Hopf Dreams

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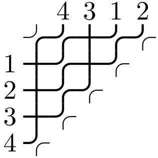
April 12, 2018

Pipe dreams

Fill a triangular shape with crosses $+$ and elbows ╯╭:



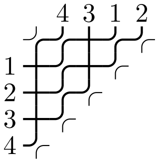
A pipe dream $P \in \Pi_4$ where $\omega_P = [4, 3, 1, 2]$.

Conditions:

- pipes entering on the left exit on the top.
- two pipes cross at most once.
- the top left corner is an elbow ╯╭.

Pipe dreams

Fill a triangular shape with crosses ┼ and elbows ╯╭:

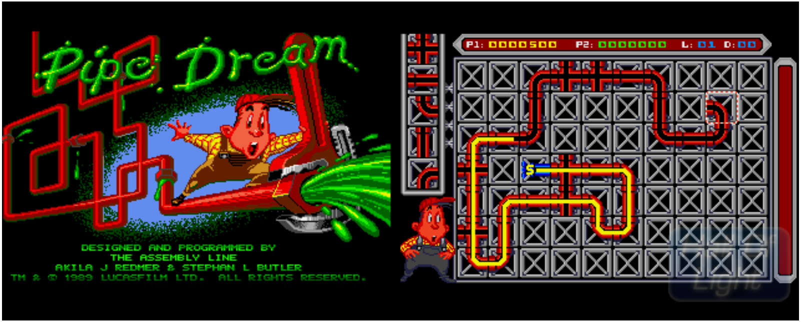


A pipe dream $P \in \Pi_4$ where $\omega_P = [4, 3, 1, 2]$.

Introduced and studied by:

- S. Fomin and A. N. Kirillov. The Yang-Baxter equation, symmetric functions, and Schubert polynomials. (FPSAC 1993)
- N. Bergeron and S. Billey. RC-graphs and Schubert polynomials. (1993)
- A. Knutson and E. Miller. Gröbner geometry of Schubert polynomials. (2005)
- ...

Pipe dreams



Pipe Dream
DESIGNED AND PROGRAMMED BY
THE ASSEMBLY LINE
AKILA J REDMER & STEPHAN L BUTLER
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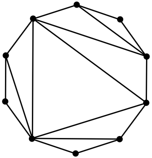
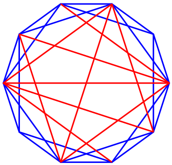
P1: 0000500
P2: 0000000
L: 01
D: 00

Pipe dreams: why are they interesting?

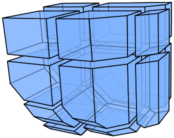
1. They give a combinatorial understanding of Schubert polynomials in the study of Schubert varieties.
2. Pipe dreams of certain families of permutations encode interesting combinatorial objects:



triangulations



multitriangulations



ν-Tamari lattices

Goal

Introduce a Hopf algebra structure on pipe dreams with some remarkable applications.

Hopf algebras

Hopf algebras

Hopf algebra: Vector space whose generators can be multiplied and comultiplied in a compatible way. Also there is an antipode.

Example

$\mathbf{k}G$: $\Delta(g) = g \otimes g \quad m(g \otimes h) = gh.$

- Polynomial rings
- Permutations
- Cohomology of Lie groups
- Universal enveloping algebra of Lie algebras
- Quantum groups
- Many more ...

Examples: Hopf algebra on permutations

$\mathfrak{S}_n$: collection of permutations of $[n]$
$\mathbf{k}\mathfrak{S}$: vector space spanned by all permutations

Theorem (Malvenuto, 1994, Malvenuto–Reutenauer, 1995)

$\mathbf{k}\mathfrak{S}$ *may be equipped with a structure of graded Hopf algebra.*

Comultiplication: sum of pairs obtained by cuttin a permutation in two

$$\Delta(312) = 312 \otimes \emptyset + 21 \otimes 1 + 1 \otimes 12 + \emptyset \otimes 312$$

Multiplication: sum of all possible shuffles between two permutations

$$12 \cdot 21 = 1243 + 1423 + 1432 + 4123 + 4132 + 4312$$

Examples: Hopf algebra on binary trees

Y_n: collection of planar binary trees with n leaves
$\mathbf{k}Y$: vector space spanned by all planar binary trees

Theorem (Loday–Ronco, 1998)

$\mathbf{k}Y$ may be equipped with a structure of graded Hopf algebra.

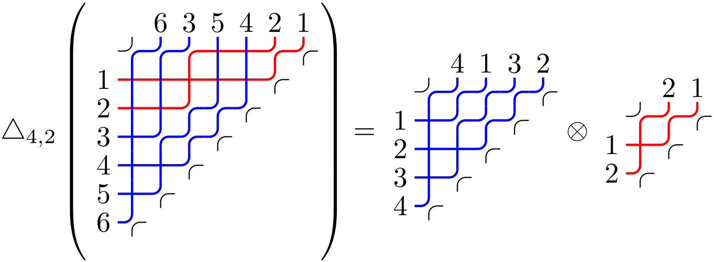
Comultiplication

Multiplication

A Hopf algebra on pipe dreams

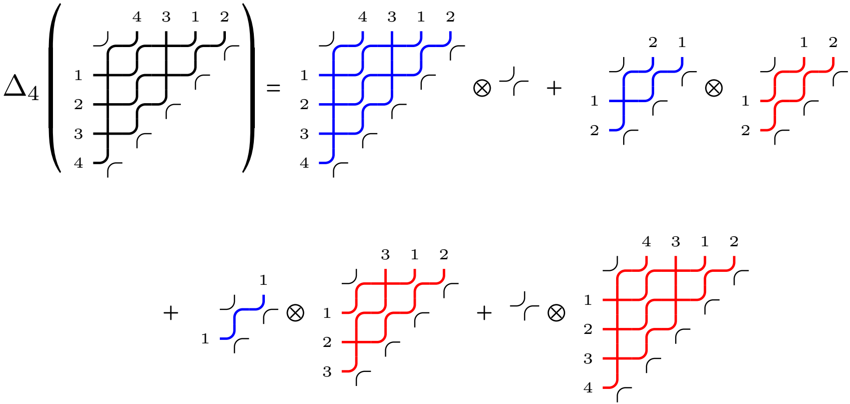
Comultiplication

$$\begin{aligned}\triangle_n : \quad \Pi_n &\longrightarrow \bigoplus_{\gamma=0}^{n} \Pi_\gamma \otimes \Pi_{n-\gamma} \\ P &\longmapsto \sum_{\gamma \in GD(\omega_P)} \triangle_{\gamma,n-\gamma}(P).\end{aligned}$$



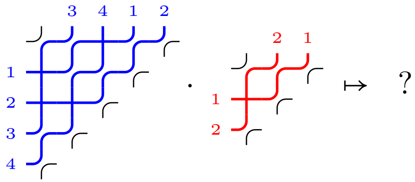
The sum ranges over allowable cuts of the permutation: global descents.

Comultiplication

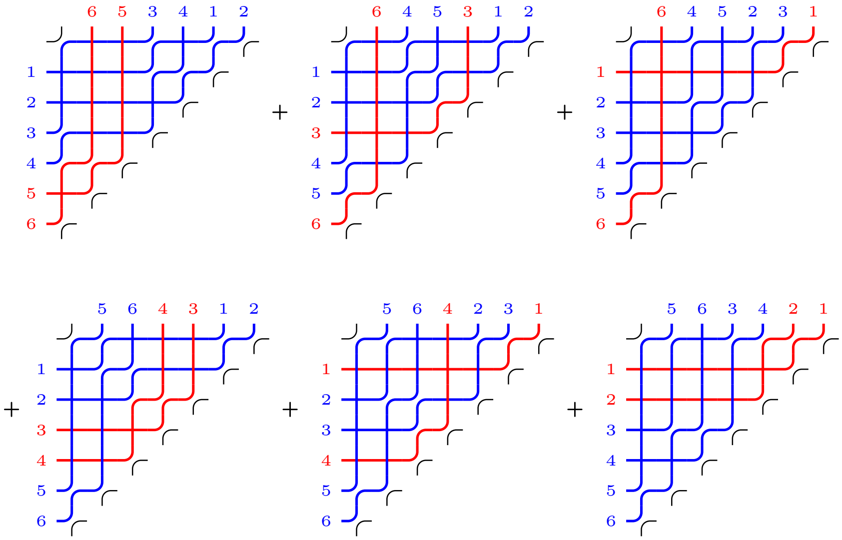


Multiplication

$$\begin{aligned} \mu_{r,s}: \quad \Pi_r \otimes \Pi_s &\longrightarrow \Pi_{r+s} \\ P \cdot Q &\longmapsto ? \end{aligned}$$



Multiplication



A Hopf algebra on pipe dreams

Π_n: collection of pipe dreams of permutations in $\mathfrak{S}_n$
$\mathbf{k}\Pi$: vector space spanned by pipe dreams

Theorem

These operations endow $\mathbf{k}\Pi$ with a graded Hopf algebra structure.
This Hopf algebra is free and cofree.

Hopf subalgebras

Hopf subalgebras from atom sets

Every permutation can be factorized into atomics (permutations with no global descents):

$$2431 = 132 \bullet 1 \quad \text{and} \quad 312 = 1 \bullet 12$$

Given a set of atomics S

$$\Pi_S = \{P \in \Pi : \text{atomics}(\omega_P) \subseteq S\}$$

Theorem

$\mathbf{k}\Pi_S$ *is a Hopf subalgebra of* $\mathbf{k}\Pi$.

Hopf subalgebras from atom sets

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Given a set of atomics S

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Example

$S = \{1\}$: $\mathbf{k}\Pi_{\{1\}} \cong$ Loday–Ronco Hopf algebra

- dim deg $n = C_n$.
- number of generators deg $n = C_{n-1}$.

Hopf subalgebras from atom sets

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$$2431 = 132 \bullet 1 \quad \text{and} \quad 312 = 1 \bullet 12$$

Given a set of atomics S

$$\Pi_S = \{P \in \Pi : \text{atomics}(\omega_P) \subseteq S\}$$

Example

$S = \{12\}$: $\mathbf{k}\Pi_{\{12\}}$

- number of generators deg $n = C_{2n-1}$.

Hopf subalgebras from atom sets

Every permutation can be factorized into atomics (permutations with no global descents):

$$2431 = 132 \bullet 1 \quad \text{and} \quad 312 = 1 \bullet 12$$

Given a set of atomics S

$$\Pi_S = \{P \in \Pi : \text{atomics}(\omega_P) \subseteq S\}$$

Example

$S = \{213\}$: $\mathbf{k}\Pi_{\{213\}}$

- number of generators deg $n = C_{3n-1}$.

Hopf subalgebras from atom sets

Every permutation can be factorized into atomics (permutations with no global descents):

$$2431 = 132 \bullet 1 \quad \text{and} \quad 312 = 1 \bullet 12$$

Given a set of atomics S

$$\Pi_S = \{P \in \Pi : \text{atomics}(\omega_P) \subseteq S\}$$

Example

$S = \{3214\}$: $\mathbf{k}\Pi_{\{3214\}}$

- number of generators deg $n = C_{4n-1}$.

Hopf subalgebras from atom sets

Every permutation can be factorized into atomics (permutations with no global descents):

$$2431 = 132 \bullet 1 \quad \text{and} \quad 312 = 1 \bullet 12$$

Given a set of atomics S

$$\Pi_S = \{P \in \Pi : \text{atomics}(\omega_P) \subseteq S\}$$

Example

$S = \{43215\}$: $\mathbf{k}\Pi_{\{43215\}}$

- number of generators deg $n = C_{5n-1}$.

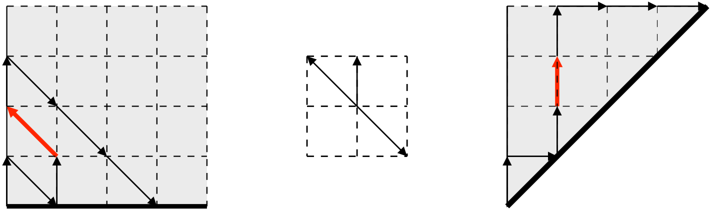
Hopf subalgebra for walks on the plane

Conjecture

$S = \{1, 12, 123, 1234, \ldots\}$: $\mathbf{k}\Pi_S$

- *dim deg n = number of walks in the quarter plane (within $\mathbb{N}^2 \subset \mathbb{Z}^2$) starting at $(0,0)$, ending on the horizontal axis, and consisting of $2n$ steps taken from $\{(-1,1),(1,-1),(0,1)\}$.*

1, 3, 12, 57, 301, 1707, 10191, 63244, 404503, 2650293, . . .



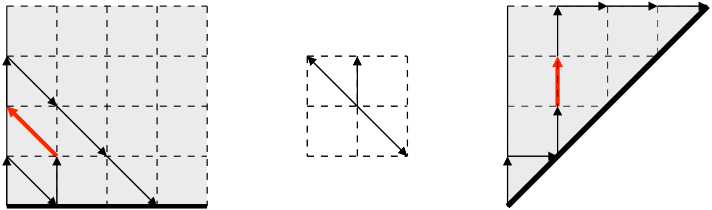
Hopf subalgebra for walks on the plane

Conjecture (refined 1)

$S = \{1, 12, 123, 1234, \dots\}$: $\mathbf{k}\Pi_S$

- *The pipe dreams of deg n with k atomic parts count the number of walks with k steps $(0, 1)$.*

1, 3, 12, 57, 301, 1707, 10191, 63244, 404503, 2650293, . . .



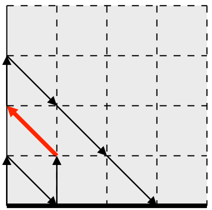
Hopf subalgebra for walks on the plane

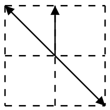
Conjecture (refined 2)

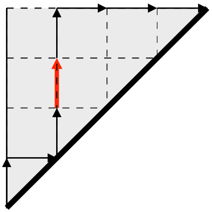
$S = \{1, 12, 123, 1234, \ldots\}$: $\mathbf{k}\Pi_S$

- *The pipe dreams of deg n with k atomic parts count the number of bicolored Dyck paths with k black north steps.*

1, 3, 12, 57, 301, 1707, 10191, 63244, 404503, 2650293, . . .



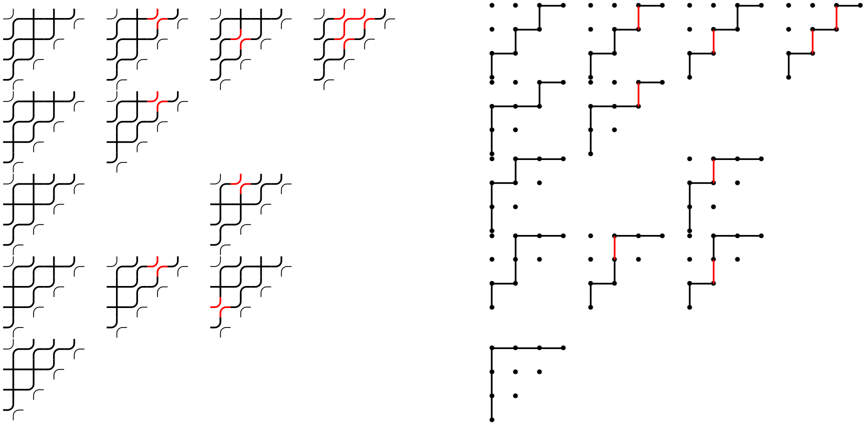




Hopf subalgebra for walks on the plane

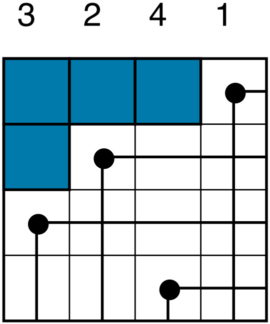
Proposition

These three conjectures are true for $k = 1, 2, n$.



Hopf subalgebra of dominant dreams

A permutation ω is called *dominant* if its Rothe diagram is a partition located at the top-left corner.



Schubert polynomials of dominant permutations are specially interesting.

Hopf subalgebra of dominant dreams

S^{dom}: Collection of all dominant permutations

Theorem

$\mathbf{k}\Pi_{S^{\mathrm{dom}}}$ *is a Hopf subalgebra of* $\mathbf{k}\Pi$.

- *dim deg* $n = \det \begin{vmatrix} C_n & C_{n+1} \\ C_{n+1} & C_{n+2} \end{vmatrix}$

Dominant pipe dreams are in bijection with pairs of nested Dyck paths.

Application to
multivariate diagonal harmonics

What is multivariate diagonal harmonics?

The story begins with the Macdonald positivity conjecture, regarding the coefficients of the Schur function expansion of Macdonald polynomials:

$$H_\mu(\mathbf{x}; q, t) = \sum_{\nu \vdash \mu} k_{\mu\nu}(q, t) s_\nu(\mathbf{x}).$$

Conjecture (Macdonald Positivity Conjecture, 1988)

$k_{\mu\nu}(q, t)$ are polynomials in q and t with non-negative coefficients.

Garsia–Haiman's combinatorial approach:
study a representation of the symmetric group on a space ∂D_μ

Garsia–Haiman's combinatorial approach

Theorem (The n! conjecture, Haiman 2001)

For any $\mu \vdash n$, we have

$$\dim_{\mathbb{C}} \partial D_\mu = n!.$$

Theorem (Haiman 2001)

$$k_{\mu\nu}(q,t) = \sum_{i,j} t^i q^j \,\mathrm{mult}(\chi^\lambda, \mathrm{ch}(D_\mu)_{i,j})$$

In particular, it is a polynomial with non-negative integer coefficients and the Macdonald positivity conjecture holds.

For $\mu = (1,1,\ldots,1)$, ∂D_μ is the *space of harmonics*.

The space of harmonics

$\mathbb{Q}[\mathbf{x}] := \mathbb{Q}[x_1, \ldots, x_n]$ is the polynomial ring in n variables,
$I :=$ ideal generated by all symmetric polynomials with no constant term,
$\partial\mathbf{x} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$.

Definition

The *space of harmonics* is defined by

$$H_n = \{h \in \mathbb{Q}[\mathbf{x}] : f(\partial\mathbf{x})h = 0,\ \forall f \in I\}.$$

Example ($n = 1$)

We want all $h(x_1) \in \mathbb{Q}[x_1]$ such that $\frac{\partial}{\partial x_1}h = 0$. Therefore

$$H_1 = \text{span}\{1\}.$$

The space of harmonics

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Definition

The *space of harmonics* is defined by

$$H_n = \{h \in \mathbb{Q}[\mathbf{x}] : f(\partial\mathbf{x})h = 0, \ \forall f \in I\}.$$

Example ($n = 2$)

We want all $h(x_1, x_2) \in \mathbb{Q}[x_1, x_2]$ such that $f(\partial\mathbf{x})h = 0, \ \forall f \in I$.
One can check that

$$H_2 = \text{span}\{1, x_1 - x_2\}.$$

The space of harmonics

$\mathbb{Q}[\mathbf{x}] := \mathbb{Q}[x_1, \ldots, x_n]$ is the polynomial ring in n variables,
$I :=$ ideal generated by invariant $\mathfrak{S}_n$ polynomials with no constant term,
$\partial\mathbf{x} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$.

Definition

The *space of harmonics* is defined by

$$H_n = \{h \in \mathbb{Q}[\mathbf{x}] : f(\partial\mathbf{x})h = 0, \ \forall f \in I\}.$$

Fact

As $\mathfrak{S}_n$-modules,

$$H_n \cong \mathbb{Q}[\mathbf{x}]/I.$$

Diagonal harmonics

$\mathbb{Q}[\mathbf{x}, \mathbf{y}] := \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$
Let the symmetric group $\mathfrak{S}_n$ act diagonally on this ring:

$$\sigma(x_i) = x_{\sigma(i)} \qquad \sigma(y_i) = y_{\sigma(i)}$$

$I :=$ ideal generated by $\mathfrak{S}_n$ invariant polynomials with no constant term.

Definition

The *space of diagonal harmonics* is defined by

$$DH_n = \{h \in \mathbb{Q}[\mathbf{x}, \mathbf{y}] : f(\partial \mathbf{x}, \partial \mathbf{y})h = 0, \ \forall f \in I\}.$$

Fact

as $\mathfrak{S}_n$-modules,

$$DH_n \cong \mathbb{Q}[\mathbf{x}, \mathbf{y}]/I.$$

Diagonal harmonics

The $(n+1)^{n-1}$ conjecture by Garsia and Haiman from 1993:

Theorem (Haiman 2002)

The dimension of DH_n is equal to $(n+1)^{n-1}$.

Theorem (Haiman 2002)

The dimension of the alternating component of DH_n is equal to $\frac{1}{n+1}\binom{2n}{n}$.

This led to the now famous q, t-Catalan polynomials!

Multivariate diagonal harmonics

The space DH_n can be generalized to three, or more sets of variables.

Conjecture (Haiman 1994)

In the trivariate case,

- *the dimension of* DH_n *is* $2^n(n+1)^{n-2}$.
- *the dimension of its alternating component is*

$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}.$$

These two numbers can be combinatorially interpreted as the number of labeled and unlabeled intervals in the Tamari lattice.

Multivariate diagonal harmonics

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These two numbers can be combinatorially interpreted as the number of labeled and unlabeled intervals in the Tamari lattice.

No conjectural formulas are known for more sets of variables.

In summary

The dimensions of the spaces of multivariate diagonal harmonics and their alternating components are

one set of variables	two sets of variables	three sets of variables	more sets of variables
$n!$	$(n+1)^{n-1}$	**Tamari lattice labelled intervals**	**Unknown**
1	$\frac{1}{n+1}\binom{2n}{n}$	**Tamari lattice intervals**	**Unknown**
		Open problems	

One may expect that dimensions for r sets of variables are counted by labeled and unlabeled chains $(\pi_1, \ldots, \pi_{r-1})$ in the Tamari lattice. But this is not true in general.

Back to pipe dreams

Pipe dreams have a natural poset structure.
The number of intervals in the graded dimensions of $\mathbf{k}\Pi_{\mathcal{S}^{\text{dom}}}$ is:

$$1, 4, 29, 297, 3823, 57956, \ldots$$

They correspond to certain triples of Dyck paths.

Definition (Hopf chains)

A Hopf chain of length r and size n is a tuple $(\pi_1, \ldots, \pi_r)$ of Dyck paths of size n such that

- π_1 is the bottom diagonal path,
- every triple comes from an interval in the Hopf algebra of dominant dreams.

Counting Hopf chains

Example (n=4)

The number of Hopf chains $(\pi_1, \ldots, \pi_r)$ of Dyck paths of size $n = 4$ is

$$1, 14, 68, 217, 549, 1196, 2345, \ldots$$

Counting Hopf chains

Example (n=4)

The number of Hopf chains $(\pi_1, \ldots, \pi_r)$ of Dyck paths of size $n = 4$ is

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Example (n=4)

The dimension of the alternating component of the space of diagonal harmonics DH_n for fixed $n = 4$ and r variables is equal to

$$1, 14, 68, 217, 549, 1196, 2345, \ldots$$

Counting Hopf chains

Theorem

For degree $n \leq 4$ and any number r of sets of variables, the Frobenius image of the character of $DH_{n,r}$ expanded in the elementary basis is

$$\Psi_{n,r} = \sum_{\textit{Hopf chains } (\pi_1,\dots,\pi_r)} e_{\mathrm{type}(\pi_r)}, \tag{1}$$

where $\mathrm{type}(\pi_r)$ *is the partition of the up steps lengths in* π_r.

Counting Hopf chains

Corollary

For degree $n \leq 4$ and any number r of sets of variables:

1. *The dimension of the alternating component of $DH_{n,r}$ is equal to the number of Hopf chains of length r and size n.*
2. *The dimension of $DH_{n,r}$ is equal to the number of labelled Hopf chains of length r and size n.*

Counting Hopf chains

The dimensions of the alternating and full component for fixed $n \leq 4$ and arbitrary r are given in the following table:

n	number of Hopf chains	number of laballed Hopf chains
$n = 1$	$\binom{r}{0}$	$\binom{r+1}{0}$
$n = 2$	$\binom{r}{1}$	$\binom{r+1}{1}$
$n = 3$	$\binom{r}{1} + 3\binom{r}{2} + \binom{r}{3}$	$\binom{r+1}{1} + 4\binom{r+1}{2} + \binom{r+1}{3}$
$n = 4$	$\binom{r}{1} + 12\binom{r}{2} + 29\binom{r}{3}$	$\binom{r+1}{1} + 22\binom{r+1}{2} + 56\binom{r+1}{3}$
	$+25\binom{r}{4} + 9\binom{r}{5} + \binom{r}{6}$	$+40\binom{r+1}{4} + 11\binom{r+1}{5} + \binom{r+1}{6}$

Counting Hopf chains

For $n = 5$ the result is almost true:

$$\text{Excess}_{n=5} = \binom{k+4}{9} e_{[5]} + \binom{k+4}{8} e_{[4,1]}. \tag{2}$$

We have a few possible candidates that kill this excess but do not have a combinatorial rule to describe them at the moment.

To be continued . . .

Thank you!