numbers [Knuth ∼1970]
- Av(123) also is [MacMahon 1915]

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Marcus-Tardos theorem (2004) (Stanley-Wilf ex-conjecture):
For any π, there is a constant c_π such that
$\forall n$, the number of permutations of size n in Av(π) is $\leq c_\pi^n$

Wilf-equivalences

- $\{\pi, \pi', \ldots\}$ and $\{\tau, \tau', \ldots\}$ are Wilf-equivalent when Av(π, $\pi', \ldots$) and Av($\tau, \tau', \ldots$) are enumerated by the same sequence.

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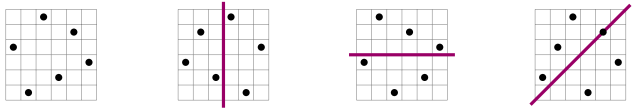
Actually, the six permutations of size 3 are all Wilf-equivalent.

Why? For every symmetry of the square $\alpha \in D_8$, $\pi \sim_{Wilf} \alpha(\pi)$.

D_8:



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Examples of non-trivial Wilf-equivalences:

- $1342 \sim_{Wilf} 2413$ and $1234 \sim_{Wilf} 1243 \sim_{Wilf} 1432 \sim_{Wilf} 2143$
- $12\ldots m \oplus \beta \sim_{Wilf} m \ldots 21 \oplus \beta$
- $\{123, 132\} \sim_{Wilf} \{132, 312\} \sim_{Wilf} \{231, 312\}$
- $\{132, 4312\} \sim_{Wilf} \{132, 4231\}$

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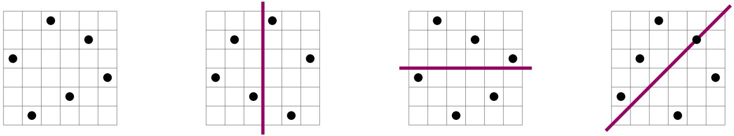

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These symmetries generate an 8-element group:

$$D_8 = \{\mathbf{id}, \mathbf{R}, \mathbf{C}, \mathbf{I}, \mathbf{R} \circ \mathbf{C}, \mathbf{I} \circ \mathbf{R}, \mathbf{I} \circ \mathbf{C}, \mathbf{I} \circ \mathbf{C} \circ \mathbf{R}\}$$

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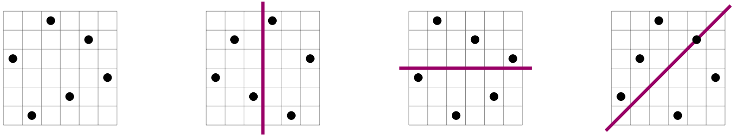

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Questions of [Claesson, Dukes, Steingrimsson]:
What are the permutations sortable by $\mathbf{S} \circ \alpha \circ \mathbf{S}$ for $\alpha \in D_8$?
And how many of each size n are there? [B., Guibert 2012]

The eight symmetries of D_8 can be paired

- The permutations that are sortable by $\mathbf{S} \circ \alpha \circ \mathbf{S}$ and those sortable by $\mathbf{S} \circ \beta \circ \mathbf{S}$ are the same, for the following pairs (α, β):

$$(\mathbf{id}, \mathbf{I} \circ \mathbf{C} \circ \mathbf{R}) \qquad (\mathbf{C}, \mathbf{I} \circ \mathbf{R}) \qquad (\mathbf{R}, \mathbf{I} \circ \mathbf{C}) \qquad (\mathbf{I}, \mathbf{R} \circ \mathbf{C}).$$

Such operators sort exactly the same sets of permutations.

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• Some operators do not sort the same sets of permutations, but still the same number of permutations of any size.

We say that they have equivalent sorting power.

Enumeration of permutations sortable by $\mathbf{S} \circ \alpha \circ \mathbf{S}$

$\alpha = \mathbf{id}$	$\frac{2(3n)!}{(n+1)!(2n+1)!}$	[West, Zeilberger]
$\alpha = \mathbf{R}$	$\frac{2(3n)!}{(n+1)!(2n+1)!}$	[B., Guibert]
$\alpha = \mathbf{C}$	Catalan numbers	[B., Guibert]
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Both bijections are *via* common generating trees in which it is possible to plug many permutation statistics

Why don't we try more stacks and symmetries?

Theorem (B., Guibert)

There are as many permutations of size n sortable by $\mathbf{S} \circ \mathbf{S}$ as permutations of size n sortable by $\mathbf{S} \circ \mathbf{R} \circ \mathbf{S}$. Moreover, many permutation statistics are equidistributed across these two sets.

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After some computer experiments, counting permutations sortable by $\mathbf{S} \circ \alpha \circ \mathbf{S} \circ \beta \circ \mathbf{S}$, $\quad \mathbf{S} \circ \alpha \circ \mathbf{S} \circ \beta \circ \mathbf{S} \circ \gamma \circ \mathbf{S}$, ...
Olivier Guibert formulated a conjecture:

Conjecture

For any operator $\mathbf{A}$ which is a composition of operators $\mathbf{S}$ and $\mathbf{R}$, there are as many permutations of size n sortable by $\mathbf{S} \circ \mathbf{A}$ as permutations of size n sortable by $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$.

$\mathbf{S} \circ \mathbf{A}$ and $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$ have equivalent sorting power

Theorem (B., Albert)

For any operator $\mathbf{A}$ *which is a composition of operators* $\mathbf{S}$ *and* $\mathbf{R}$*, there are as many permutations of size* n *sortable by* $\mathbf{S} \circ \mathbf{A}$ *as permutations of size* n *sortable by* $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$*.*

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Main ingredients for the proof:

- the characterization of preimages of permutations by **S**;
  [Bousquet-Mélou, 2000]
- the little known bijection P between Av(231) and Av(132).
  [Dokos, Dwyer, Johnson, Sagan, Selsor, 2012]

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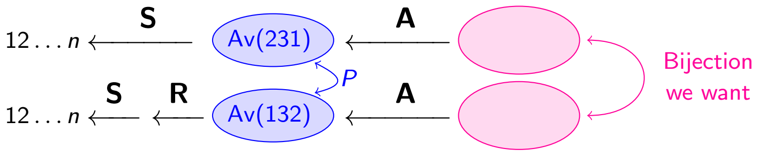
But... How does the theorem relate to these ingredients?

An equivalent statement



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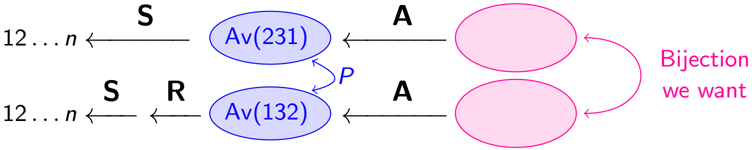

Theorem

For any operator **A** *which is a composition of operators* **S** *and* **R**, *P is a size-preserving bijection between*

- *permutations of* Av(231) *that belong to the image of* **A**, *and*
- *permutations of* Av(132) *that belong to the image of* **A**,

that preserves the number of preimages under **A**.

A simple remark about stack sorting and trees

The stack sorting of θ is equivalent to the post-order reading of the in-order tree $\mathsf{T}_{\mathsf{in}}(\theta)$ of θ: $\mathbf{S}(\theta) = \mathbf{Post}(\mathsf{T}_{\mathsf{in}}(\theta))$

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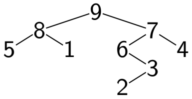
$T_{in}(\theta) =$ 

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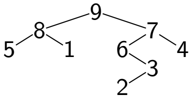
$\mathsf{T}_{\text{in}}(\theta) =$ 

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Proof: $\mathbf{S}$ and $\mathbf{Post} \circ \mathsf{T}_{\text{in}}$ are defined by the same recursive equation: $\mathbf{S}(LnR) = \mathbf{S}(L)\mathbf{S}(R)n$.

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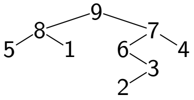
$\mathsf{T}_{\mathsf{in}}(\theta) =$ 

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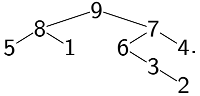
Consequence:
For π in the image of $\mathbf{S}$, $\theta \in \mathbf{S}^{-1}(\pi)$ iff $\mathbf{Post}(\mathsf{T}_{\mathsf{in}}(\theta)) = \pi$.
Preimages of π correspond to in-order trees T s.t. $\mathbf{Post}(T) = \pi$.

A canonical representative $\mathbf{S}^{-1}(\pi)$

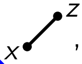
Lemma (Bousquet-Mélou, 2000)

For any permutation π in the image of $\mathbf{S}$, there is a unique canonical tree $\mathcal{T}_\pi$ whose post-order reading is π.

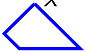
Example: For $\pi = 5\,1\,8\,2\,3\,6\,4\,7\,9$,

$\mathcal{T}_\pi =$ 

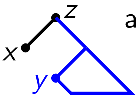
Canonical tree:

For every edge 

,

there exists  $\neq \emptyset$

and y such that



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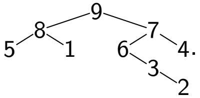
$\mathcal{T}_\pi =$ 

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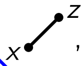
For every edge 

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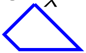
there exists  $\neq \emptyset$

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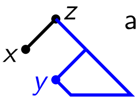

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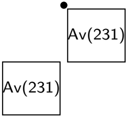
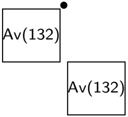
Theorem (Bousquet-Mélou, 2000)

$\mathcal{T}_\pi$ determines $\mathbf{S}^{-1}(\pi)$.
Moreover $|\mathbf{S}^{-1}(\pi)|$ is determined only by the shape of $\mathcal{T}_\pi$.

Bijection Av(231) $\xleftrightarrow{P}$ Av(132)

Representing permutations as diagrams, we have

Av(231) $= \varepsilon +$ 

and Av(132) $= \varepsilon +$ 

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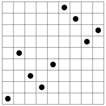
Example: $\pi =$ 

1 5 3 2 4 9 8 6 7

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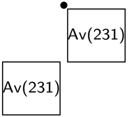
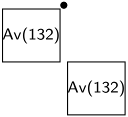
$\mathrm{Av}(231) = \varepsilon +$ 

and $\mathrm{Av}(132) = \varepsilon +$ 

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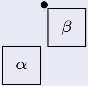
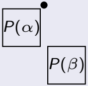
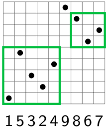
Definition

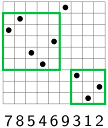
We define $P : \mathrm{Av}(231) \to \mathrm{Av}(132)$ recursively as follows:



$\xrightarrow{P}$


, with $\alpha, \beta \in \mathrm{Av}(231)$

Example: For $\pi =$ 

, we obtain $P(\pi) =$ 

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Bijection $\Phi_{\mathbf{A}}$ between $\mathbf{S} \circ \mathbf{A}$- and $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$-sortables

For $\pi \in \mathrm{Av}(231)$, write $P(\pi) \in \mathrm{Av}(132)$ as $P(\pi) = \lambda_\pi \circ \pi$.

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$$12\dots n \xleftarrow{\mathbf{S}} \overbrace{\pi \xleftarrow{\mathbf{S}} \tau \xleftarrow{\mathbf{S} \text{ or } \mathbf{R}} \gamma \xleftarrow{\mathbf{S}} \cdots \quad \rho \xleftarrow{\mathbf{S}} \theta}^{\mathbf{A}}$$

$$12\dots n \xleftarrow{\mathbf{S}} \xleftarrow{\mathbf{R}} \lambda_\pi \circ \pi = P(\pi)$$

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$$= P(\pi)$$

$$\lambda_\pi(\mathcal{T}_\pi) = \mathcal{T}_{\lambda_\pi \circ \pi}$$

Bijection $\Phi_{\mathbf{A}}$ between $\mathbf{S} \circ \mathbf{A}$- and $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$-sortables

For $\pi \in Av(231)$, write $P(\pi) \in Av(132)$ as $P(\pi) = \lambda_\pi \circ \pi$.

For θ sortable by $\mathbf{S} \circ \mathbf{A}$, set $\pi = \mathbf{A}(\theta)$.
Because $\pi \in Av(231)$, we may define $\Phi_{\mathbf{A}}(\theta) = \lambda_\pi \circ \theta$.

Theorem

$\Phi_{\mathbf{A}}$ is a size-preserving bijection between permutation sortable by $\mathbf{S} \circ \mathbf{A}$ and those sortable by $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$.

$$
\begin{tikzcd}
12\dots n & \pi \arrow[l, "\mathbf{S}"'] & \tau \arrow[l, "\mathbf{S}"'] & \gamma \arrow[l, "\mathbf{S}\text{ or }\mathbf{R}"'] & \cdots \arrow[l, "\mathbf{S}"'] & \rho & \theta \arrow[l, "\mathbf{S}"'] \arrow[lllll, bend right, "\mathbf{A}"']
\end{tikzcd}
$$

$$
12\dots n \xleftarrow{\mathbf{S}} \xleftarrow{\mathbf{R}} \lambda_\pi \circ \pi \xleftarrow{\mathbf{S}} \lambda_\pi \circ \tau \xleftarrow{\mathbf{S}\text{ or }\mathbf{R}} \lambda_\pi \circ \gamma \xleftarrow{\mathbf{S}} \cdots \quad \lambda_\pi \circ \rho \xleftarrow{\mathbf{S}} \lambda_\pi \circ \theta
$$

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$$\begin{array}{ccccccccccc}
 & & & & & & & & & \overset{\mathbf{A}}{\longleftarrow\!\!\!-\!\!\!-\!\!\!-} & \\
12\dots n & \xleftarrow{\mathbf{S}} & \pi & \xleftarrow{\mathbf{S}} & \tau & \xleftarrow{\mathbf{S} \text{ or } \mathbf{R}} & \gamma & \xleftarrow{\mathbf{S}} & \cdots \quad \rho & \xleftarrow{\mathbf{S}} & \theta \\
12\dots n & \xleftarrow{\mathbf{S}}\xleftarrow{\mathbf{R}} & \lambda_\pi \circ \pi & \xleftarrow{\mathbf{S}} & \lambda_\pi \circ \tau & \xleftarrow{\mathbf{S} \text{ or } \mathbf{R}} & \lambda_\pi \circ \gamma & \xleftarrow{\mathbf{S}} & \cdots \quad \lambda_\pi \circ \rho & \xleftarrow{\mathbf{S}} & \lambda_\pi \circ \theta \\
 & & = P(\pi) & & & & & & & & = \Phi_{\mathbf{A}}(\theta) \\
 & & \lambda_\pi(\mathcal{T}_\pi) = \mathcal{T}_{\lambda_\pi \circ \pi} & & \lambda_\pi(\mathcal{T}_\tau) = \mathcal{T}_{\lambda_\pi \circ \tau} & & & & \lambda_\pi(\mathcal{T}_\rho) = \mathcal{T}_{\lambda_\pi \circ \rho} & &
\end{array}$$

Who is Φ_S?

- Φ_S provides a bijection between the set of permutations sortable by $\mathbf{S} \circ \mathbf{S}$ and those sortable by $\mathbf{S} \circ \mathbf{R} \circ \mathbf{S}$.
- With O. Guibert, we gave a common generating tree for those two sets, providing a bijection between them.

Question

Are these two bijections the same one?

P and Wilf-equivalences

$\{\pi, \pi', \ldots\}$ and $\{\tau, \tau', \ldots\}$ are Wilf-equivalent when Av$(\pi, \pi', \ldots)$ and Av$(\tau, \tau', \ldots)$ are enumerated by the same sequence.

P and Wilf-equivalences

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Theorem

Description of the patterns $\pi \in \mathrm{Av}(231)$ *such that* P *provides a bijection between* $\mathrm{Av}(231, \pi)$ *and* $\mathrm{Av}(132, P(\pi))$

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Theorem

Computation of the generating function of such classes $\mathrm{Av}(231, \pi)$ *... and it depends only on* $|\pi|$*.*

$\Rightarrow$ Even more Wilf-equivalences!

The families of patterns (λ_n) and (ρ_n)

Sum:

$$\alpha \oplus \beta = \alpha\,(\beta + a) = \begin{array}{cc} & \boxed{\beta} \\ \boxed{\alpha} & \end{array}$$

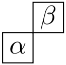
Skew sum:

$$\alpha \ominus \beta = (\alpha + b)\,\beta = \begin{array}{cc} \boxed{\alpha} & \\ & \boxed{\beta} \end{array}$$

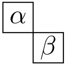
where α and β are permutations of size a and b, respectively

The families of patterns (λ_n) and (ρ_n)

Sum:

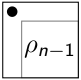
$\alpha \oplus \beta = \alpha\,(\beta + a) =$ 

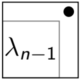
Skew sum:

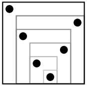
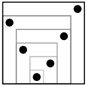
$\alpha \ominus \beta = (\alpha + b)\,\beta =$ 

where α and β are permutations of size a and b, respectively

- $\lambda_0 = \rho_0 = \varepsilon$ (or $\lambda_1 = \rho_1 = 1$)
- $\lambda_n = 1 \ominus \rho_{n-1}$
- $\rho_n = \lambda_{n-1} \oplus 1$

$\lambda_n =$ 

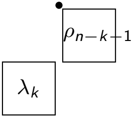
, $\rho_n =$ 

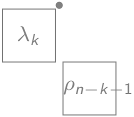
; $\lambda_6 =$ 
, $\rho_6 =$ 

Patterns π such that $\mathrm{Av}(231, \pi) \overset{P}{\longleftrightarrow} \mathrm{Av}(132, P(\pi))$

Theorem

A pattern $\pi \in \mathrm{Av}(231)$ *is such that* P *provides a bijection between* $\mathrm{Av}(231, \pi)$ *and* $\mathrm{Av}(132, P(\pi))$ *if and only if* $\pi = \lambda_k \oplus (1 \ominus \rho_{n-k-1})$.

$\pi =$ 

hence $P(\pi) =$ 

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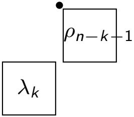
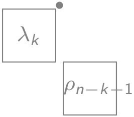
$\pi =$ 

hence $P(\pi) =$ 

Consequence: For all $\pi = \lambda_k \oplus (1 \ominus \rho_{n-k-1})$, $\{231, \pi\}$ and $\{132, P(\pi)\}$ are Wilf-equivalent.

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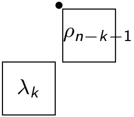
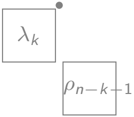
$\pi =$ 

hence $P(\pi) =$ 

Consequence: For all $\pi = \lambda_k \oplus (1 \ominus \rho_{n-k-1})$, $\{231, \pi\}$ and $\{132, P(\pi)\}$ are Wilf-equivalent.

Example: $\lambda_3 \oplus (1 \ominus \rho_1) = 31254 \in \mathrm{Av}(231)$ and $P(31254) = 42351$

- $\Rightarrow$ P is a bijection between $\mathrm{Av}(231, 31254)$ and $\mathrm{Av}(132, 42351)$
- $\Rightarrow$ $\{231, 31254\}$ and $\{132, 42351\}$ are Wilf-equivalent

Known Wilf-equivalences that we recover (or not)

☺ We recover

- for $\pi = 312$, $\{231, 312\} \sim_{Wilf} \{132, 312\}$,
- for $\pi = 3124$, $\{231, 3124\} \sim_{Wilf} \{132, 3124\}$,
- for $\pi = 1423$, $\{231, 1423\} \sim_{Wilf} \{132, 3412\}$,

which are (up to symmetry) referenced in Wikipedia.

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With $|\pi| = 3$ or 4, there are five more non-trivial Wilf-equivalence of the form $\{231, \pi\} \sim_{Wilf} \{132, \pi'\}$ (up to symmetry).
☹ We do not recover them.

More Wilf-equivalences that we obtain

Patterns π such that $\{231, \pi\} \sim_{Wilf} \{132, P(\pi)\}$ and $\text{Av}(231, \pi) \overset{P}{\longleftrightarrow} \text{Av}(132, P(\pi))$ i.e. $\pi = \lambda_k \oplus (1 \ominus \rho_{n-k-1})$:

π	$P(\pi)$
42135	42135
21534	43512
53124	53124
31254	42351
15324	45213

π	$P(\pi)$
216435	546213
531246	531246
312645	534612
642135	642135
421365	532461
164235	563124

π	$P(\pi)$
6421357	6421357
3127546	6457213
7531246	7531246
4213756	6435712
1753246	6742135
5312476	6423571
2175346	6573124

π	$P(\pi)$
31286457	75683124
75312468	75312468
64213587	75324681
53124867	75346812
86421357	86421357
21864357	76842135
42138657	75468213
18642357	78531246

Except two they are non-trivial.

More Wilf-equivalences that we obtain

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Except two they are non-trivial.

But because of symmetries, there are some redundancies.

Common generating function when $\mathrm{Av}(231, \pi) \overset{P}{\longleftrightarrow} \mathrm{Av}(132, P(\pi))$

Definition: $F_1(t) = 1$ and $F_{n+1}(t) = \frac{1}{1-tF_n(t)}$.

Theorem

For $\pi \in \mathrm{Av}(231)$ such that $\mathrm{Av}(231, \pi) \overset{P}{\longleftrightarrow} \mathrm{Av}(132, P(\pi))$, denoting $n = |\pi|$, the generating function of $\mathrm{Av}(231, \pi)$ is F_n.

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$$F_5(t) = \frac{t^2 - 3t + 1}{3t^2 - 4t + 1}.$$

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Example: The common generating function of $\mathrm{Av}(231, 31254)$ and $\mathrm{Av}(132, 42351)$ is

$$F_5(t) = \frac{t^2 - 3t + 1}{3t^2 - 4t + 1}.$$

F_5 is also the generating function of $\mathrm{Av}(231,\pi)$ for $\pi = 53124$ or 15324 or 21534 or 42135.

Many Wilf-equivalent classes

Theorem

$\{231, \pi\}$ *and* $\{132, P(\pi)\}$ *are* *all* *Wilf-equivalent when* $|\pi| = |\pi'| = n$ *and* π *and* π' *are of the form* $\lambda_k \oplus (1 \ominus \rho_{n-k-1})$. *Moreover, their generating function is* F_n.

Many Wilf-equivalent classes ... and even more?

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In future: For classes recursively described (like Av(231) and Av(132)), define recursive bijections (like P), to find or explain more Wilf-equivalences.

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Merci !