Helly-type theorems, intersection patterns, and topological combinatorics

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Joint work with

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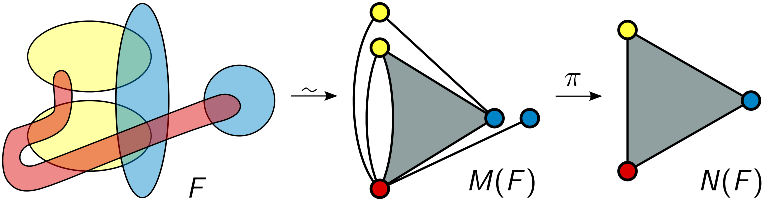


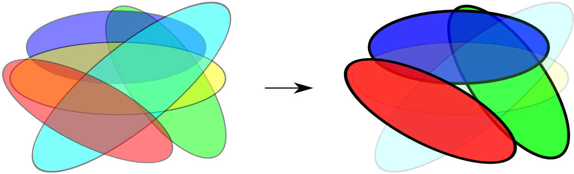
Helly's original theorem

Helly's original theorem (1923)

Let F be a finite family of convex sets in $\mathbb{R}^d$.
If every $G \subseteq F$ with $|G| \leq d+1$ has non-empty intersection,
then F has non-empty intersection.

Small-sized certificate of empty intersection

If F has empty intersection, some subfamily of size $\leq d+1$ has empty intersection.

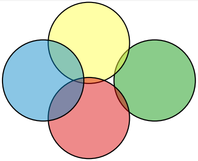


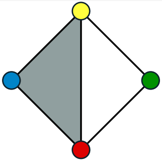
Intersection patterns and simplicial complexes

Intersection patterns

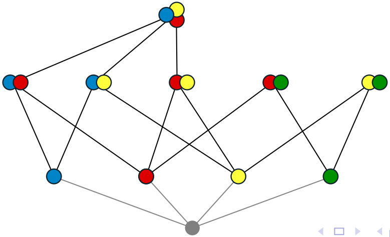
- Let F be a (finite) family of subsets of an arbitrary ground set.
- The nerve $N(F)$ of F is $\{G \subseteq F \mid \bigcap_G \neq \emptyset\}$.
- It is a simplicial complex (stable under taking subsets / a.k.a. a monotone hypergraph / a monotone set system).

F



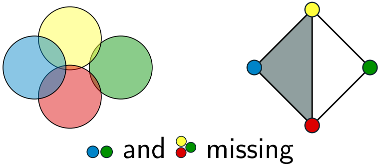


$N(F)$



Definition

Missing face F' of $N(F)$:
$$\begin{cases} F' \notin N(F), \\ \forall F'' \subsetneq F', \quad F'' \in N(F). \end{cases}$$



Definition

F is k-Helly if all the missing faces of $N(F)$ have size $\leq k$.

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If F has empty intersection,
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Helly-type theorems

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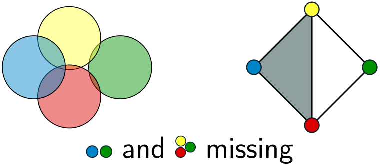

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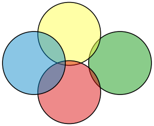

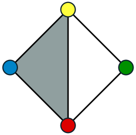

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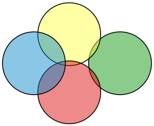

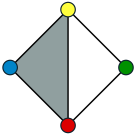

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In this talk. . .

Results

- A new topological Helly-type theorem for families of disconnected geometric objects
- based on a generalization of the nerve theorem from topological combinatorics
- with applications to geometric transversal theory.

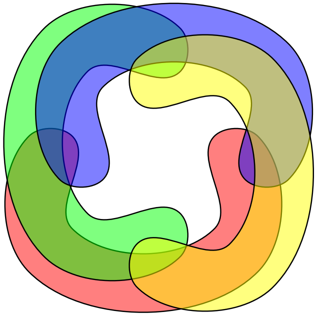
Warm-Up

Wanted!

Every "convex-like" family in $\mathbb{R}^d$ is $(d+1)$-Helly.

Wrong statement $\longrightarrow$

Replace "convex-like" with "contractible" ("without hole"; e.g., homeomorphic to a convex set).



Definition

A (finite) family F of (open) geometric objects is acyclic (a.k.a. a *good cover*) if: For every $G \subseteq F$, $\bigcap_G$ is either empty or contractible.

Topological Helly theorem

Every acyclic family in $\mathbb{R}^d$ is $(d+1)$-Helly [Helly, 1930].

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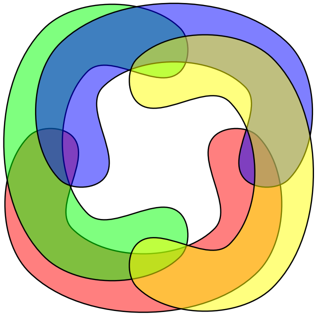

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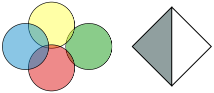
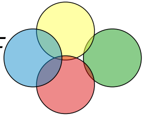
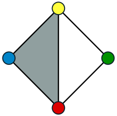
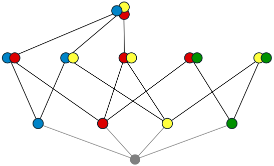
Nerve theorem

Nerves as topological spaces

- Vertices in general position in $\mathbb{R}^d$, d large;
- attach segments, triangles, tetrahedra, ...



F  $\xrightarrow{\sim}$  $N(F)$



Nerve theorem

If F is acyclic, then $\bigcup_F \simeq N(F)$: they have "holes" in the same dimensions [Borsuk, 1948].

Proof(s)

- Follows "trivially" from algebraic topology arguments;
- more "hands-on" (homotopic) combinatorial proof [Björner, 2003].

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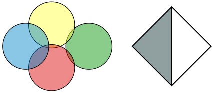

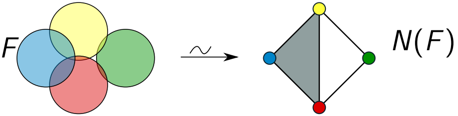

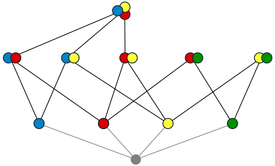

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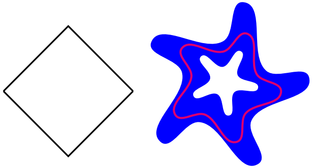
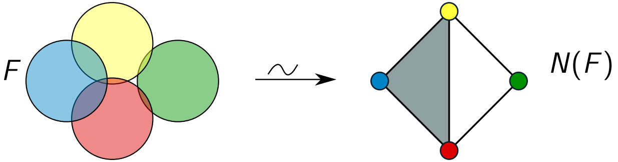
Topological interlude: "holes"



Holes in a topological space S

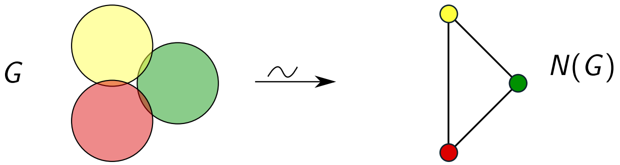
- Formally, S has a k-hole if the kth dimensional reduced homology of S is nonzero: $\dim \tilde{H}_k(S, \mathbb{Q}) > 0$.
- Intuitively, S has a k-hole if some k-dimensional "closed part" of S is the boundary of no $(k+1)$-dimensional subset of S.
- Examples:
  - S has a 0-hole if it is not connected;
  - S has a 1-hole if it contains a closed curve that is not the boundary of a surface in S;
  - S has a 2-hole if it contains a "bubble"...
- Contractible means "without hole".

Nerve theorem ⇒ topological Helly theorem



- Let F be an acyclic family in $\mathbb{R}^d$.
- Let G be a missing face of $N(F)$.
- $N(G)$ has a $(|G|-2)$-hole.
- On the other hand, we have $N(G) \simeq \bigcup_G \ldots$
- and $\bigcup_G \subseteq \mathbb{R}^d$, so $\bigcup_G$ has no hole in dimension $\geq d$.
- So $|G| - 2 < d$, i.e., $|G| \leq d+1$. $\square$

Nerve theorem $\Rightarrow$ topological Helly theorem



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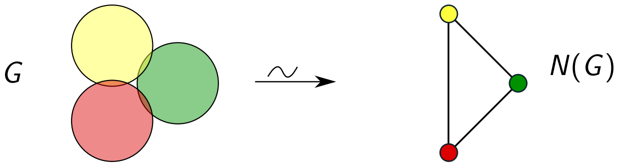

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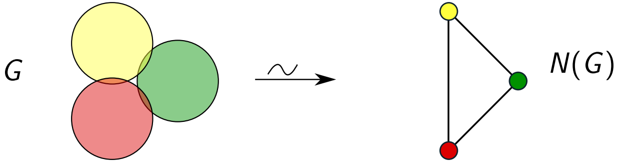

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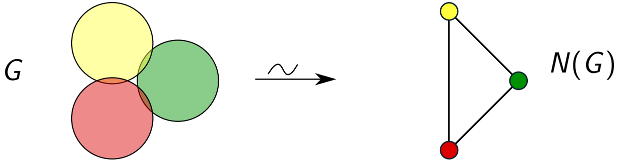

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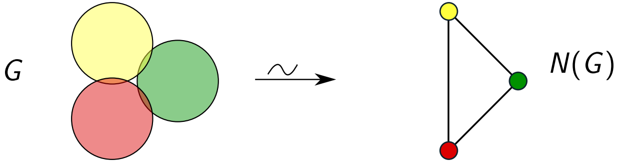

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Results

Generalized version

Definition

A family F of sets in $\mathbb{R}^d$ is *r-acyclic* if $\forall G \subseteq F$, $\bigcap_G$ is the disjoint union of at most r contractible sets.

Topological Helly theorem

Let F be a 1-acyclic family in $\mathbb{R}^d$.
Then F is $(d+1)$-Helly.

Remarks

- The value $(d+1)r$ cannot be lowered;
- strengthens a result by [Kalai and Meshulam, 2008] on r-families of acyclic families (also [Amenta, 1996]);
  r-family F of a "ground" family G: The intersection of a subfamily of F is the disjoint union of at most r elements in G.
- [Matoušek, 1997] had proved that F is k-Helly for some (*large*) k.

Definition

A family F of sets in $\mathbb{R}^d$ is r-acyclic if $\forall G \subseteq F$, $\bigcap_G$ is the disjoint union of at most r contractible sets.

New topological Helly-type theorem: Let $r \geq 1$

Let F be an $\boldsymbol{r}$-acyclic family in $\mathbb{R}^d$.
Then F is $(d+1) \times \boldsymbol{r}$-Helly.

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Comparison with other results

convex sets in $\mathbb{R}^d$
[Helly, 1923]
$d + 1$

acyclic families in $\mathbb{R}^d$
[Helly, 1930]
$d + 1$

r-family of convex sets in $\mathbb{R}^d$
[Amenta, 1996]
$(d + 1)r$

r-family of an acyclic family in $\mathbb{R}^d$
[Kalai and Meshulam, 2008]
$(d + 1)r$

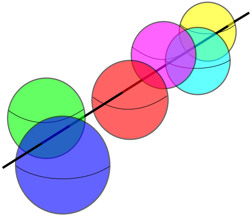
topological condition
[Matoušek, 1997]
no explicit bound

r-family of a non-additive family G closed under $\bigcap$
[Eckhoff and Nischke, 2009]
$r \times h(G)$

r-acyclic family
[CdV, G, and G]
$(d + 1)r$

Application to geometric transversal theory

- Let $C_1, \ldots, C_n$ be disjoint convex sets in $\mathbb{R}^d$.
- For each i, let F_i be the set of lines meeting C_i.
- Let $F := \{F_1, \ldots, F_n\}$.



- In which cases is F k-Helly?
- Central question in geometric transversal theory.

Shape		k-Helly for $k = \ldots$	
		previous bound	our bound
parallelotopes in $\mathbb{R}^d$ $(d \geq 2)$		$2^{d-1}(2d-1)$ [Santaló, 1940]	$2^{d-1}(2d-1)$
disjoint translates of a convex in $\mathbb{R}^2$		5 [Tverberg, 1989]	10
disjoint unit balls in $\mathbb{R}^d$	$d = 2$	5 [Danzer, 1957]	12
	$d = 3$	11 [Cheong et al., 2008]	15
	$d = 4$	15 [Cheong et al., 2008]	20
	$d = 5$	19 [Cheong et al., 2008]	20
	$d \geq 6$	$4d - 1$ [Cheong et al., 2008]	$4d - 2$

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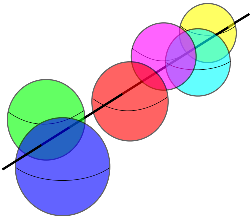

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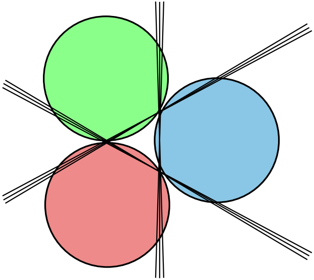
Why does our result apply?



New topological Helly-type theorem

Let F be a family of sets in $\mathbb{R}^d$ such that $\forall G \subseteq F$, $\bigcap_G$ is the disjoint union of at most r contractible sets. Then F is $((d+1)r)$-Helly.

Idea: Apply the main result in the space of lines of $\mathbb{R}^d$

- Good: Often, if $G \subseteq F$, each connected component of $\bigcap_G$ corresponds to a *geometric permutation* of the objects C_i.
- Bad: The space of lines in $\mathbb{R}^d$ is a $(2d-2)$-manifold.
  $\rightarrow$ Extension to arbitrary topological spaces (d=dimension of vanishing homology of open sets).
- Bad: Some components of $\bigcap_G$ are not contractible.
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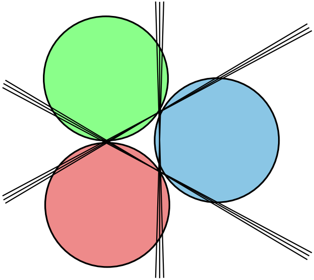

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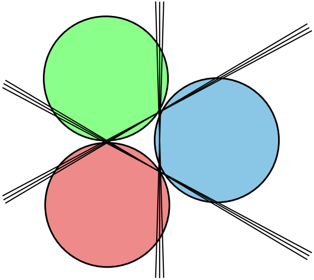

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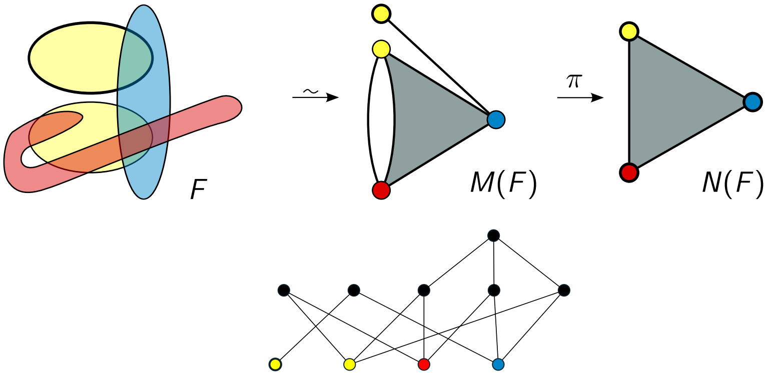
Sketch of Proof

Main new object: the multinerve



The multinerve $M(F)$ of a family F of sets is a blown-up version of the nerve $N(F)$: (roughly,) order the connected components of the intersecting subfamilies by inclusion.

- $M(F)$ is a more general simplicial poset [Björner, Stanley, ...];
- every "lower interval" is a simplex.

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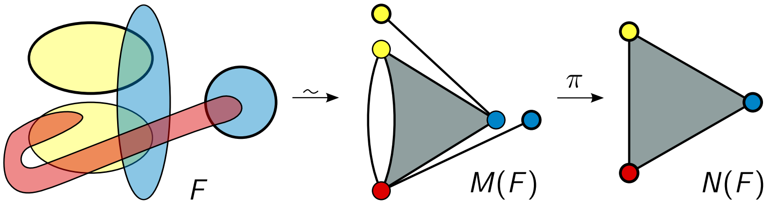

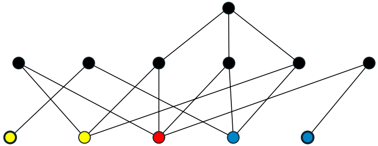

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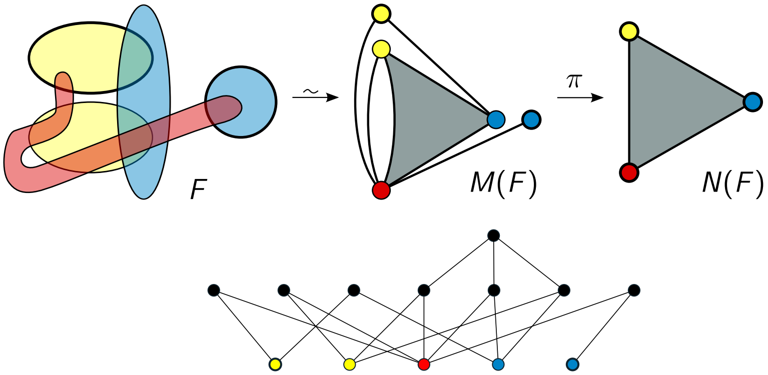

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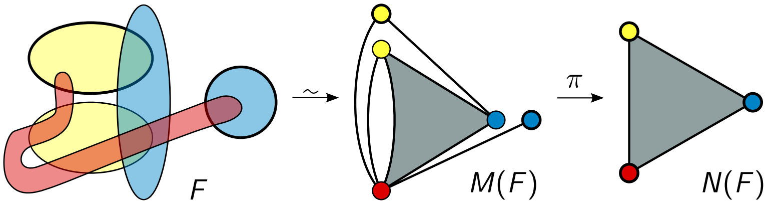

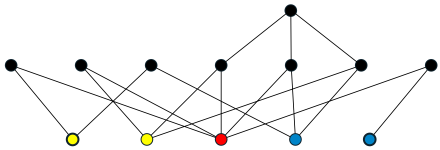

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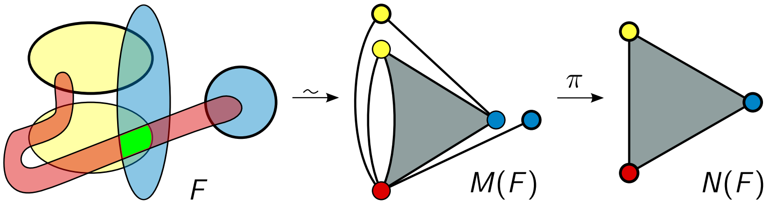

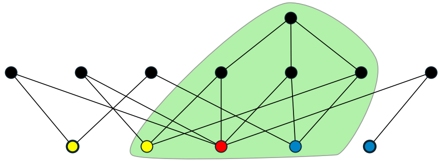

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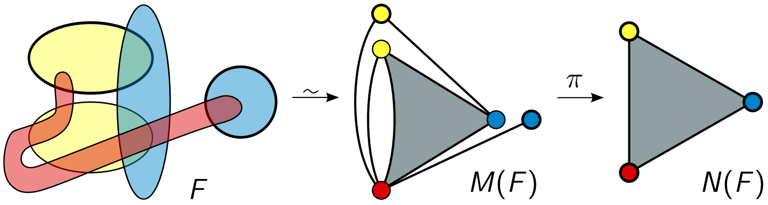
Multinerve theorem



Multinerve theorem

Let F be a family of sets in $\mathbb{R}^d$ such that
$\forall G \subseteq F$, $\bigcap_G$ is the disjoint union of finitely many contractible sets.
Then $M(F)$ and $\bigcup_F$ have holes in the same dimensions.

Proof

- Spectral sequences with Leray's acyclic cover theorem;
- alternatively, variation on [Björner, 2003].

New topological Helly theorem: proof sketch

- We know that $M(F)$ has no hole in dimension $\geq d$;
- we want to infer that $N(F)$ has no hole in dimension $\geq (d+1)r-1$.

Theorem [Kalai and Meshulam, 2008]

- Let M and N be simplicial complexes.
- Let $\pi : M \to N$ be simplicial, size-preserving, at most r-to-one, and onto.
- Assume (roughly) that M has no hole in dim. $\geq d$.
  Assume that some suitably defined subcomplexes of $\mathrm{sd}(M)$ have no hole in dim. $\geq d$.

Then N has no hole in dimension $\geq (d+1)r-1$.

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- Let M be a simplicial poset and N a simplicial complex.
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Theorem (generalizing [Kalai and Meshulam, 2008])

- Let M be a simplicial poset and N a simplicial complex.
- Let $\pi : M \rightarrow N$ be simplicial, size-preserving, at most r-to-one, and onto.
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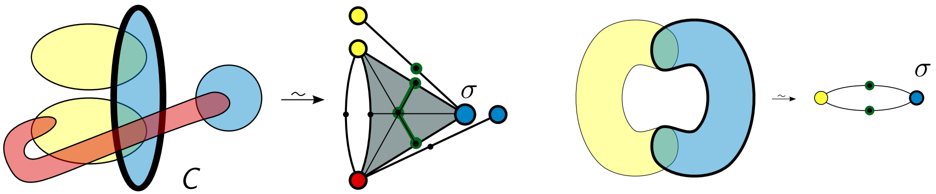
Tools

Algebraic topology (spectral sequences, multiple point set, etc.).

Proof sketch (continued)



Definition

If σ is a simplex of a simplicial poset X, then $\text{barylink}_X(\sigma)$ is the subcomplex of sd_X that is the order complex of $(\sigma, \cdot]$ in X.

Lemma

For any acyclic family F in $\mathbb{R}^d$, $\text{barylink}_{M(F)}(\sigma)$ has no hole in dimension $\geq d$.

Proof sketch (end)

Multiple point set
$M_k := \{m_1, \ldots, m_k \in |M|^k \mid \pi(m_1) = \ldots = \pi(m_k)\}.$

Consequence of [Goryunov and Mond, 1993]

Some spectral sequence $(E^\bullet_{p,q})$ converging to $H_*(N)$ satisfies:
If, for all q, for all $p \leq r-1$, and for all $p+q \geq (d+1)r-1$, we have $H_q(M_{p+1}) = 0$,
then $E^1_{p,q} = 0$ (and therefore $H_k(N) = 0$ for all $k \geq (d+1)r-1$).

Rephrasing [Kalai and Meshulam, 2008]

Some spectral sequence $(E'^\bullet_{p,q})$ converging to $H_*(M_{p+1})$ satisfies

$$E'^1_{p,q} \cong \bigoplus_{\substack{(\sigma_2,\ldots,\sigma_k) \\ \in \mathcal{S}_p}} \bigoplus_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\ldots+i_k = p+q}} H_{i_1}\left(M\left[\bigcap_{i=2}^{k} \tilde{\sigma}_i\right]\right) \otimes \bigotimes_{j=2}^{k} \tilde{H}_{i_j - 1}\left(\mathrm{barylink}_M(\sigma_j)\right)$$

Thus in our setting $H_q(M_{p+1}) = 0\ldots$

Conclusion

Fractional Helly-type theorems

Definition

F is k-fractional Helly if the following holds: If "many" k-tuples of F have non-empty intersection, then there exists a "large" subfamily of F that has non-empty intersection.

More precisely: If a fraction x of the k-tuples have non empty intersection, then a fraction $f(x)$ of the elements in F have non-empty intersection, where $f(x)$ tends to one as x tends to one.

More theorems for free!

Using [Alon, Kalai, Matoušek, Meshulam, 2002], we obtain immediately such fractional Helly theorems for r-acyclic families.

Conclusion: Get rid of topology?

Another proof without topology?

[Eckhoff and Nischke, 2009] reproves [Kalai and Meshulam, 2008] in a purely combinatorial way ("generalized pigeonhole principle").
Can we use that proof technique instead?

Core of their proof

- Let M, N be simplicial complexes.
- Let $\pi : M \to N$ be simplicial, size-preserving, at most r-to-one, and onto.
- If N contains all the strict subfamilies of a set S of size $k+1$, then $\pi^{-1}(2^S)$ contains all the subfamilies of size $\leq \lfloor \frac{k}{r} \rfloor$ of a set of size $k+1$.

Can we allow M to be a simplicial poset? Under which conditions?


Thanks!
chain complex
line
deformation retract
slack
connected
geometric realization
Kalai
differential
isomorphism
contractible
link
acyclic
convex
reduced homology
Cech
spectral sequence
Mond
sheaf
Alon
subfamily
intersection
barycentric subdivision
Amenta
orientation
Borsuk
Meshulam
page
homology
Leray number
simplicial complex
multiple point space
projection
order complex
simplicial poset
simplex
geometric transversal
Goryunov
manifold
good cover
geometric permutation
nerve
Helly
multinerve
fiber
homeomorphism
induced complex
Matousek
dimension

Most general results

Common hypotheses

- Let Γ be a locally arcwise connected topological space.
- Let F be a finite family of open subsets of Γ that is r-acyclic with slack $\boldsymbol{d}$: for every subfamily $G \subseteq F$, $G \neq \emptyset$,
  - if $|G| \geq d$, then G intersects in at most r connected components.
  - for every $i \geq \max\{1, d - |G|\}$, we have $\tilde{H}_i(\bigcap_G, \mathbb{Q}) = 0$.

General multinerve theorem

For every $i \geq d$, $\tilde{H}_i(M(F), \mathbb{Q}) \simeq \tilde{H}_i(\bigcup_F, \mathbb{Q})$.

General topological Helly theorem

Assume moreover that every open set of Γ has trivial homology in dimension $\geq d$. Then F is $((d+1)r)$-Helly.