Inhomogeneous Multispecies TASEP on a ring

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The Asymmetric Simple exclusion Process

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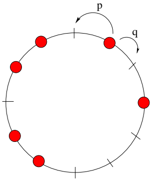

L SITES

N PARTICLES

$$\Omega = \binom{L}{N}$$

CONFIGURATIONS

Since its introduction in the '60 as a biophysical model for protein synthesis of RNA, ASEP has found several very different applications as a (toy) model for traffic flow, formation of shocks, etc... It is fair to say that it plays a fundamental role in our understanding of non-equilibrium processes.

Many approaches has been developed and exact results derived since its introduction and its rich combinatorial structure has been explored.

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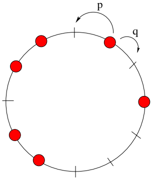

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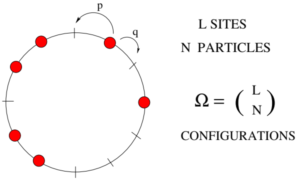

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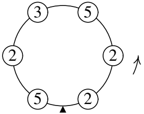
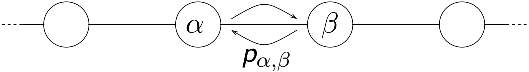
Multispecies ASEP general framework

We consider a periodic lattice $\mathbb{Z}/L\mathbb{Z}$ on which we have for $1 \leq \alpha \leq N$, m_α particles of species α, $\sum_{\alpha=1}^{N} m_\alpha = L$



The rates $p_{\alpha,\beta}$ for a local exchange $\alpha \leftrightarrow \beta$ depends on the species involved.



The case that we are interested in is

$$p_{\alpha,\beta} = \begin{cases} 0 & \text{for} \quad \alpha \geq \beta \\ \tau_\alpha + \nu_\beta & \text{for} \quad \alpha < \beta \end{cases}$$

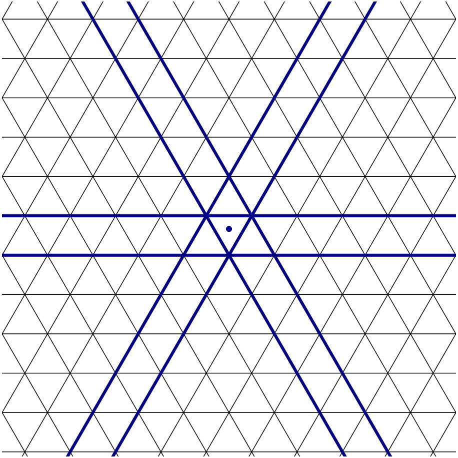
We'll see later where this choice comes from.

Random walk in an affine Weyl group [Lam]

The homogeneous TASEP process ($\nu_\beta = 0$ and $\tau_\alpha = \tau$) has appeared recently in a work by Lam.

He was interested in certain infinite random reduced words in affine Weyl groups that can be defined as random walks on the affine Coxeter arrangement, conditioned never to cross the same hyperplane twice.

Random walk in an affine Weyl group [Lam]



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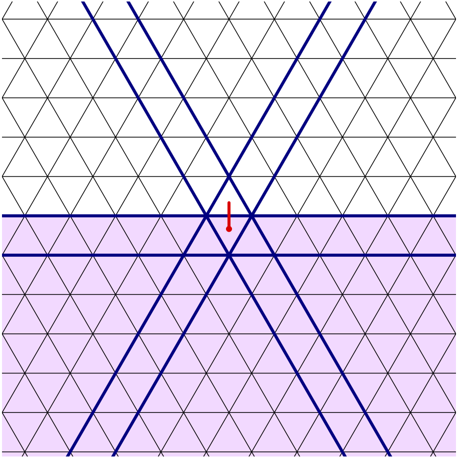

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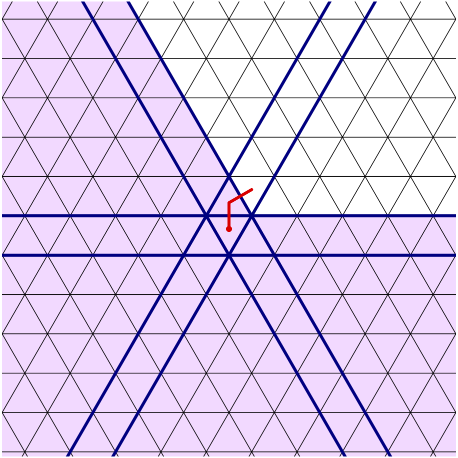

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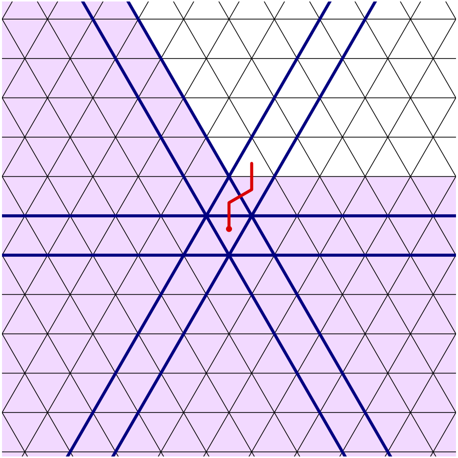

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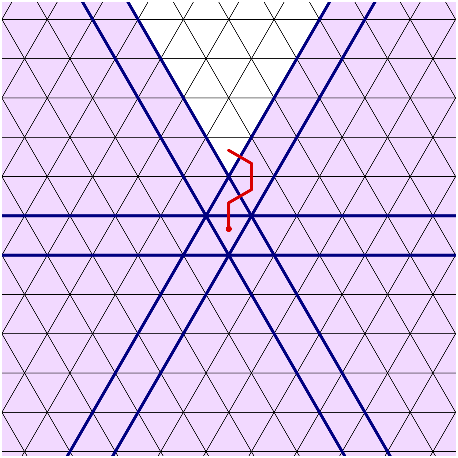

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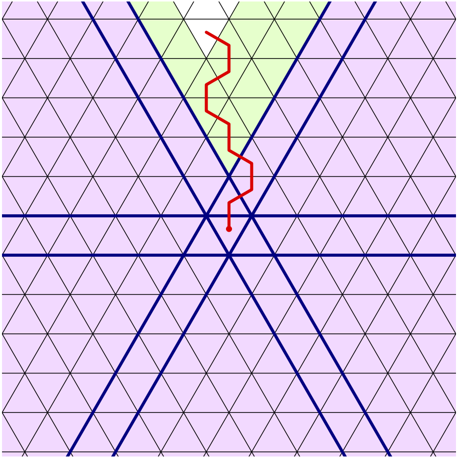

Random walk in an affine Weyl group [Lam]

Among other remarkable results, he proved that the walk almost surely gets stuck in a Weyl chamber and that the walk will almost surely tend to a certain direction in that chamber.

For the case of $\tilde{A}_n$ the probability of getting stuck in the Weyl chamber C_σ is

$$P(C_\sigma) = P_{\sigma^{-1}\sigma_0}$$

where P_σ is the stationary probability of being in the state $\{\sigma(1), \dots, \sigma(N)\}$ for the homogeneous TASEP with N species on a $\mathbb{Z}/N\mathbb{Z}$

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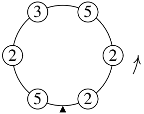
Multispecies TASEP on a ring

We consider the M-TASEP on a ring $\mathbb{Z}/L\mathbb{Z}$. A state of this system is just a periodic word w of length $L(w) = L$, $w_i = w_{i+L}$.

$$w = \{2, 2, 5, 3, 2, 5\}$$



The dynamics conserves the total number of particles of a given species. We denote the species content of a word w by

$$\mathbf{m}(w) = \{\dots, m_\alpha(w), m_{\alpha+1}(w), \dots\} \in \mathbb{N}^{\mathbb{Z}}$$

which means that we have $m_\alpha(w)$ particles of species α

$$\sum_{\alpha=\mathbb{Z}} m_\alpha(w) = L(w)$$

$$\mathbf{m}(w) = \{m_{<2} = 0, m_2 = 3, m_3 = 1, m_4 = 0, m_5 = 2, m_{>5} = 0\}, \quad L = 6$$

Positivity conjectures [Lam-Williams, L.C]

For a given (periodic) word w we define the **descents number**

$$d(w) = \#\{1 \leq i \leq L | w_i > w_{i+1}\}$$

Normalizing the "probability" of a state w^* with the minimal descent number ($d(w^*) = 1$) as

$$\psi_{w^*} = \chi_{\mathbf{m}}(\tau, \nu) := \prod_{\alpha<\beta} (\tau_\alpha + \nu_\beta)^{(\beta-\alpha-1)(m_\alpha+m_\beta-1)}$$

Positivity Conjecture

The polynomials $\psi_w(\tau, \nu)$ have positive integer coefficients

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Double Schubert polynomials [Lascoux-Schützenberger]

Let $\mathbf{t} = t_1, t_2, \dots$ and $\mathbf{v} = v_1, v_2 \dots$ two infinite sets of commuting variables

Definition: double Schubert polynomials

For the longest permutation $w_0 \in S_n$

$$\mathfrak{S}_{w_0}(\mathbf{t}, \mathbf{v}) := \prod_{i+j \leq n} (t_i - v_j)$$

for generic $w \in S_n$

$$\mathfrak{S}_w(\mathbf{t}, \mathbf{v}) = \partial_{w^{-1}w_0} \mathfrak{S}_{w_0}(\mathbf{t}, \mathbf{v})$$

where $\partial_{w^{-1}w_0} = \partial_{s_{i_1}} \partial_{s_{i_2}} \dots \partial_{s_{i_\ell}}$, ($s_{i_1} \cdot s_{i_2} \cdots s_{i_\ell}$ is a reduced decomposition of $w^{-1}w_0$) and

$$\partial_{s_{i_1}} = \frac{1 - s_i^{\mathbf{t}}}{t_i - t_{i+1}}, \qquad s_i^{\mathbf{t}} : t_i \leftrightarrow t_{i+1}.$$

Conjecture

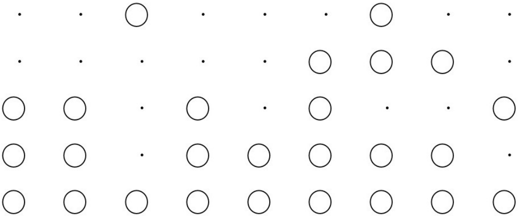
- The functions $\psi_w(\tau,\nu)$ can be expressed as polynomials of double Schubert polynomials with positive integer coefficients.
- The double Schubert polynomials appearing in the expression of $\psi_w(\tau,\nu)$ correspond to permutations in $S_{L(w)}$ and the variables $\mathbf{t},\mathbf{v}$ are choosen as

$$\mathbf{t} = \overbrace{\tau_{\min(\mathbf{m})},\dots,\tau_{\min(\mathbf{m})}}^{m_{\min(\mathbf{m})}},\overbrace{\tau_{\min(\mathbf{m})+1},\dots,\tau_{\min(\mathbf{m})+1}}^{m_{\min(\mathbf{m})+1}},\dots$$

$$\mathbf{v} = \underbrace{-\nu_{\max(\mathbf{m})},\dots,-\nu_{\max(\mathbf{m})}}_{m_{\max(\mathbf{m})}},\underbrace{-\nu_{\max(\mathbf{m})-1},\dots,-\nu_{\max(\mathbf{m})-1}}_{m_{\max(\mathbf{m})-1}},\dots$$

The case $\nu_\alpha = 0$ and multiline queues [Ayyer-Linusson, Arita-Mallick]

The positivity conjecture has been settled by Arita and Mallick in the case $\nu_\alpha = 0$ in terms of *multiline queus* as conjectured by Ayyer and Linusson.
Multiline queues have been introduced by Ferrari and collaborators.
A multiline queue of type **m** is a $\mathbb{Z} \times L$ array ($L = \sum m_i$), which has $\sum_{j \leq i} m_j$ particles on the i-th row.



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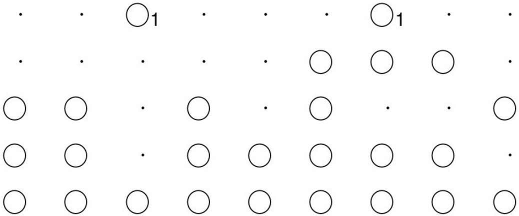

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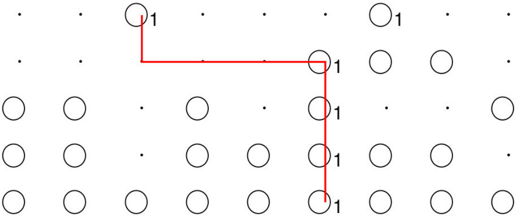

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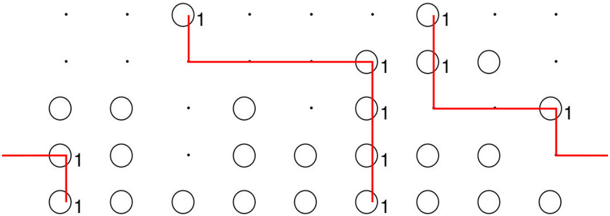

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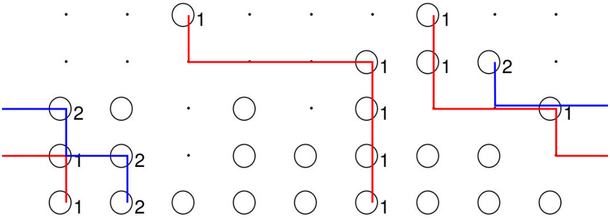

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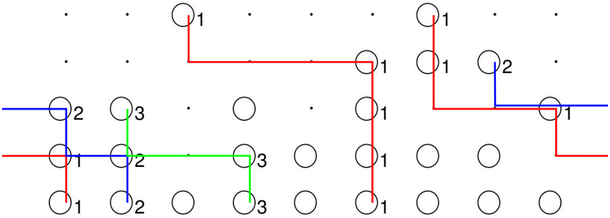

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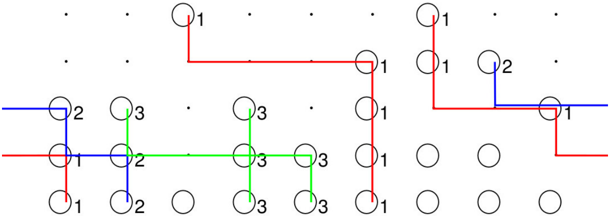

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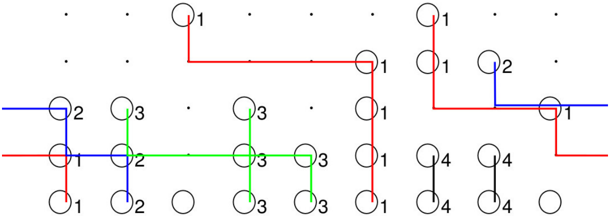

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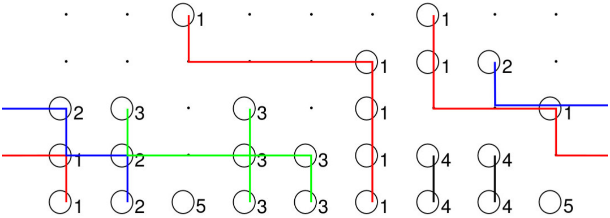

The case $\nu_\alpha = 0$ and multiline queues

To a multiline queue q one can associate a M-TASEP state of content $\mathbf{m}$ through the Bully Path (BP) algorithm.

Theorem [Arita Mallick]

$$\psi_w \propto \sum_{q|BP(q)=w} \prod_{\alpha<\beta} \left(\frac{\tau_\beta}{\tau_\alpha} \right)^{z_{\alpha,\beta}(q)}$$

where $z_{\alpha,\beta}(q)$ is the number of vacancies on row j that are covered by a i Bully Path.

Open question

Does such a construction extend to the general case $\nu_\alpha \neq 0$?

Integrability

The master equation for the time evolution of the probability of a configuration is

$$\frac{d}{dt}P_w(t) = \sum_{w'|w'\to w} \mathcal{M}_{w,w'}P_w(t) - \sum_{w'|w\to w'} \mathcal{M}_{w',w}P_w(t)$$

$$\frac{d}{dt}P(t) = \mathcal{M}P(t)$$

The important point to remark here is that *the Markov matrix $\mathcal{M}$ is the sum of local terms* acting on $V_{\mathbf{m}}$, the vector space with a basis labeled by periodic words of content $\mathbf{m}$

$$\mathcal{M} = \sum_{i=1}^{L} M^{(i)}, \qquad M^{(i)} = \sum_{1\leq\alpha\neq\beta\leq N} p_{\alpha,\beta}M^{(i)}_{\alpha,\beta}$$

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Integrability

Now suppose that we have a matrix $\check{R}(x,y)$ depending on two formal commuting variables, such that

$$\check{R}(x,x) = \mathbf{1}, \qquad \frac{d}{dx}\check{R}(x,y)|_{x=y=0} \propto \sum_{1\leq\alpha\neq\beta,N} p_{\alpha,\beta} M_{\alpha,\beta}$$

and a vector

$$\psi(\mathbf{z}) \in V_{\mathbf{m}} \otimes \mathbb{Z}[\mathbf{z}], \qquad \mathbf{z} = \{z_1, \ldots, z_{L(\mathbf{m})}\}$$

such that the following Exchange equations

$$\check{R}_i(z_i, z_{i+1})\psi(\mathbf{z}) = s_i \circ \psi(\mathbf{z})$$

where s_i acts on $\mathbb{Z}[\mathbf{z}]$ by the exchange $z_i \leftrightarrow z_{i+1}$.

Integrability

Then I claim that $\psi(\mathbf{0})$ is proportional to the TASEP stationary probability

$$\mathcal{M}\psi(\mathbf{0}) = 0$$

The consistency of the exchange equations is ensured by the unitarity relation

$$\check{R}_i(x,y)\check{R}_i(y,x) = \mathbf{1}$$

and the braid Yang-Baxter equation

$$\check{R}_i(y,z)\check{R}_{i+1}(x,z)\check{R}_i(x,y) = \check{R}_{i+1}(x,y)\check{R}_i(x,z)\check{R}_{i+1}(y,z)$$

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Multispecies TASEP: baxterized form of R-matrix

Theorem

The most general solution of the unitarity and Yang Baxter equations of the Baxterized form

$$\check{R}(x,y) = \mathbf{1} + \sum_{1\leq\alpha<\beta\leq N} g_{\alpha,\beta}(x,y) M_{\alpha,\beta}$$

is given by

$$g_{\alpha,\beta}(x,y) = \frac{(y-x)(\tau_\alpha+\nu_\beta)}{(\tau_\alpha y-1)(\nu_\beta x+1)} \to p_{\alpha<\beta} = \tau_\alpha + \nu_\beta$$

Theorem

The exchange equations corresponding the the $\check{R}$ matrix of the Multispecies TASEP admit a polynomial solution, unique up to multiplication of a completely symmetric polynomial in the $\mathbf{z}$.

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Exchange equations in components

Once expanded in components, the exchange equations read as follows

$$\psi_{\dots,w_i=w_{i+1},\dots}(\mathbf{z}) = s_i \circ \psi_{\dots,w_i=w_{i+1},\dots}(\mathbf{z})$$

$$\psi_{\dots,w_i>w_{i+1},\dots}(\mathbf{z}) = \hat{\pi}_i(w_i, w_{i+1})\psi_{\dots,w_{i+1},w_i,\dots}(\mathbf{z})$$

and

$$\hat{\pi}_i(\alpha,\beta) = \frac{(\tau_\alpha z_{i+1}-1)(\nu_\beta z_i+1)}{\tau_\alpha+\nu_\beta}\frac{1-s_i}{z_i-z_{i+1}}$$

This system of equation is cyclic: given $\psi_w(\mathbf{z})$ for a word w one can obtain $\psi_{w'}(\mathbf{z})$ for any other w' by acting with the $\hat{\pi}$ operators.

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Affine 0-Hecke algebra with spectral parameters

The operators $\hat{\pi}_i(\alpha,\beta)$ satisfy a spectral parameter deformation (not baxterization!) of the 0-Hecke algebra (recovered for t_α and ν_α independent of α)

$$\hat{\pi}_i^2(\alpha,\beta) = -\hat{\pi}_i(\alpha,\beta)$$

$$\hat{\pi}_i(\beta,\gamma)\hat{\pi}_{i+1}(\alpha,\gamma)\hat{\pi}_i(\alpha,\beta) = \hat{\pi}_{i+1}(\alpha,\beta)\hat{\pi}_i(\alpha,\gamma)\hat{\pi}_{i+1}(\beta,\gamma)$$

$$[\hat{\pi}_i(\alpha,\beta),\hat{\pi}_j(\gamma,\delta)] = 0 \qquad |i-j|>2$$

Simple consequences of the exchange equations

- If the word w has a sub-sequence $w_\ell \leq w_{\ell+1} \leq \cdots \leq w_{k-1} \leq w_k$ then

$$\psi_w(\mathbf{z}) = \prod_{i=\ell}^{k} \left(\prod_{\substack{\alpha \in w_{\ell,k} \\ \alpha < w_i}} (\tau_\alpha z_i - 1) \prod_{\substack{\alpha \in w_{\ell,k} \\ \beta > w_i}} (\nu_\beta z_i + 1) \right) \tilde{\psi}_w(\mathbf{z})$$

where $\tilde{\psi}_w(\mathbf{z})$ is symmetric in the variable $\{z_\ell, \ldots, z_k\}$

- In particular if $w = w^*$ has minimum number of descents $w_\ell \leq w_\ell \leq \cdots \leq w_{\ell-2} \leq w_{\ell-1}$ then $\tilde{\psi}_{w^*}(\mathbf{z})$ is symmetric in the whole set of variables $\mathbf{z}$ and by cyclicity is a common factor of all the $\psi_w(\mathbf{z})$.

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Simple consequences of the exchange equations

- The solution of the exchange equation of minimal degree in the sector **m** has degree

$$\deg_{z_i} \psi^{(\mathbf{m})}(\mathbf{z}) = \#\{\alpha | m_\alpha \neq 0\} - 1$$

- Normalization choice

$$\psi_{w^*}(\mathbf{z}) = \chi_{\mathbf{m}}(\tau, \nu) \prod_{i=1}^{L} \left(\prod_{\alpha < w_i^*} (1 - \tau_\alpha z_i) \prod_{\beta > w_i^*} (1 + \nu_\beta z_i) \right)$$

Conjecture

With the normalization given above, the components ψ_w are polynomials in all their variables $(\mathbf{z}, \tau, \nu)$ with no common factors.

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Recursions

Proposition

By specializing $z_L = \tau^{-1}_{\min(\mathbf{m})}$ or $z_L = -\nu^{-1}_{\max(\mathbf{m})}$ we have the following recursion

$$\psi_w(\mathbf{z})\big|_{z_L=\tau^{-1}_{\min(\mathbf{m})}} = \begin{cases} 0 & w_L \neq \min(\mathbf{m}) \\ K^{-}(\mathbf{z}\setminus z_L)\psi_{w\setminus w_L}(\mathbf{z}\setminus z_L) & w_L = \min(\mathbf{m}) \end{cases}$$

$$\psi_w(\mathbf{z})\big|_{z_L=-\nu^{-1}_{\max(\mathbf{m})}} = \begin{cases} 0 & w_L \neq \max(\mathbf{m}) \\ K^{+}(\mathbf{z}\setminus z_L)\psi_{w\setminus w_L}(\mathbf{z}\setminus z_L) & w_L = \max(\mathbf{m}) \end{cases}$$

where the factors $K^{\pm}(\mathbf{z}\setminus z_L)$ can be easily computed by inspection of $\psi_{w^*}(\mathbf{z})$.

Simplest non trivial component

Let $w^{(\alpha)}$ such that for $i \leq j \leq L - m_\alpha$

$$w_i \neq \alpha \qquad \text{and} \qquad w_i \leq w_j$$

For example

$$w^{(6)} = 2\ 2\ 3\ 5\ 5\ 5\ 7\ 9\ 9\ 6\ 6\ 6$$

Then

$$\psi^{(\mathbf{m})}_{w^{(\alpha)}}(\mathbf{z}) = (\text{Trivial Factors}) \times \phi^{(\mathbf{m})}_{\alpha}(z_1, \dots, z_{L-m_\alpha})$$

where $\phi^{(\mathbf{m})}_{\alpha}(z_1, \dots, z_{L-m_\alpha})$ is a symmetric polynomial in $z_1, \dots, z_{L-m_\alpha}$ of degree 1 in each variable separately.

- These polynomials turn out to be the building blocks of more general components
- Thanks to the recursion relations they can be computed explicitly

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For any $n > 0$ and $1 \leq \beta \leq n$ define the following polynomials

$$\Phi^n_\beta(\mathbf{z};\mathbf{t};\mathbf{v}) := \Delta(\mathbf{t},\mathbf{v}) \oint_{\mathbf{t}} \frac{dw}{2\pi i} \frac{\prod_{i=1}^{n-1}(1 - wz_i)}{\prod_{1\leq\rho\leq\beta}(w - t_\rho)\prod_{1\leq\sigma\leq n-\beta+1}(w - v_\sigma)}$$

Notice that these specialize to double Schubert Polynomials

$$\Phi^n_\beta(\mathbf{0};\mathbf{t};\mathbf{v}) = \mathfrak{S}_{1,\beta+1,\beta+2,\ldots n,2,3,\ldots,\beta}(\mathbf{t};\mathbf{v})$$

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Proposition

$$\phi_\alpha^{(\mathbf{m})}(z_1, \dots, z_{L-m_\alpha}) = \Phi_\beta^{L-m_\alpha}(\mathbf{z}; \mathbf{t}; \mathbf{v})$$

with $\beta = 1 + \sum_{\gamma<\alpha} m_\gamma$, and

$$\mathbf{t} = \{\dots, \overbrace{\tau_\gamma, \dots, \tau_\gamma}^{m_\gamma}, \dots, \overbrace{\tau_{\alpha-1}, \dots, \tau_{\alpha-1}}^{m_{\alpha-1}}, \tau_\alpha\}$$

$$\mathbf{v} = \{-\nu_\alpha, \underbrace{-\nu_{\alpha+1}, \dots, -\nu_{\alpha+1}}_{m_{\alpha+1}}, \dots, \underbrace{-\nu_\gamma, \dots, -\nu_\gamma}_{m_\gamma}, \dots\}$$

Factorization of components with least ascending

We have seen that to each "ascent" in a word w one has a bunch of trivial factors, therefore the intuition is that the more ascents w has the "simpler" is its component ψ_w.

Actually the words $\tilde{w}$ which have minimal number of ascent are also computable

Exm

$$\tilde{w} = 9\ 9\ 7\ 6\ 6\ 6\ 5\ 5\ 5\ 3\ 2\ 2$$

Conjecture

Calling $\mathbf{z}_\alpha = \{z_i | w_i = \alpha\}$

$$\psi_{\tilde{w}} = \prod_\alpha \phi_\alpha^{(\mathbf{m})}(\mathbf{z} \setminus \mathbf{z}_\alpha)$$

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Factorization of components with least ascending: corollaries

Corollary I

The formula for the least ascending component implies and generalizes a formula conjectured by Lam and Williams which expresses $\psi_{\tilde{w}}$ in the case $\mathbf{m} = \{\dots, 0, 1, 1 \dots, 1, 0 \dots\}$ as a product of double-Schubert Polynomials of τ, ν

$$\psi_{L,L-1,\dots,1} = \mathfrak{S}_{1,2,3\dots,L}\mathfrak{S}_{1,3,4\dots,L,2}\mathfrak{S}_{1,4,5,\dots,L,2,3}\mathfrak{S}_{1,L,2,3\dots,L-1}$$

Corollary II

Suppose that we condition w to split as $w^{(k)}w^{(k-1)}\dots w^{(2)}w^{(1)}$, with $w^{(j)}$ of fixed length L_j (possibly 0) and

$$w_i^{(r)} < w_j^{(s)} \qquad \text{for} \qquad r < s$$

then the events $w^{(j)}$ are independent.

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Normalization

In order to compute actual probabilities we need the normalization

$$\mathcal{Z}^{(\mathbf{m})}(\mathbf{z}) = \sum_{w|\mathbf{m}(w)=\mathbf{m}} \psi_w(\mathbf{z})$$

Thanks to the exchange relations this polynomial turns out to be symmetric in $\mathbf{z}$ and satisfies the recursion relation induced by $\psi(\mathbf{z})$ itself.

Unfortunately in the general case we are not able to provide a formula for $\mathcal{Z}^{(\mathbf{m})}(\mathbf{z})$.

What we can solve is the case $\nu_\alpha = \nu$ for $\alpha \leq \gamma$, $\tau_\alpha = \tau$ for $\alpha \geq \gamma$ for some γ.

Projections

Let us define the idempotent operators $\Pi^{\beta}_{\alpha_1,\ldots,\alpha_k}$ (by convention I assume $\beta \notin \{\alpha_1,\ldots,\alpha_k\}$)

$$\Pi^{\beta}_{\alpha_1,\ldots,\alpha_k}(w)_i = \begin{cases} w_i & \text{if} \quad w_i \notin \{\alpha_1,\ldots,\alpha_k\} \\ \beta & \text{if} \quad w_i \in \{\alpha_1,\ldots,\alpha_k\} \end{cases}$$

and extend it by linearity. We get projectors on the Hilbert space of the configurations of the M-TASEP.

Lemma

Suppose that $(\tau_\alpha, \nu_\alpha) = (\tau_{\alpha+1}, \nu_{\alpha+1})$, then we have that the $\check{R}$ matrix commutes with the projection $\Pi^{\alpha}_{\alpha+1}$

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Projections

Proposition

This means that if $(\tau_\alpha, \nu_\alpha) = (\tau_{\alpha+1}, \nu_{\alpha+1})$, then $\Pi^\alpha_{\alpha+1}\psi^{(\mathbf{m})}(\mathbf{z})$ is again solution of the exchange equations and therefore

$$\Pi^\alpha_{\alpha+1}\psi^{(\mathbf{m})}(\mathbf{z}) = \rho^{(\mathbf{m},\alpha)}(\mathbf{z})\psi^{(\Pi^\alpha_{\alpha+1}(\mathbf{m}))}(\mathbf{z})$$

where

$$\Pi^\alpha_\gamma(\mathbf{m})_\beta = \begin{cases} m_\beta & \text{for} \quad \beta \neq \alpha, \gamma \\ m_\alpha + m_\gamma & \text{for} \quad \beta = \alpha \\ 0 & \text{for} \quad \beta = \gamma \end{cases}$$

and $\rho^{(\mathbf{m}',\alpha)}(\mathbf{z})$ is a symmetric function of degree 1 in each z_i which is given by a specialization of $\Phi^n_\beta(\mathbf{z}; \mathbf{t}; \mathbf{v})$.

Factorization of the sum rule

If for some γ we have

$$\begin{aligned} \nu_\alpha &= \nu \quad \text{for} \quad \alpha \leq \gamma \\ \tau_\alpha &= \tau \quad \text{for} \quad \alpha \geq \gamma \end{aligned}$$

(and $m_\alpha > 0$ for $\min(\mathbf{m}) \leq \alpha \leq \max(\mathbf{m})$), by projecting "downward" from $\max(\mathbf{m})$ and "upward" from $\min(\mathbf{m})$ until γ

Theorem

$$\mathcal{Z}^{(\mathbf{m})}(\mathbf{z}) = \prod_{\alpha=\min(\mathbf{m})}^{\gamma-1} \phi_\alpha^{(\mathbf{m}_\alpha^\uparrow)}(\mathbf{z}) \prod_{\alpha=\gamma+1}^{\max(\mathbf{m})} \phi_\alpha^{(\mathbf{m}_\alpha^\downarrow)}(\mathbf{z})$$

where

$$\mathbf{m}_\alpha^\downarrow = \Pi^{\alpha-1}_{\alpha,\alpha+1,\dots}\mathbf{m}, \qquad \mathbf{m}_\alpha^\uparrow = \Pi^{\alpha+1}_{\dots,\alpha-1,\alpha}\mathbf{m}$$

Some open questions

- Correlation functions, currents, etc.
- Do the components $\psi_w(\mathbf{z})$ have a combinatorial expression?
- What is the "right" context for the 0-Hecke algebra with spectral parameters?
  The operators $\hat{\pi}(\alpha, \beta)$ can be used for example to define a family of deformed Grothendieck "polynomials" which depend on the parameters τ, ν. Do they have any geometric meaning?
- Deal with others Weyl groups.

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