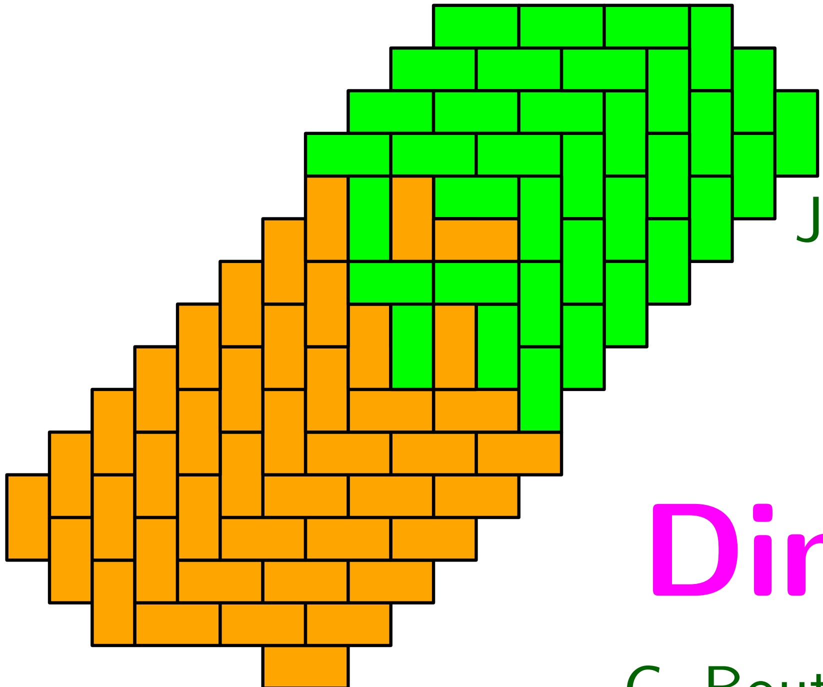
Steep tilings

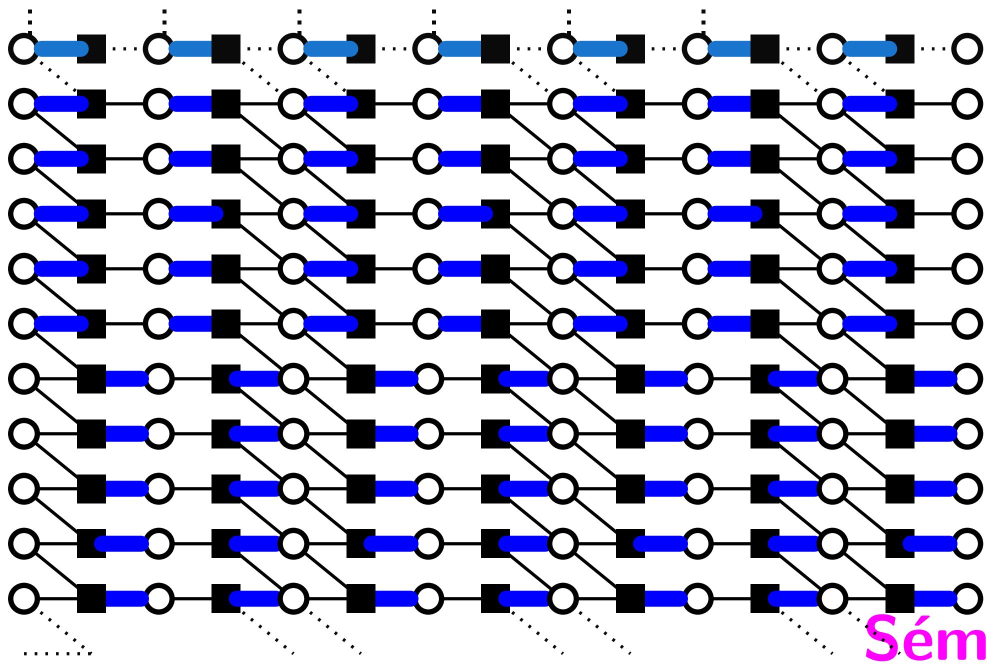


Sylvie Corteel (CNRS Paris 7)

J. Bouttier (CEA ENS) and G. Chapuy (CNRS Paris 7)

Dimers on Rail Yard Graphs

C. Boutillier (Paris 6), JB, GC, SC and S. Ramassamy (Brown)



Séminaire Flajolet - 4 Décembre 2014

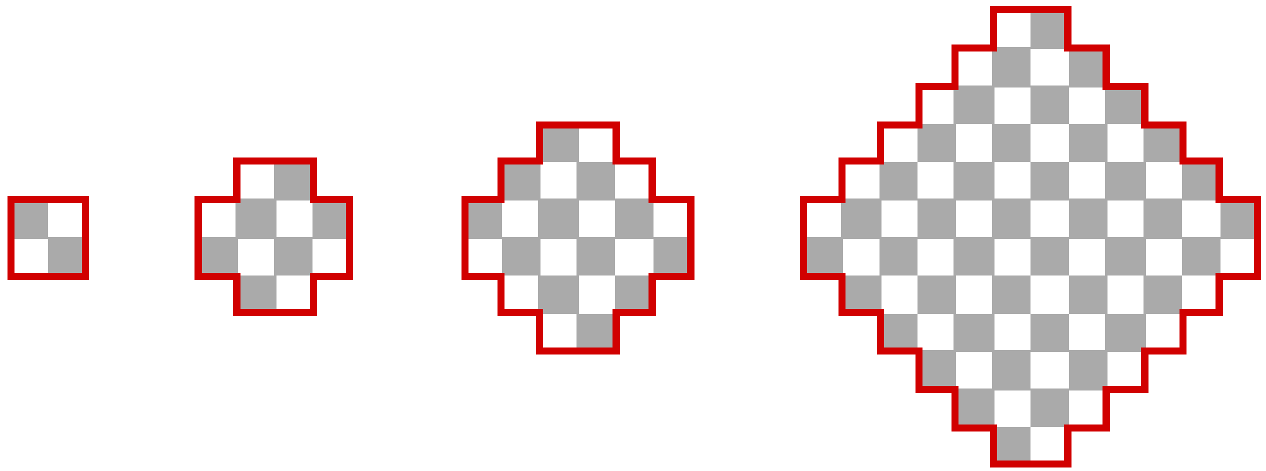
1. Aztec diamond and pyramid partitions

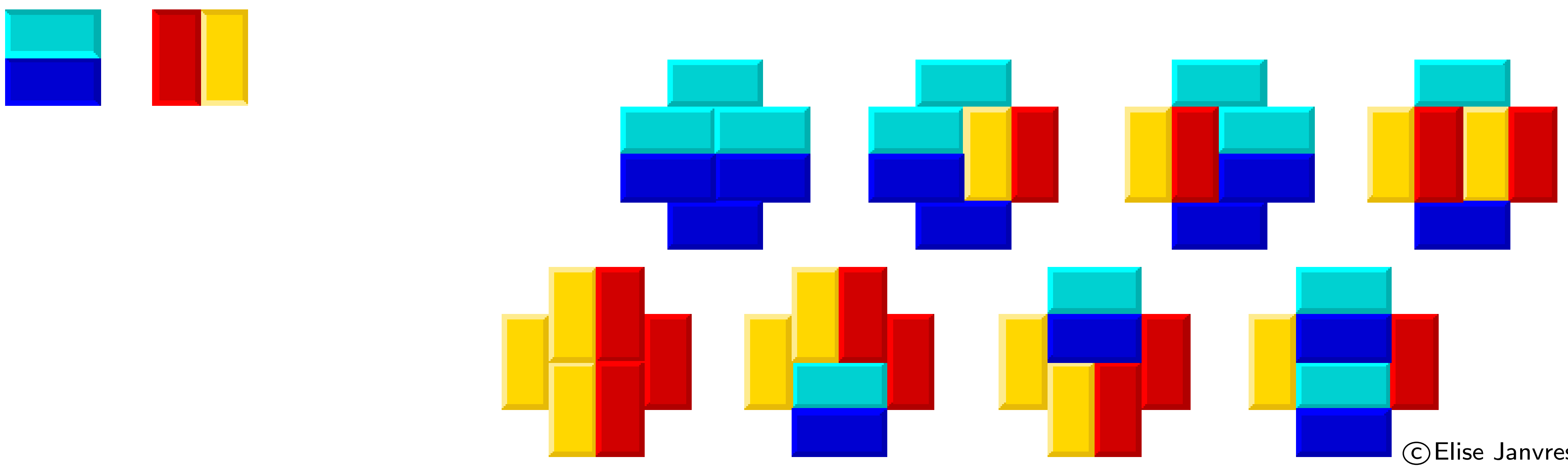
2. Steep tilings

3. Dimers on rail yard graphs

Aztec diamond

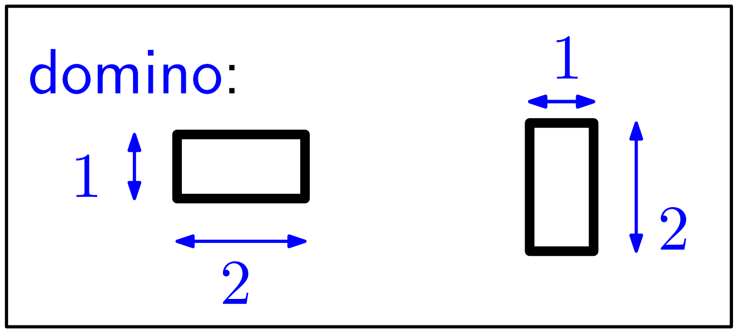
[Elkies, Kuperberg, Larsen, Propp (1992)]





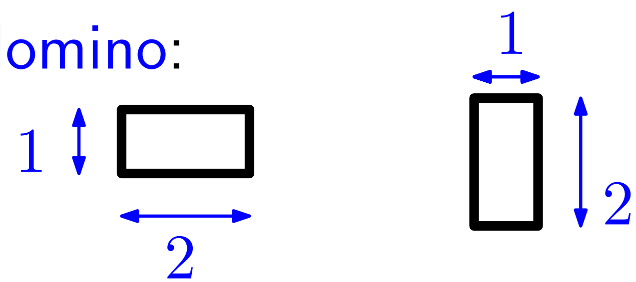
Aztec diamond [Elkies, Kuperberg, Larsen, Propp (1992)]

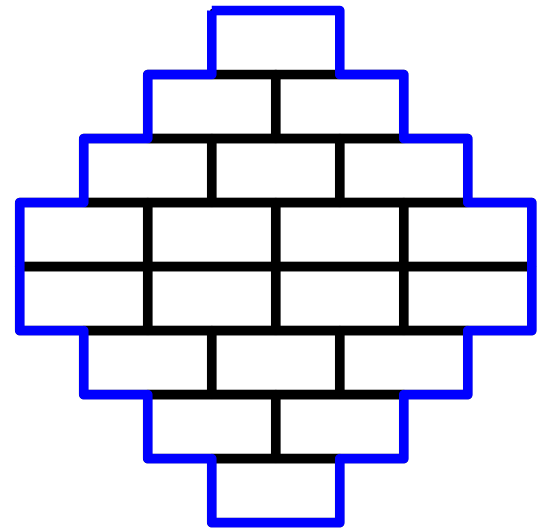
Aztec diamond [Elkies, Kuperberg, Larsen, Propp (1992)]



Aztec diamond [Elkies, Kuperberg, Larsen, Propp (1992)]

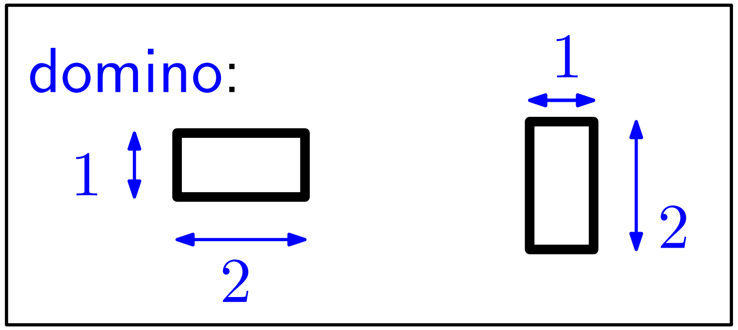
domino:

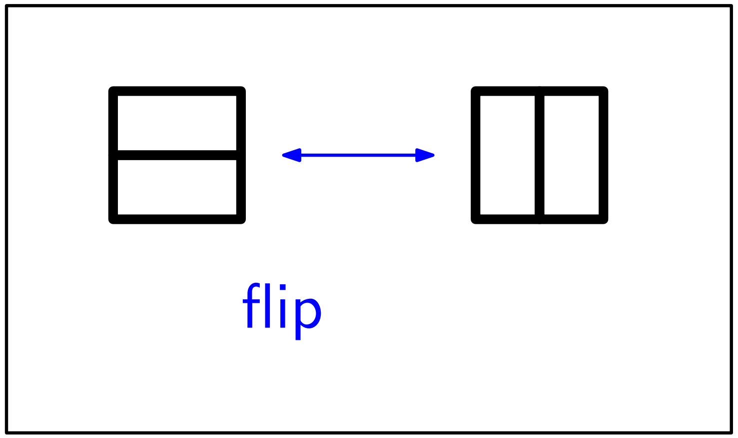


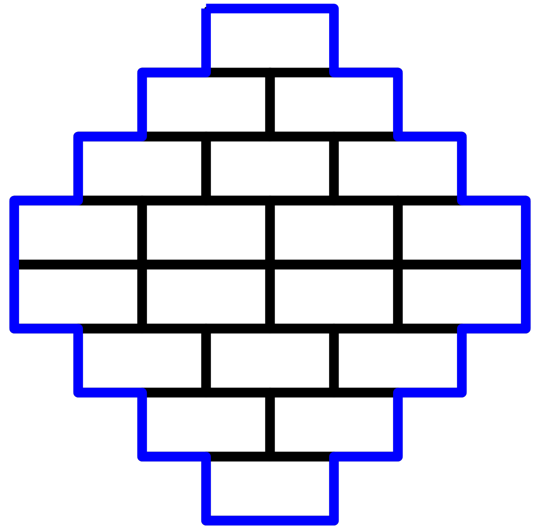


Aztec diamond of size 4)

Aztec diamond [Elkies, Kuperberg, Larsen, Propp (1992)]

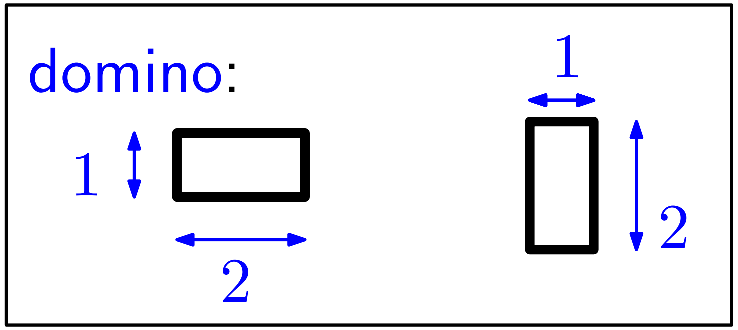


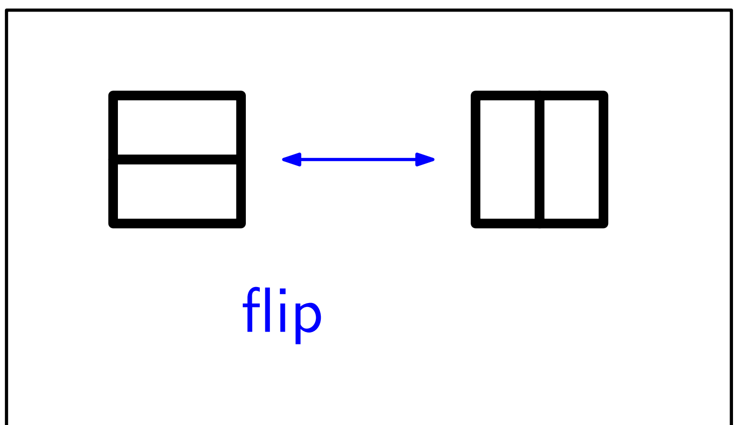


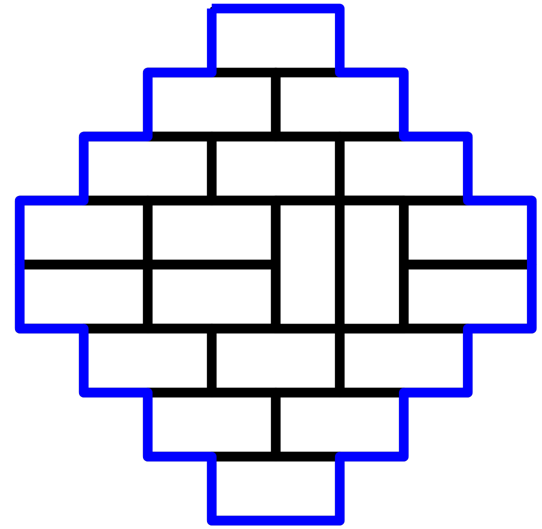


Aztec diamond of size 4)

Aztec diamond [Elkies, Kuperberg, Larsen, Propp (1992)]

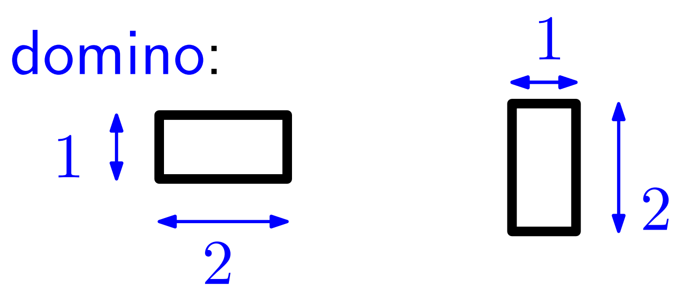


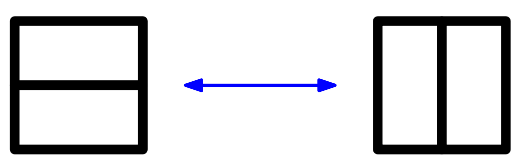




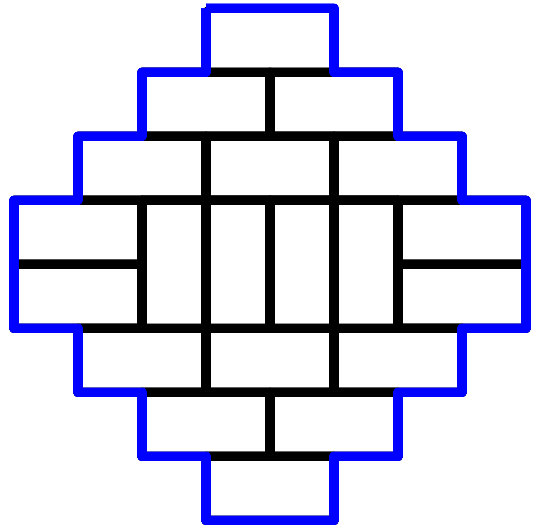
Aztec diamond of size 4)

Aztec diamond [Elkies, Kuperberg, Larsen, Propp (1992)]



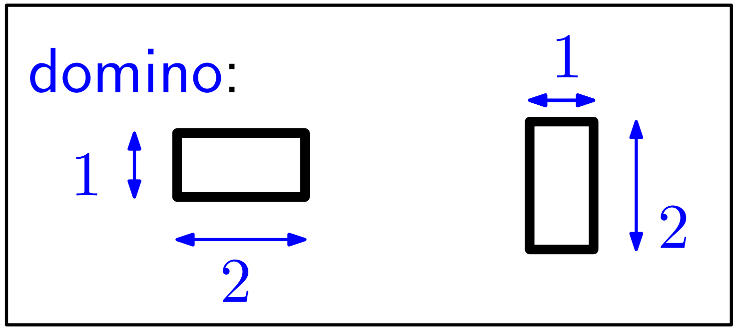


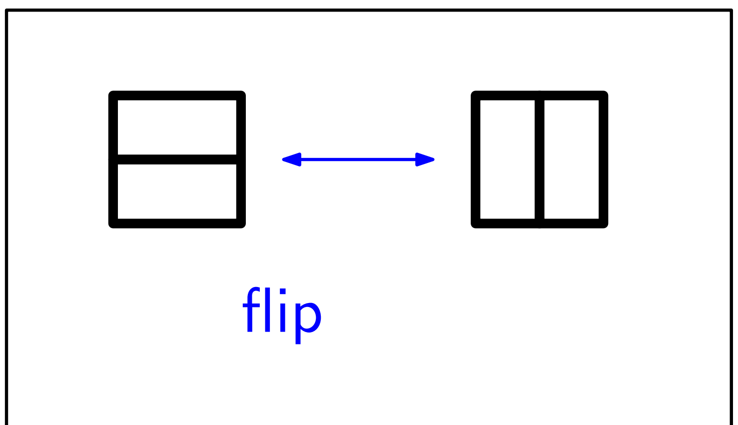
flip

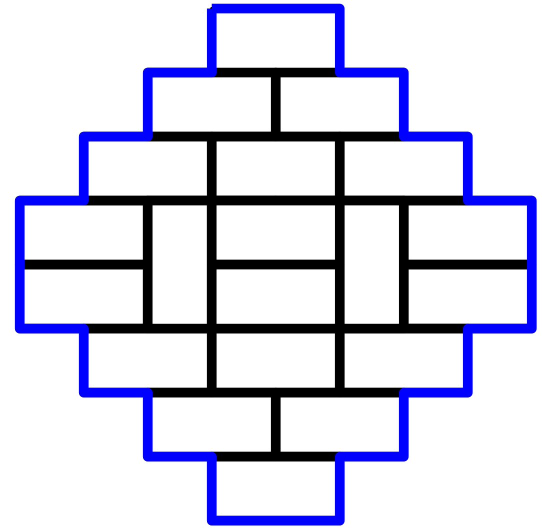


Aztec diamond of size 4)

Aztec diamond [Elkies, Kuperberg, Larsen, Propp (1992)]



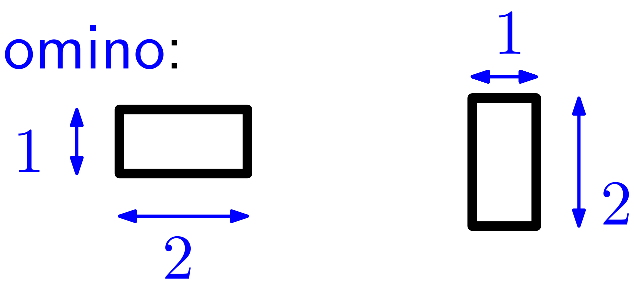


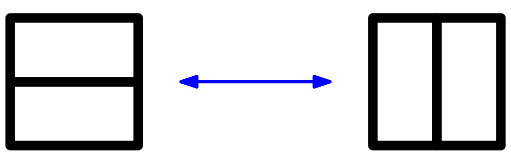


Aztec diamond of size 4)

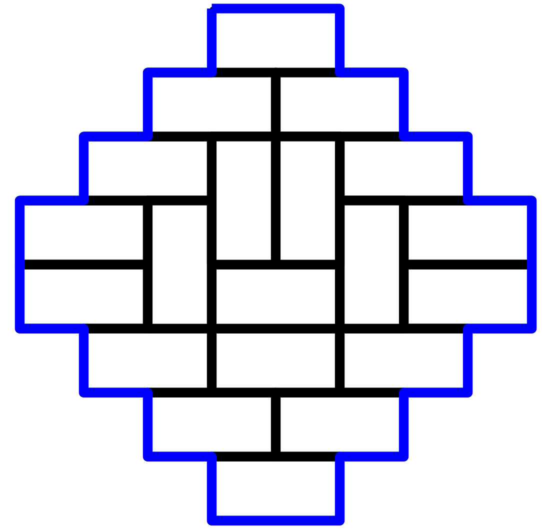
Aztec diamond [Elkies, Kuperberg, Larsen, Propp (1992)]

domino:



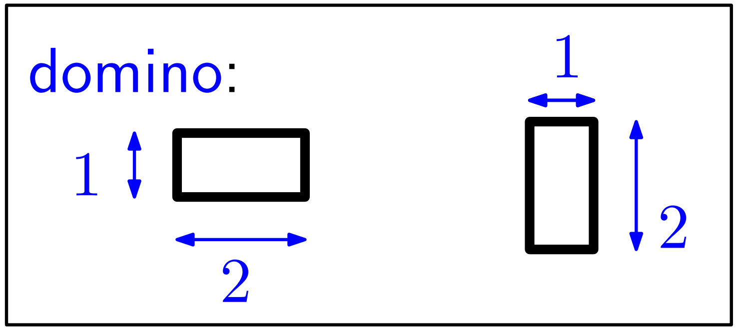


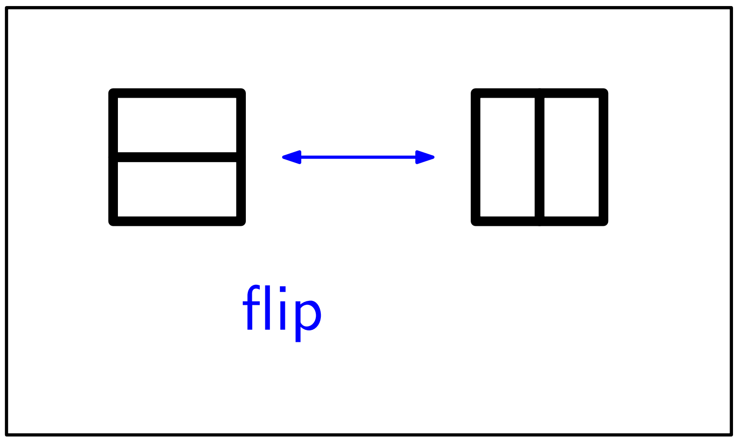
flip

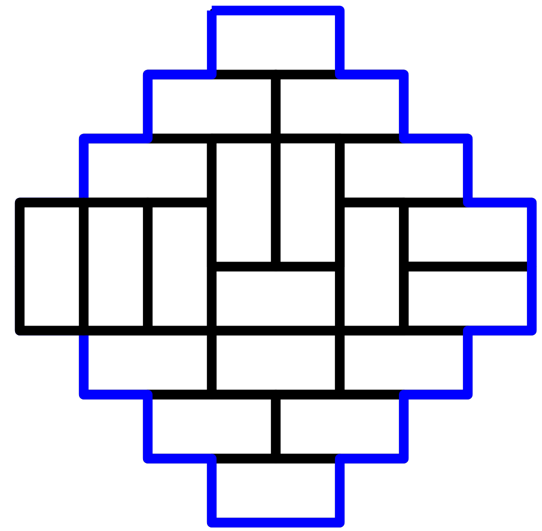


Aztec diamond of size 4)

Aztec diamond [Elkies, Kuperberg, Larsen, Propp (1992)]

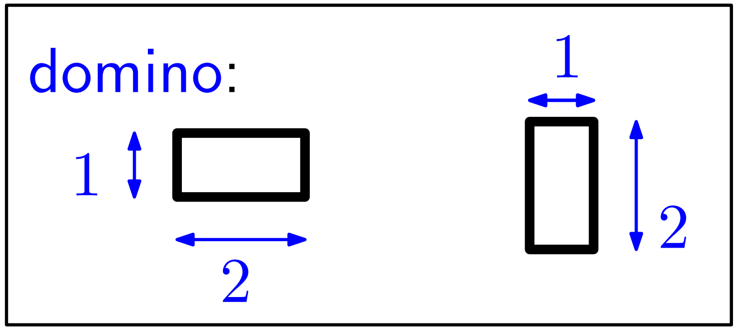


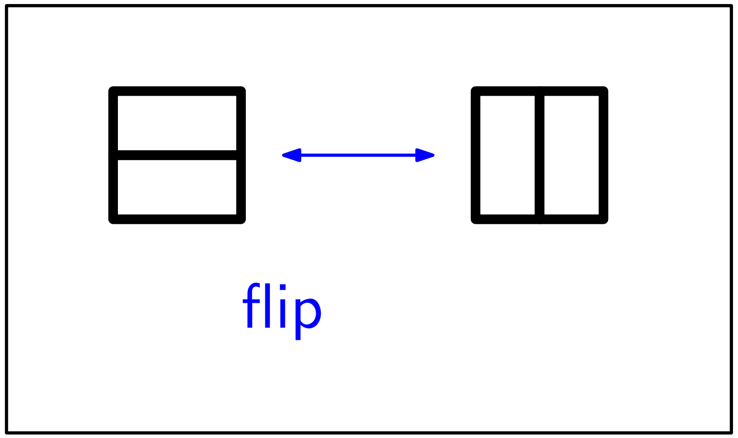


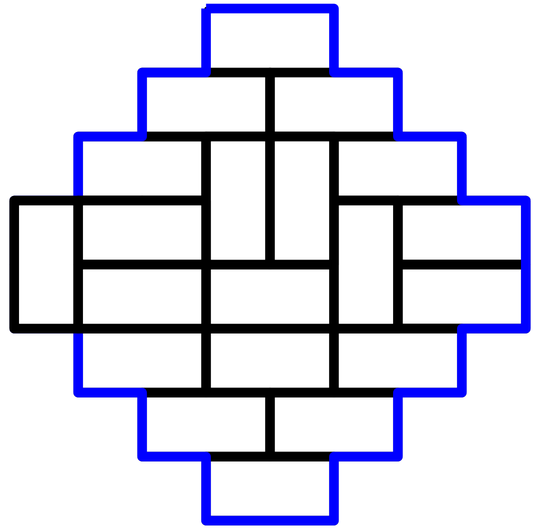


Aztec diamond of size 4)

Aztec diamond [Elkies, Kuperberg, Larsen, Propp (1992)]



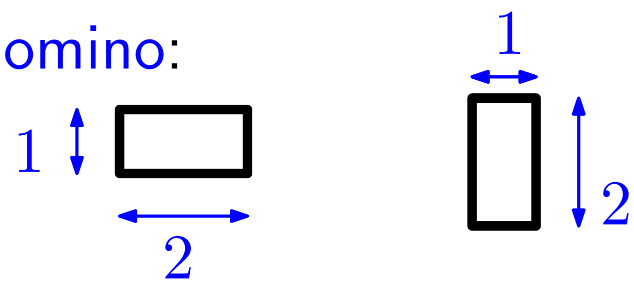


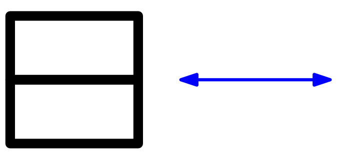


Aztec diamond of size 4)

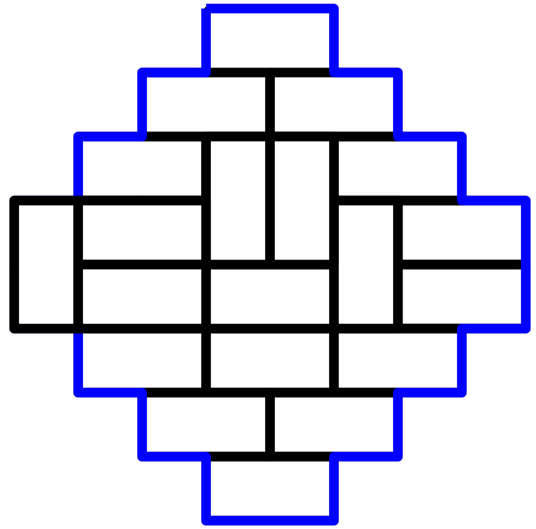
Aztec diamond [Elkies, Kuperberg, Larsen, Propp (1992)]

domino:





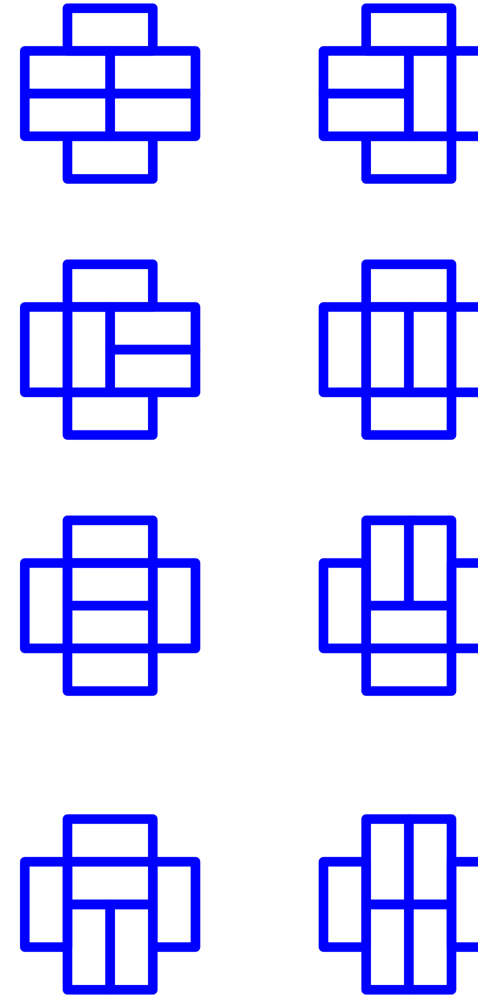
flip



Aztec diamond of size 4)

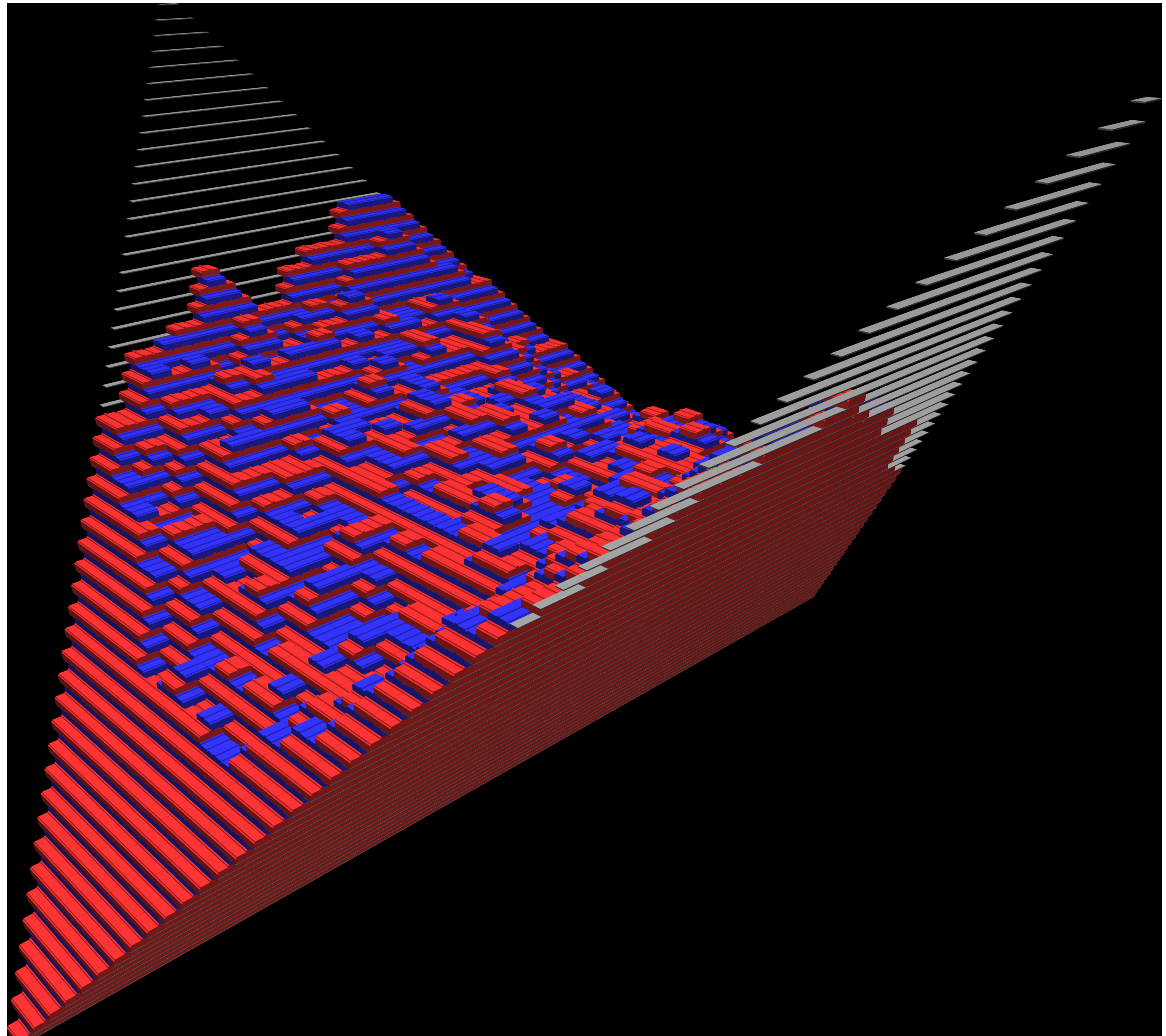
Example: $\ell = 2$

$2^3 = 8$ tilings



• Thm[EKLP92]: The aztec diamond of size ℓ has $2^{\frac{\ell(\ell+1)}{2}}$ tilings

Domino tilings↔ Piles of bricks



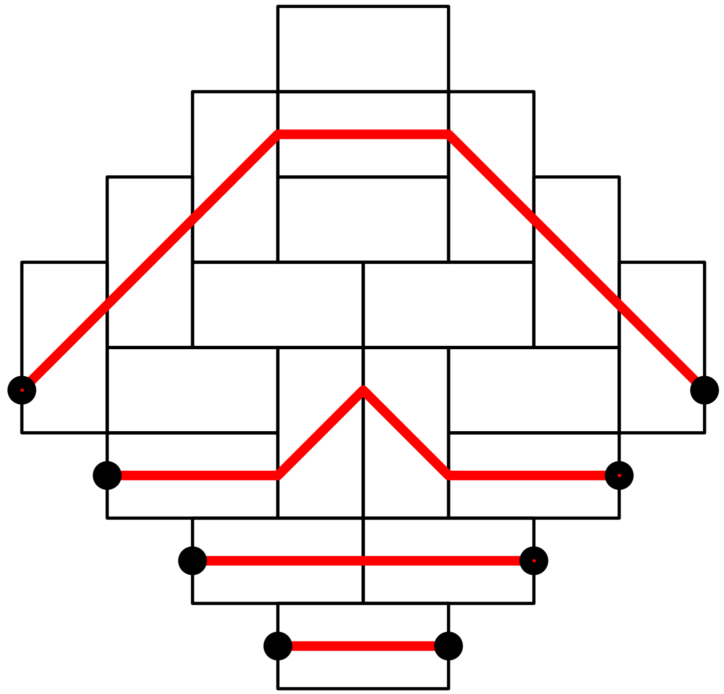
©Ben Young

Flips ↔ Remove/add a brick

Height functions [Thurston]

Aztec diamond [Eu et Fu 2005]

A_ℓ number of tilings of the diamond of size ℓ

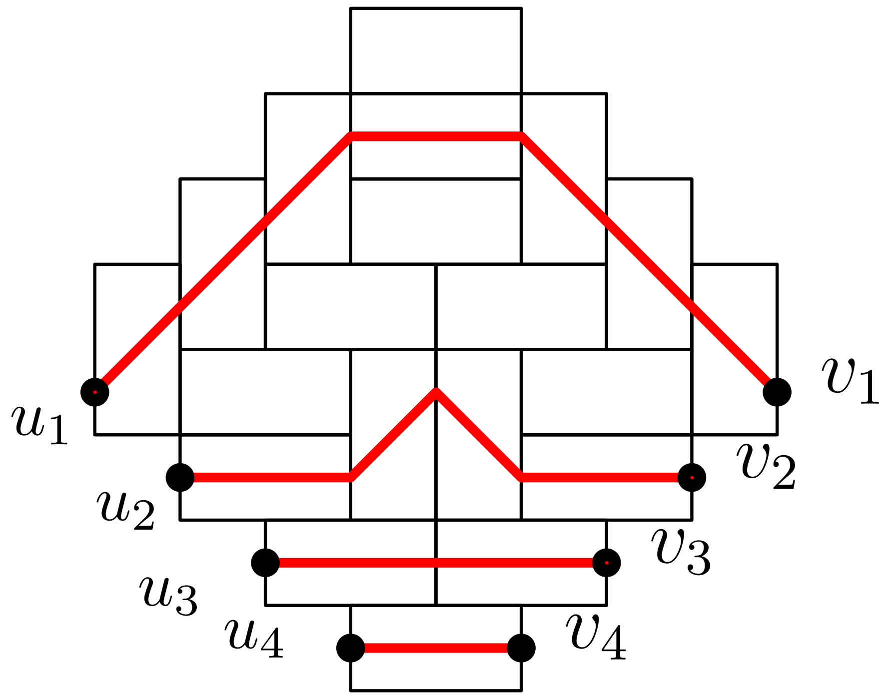


Non intersecting paths

Lindström, Gessel, Viennot (80s)

Aztec diamond [Eu et Fu 2005]

A_ℓ number of tilings of the diamond of size ℓ



Non intersecting paths

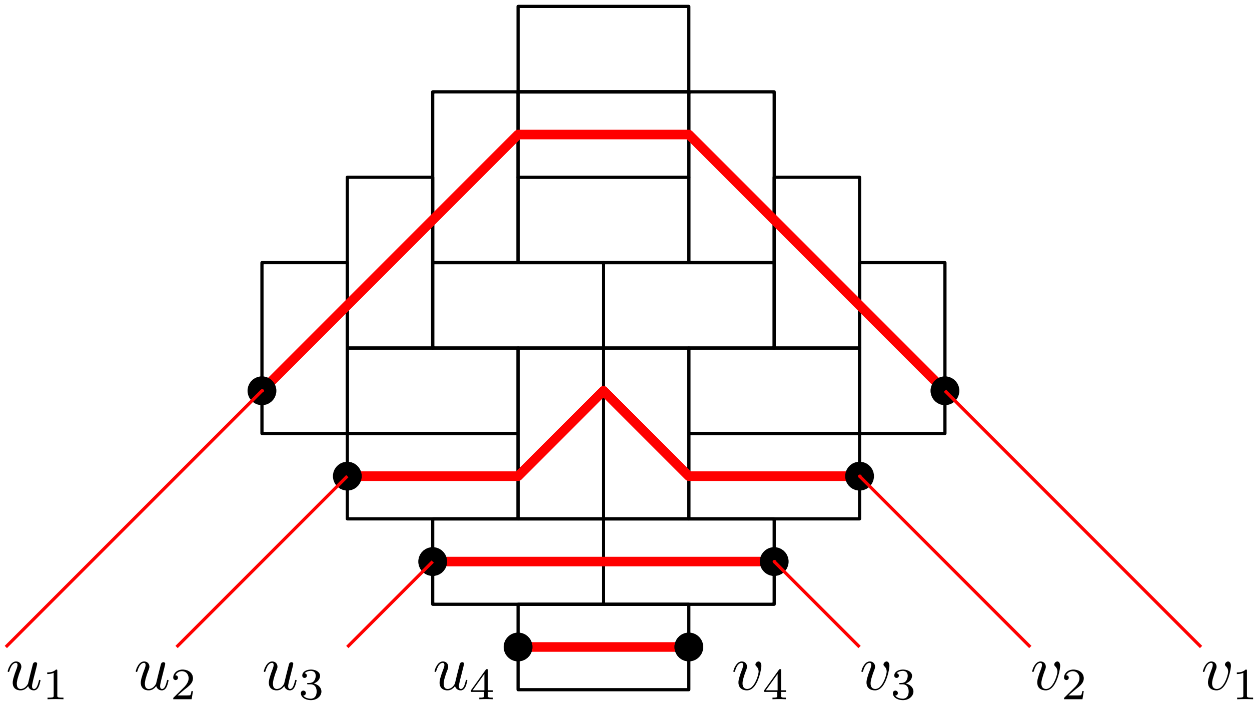
Lindström, Gessel, Viennot (80s)

$$A_n = \det(S_{i,j})_{1\leq i,j\leq \ell}$$

$S_{i,j}$: number of paths from u_i to v_j

Aztec diamond [Eu et Fu 2005]

A_ℓ number of tilings of the diamond of size ℓ



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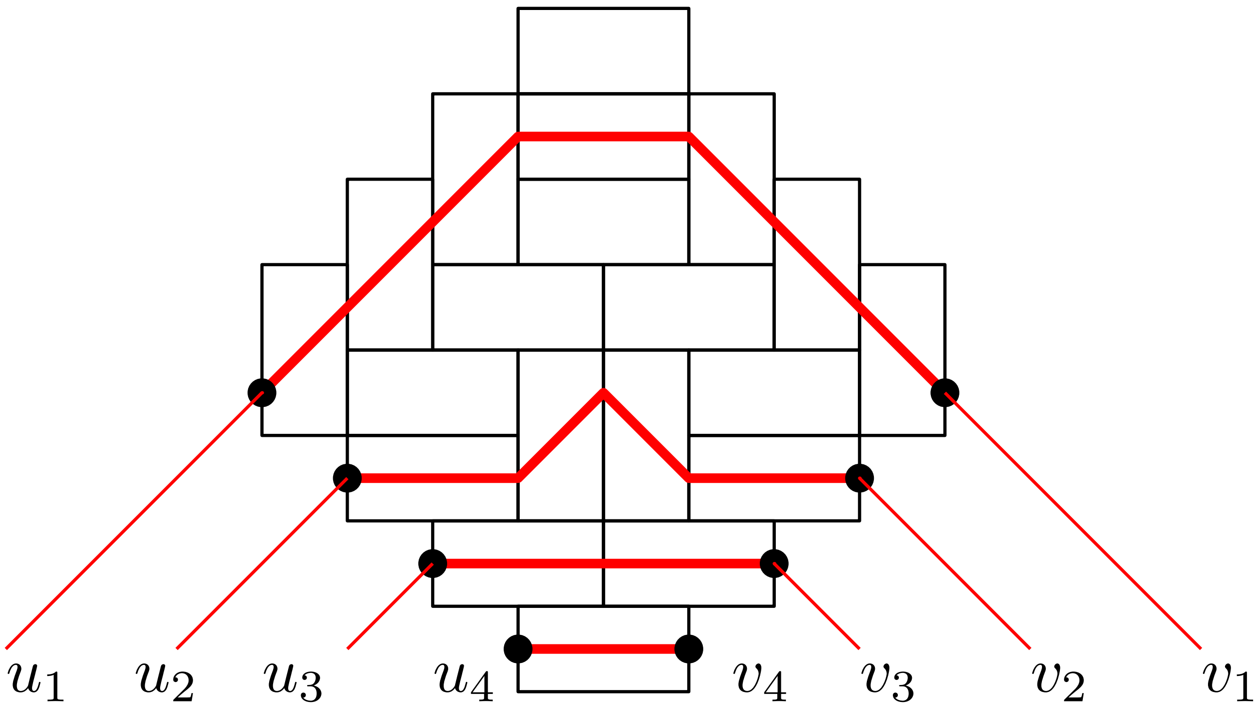

Non intersecting paths

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$$A_n = \det(S_{i,j})_{1\leq i,j\leq \ell}$$

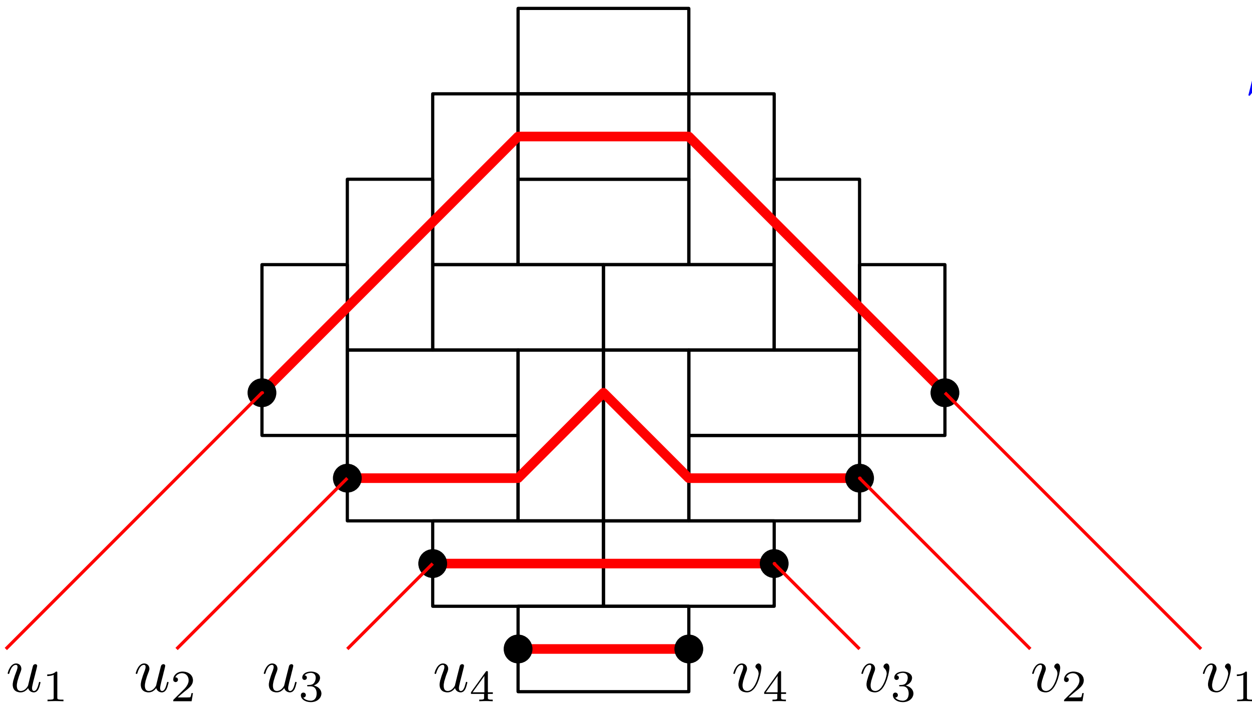
$S_{i,j}$: number of paths from u_i to v_j

Aztec diamond

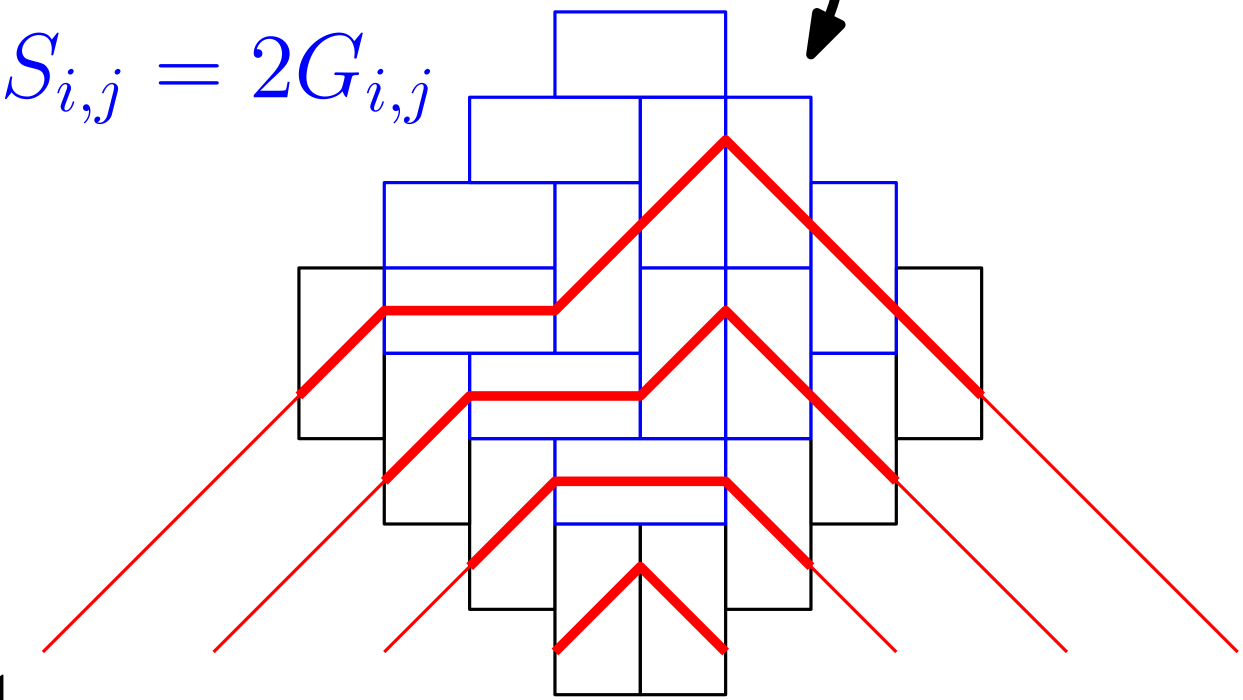
[Eu et Fu 2005]

Big/Small Schröder numbers

A_ℓ number of tilings of the diamond of size ℓ



$$S_{i,j} = 2G_{i,j}$$



Non intersecting paths

Lindström, Gessel, Viennot (80s)

$$A_n = \det(S_{i,j})_{1\leq i,j\leq \ell}$$

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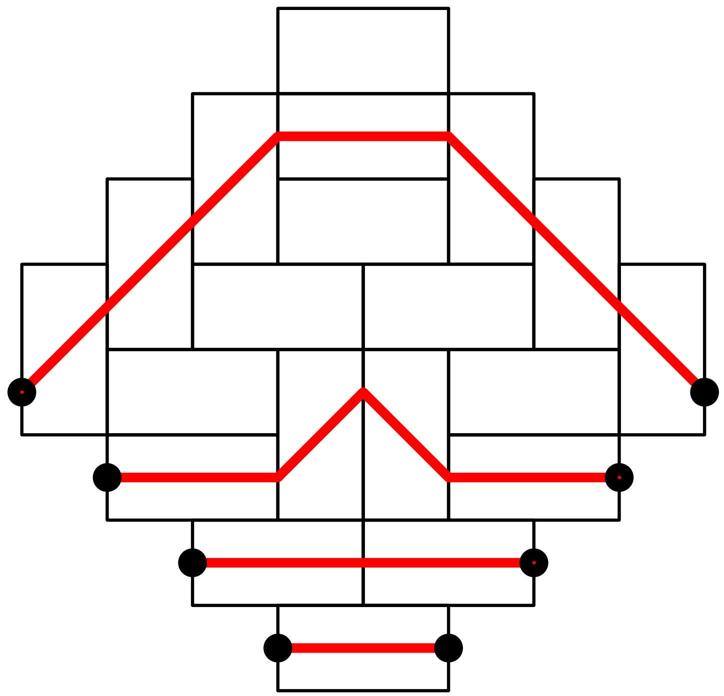
Aztec diamond

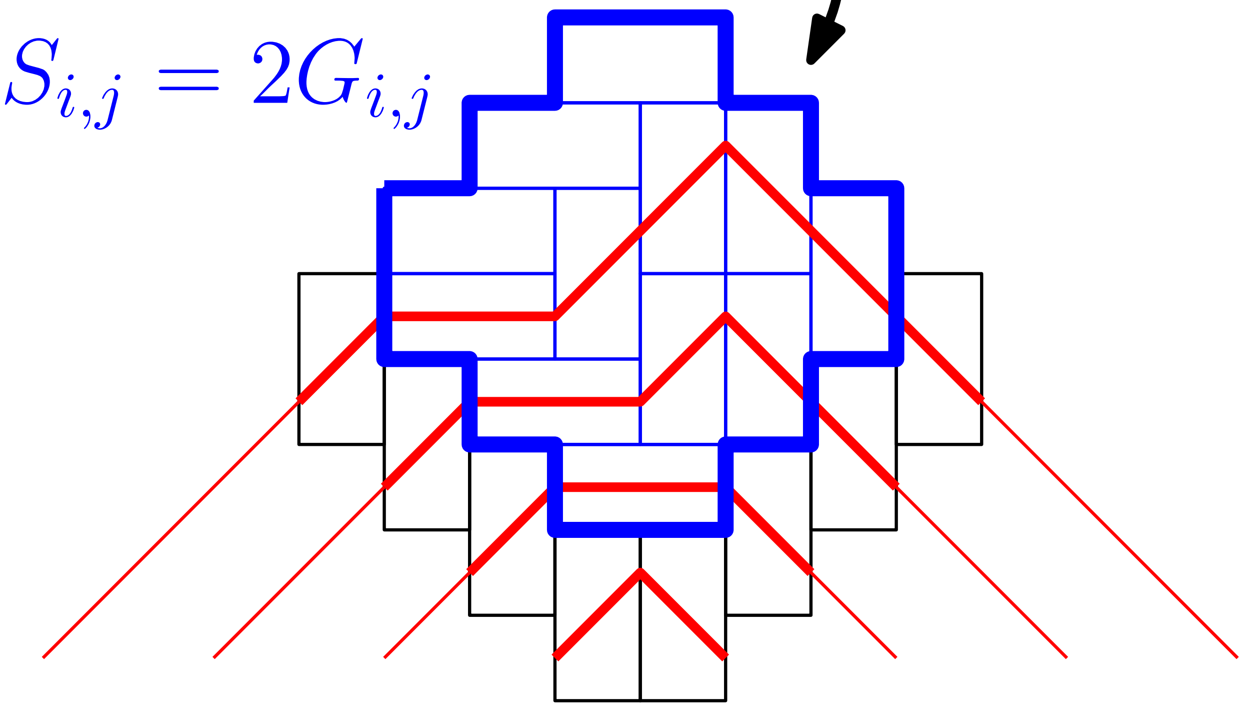
[Eu et Fu 2005]

Big/Small Schröder numbers

A_ℓ number of tilings of the diamond of size ℓ



$S_{i,j} = 2G_{i,j}$



Non intersecting paths

Lindström, Gessel, Viennot (80s)

$$A_n = \det(S_{i,j})_{1\le i,j\le \ell}$$

$$\det(G_{i,j})_{1\le i,j\le \ell} = A_{\ell-1}$$

$$\boxed{A_\ell = 2^{\binom{\ell+1}{2}}}$$

Aztec diamond

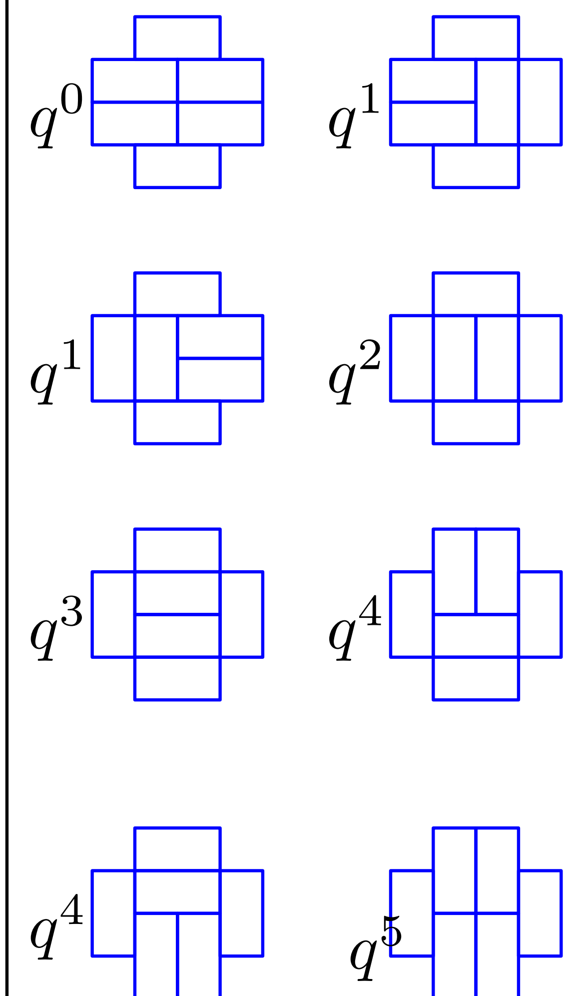
• Let P a tiling and let $h(P)$ be the minimal number of flips to obtain P from the tiling with horizontal tiles.

• Thm[EKLP92]: Tilings of the aztec diamond on size ℓ

$$\sum_{P} q^{h(P)} = \prod_{i=1}^{\ell} (1 + q^{2i-1})^{\ell-i+1}$$

$q = 1 \Rightarrow 2^{\frac{(\ell+1)\ell}{2}}$

Example: $\ell = 2$



$$= (1+q)^2(1+q^3)$$

Aztec diamond

- Let P a tiling and let $h(P)$ be the minimal number of flips to obtain P from the tiling with horizontal tiles.

- Thm[EKLP92]: Tilings of the aztec diamond on size n

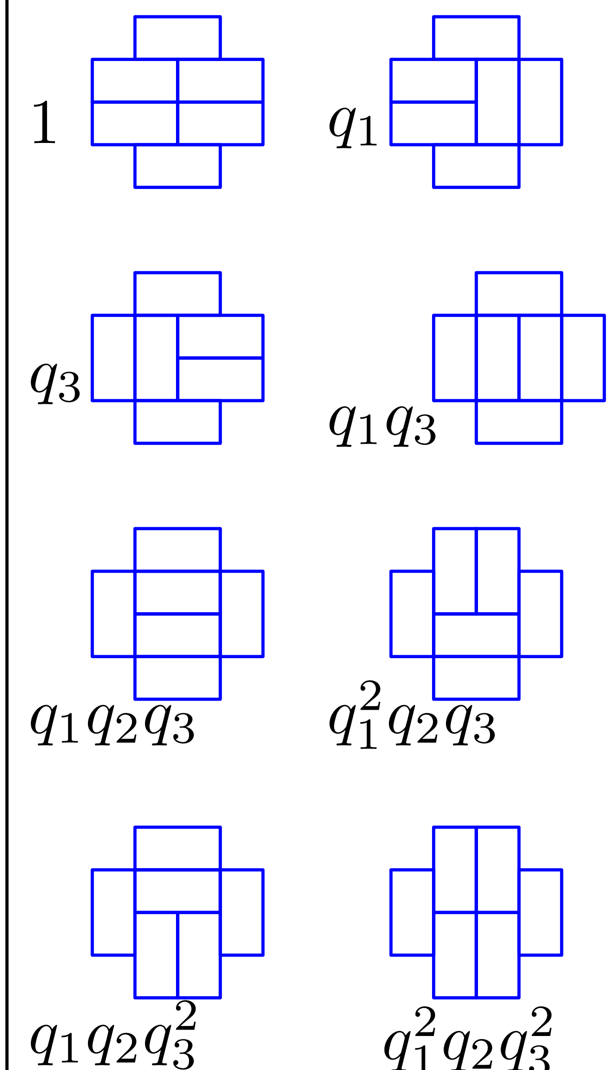
$$\sum_{P} q^{h(P)} = \prod_{i=1}^{\ell} (1+q^{2i-1})^{\ell-i+1}$$

$q=1 \Rightarrow 2^{\frac{(\ell+1)\ell}{2}}$

- Thm[Stanley]: If q_i counts the flips on the diagonal $y=x-i$

$$\sum_{P} \prod_{i} q_i^{\text{flips sur } y=x-i} = \prod_{1\le i\le j\le \ell} (1+q_{2i-1}\cdots q_{2j-1})$$

Example: $\ell = 2$



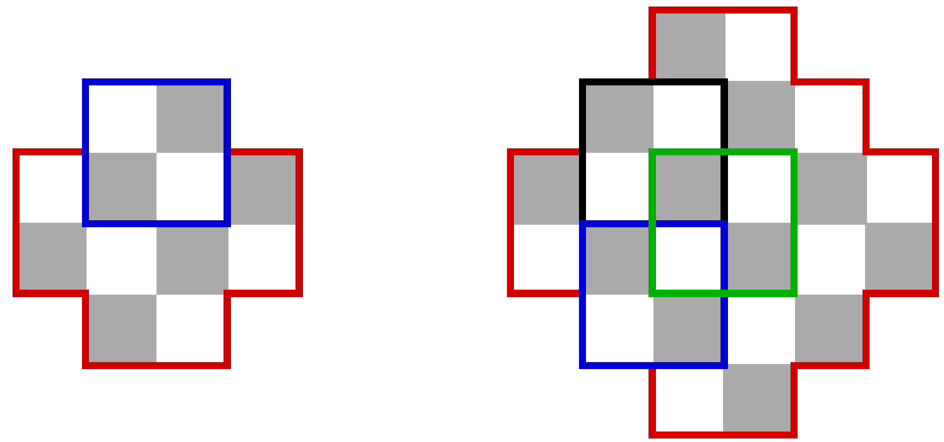
$$= (1+q_1)(1+q_1q_2q_3)(1+q_3)$$

How can we generate randomly a tiling?

Shuffling algorithm [EKLP 92]

Get a tiling of size n from a tiling of size $\ell - 1$ and ℓ bits

Cell : 2*2 square

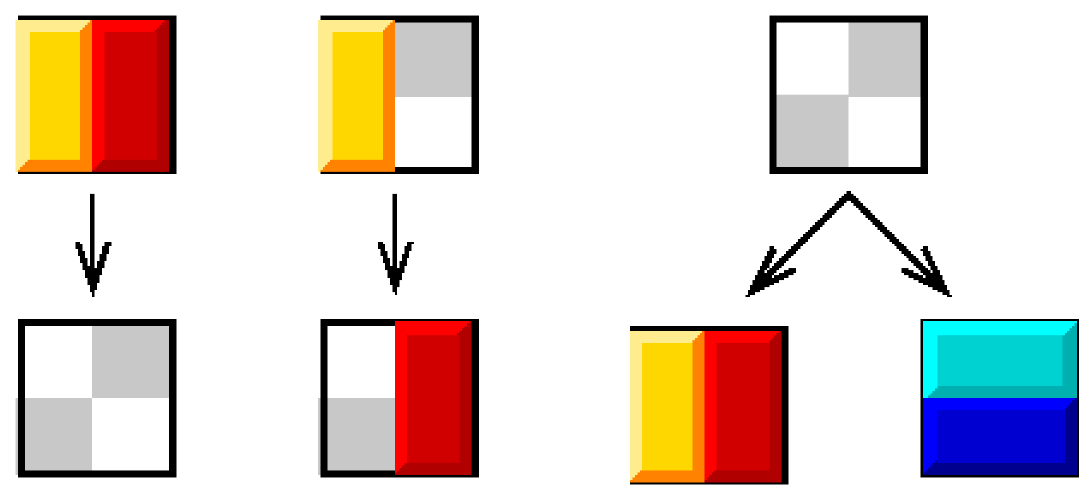


Active cell: up left square is white (ℓ even) grey (ℓ odd)

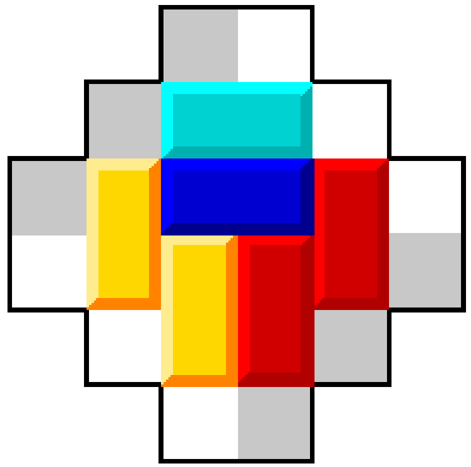
A cell has 2 dominos: erase

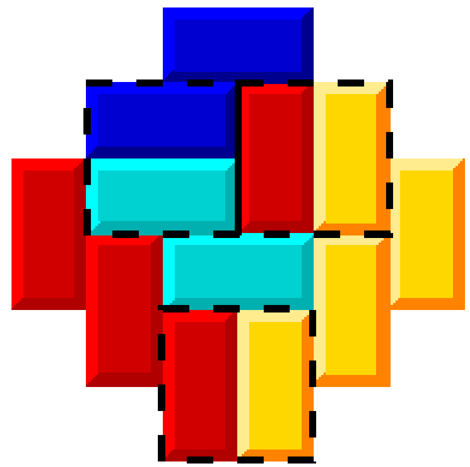
A cell has one domino: move it to the other side

A cell has no domino: flip a coin



Example $\ell = 3$





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Movie!

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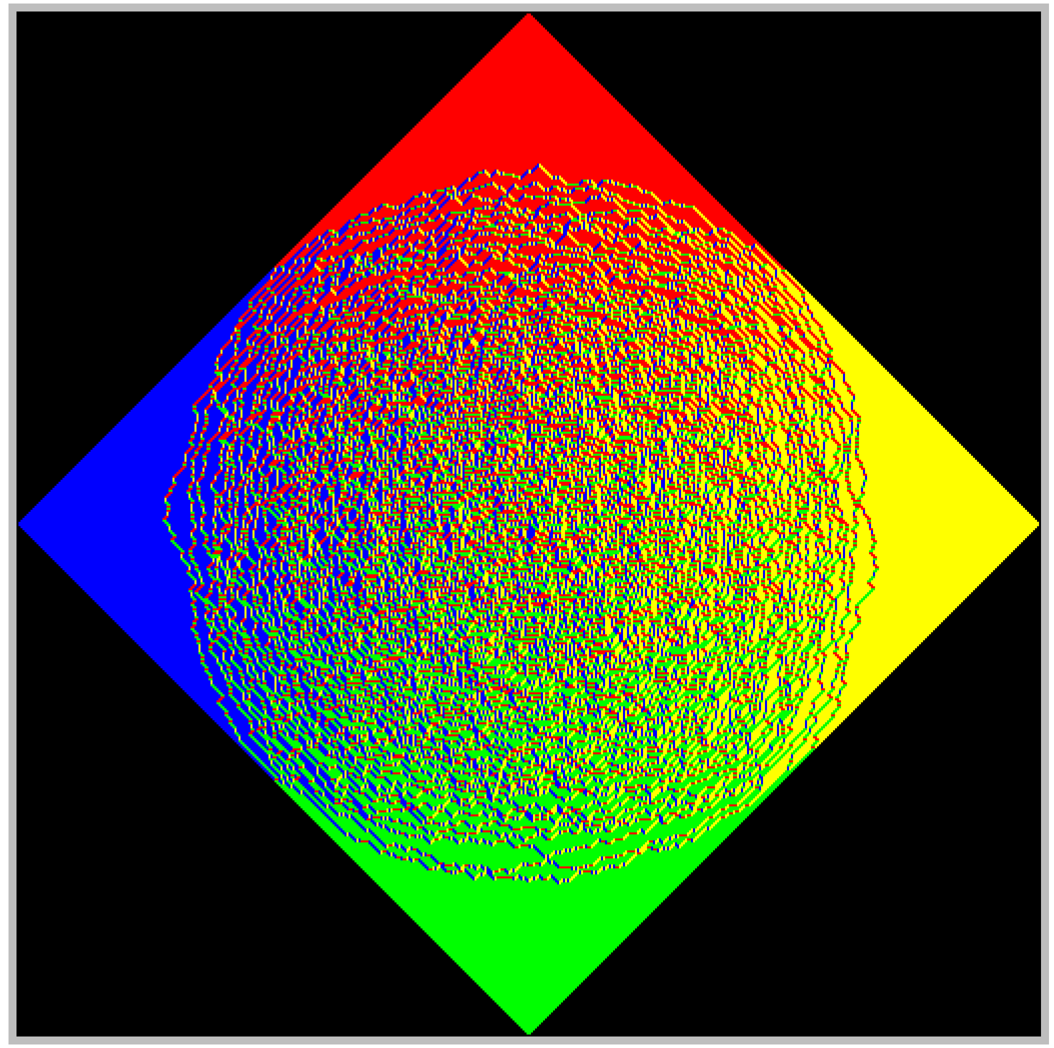
Aztec diamond: artic circle

- When ℓ is big how does a randomly chosen tiling look like?



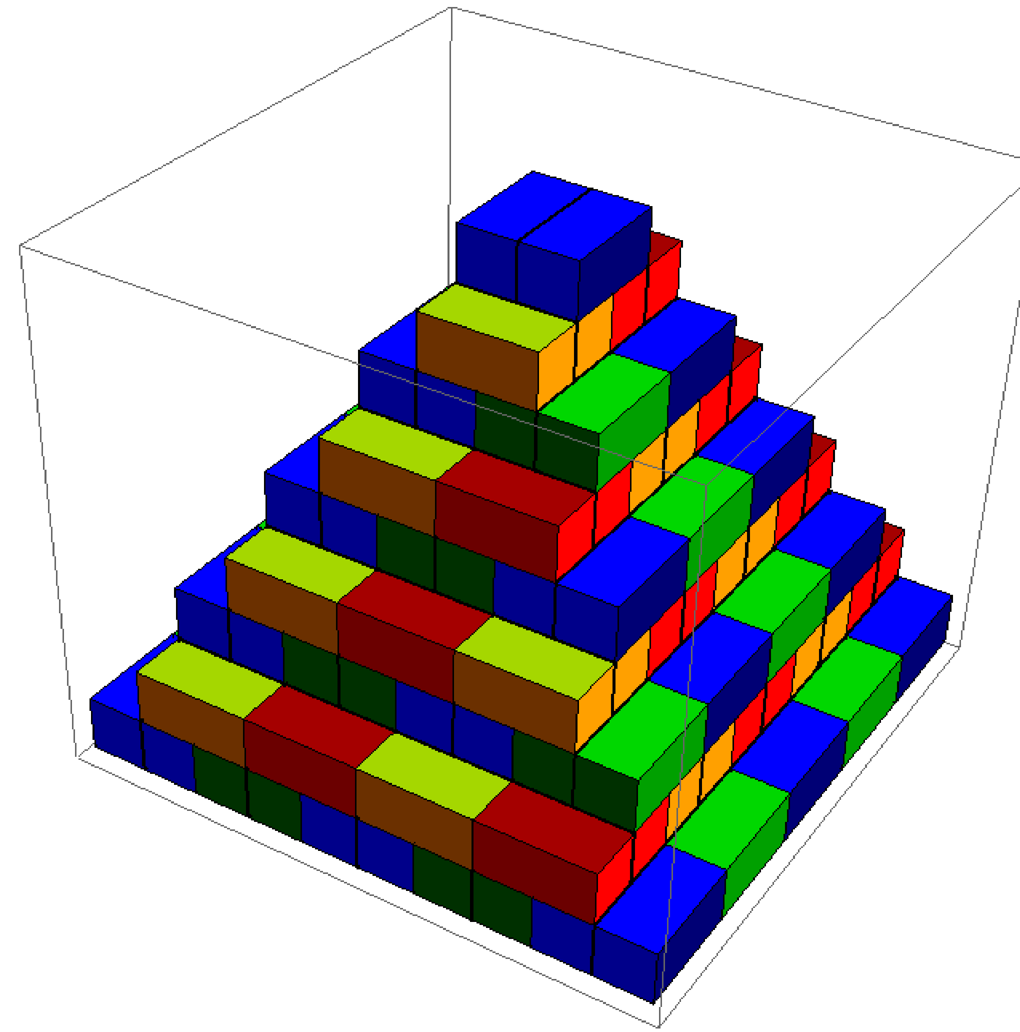
- Thm[Jockusch-Propp-Shor 95] The arctic circle theorem: A random tiling of a large Aztec diamond tends to be frozen outside a certain circle.

- Fluctuations [Johansson 05] (Airy process).

- Inverse of the Kasteleyn matrix [Chhita and Young 13]

Pyramid partitions

[Szendrõi] [Kenyon], [Young, 2006]:

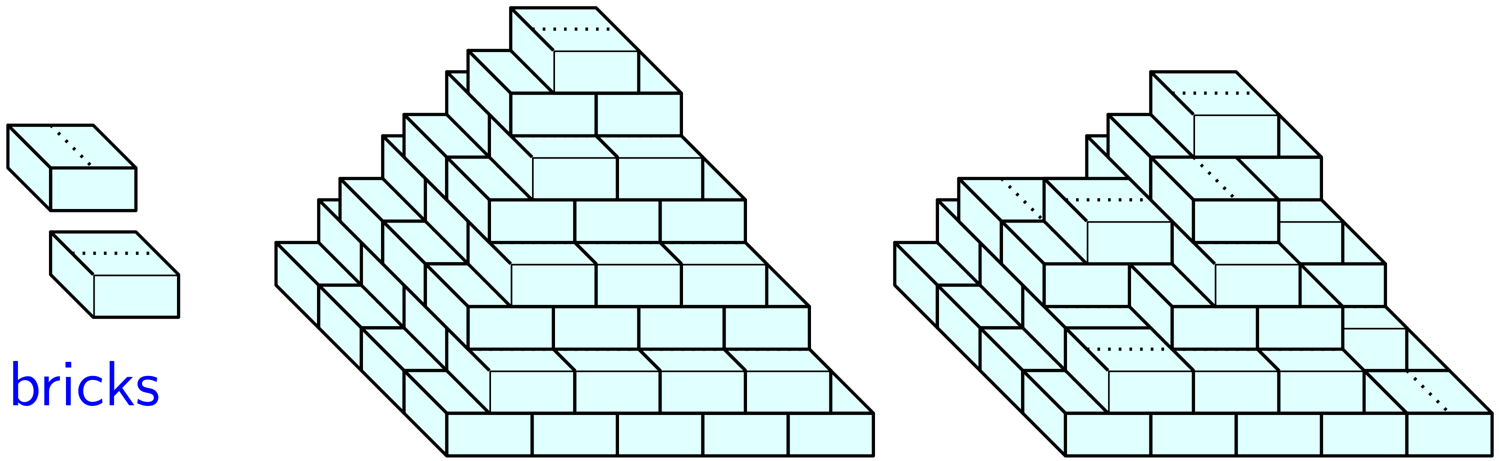


a_n number of configurations after taking off n bricks

$$a_0 = 1,\ a_1 = 1,\ a_2 = 2,\ a_3 = 5 \ldots$$

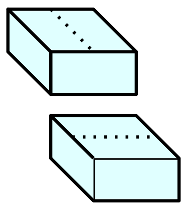
Pyramid partitions

- Partitions pyramides [Szendrõi] [Kenyon], [Young, 2006]:

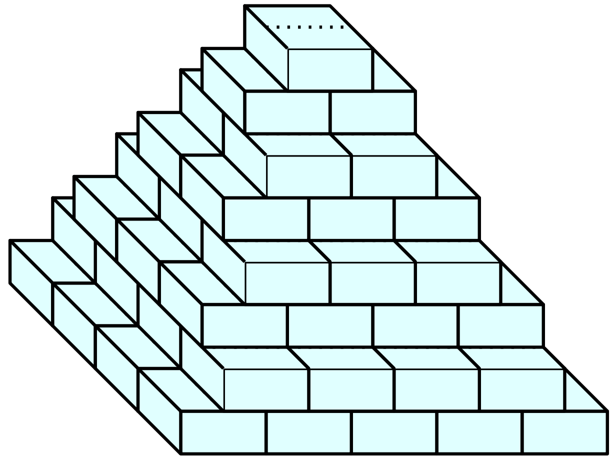


Pyramid partitions

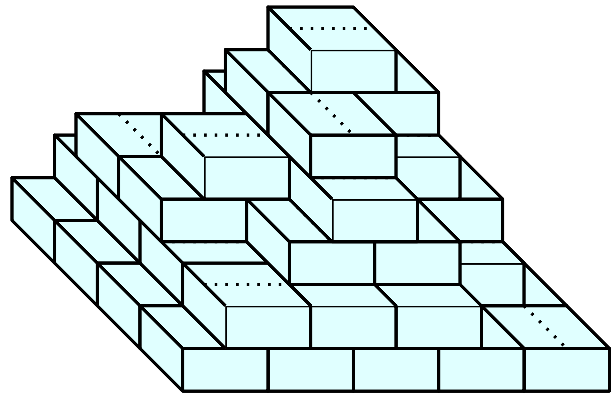
- Partitions pyramides [Szendrõi] [Kenyon], [Young, 2006]:



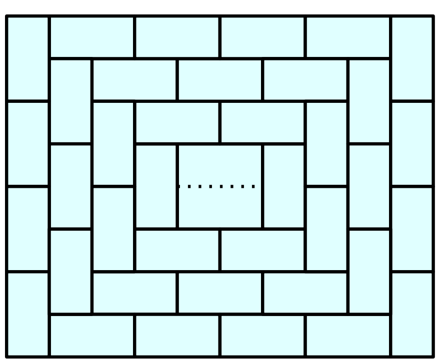
bricks



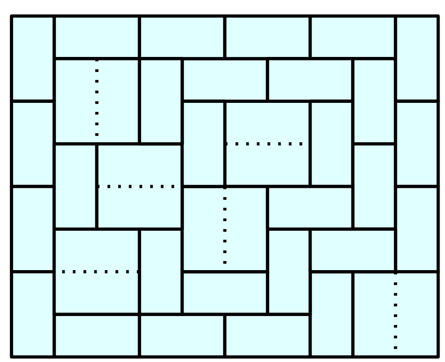
minimal pyramid



- From the top: domino tilings of the whole plane!

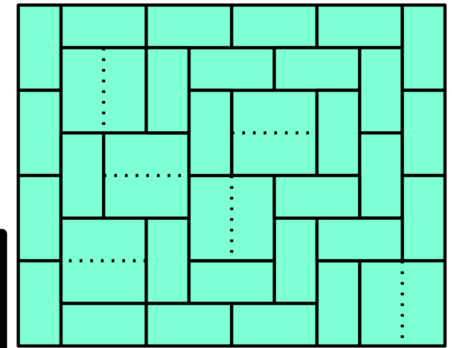


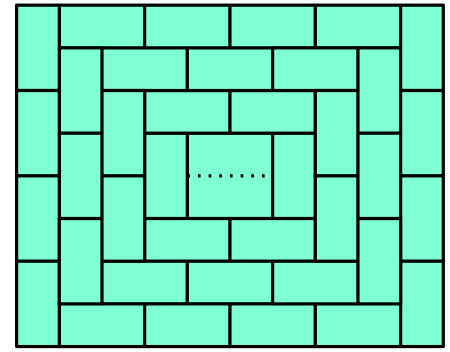
minimal tiling



a_n: number of tilings after n flips

Pyramid partitions



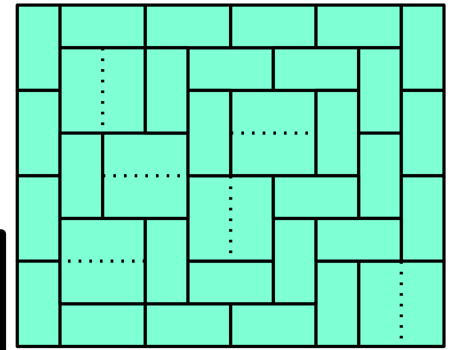


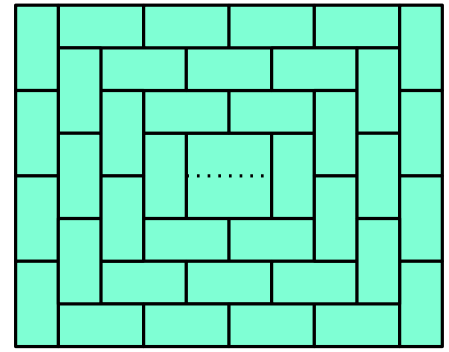
- Thm[Young 06]: Generating function

$$\boxed{\sum_{P} q^{\#\text{flips}(P)} = \prod_{i\geq 1} \frac{(1+q^{2i-1})^{2i-1}}{(1-q^{2i})^{2i}}}$$

Pyramid partitions

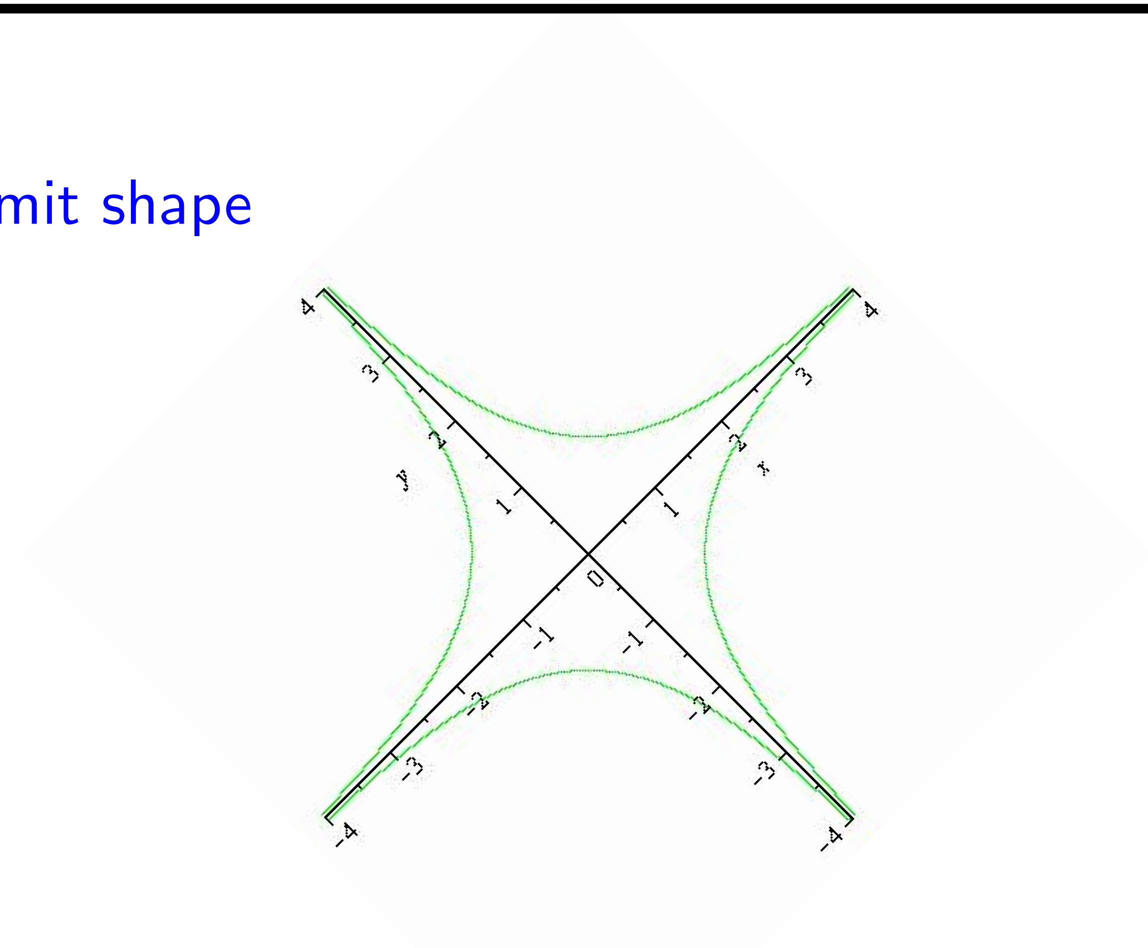
- Thm[Young 06]: Generating function





$$\boxed{\sum_{P} q^{\#\text{flips}(P)} = \prod_{i\geq 1} \frac{(1+q^{2i-1})^{2i-1}}{(1-q^{2i})^{2i}}}$$

- limit shape

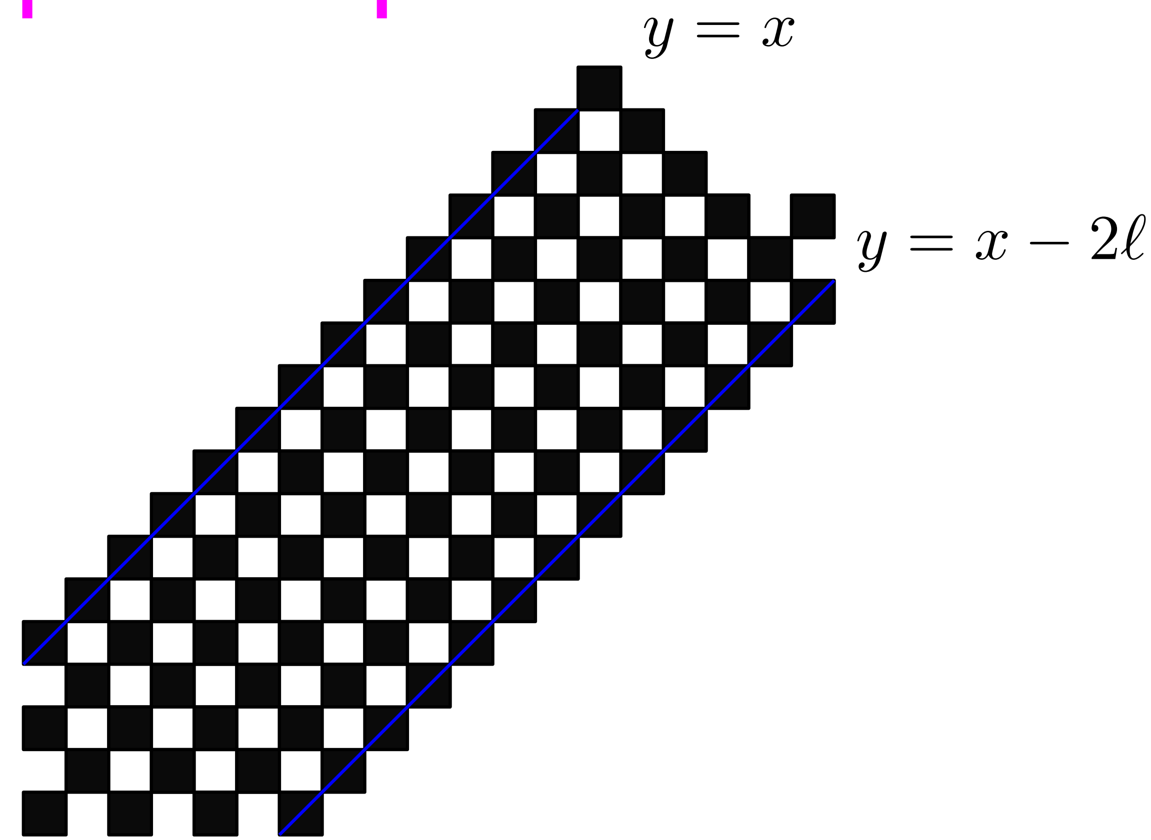


[Kenyon-Okounkov-Sheffield 03]

2. Steep tilings

Steep tilings in an oblique strip

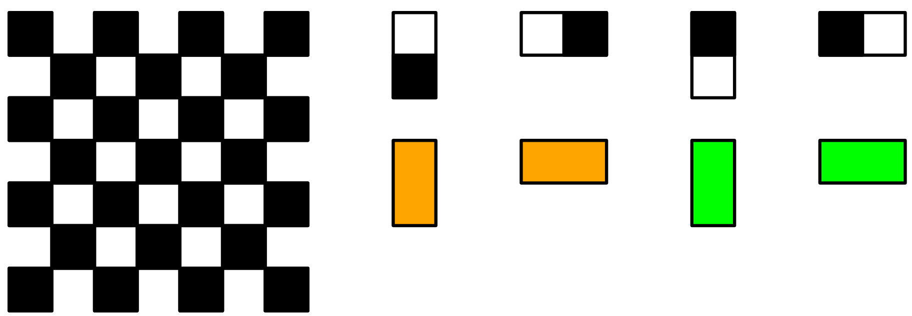
- Oblique strip of with 2ℓ

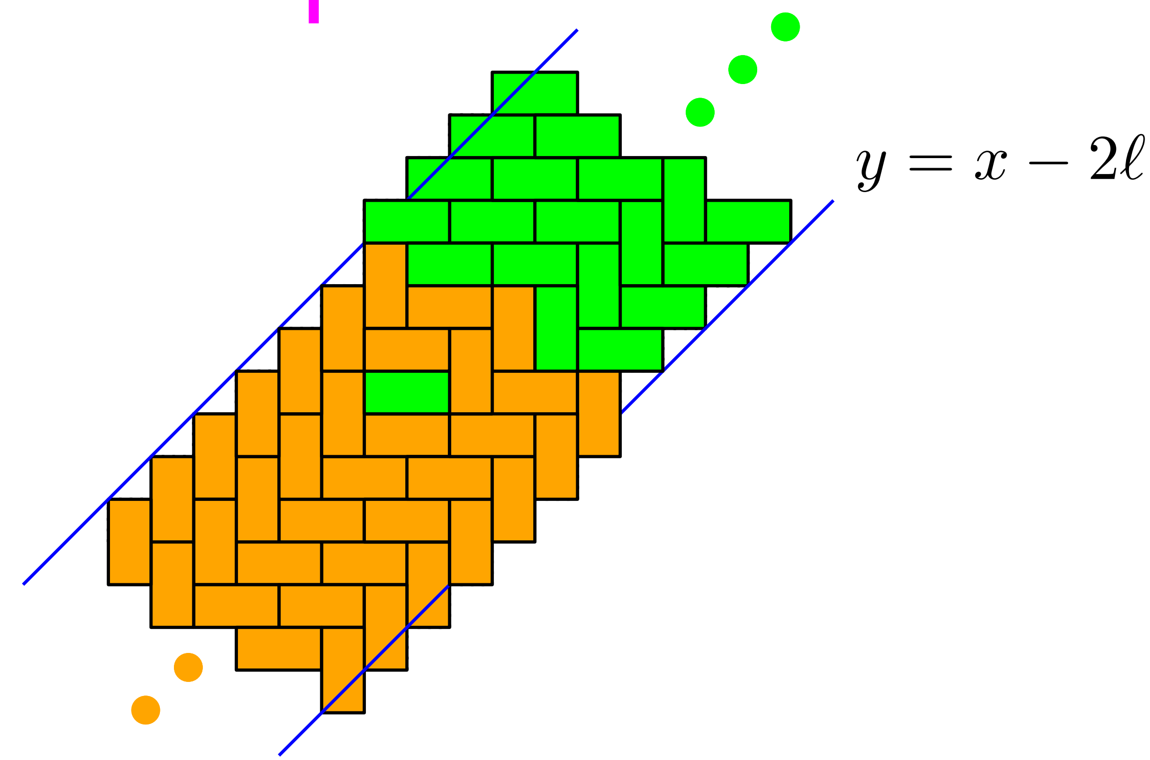


Steep tilings in an oblique strip

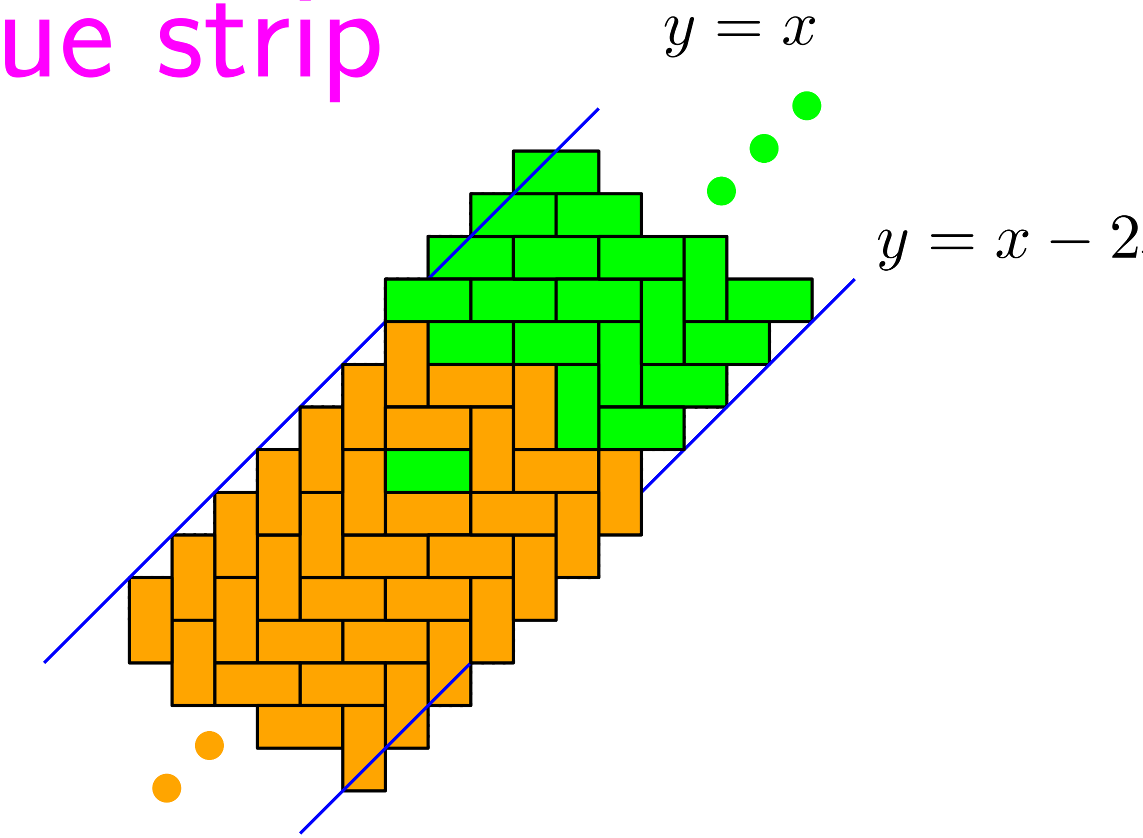
- Oblique strip of width 2ℓ

- Four types of dominos





Steep tilings in an oblique strip



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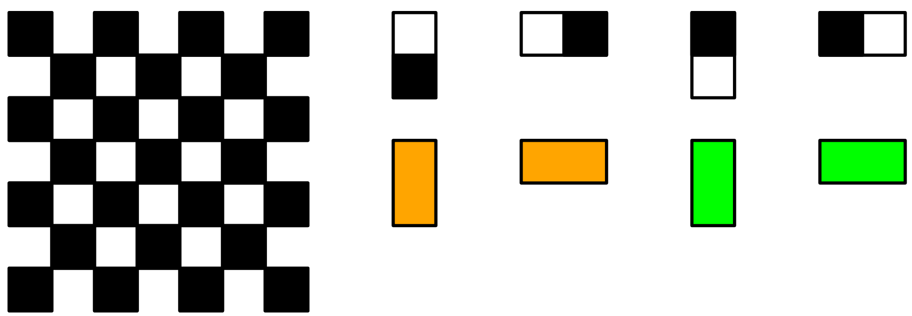

- $y = x$ no dominos and then green dominos

- A tiling is steep if it has only green dominos at $+\infty$ and orange dominos at $-\infty$

Steep tilings in an oblique strip

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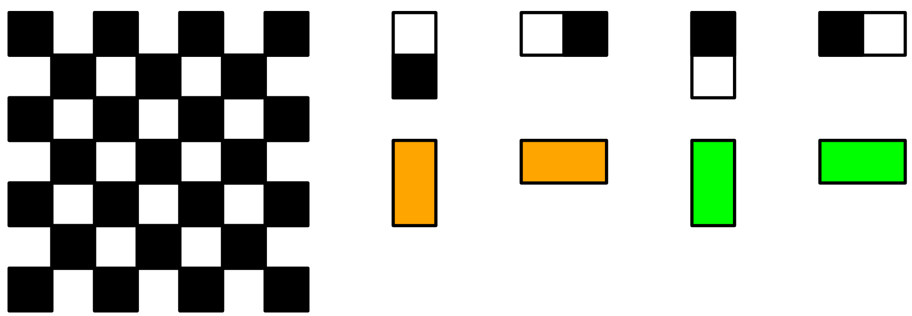

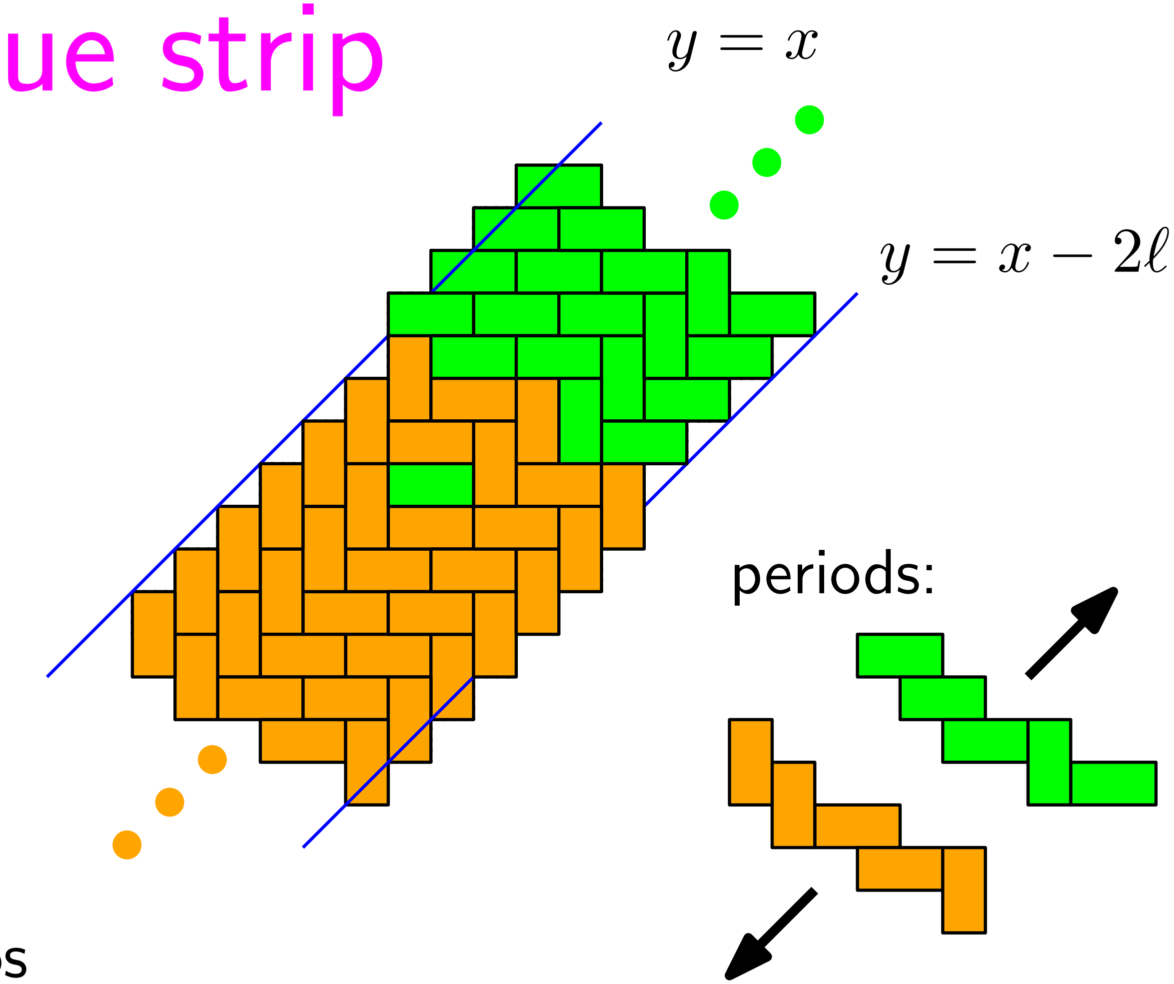

- $y = x$ no dominos and then green dominos

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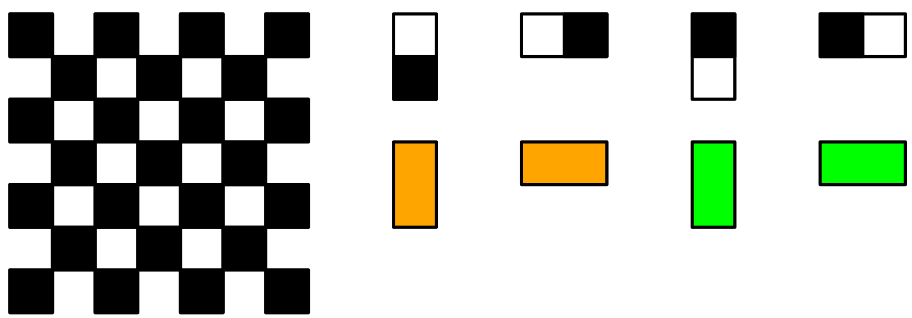
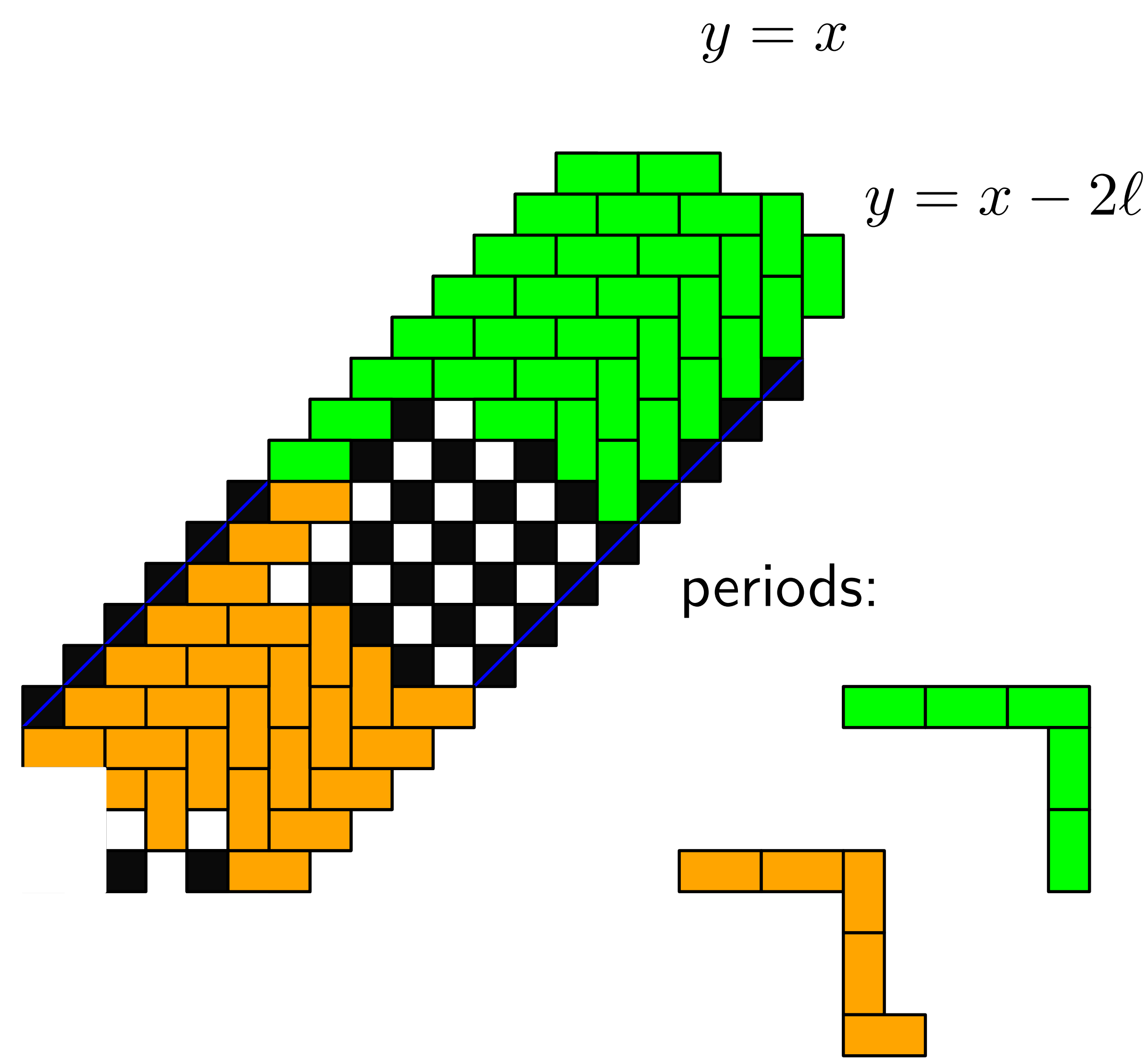
- Proposition: A steep tiling is "periodic"

Steep tilings in an oblique strip

- A steep tiling is periodic in $\pm\infty$.



- The periods are encoded by a binary word of length 2ℓ.



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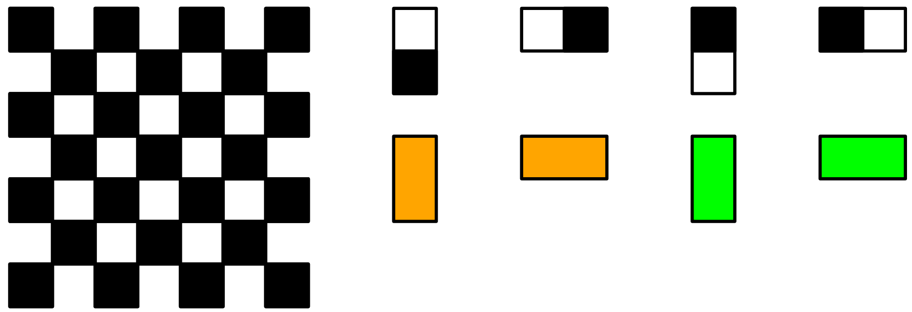

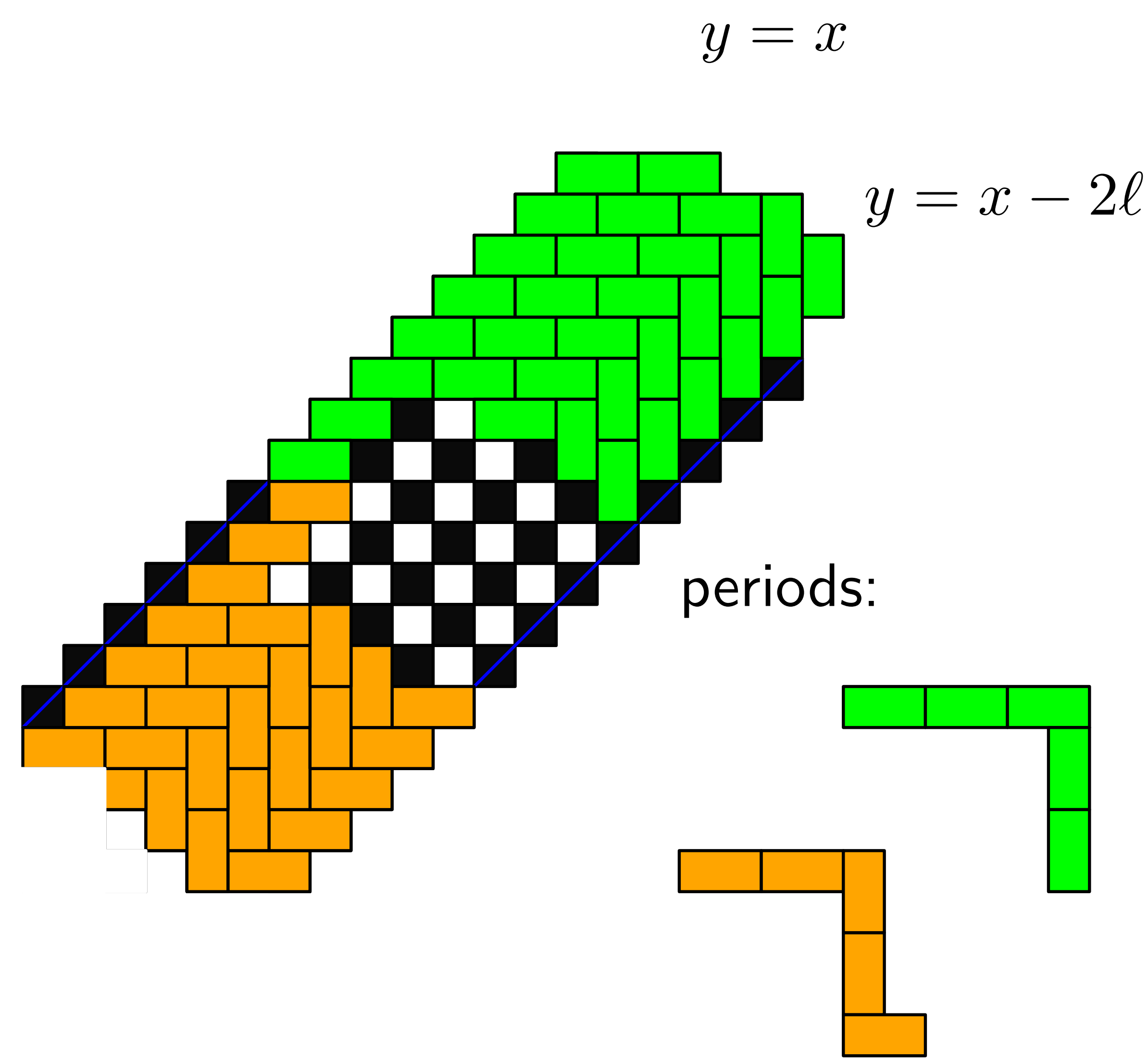

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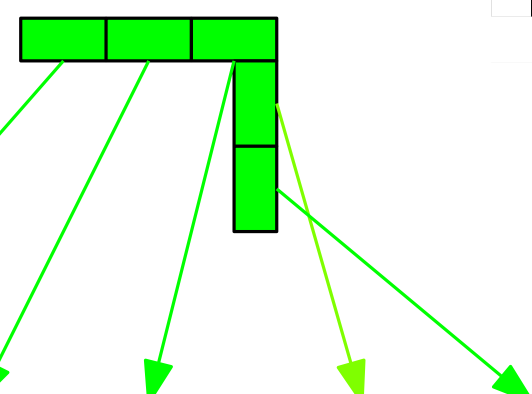

$w =$ + - + - + + - + - -

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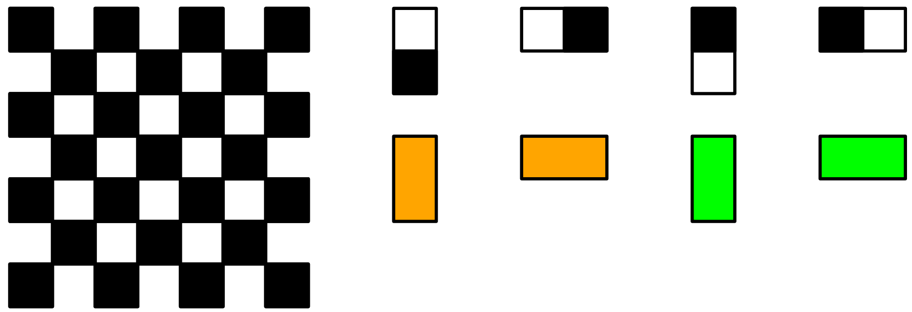

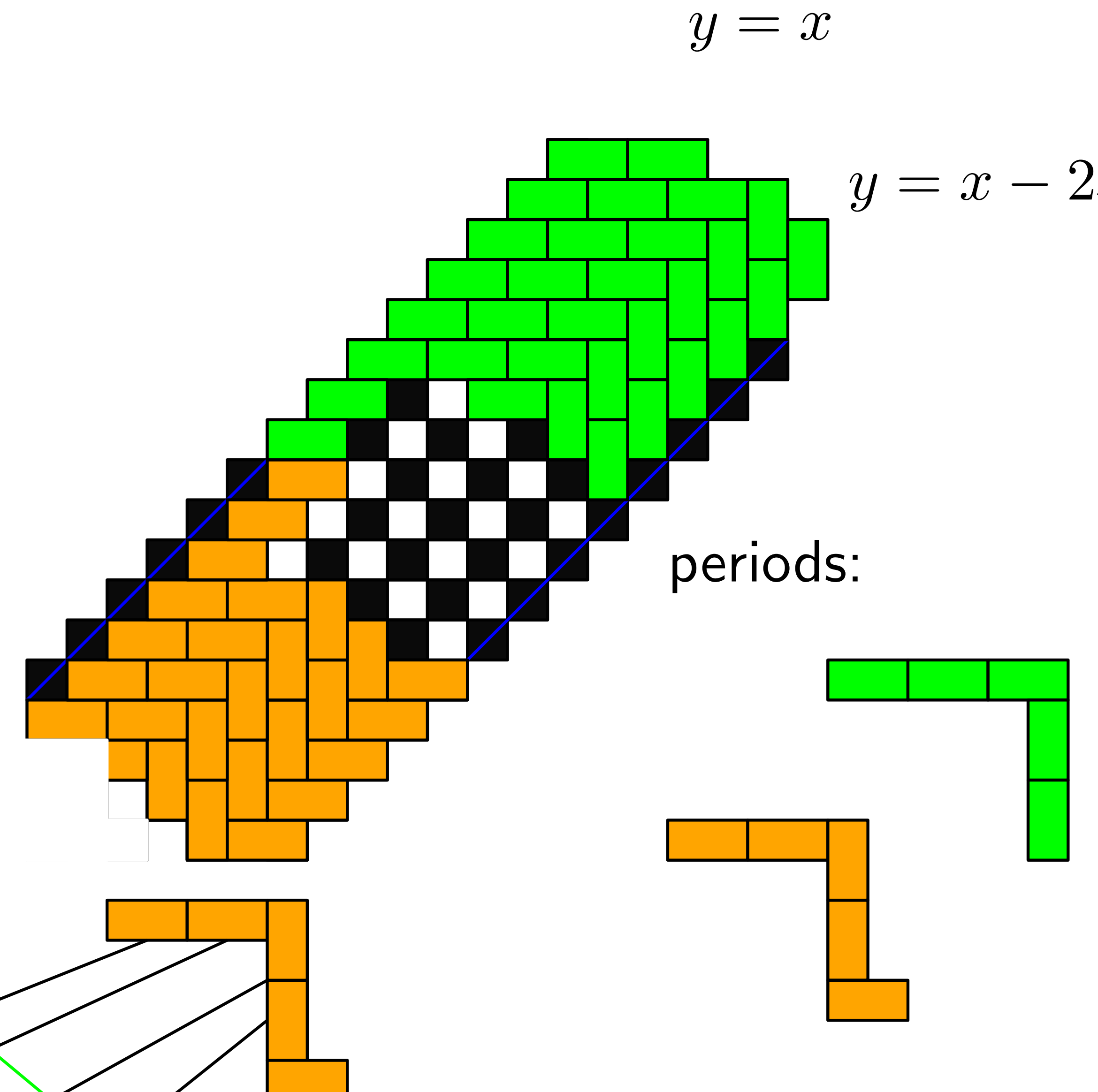

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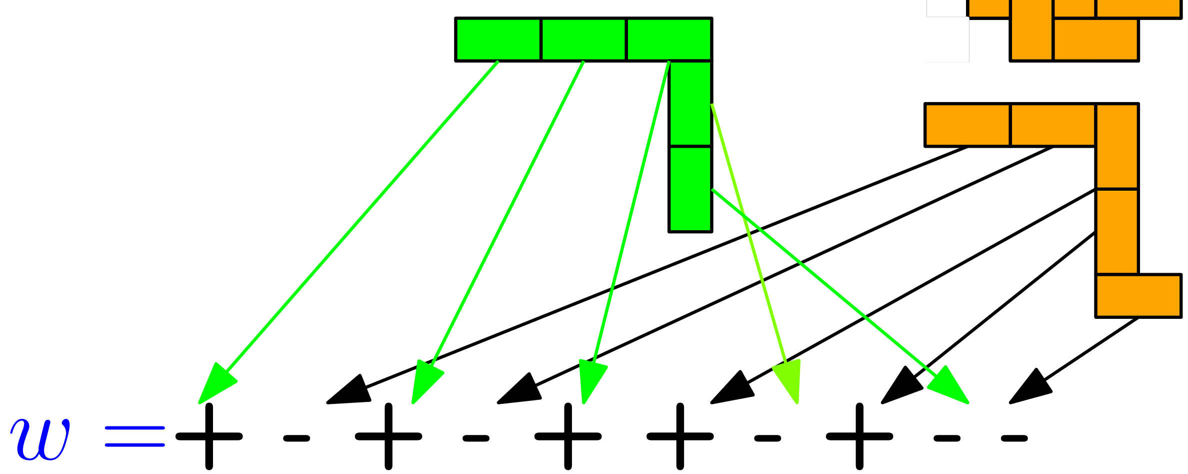

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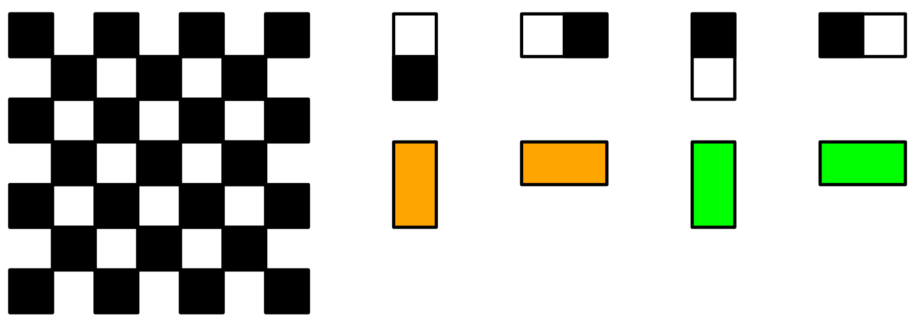

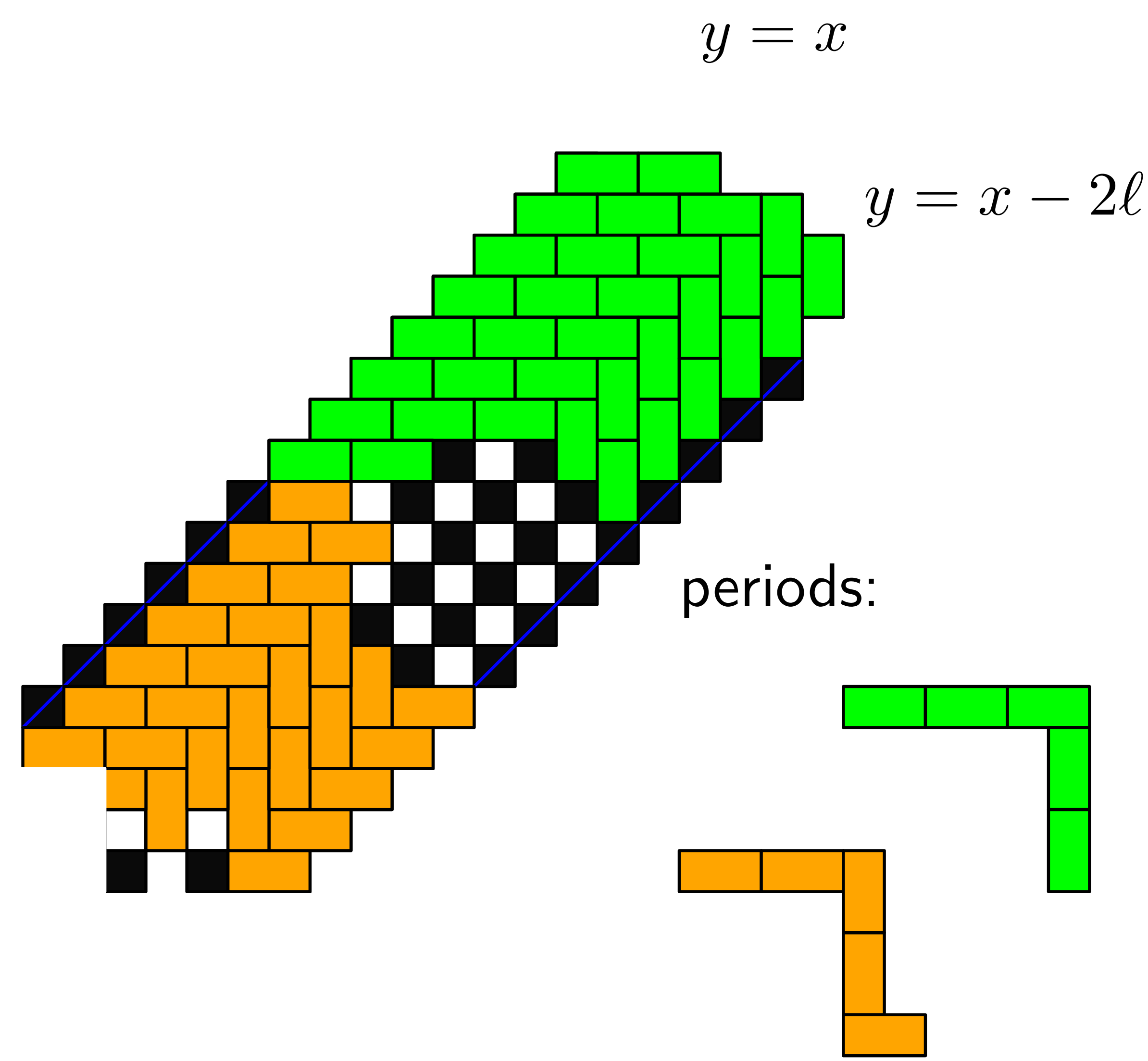

- The periods are encoded by a binary word of length 2ℓ.

- Unique minimal tiling

- All tilings with the same period can be obtained from the minimal tiling using flips

Steep tilings in an oblique strip

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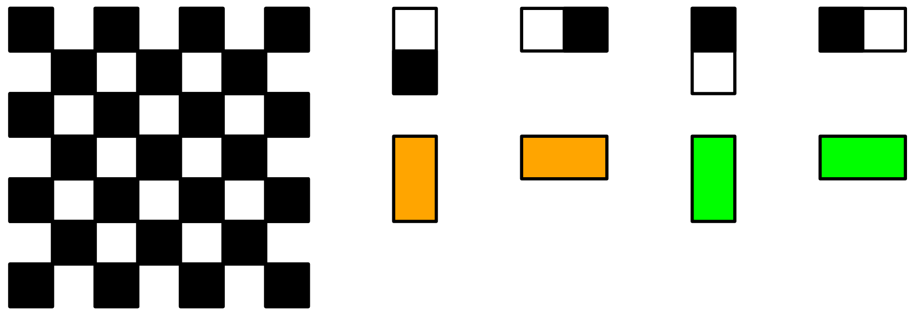

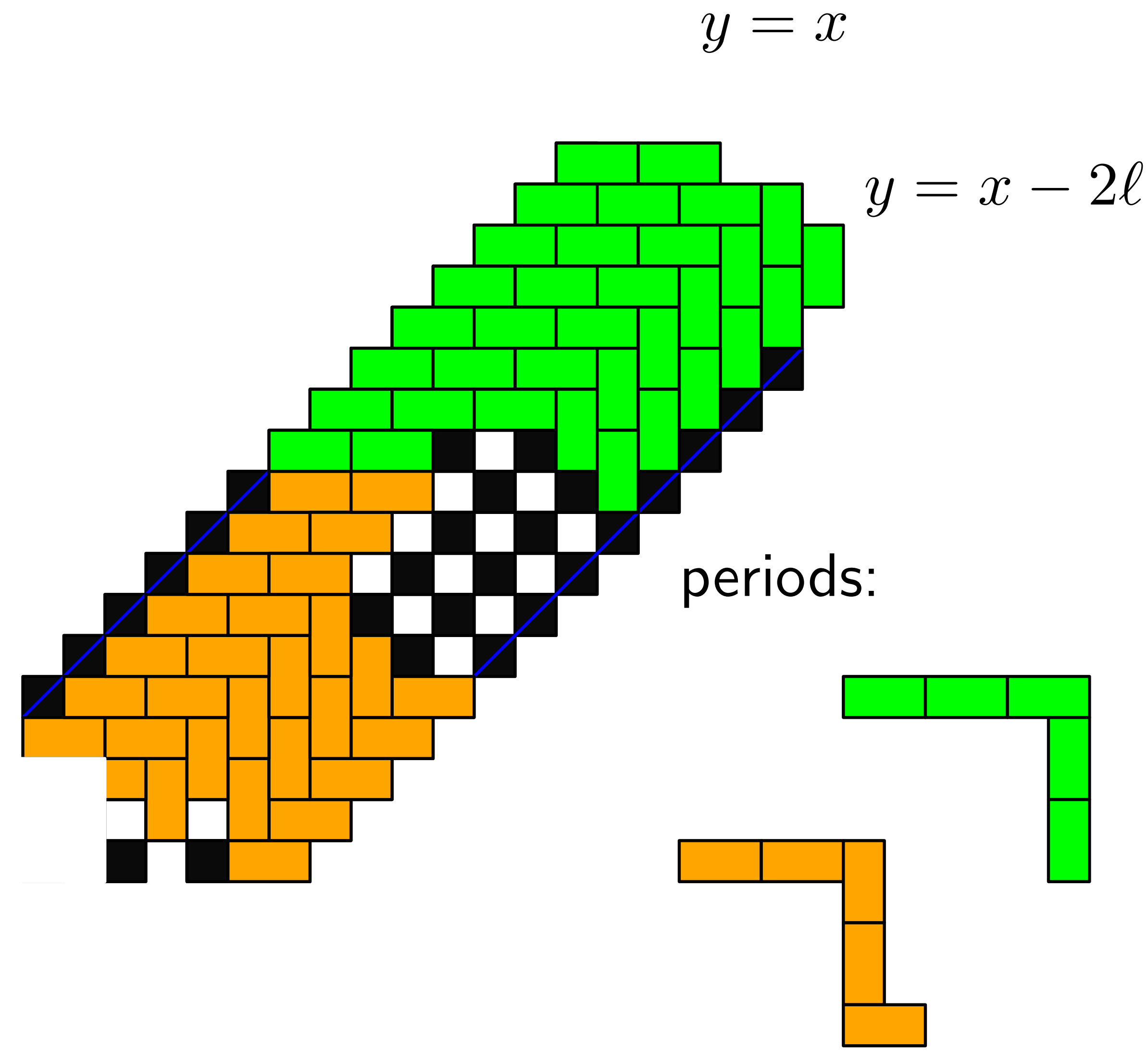

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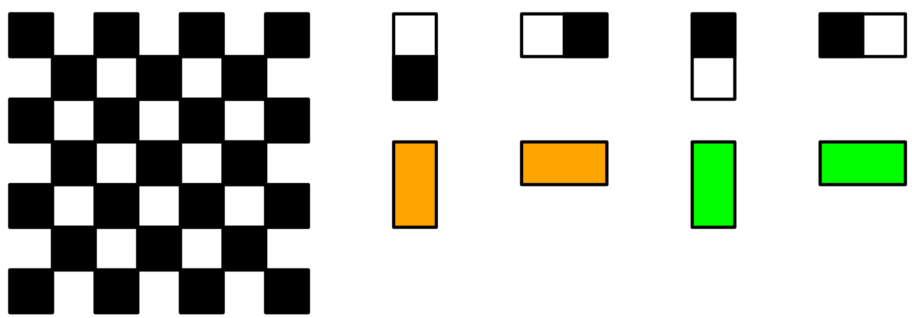

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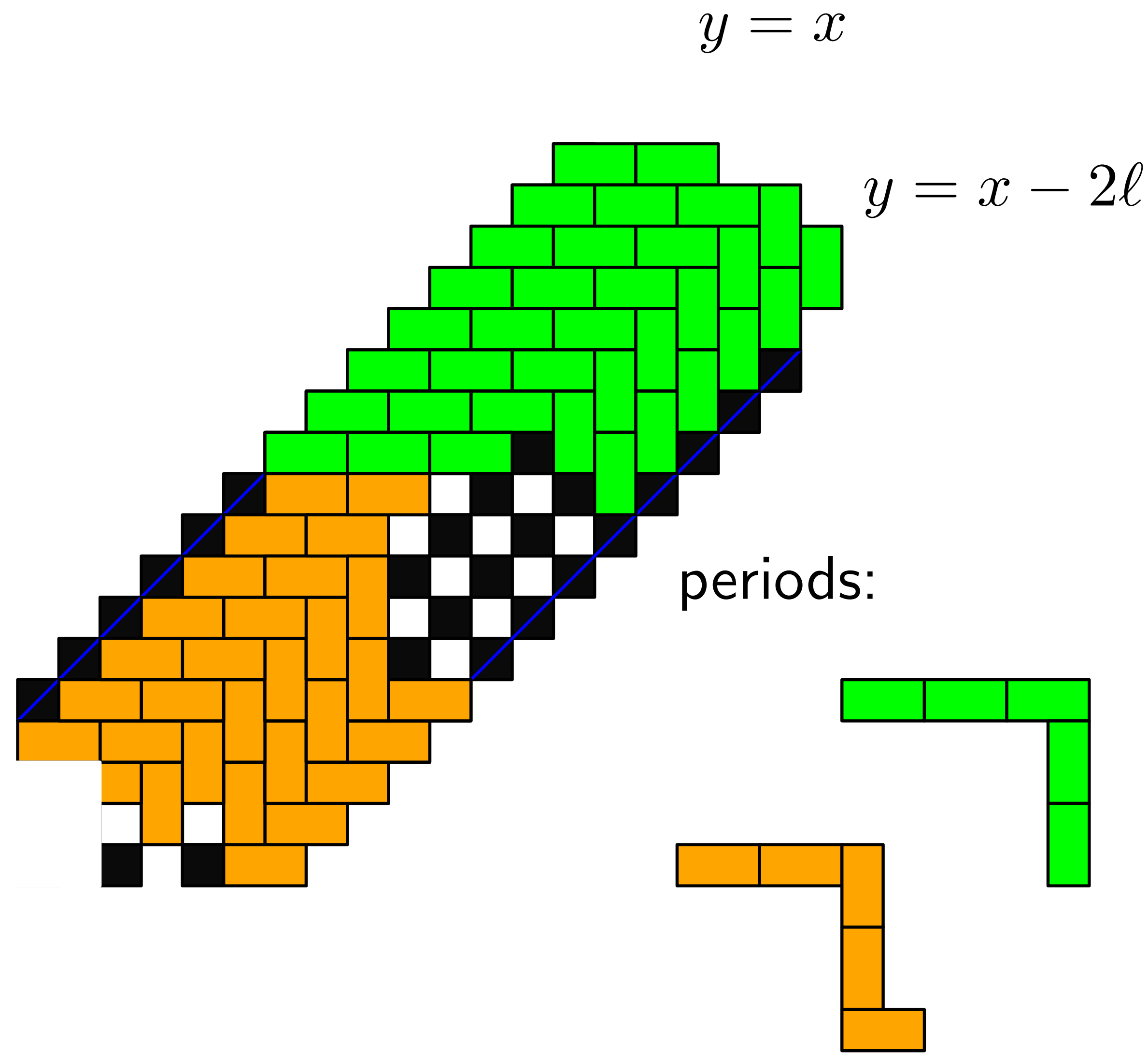

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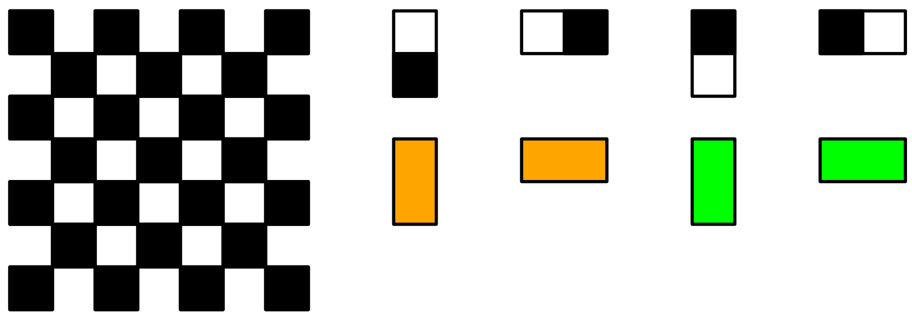

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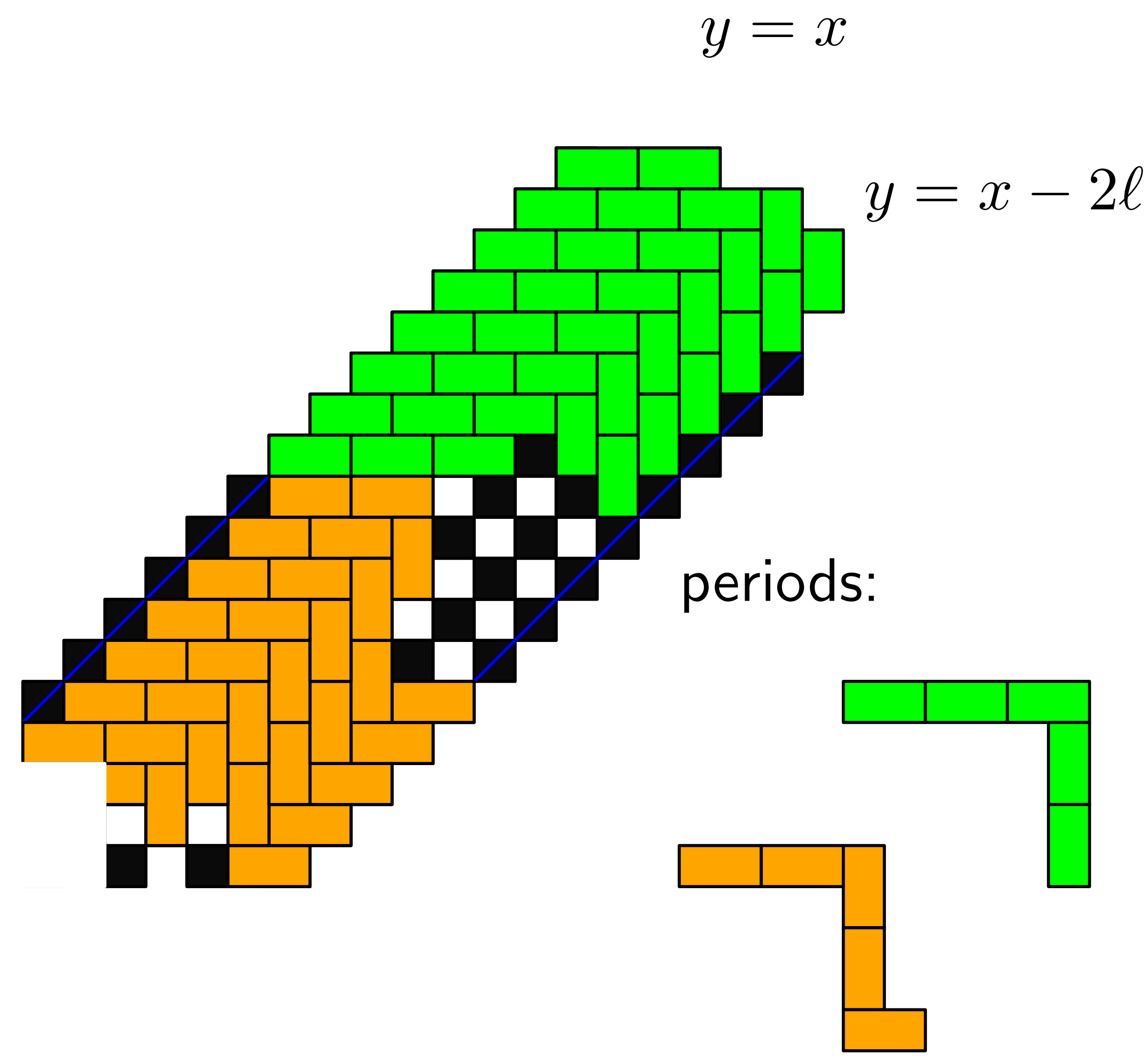

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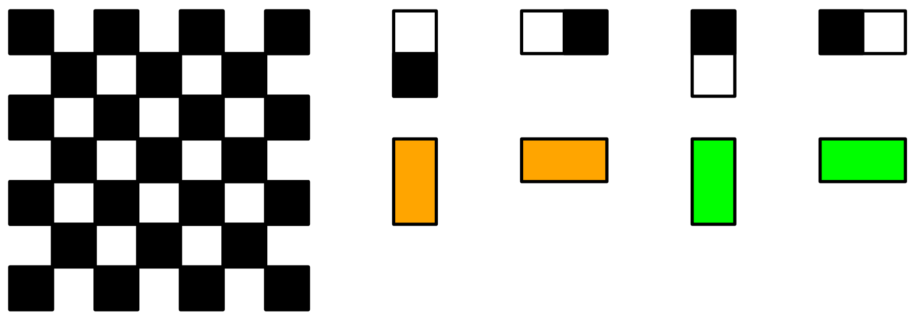

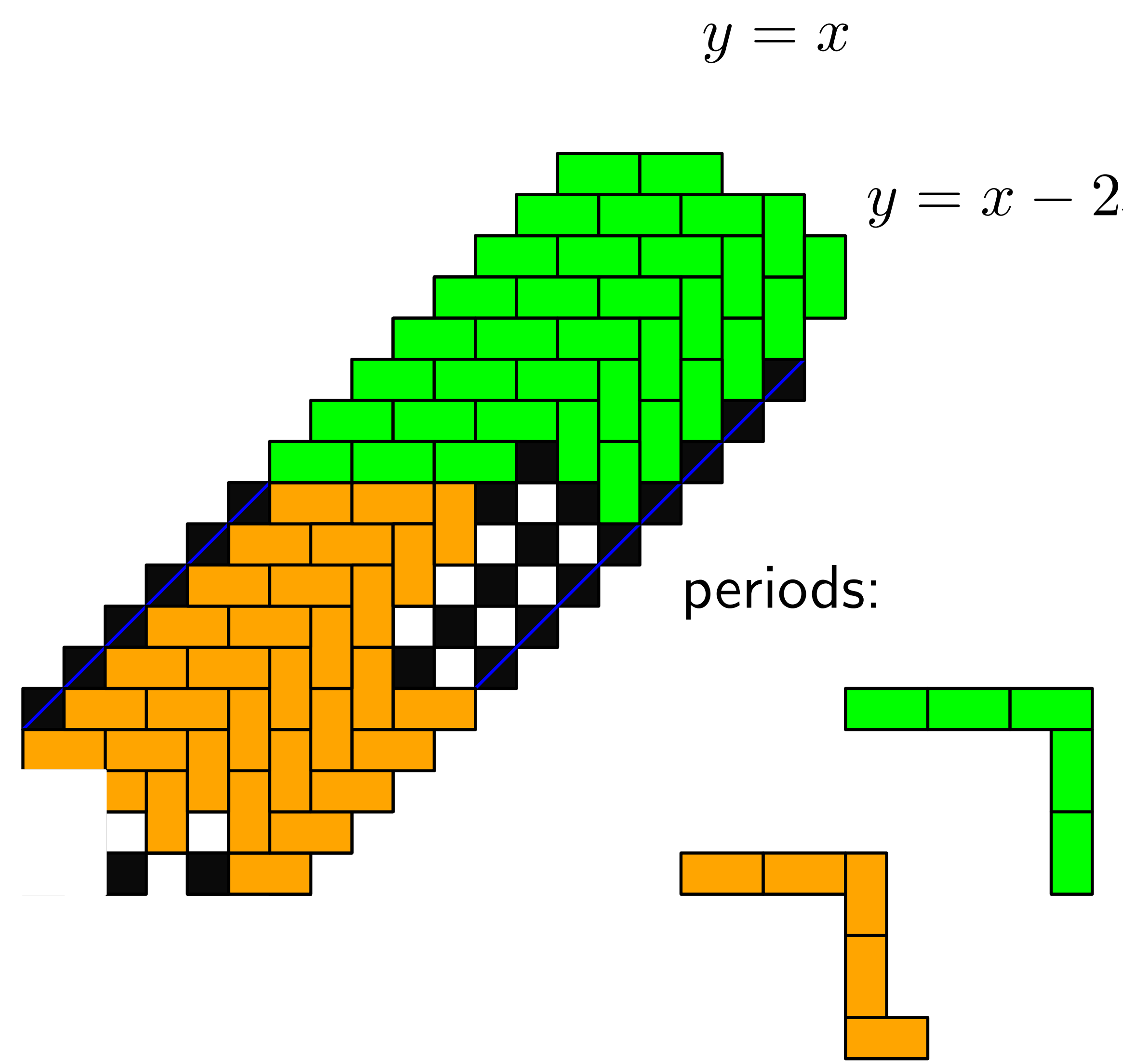

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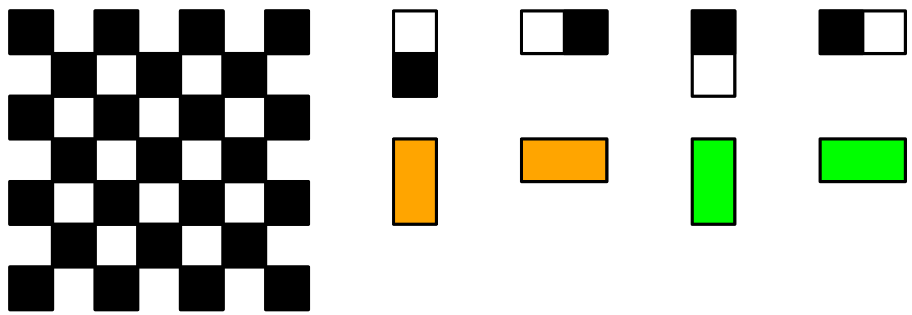

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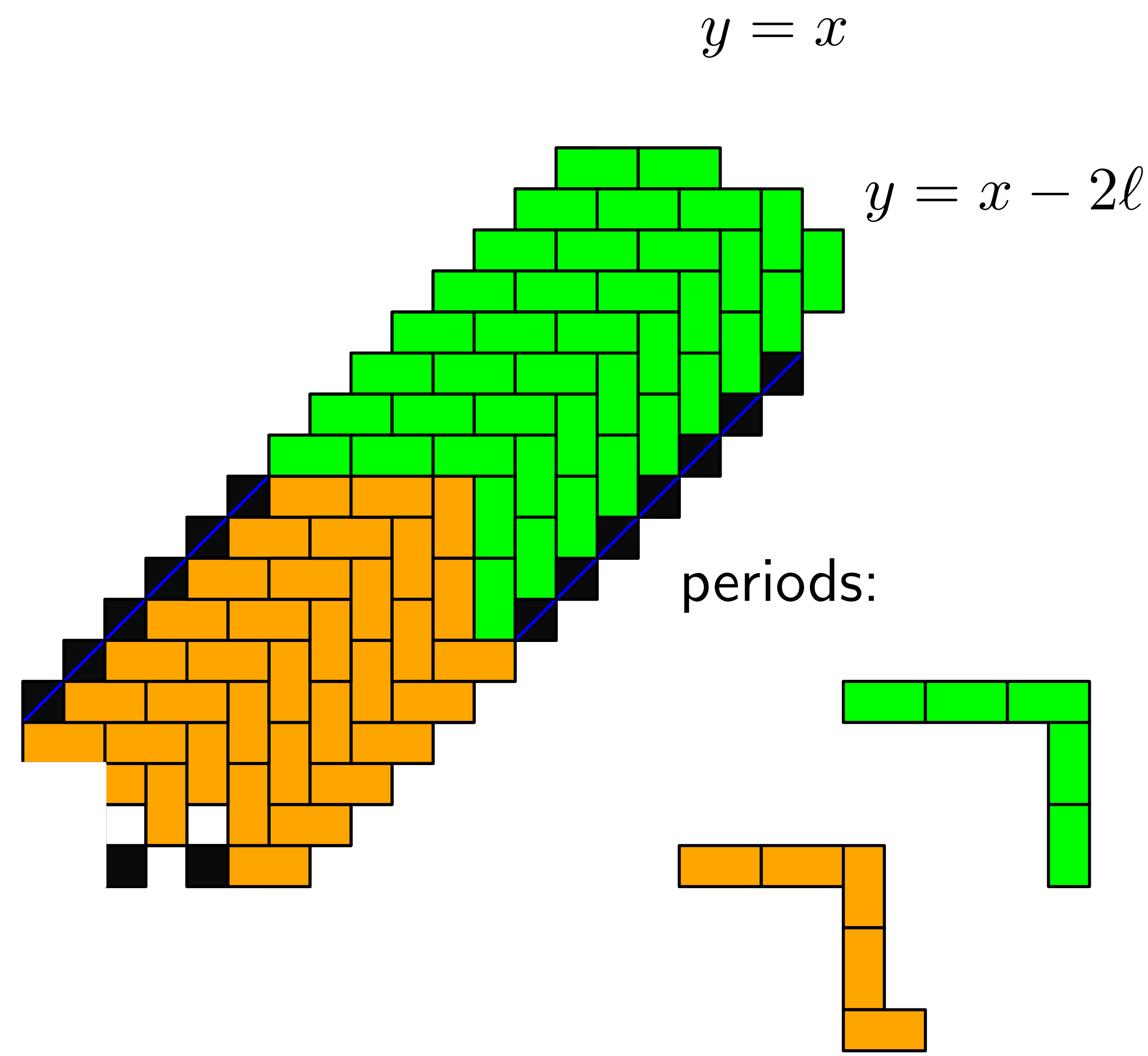

Generating function = hook formula

- **Theorem**[Bouttier-Chapuy-C 13]:

$$T_w(q) = \sum_{P} q^{\#\mathsf{flips}(P)} = \prod_{\substack{i<j \\ w_i=+,\ w_j=-}} z_{ij}$$

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$$T_w(q) = \sum_{P} q^{\#\mathsf{flips}(P)} = \prod_{\substack{i<j \\ w_i=+,\ w_j=-}} z_{ij}$$

$$j-i \text{ odd: } z_{ij} = (1+q^{j-i})$$

$$j-i \text{ even: } z_{ij} = \frac{1}{1-q^{j-i}}$$

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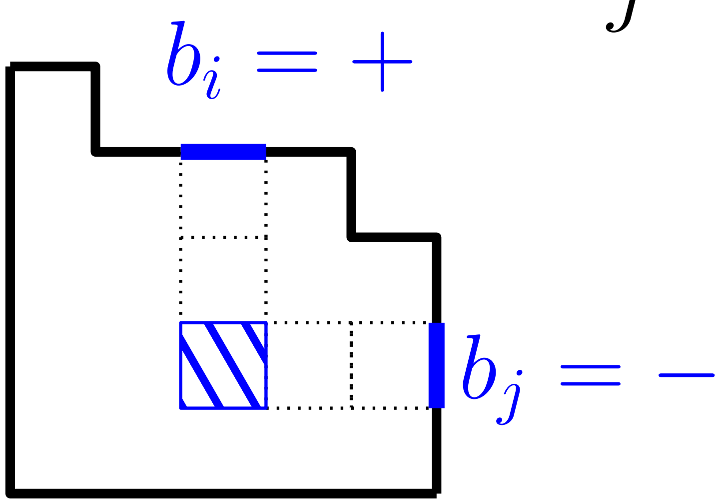

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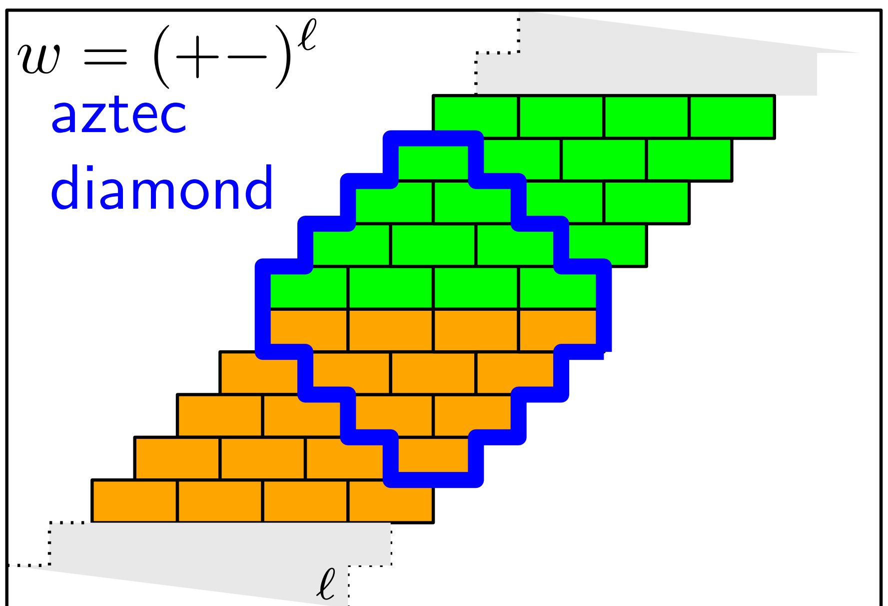

$$T_w(q) = \prod_{i=1}^{\ell} (1+q^{2i-1})^{\ell-i+1}.$$

Generating function = hook formula

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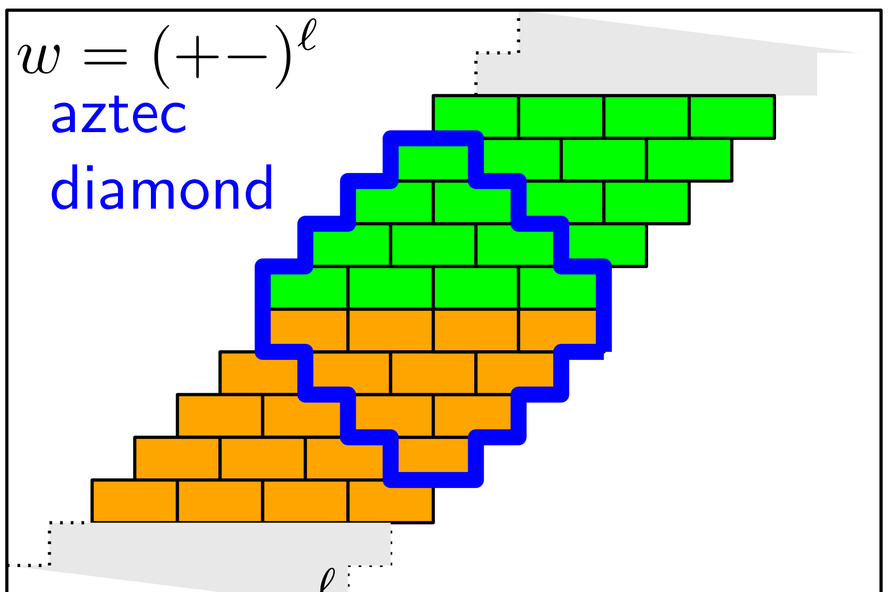

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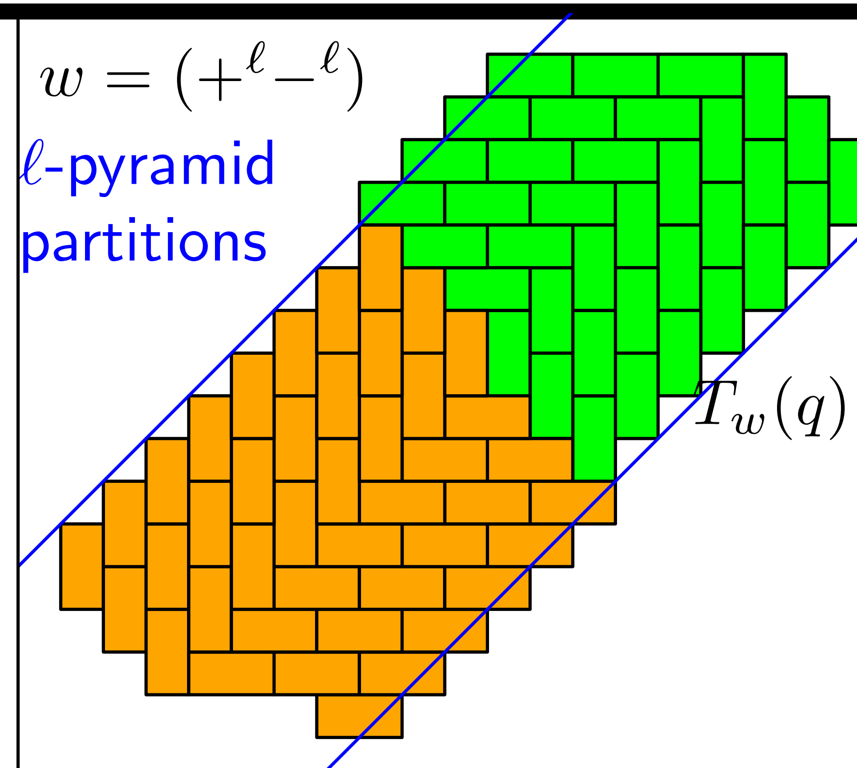


$$T_w(q) = \prod_{i=1}^{\ell} \prod_{j=\ell+1}^{2\ell} (1+\epsilon q^{j-i})^{\epsilon}$$

Generating function = hook formula

- **Theorem**[Bouttier-Chapuy-C 13]:

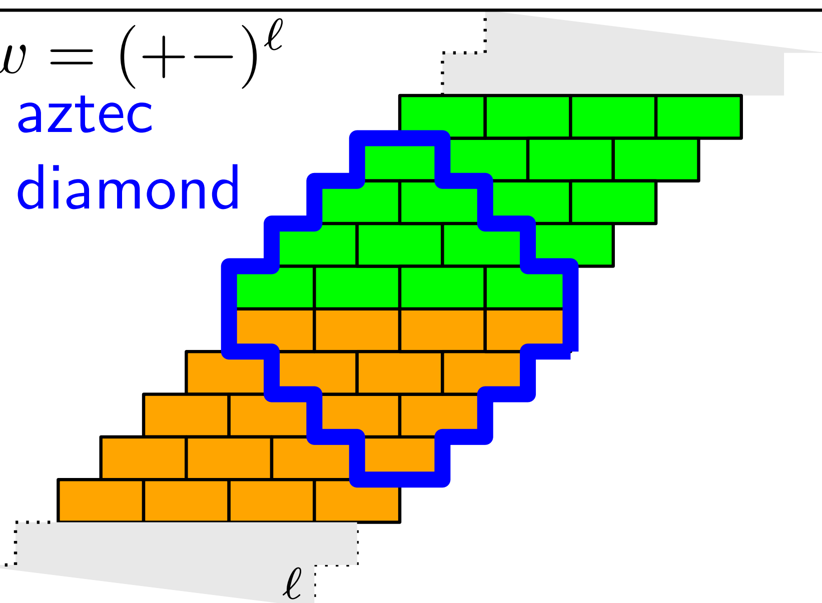
$$T_w(q) = \sum_P q^{\#\text{flips}(P)} = \prod_{\substack{i<j \\ w_i=+,\ w_j=-}} z_{ij}$$

$$\sum_P q^{\#\text{flips}(P)} = \prod_{\substack{i<j \\ w_i=+,\ w_j=-}} (1+\epsilon q_i q_{i+1} \dots q_{j-1})^{\epsilon}$$

q_i: follows the flips on diagonal $y = x - i$; $\epsilon = (-1)^{j-i-1}$

$w = (+-)^{\ell}$

aztec diamond



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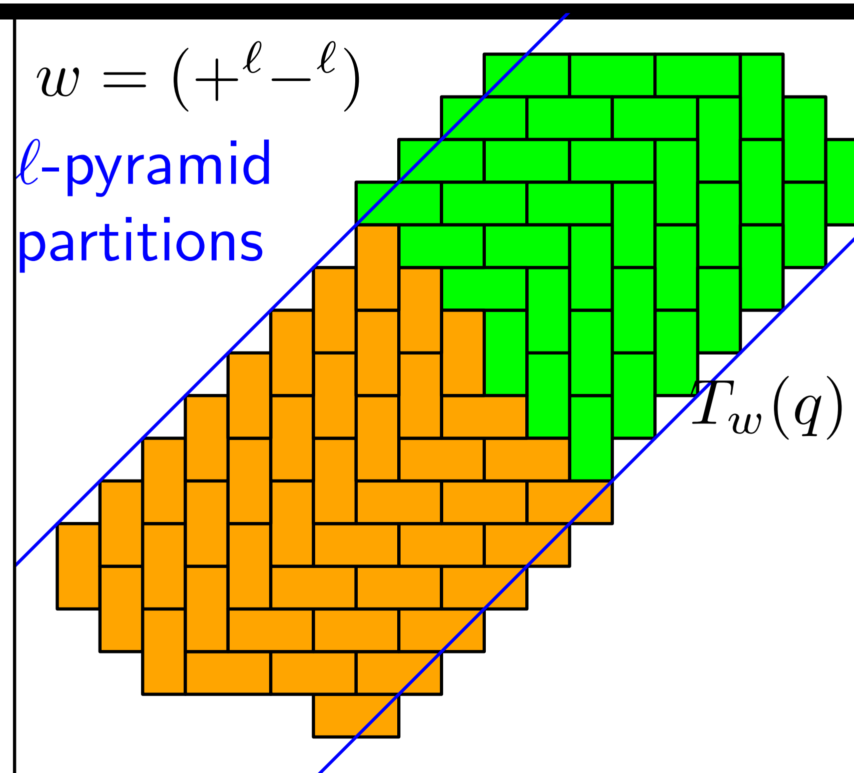
ℓ-pyramid partitions



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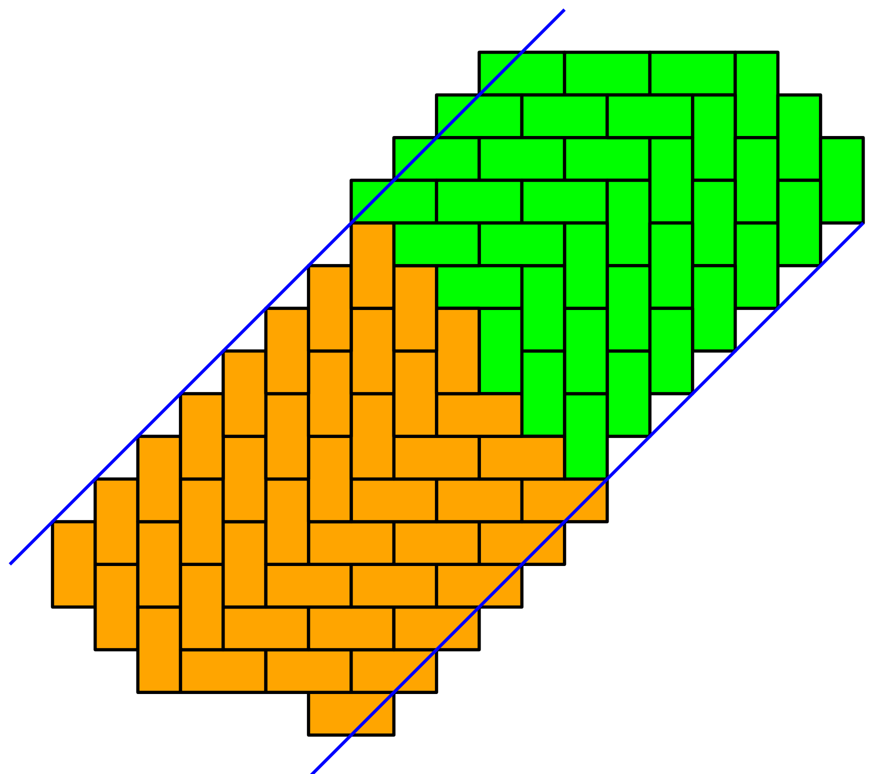
Tilings and particles

- In each domino we put two particles ○ or ●



minimal tiling

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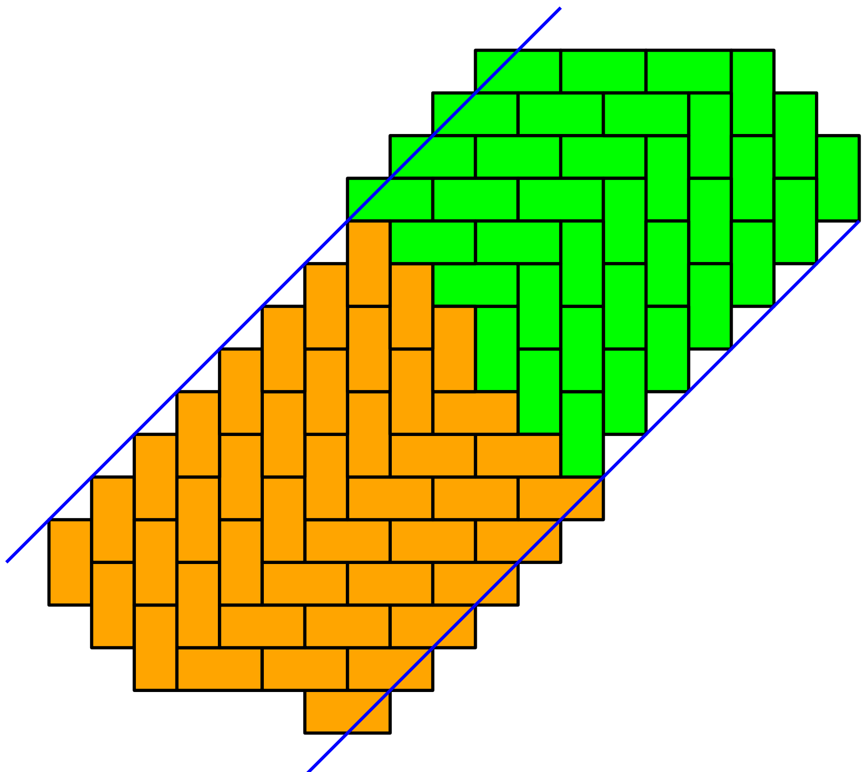

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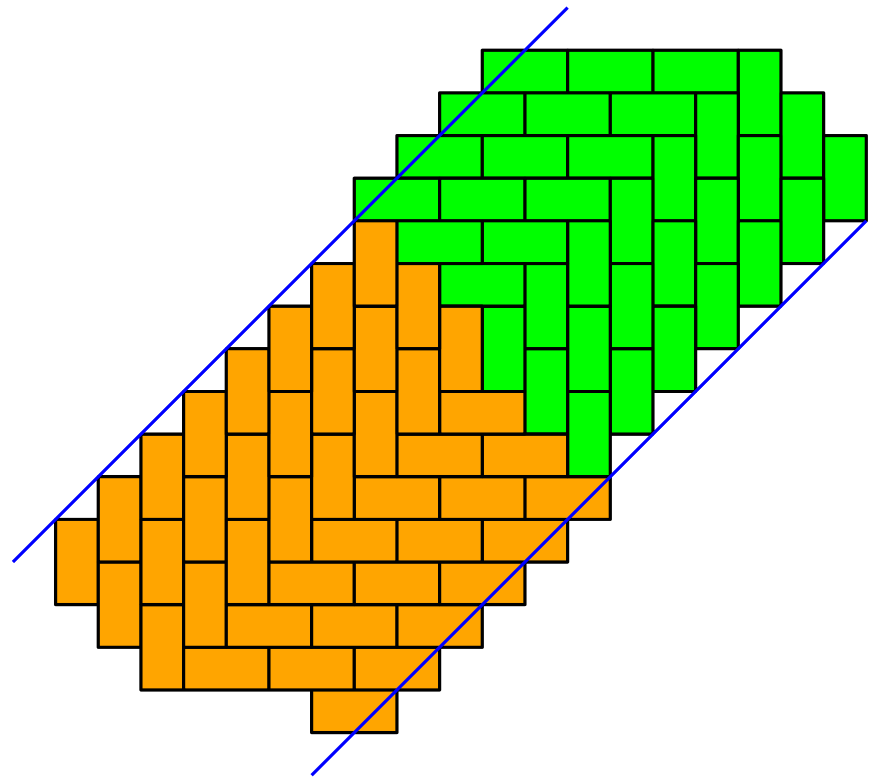

Tilings and particles

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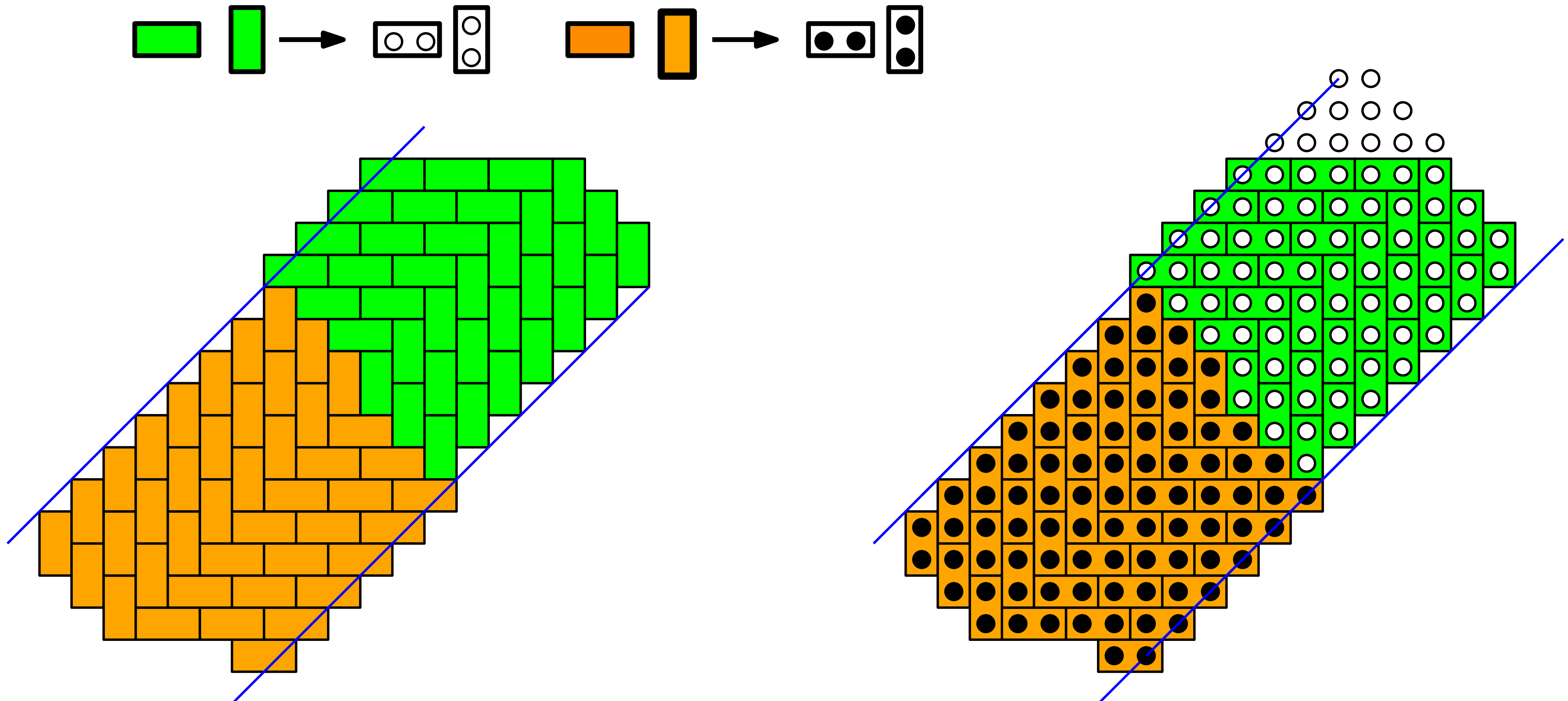

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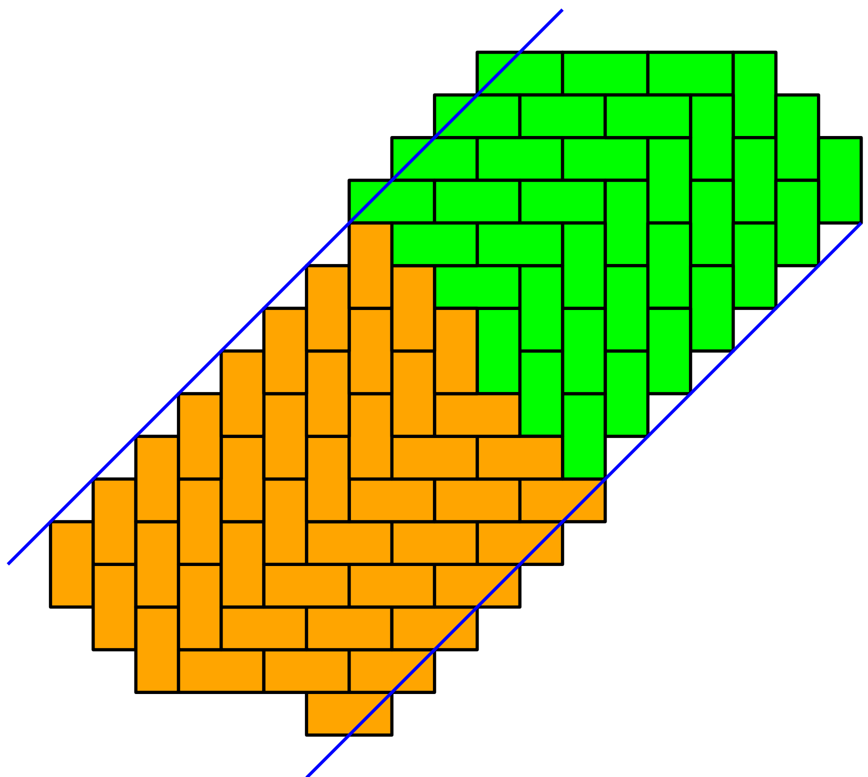

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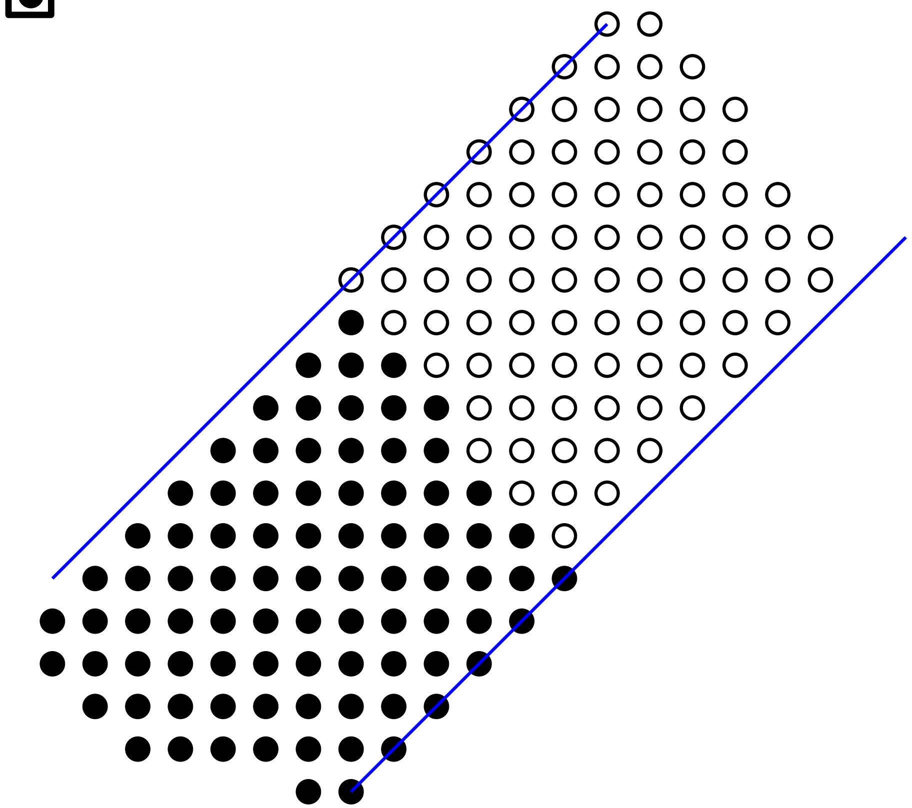

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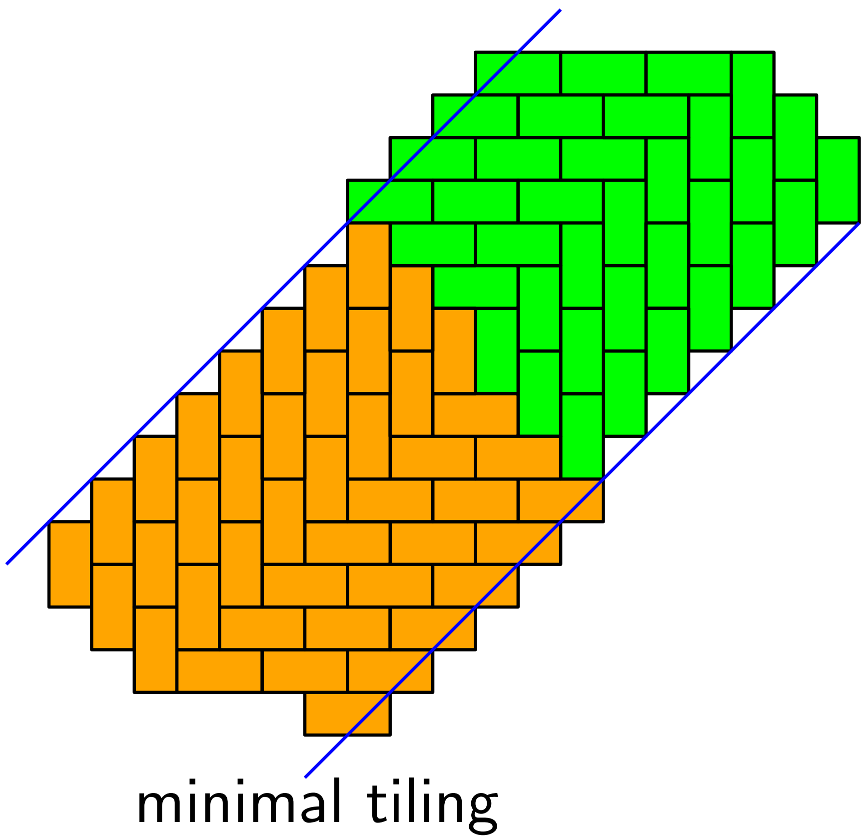

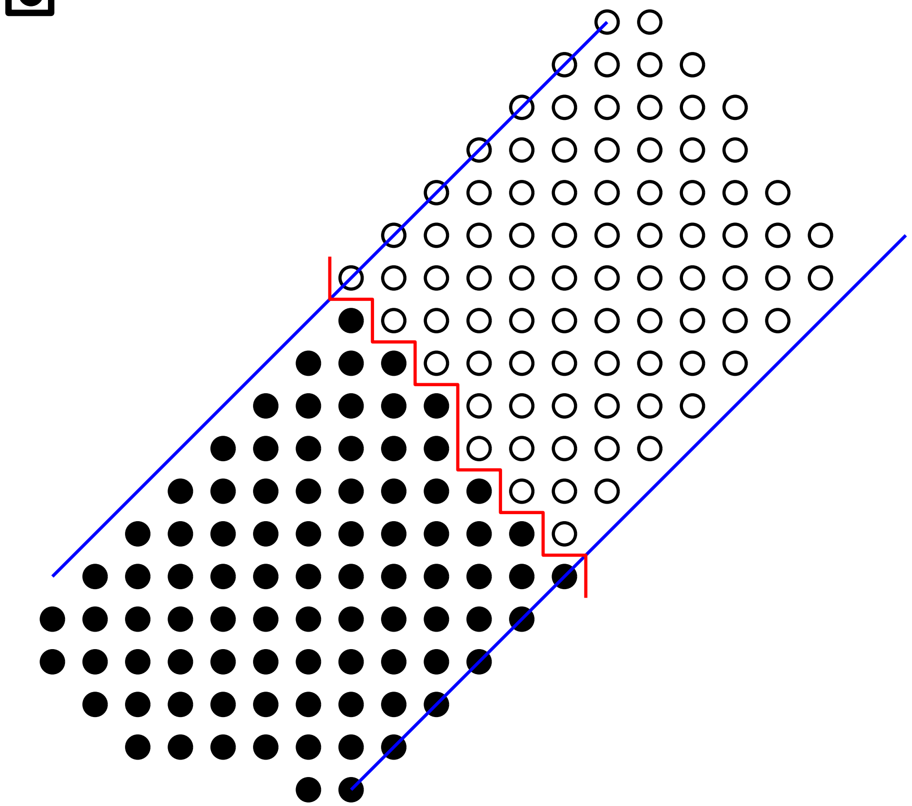

Tilings and particles

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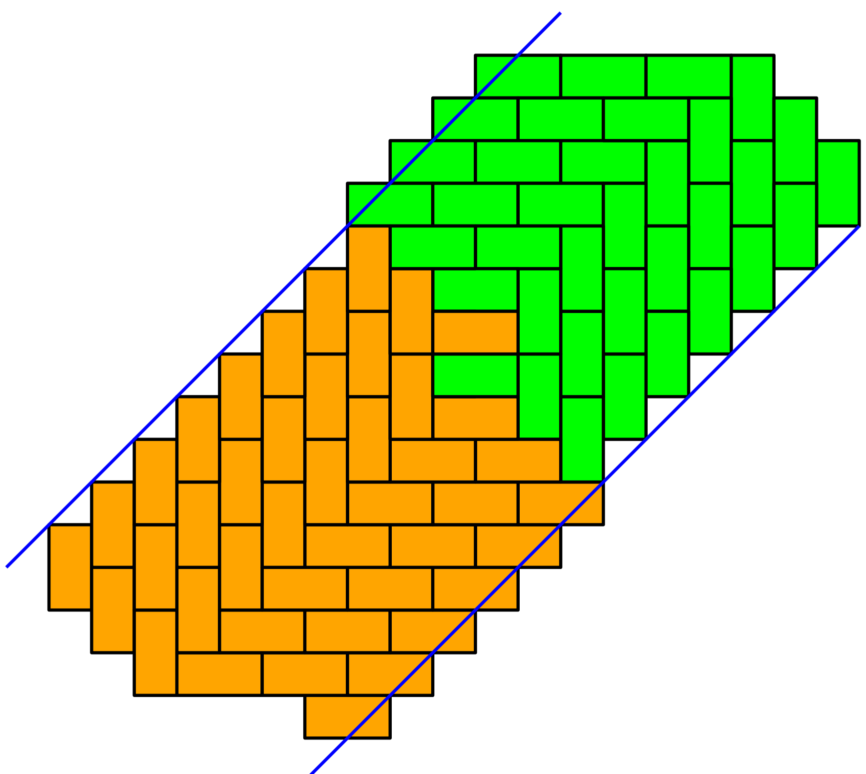
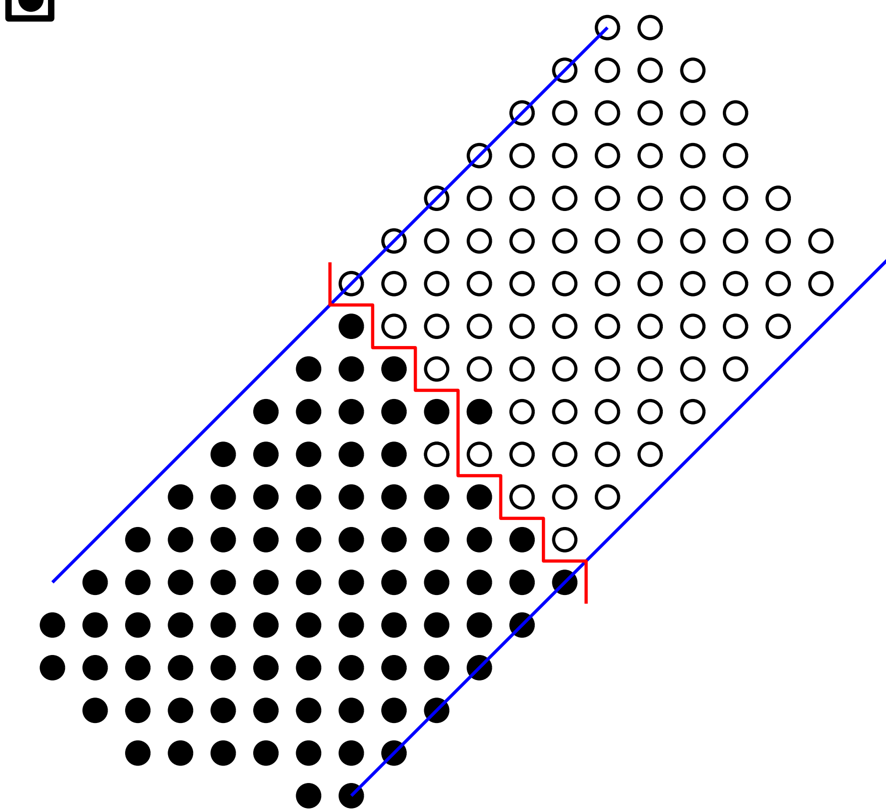


minimal tiling



- A flip makes one particle move forward

Tilings and particles

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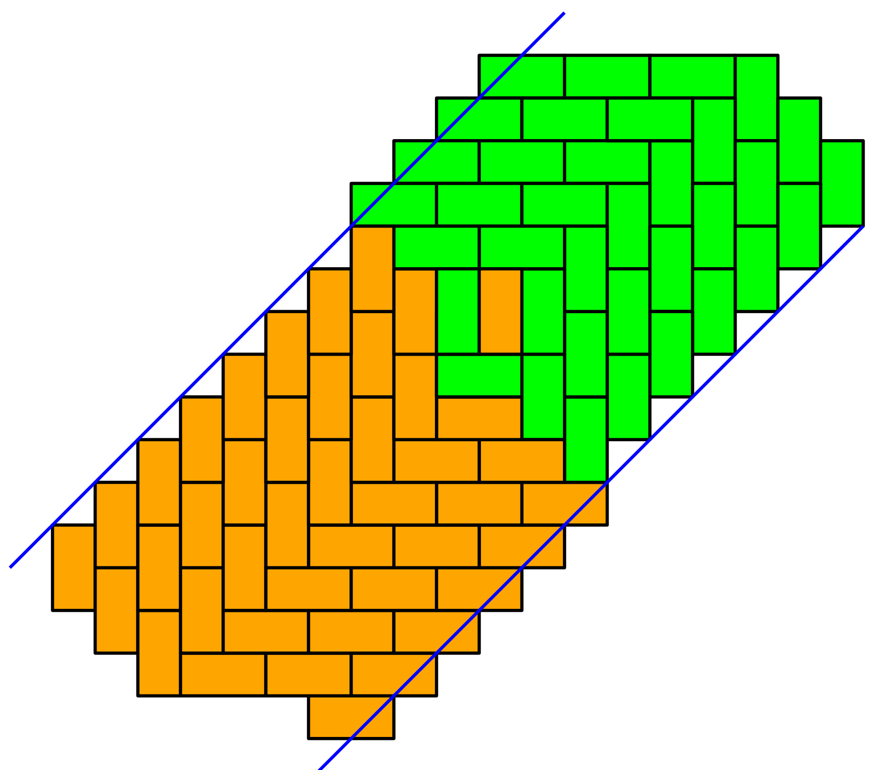

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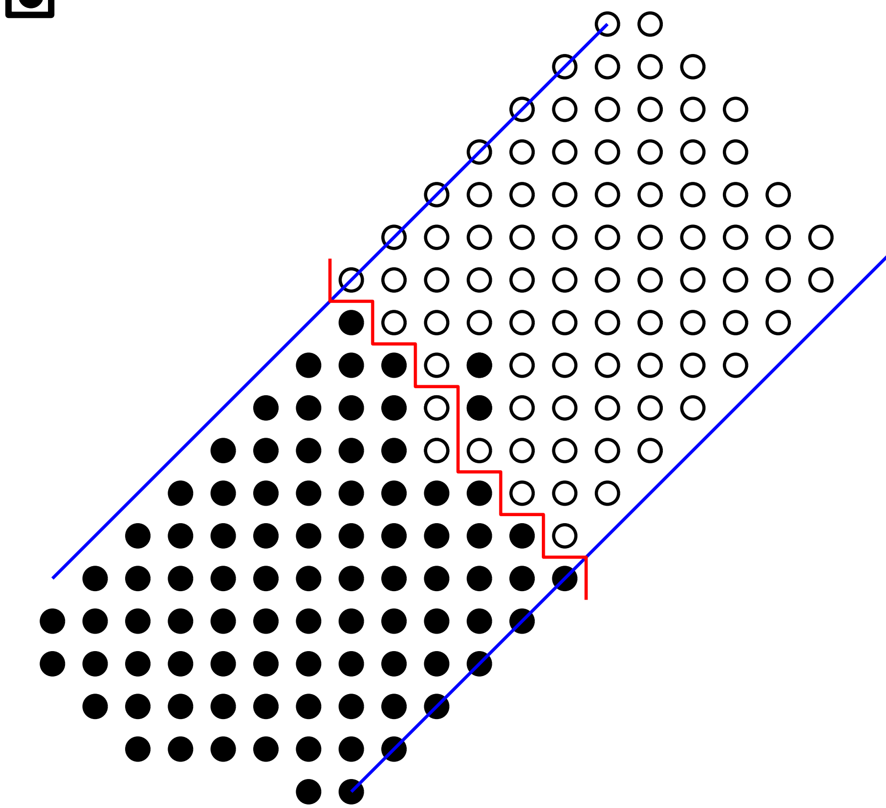

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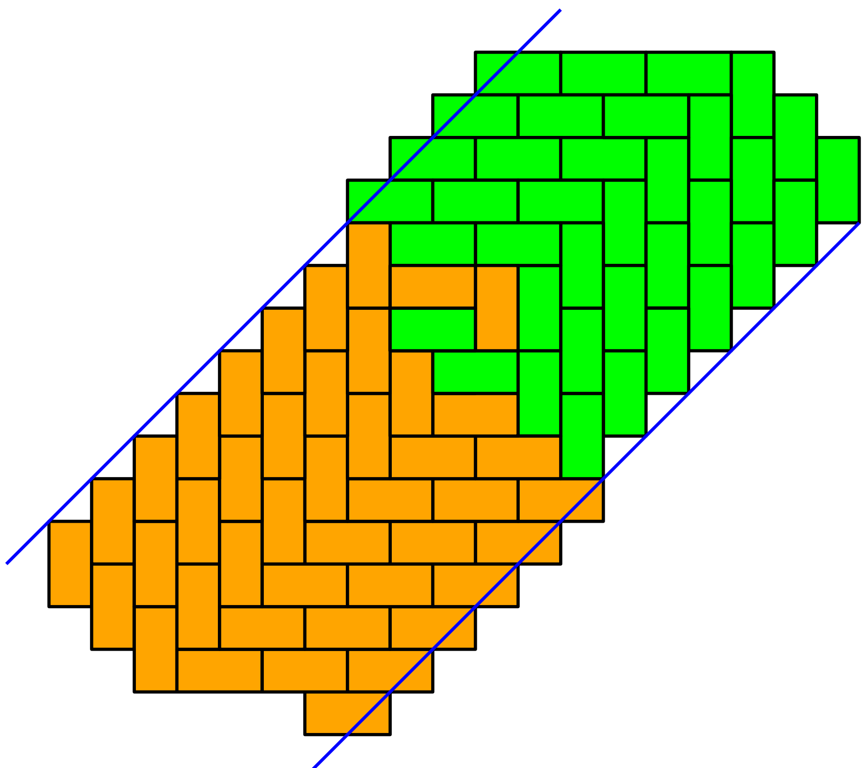

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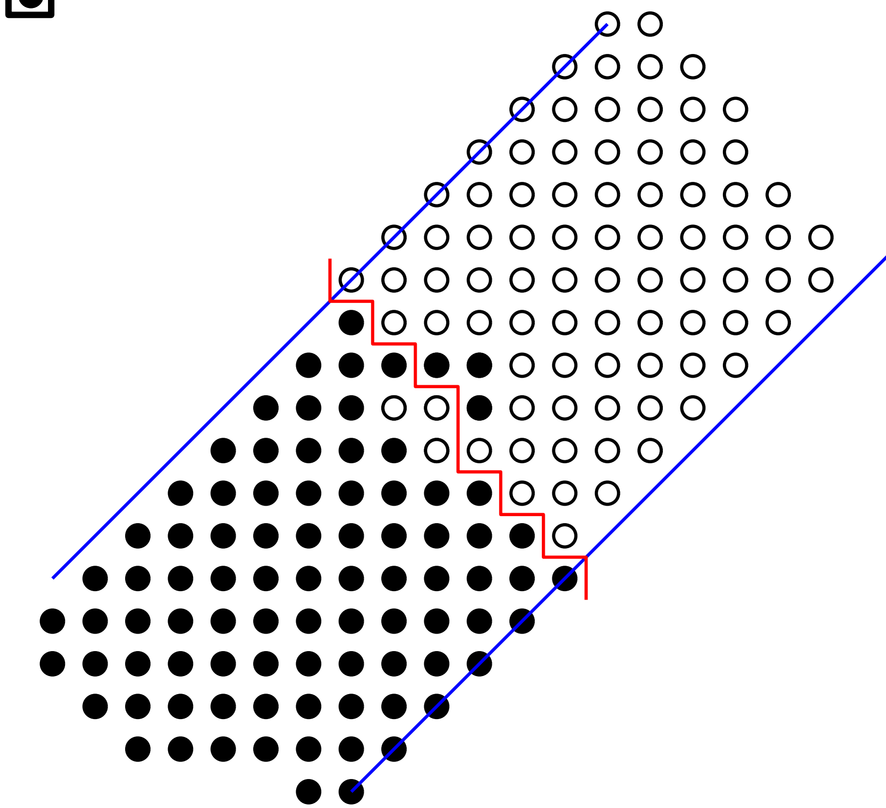

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Tilings and particles

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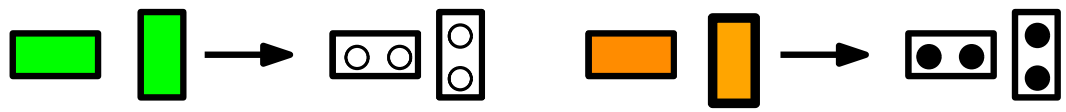

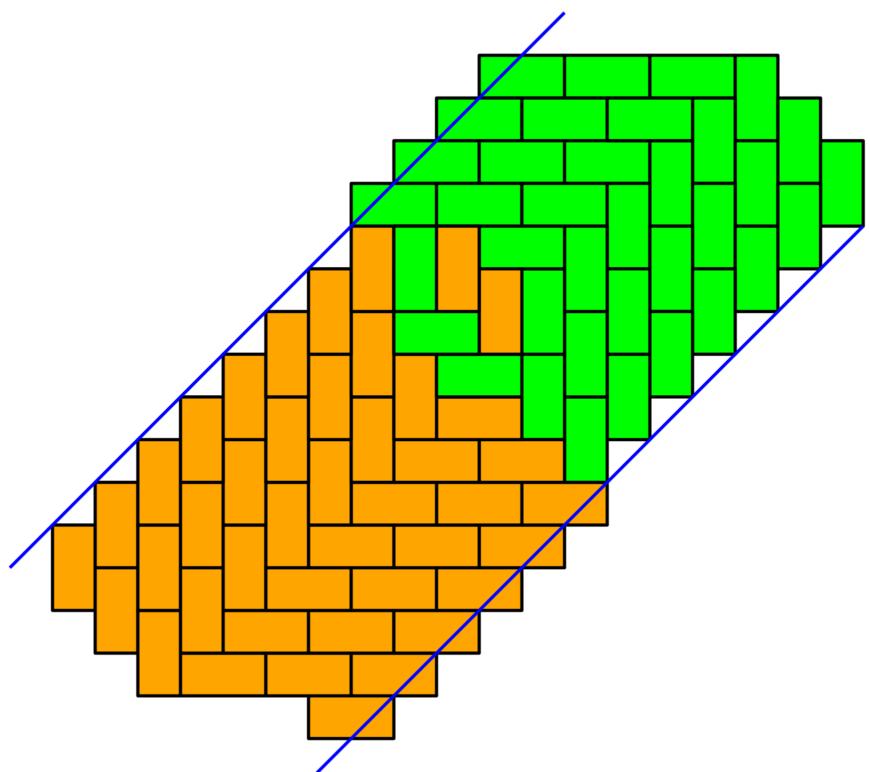

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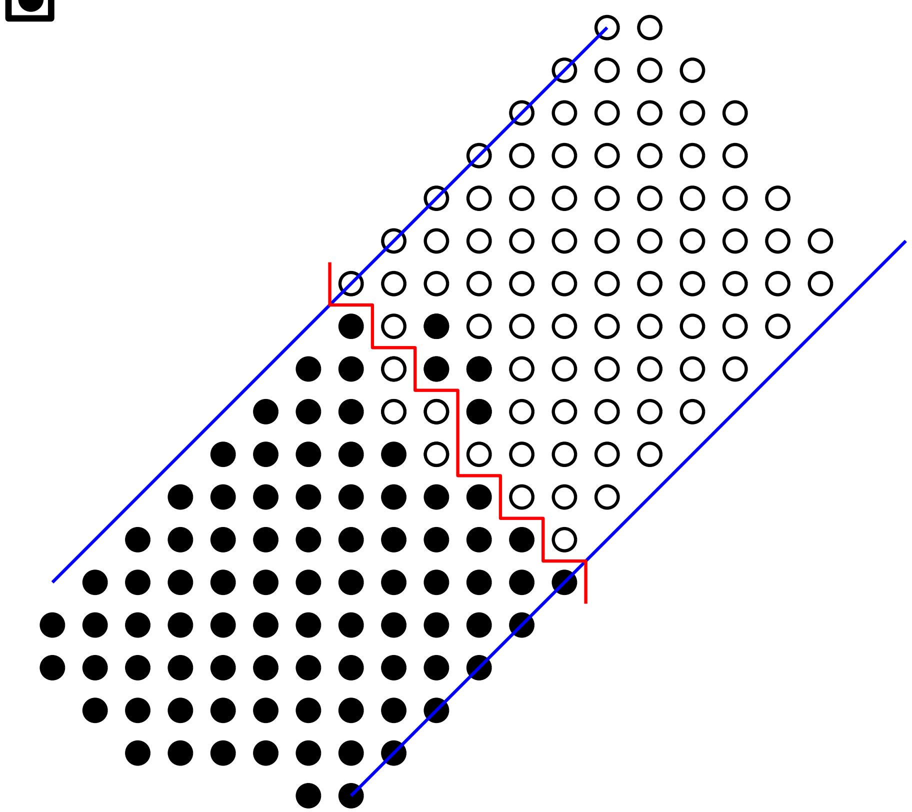

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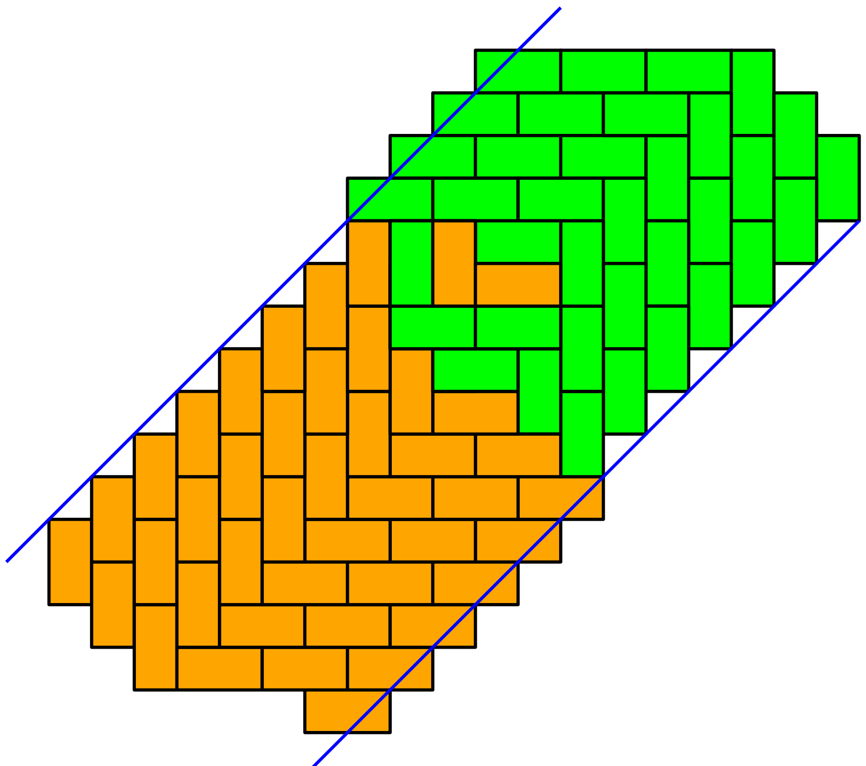
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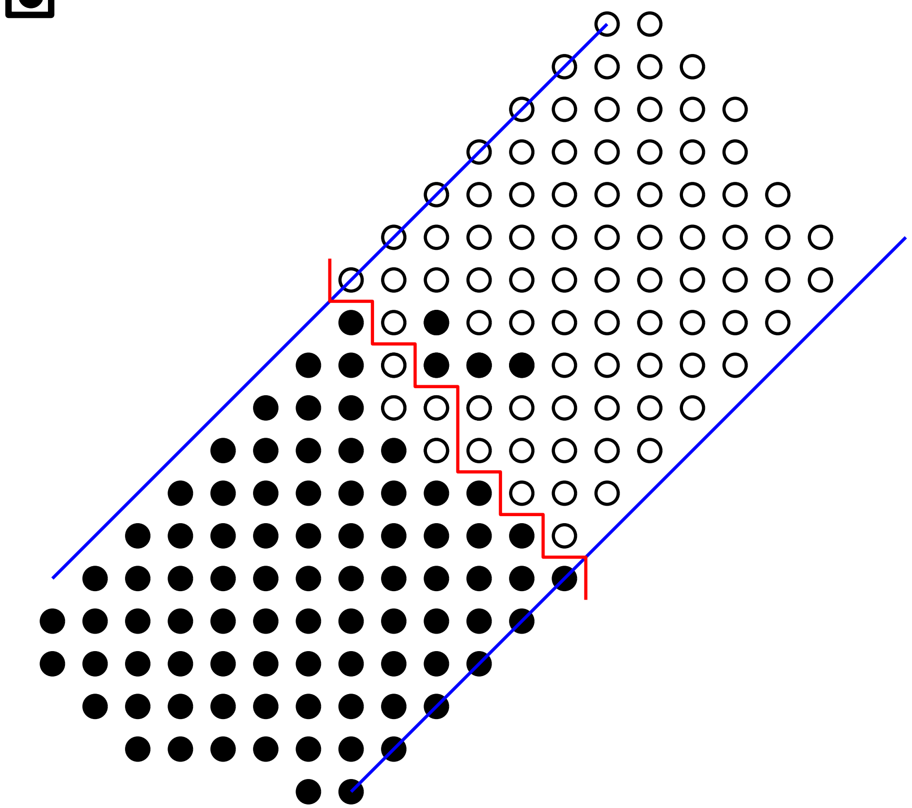

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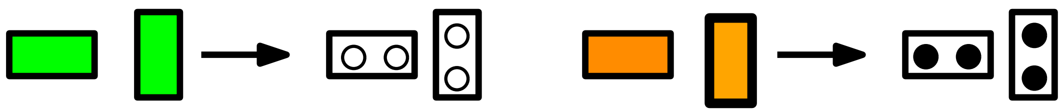

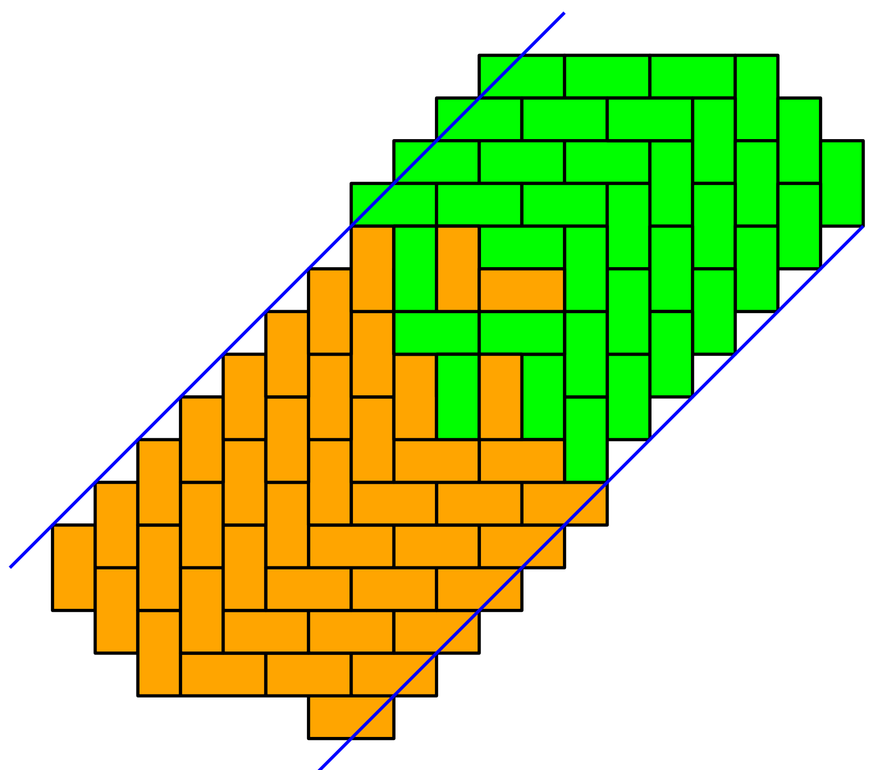

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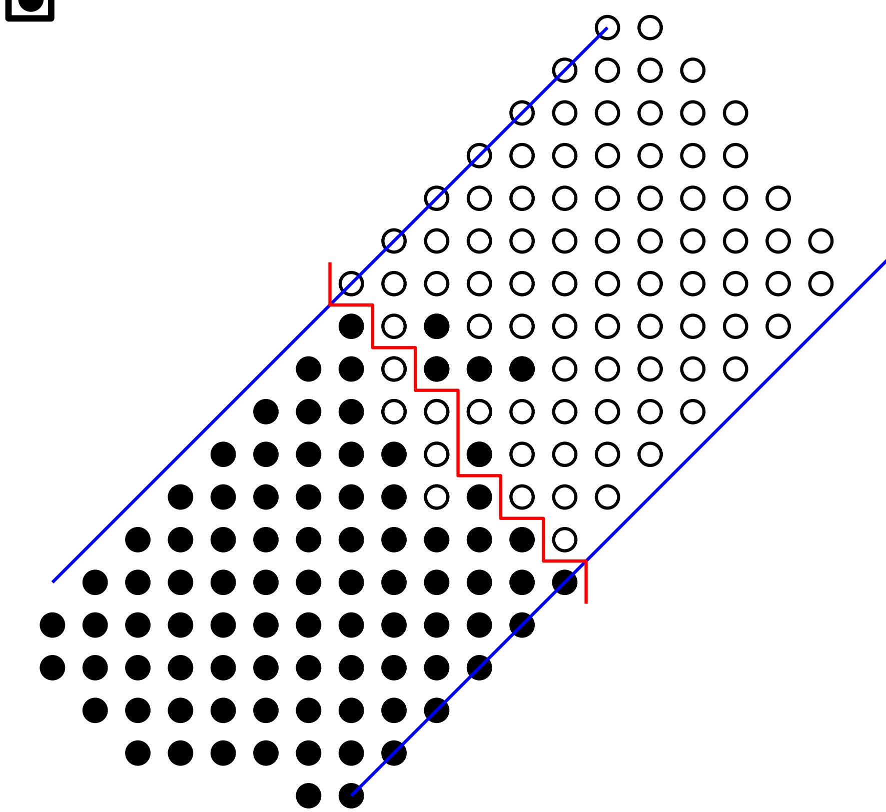
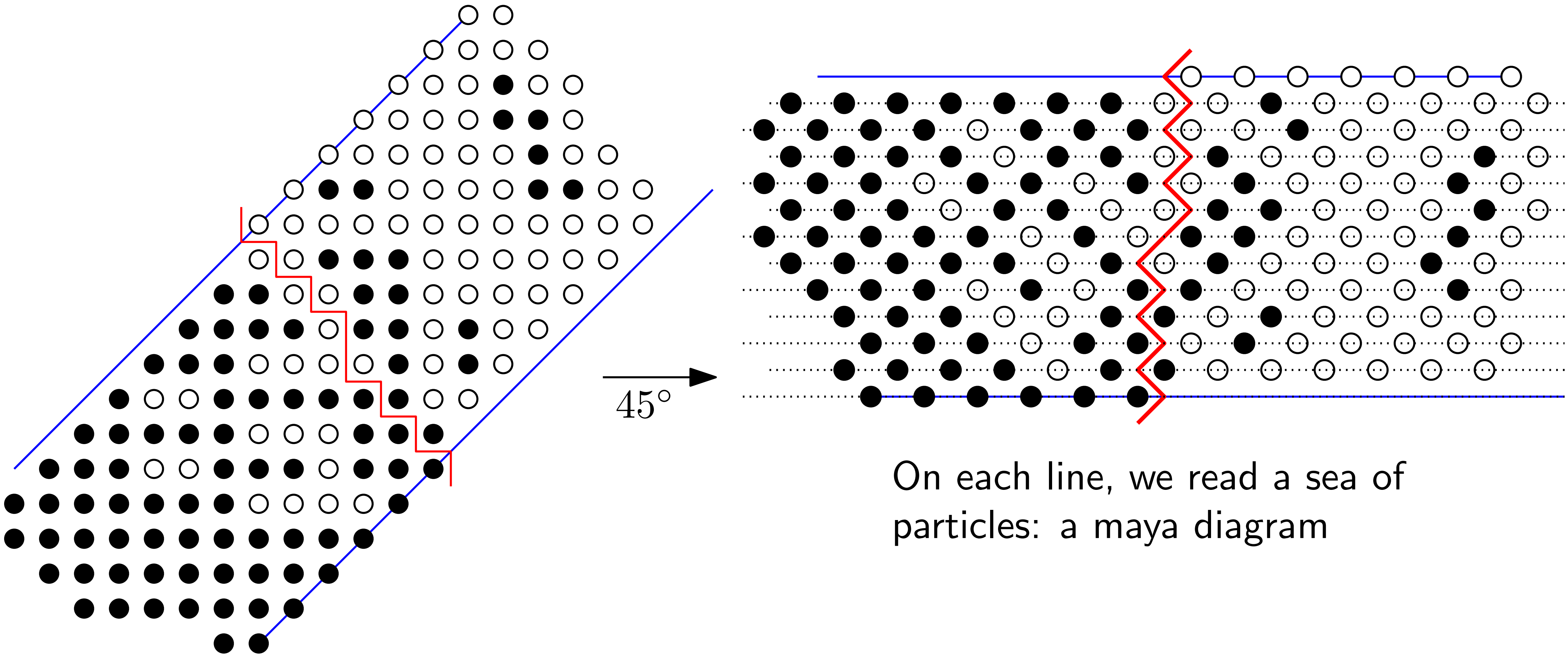


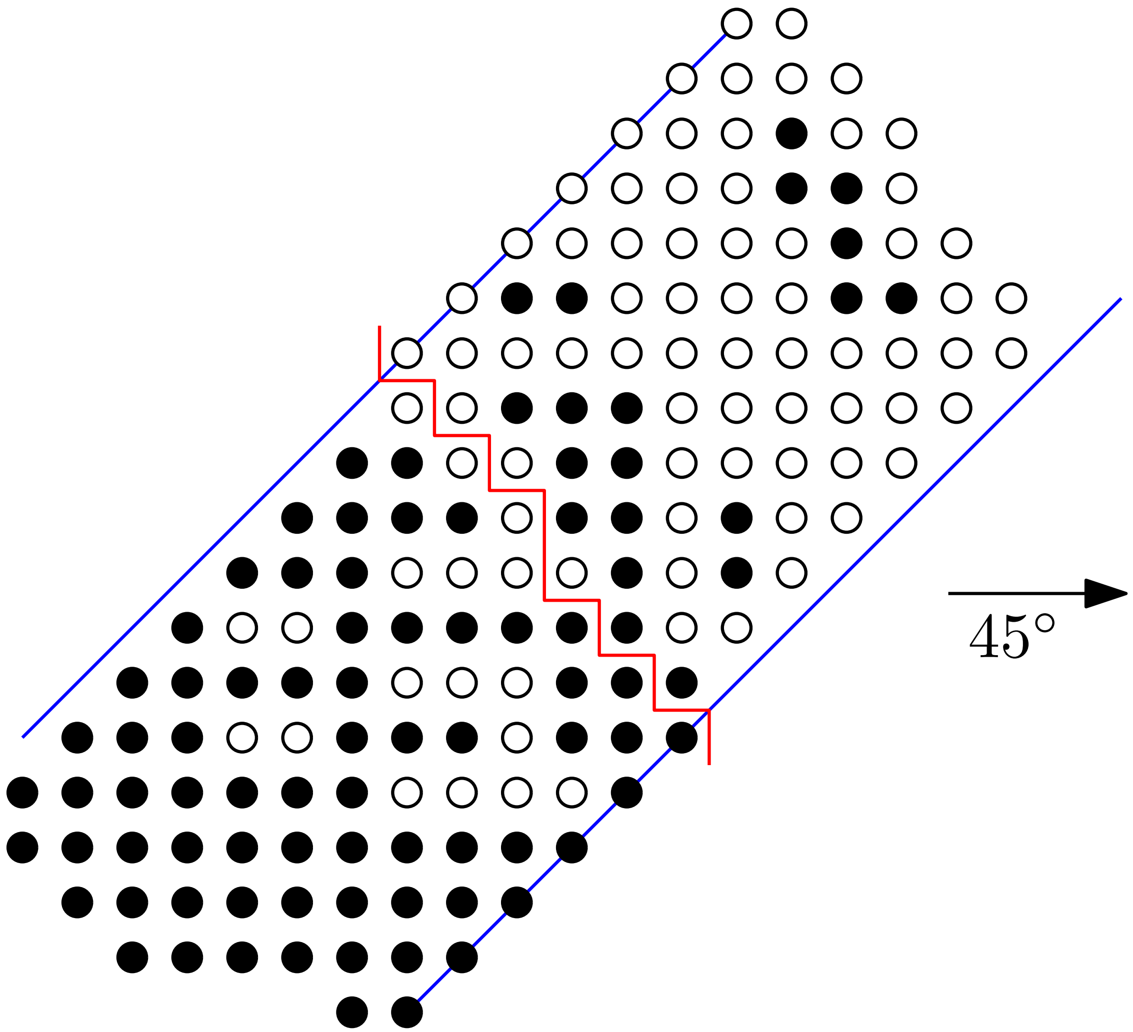
- A flip makes one particle move forward

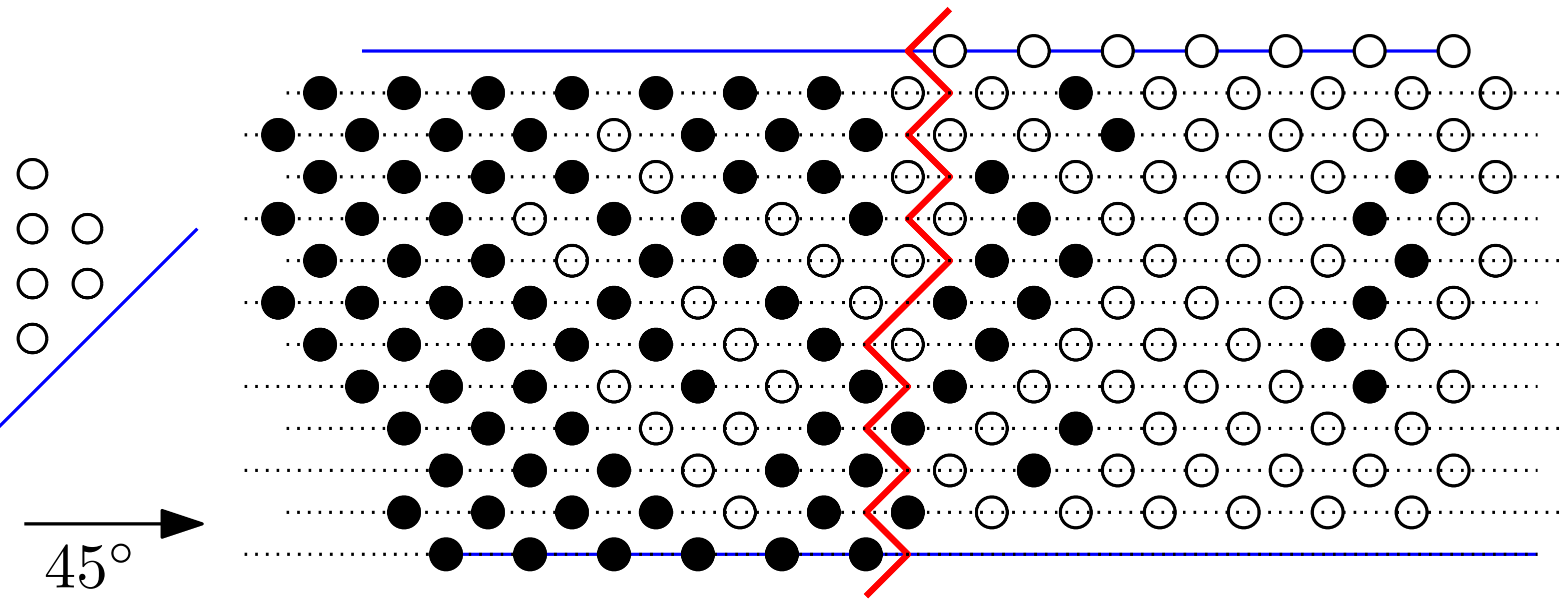
Particles



On each line, we read a sea of particles: a maya diagram

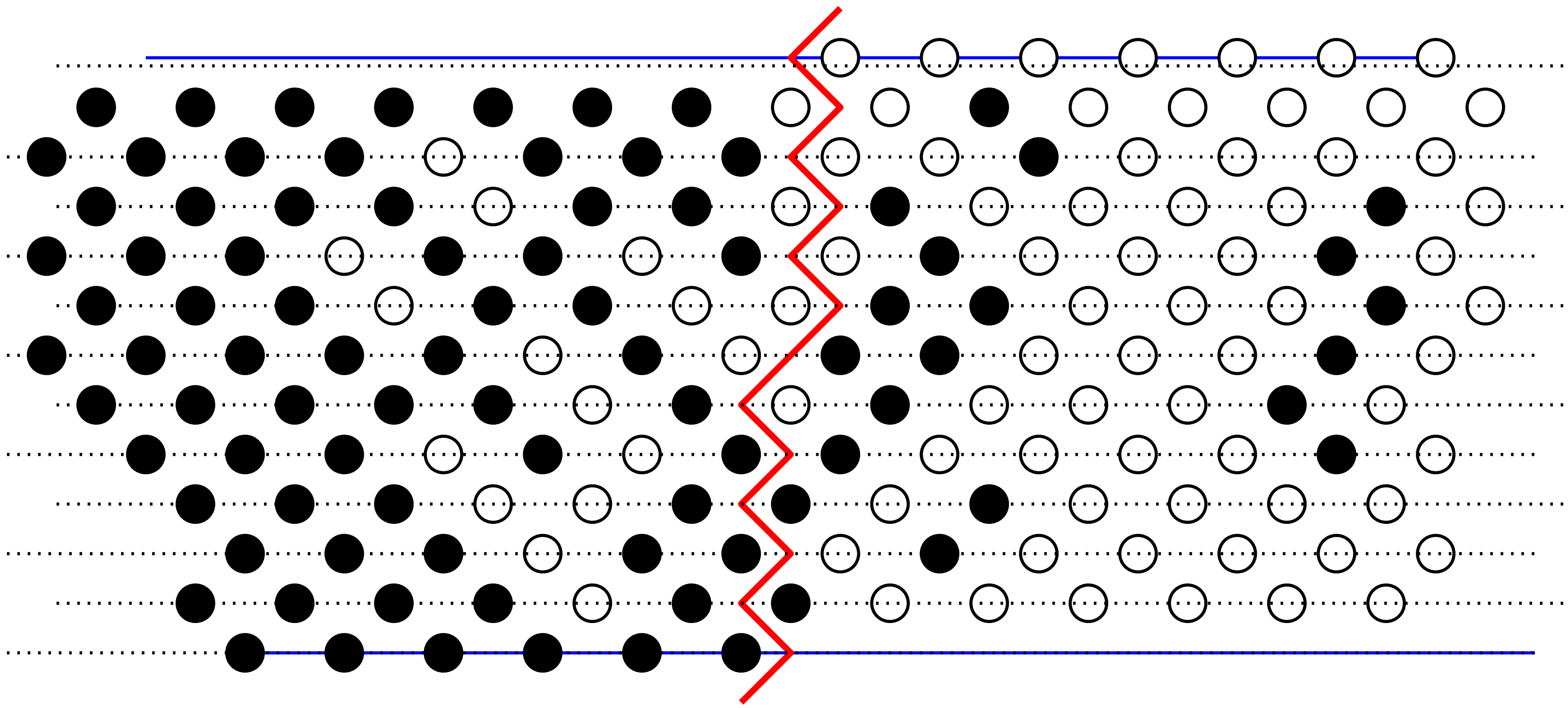
Particles



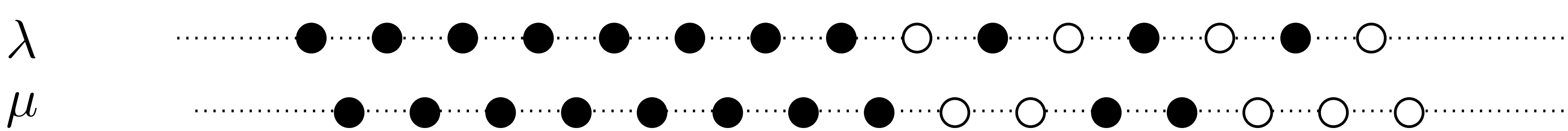


On each line, we read a sea of particles: a maya diagram

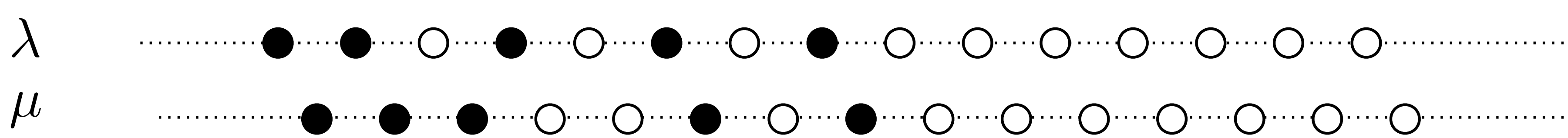
Particles



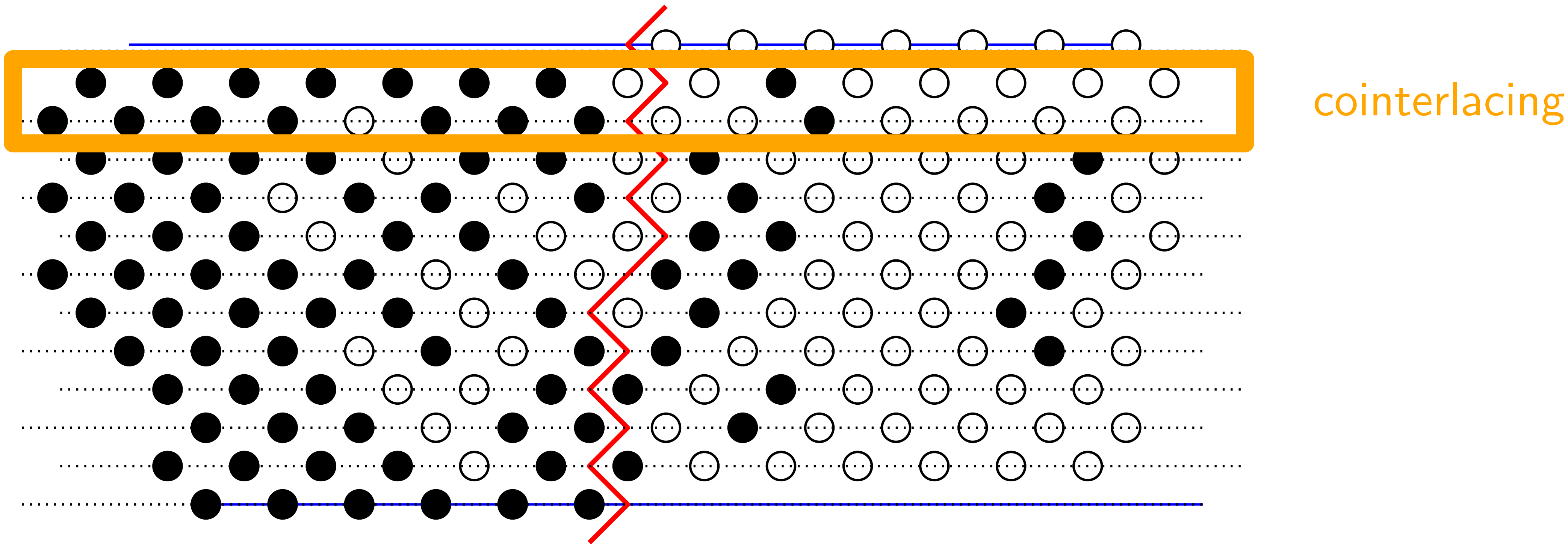
• Two lines are interlacing if the positions of black particles are alternating.



• Two lines are cointerlacing if the positions of white particles are alternating.



Particles



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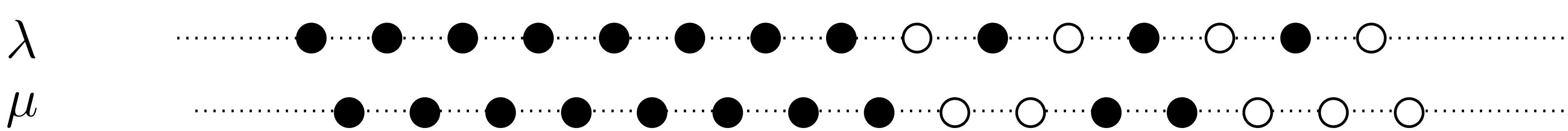

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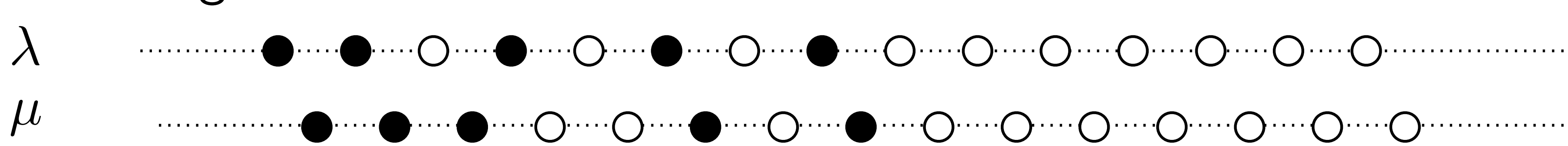
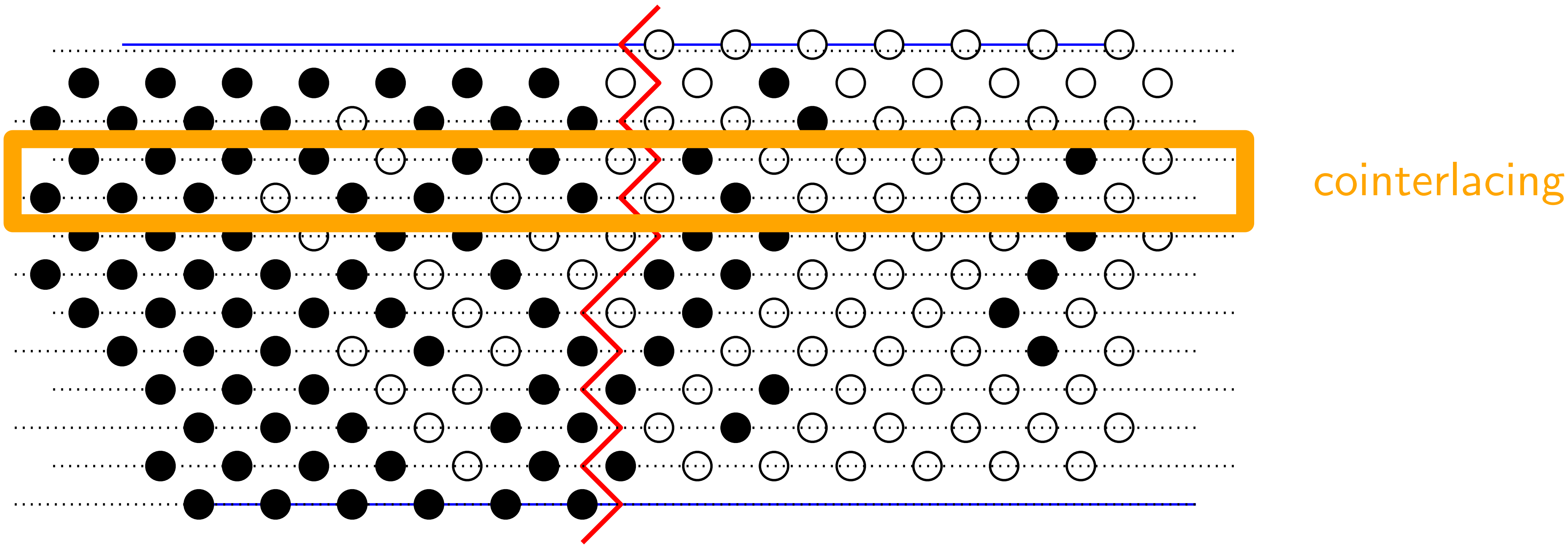


Particles



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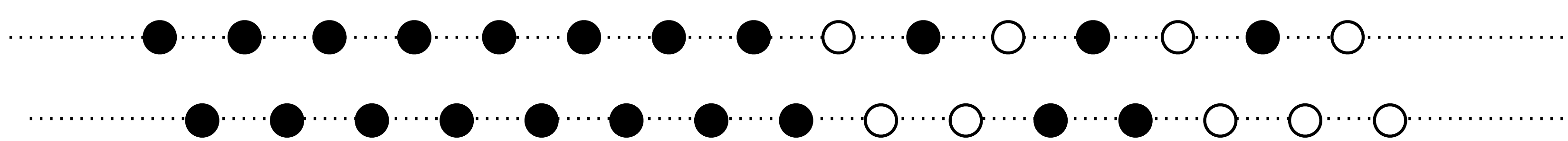
λ

μ



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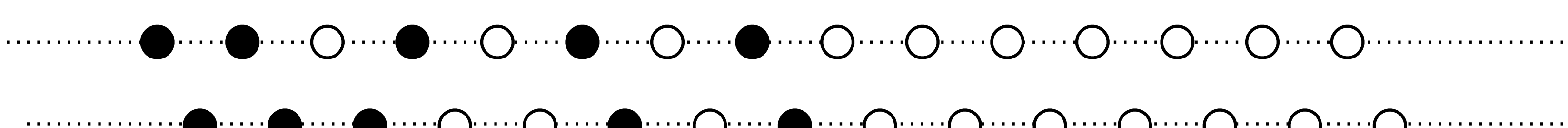
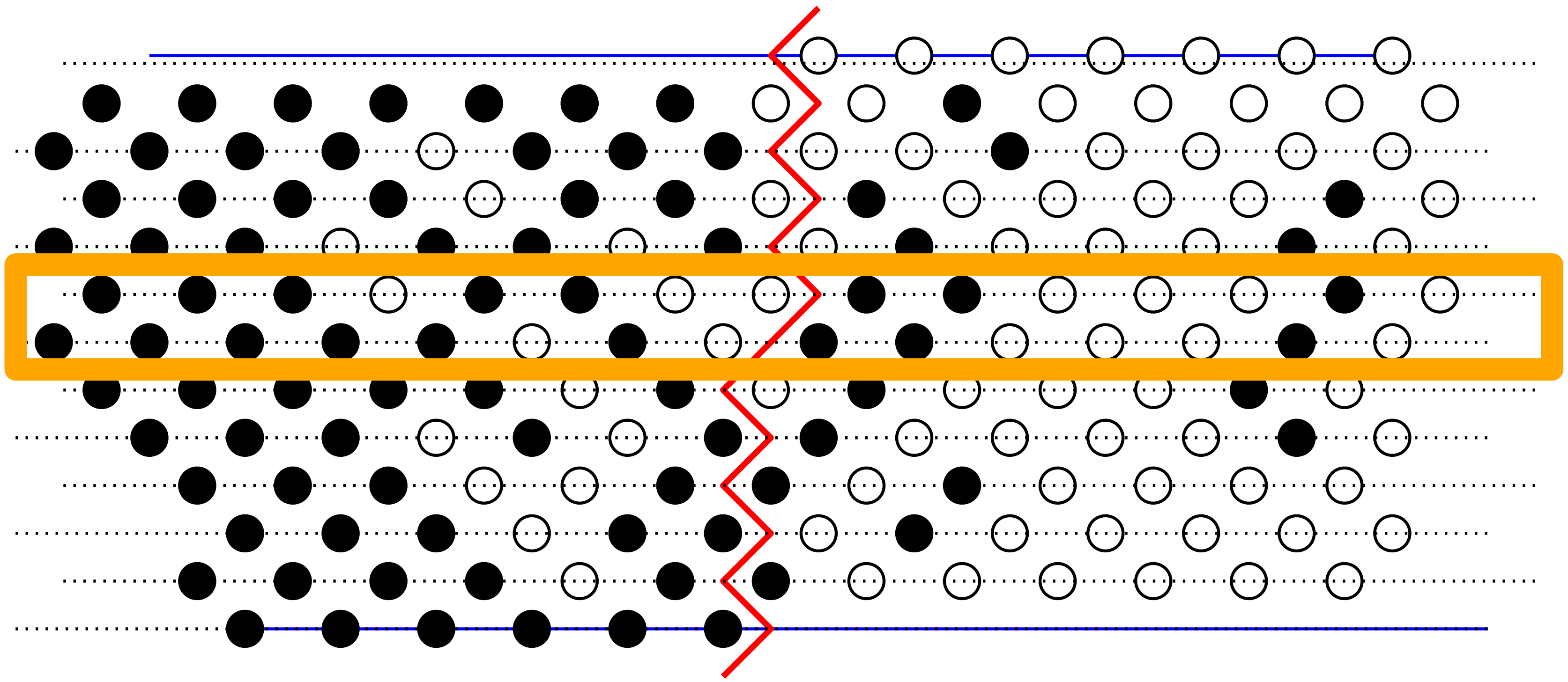
μ



Particles

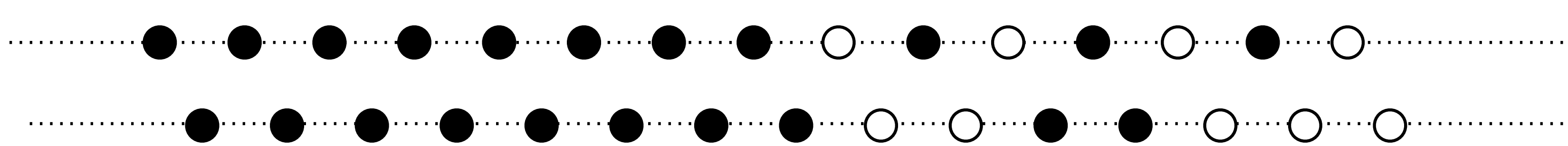


cointerlacing

• Two lines are interlacing if the positions of black particles are alternating.

λ

μ



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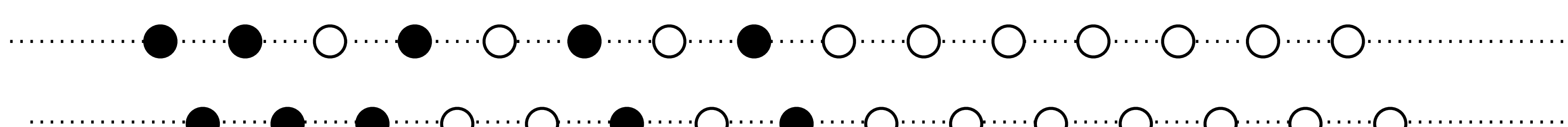
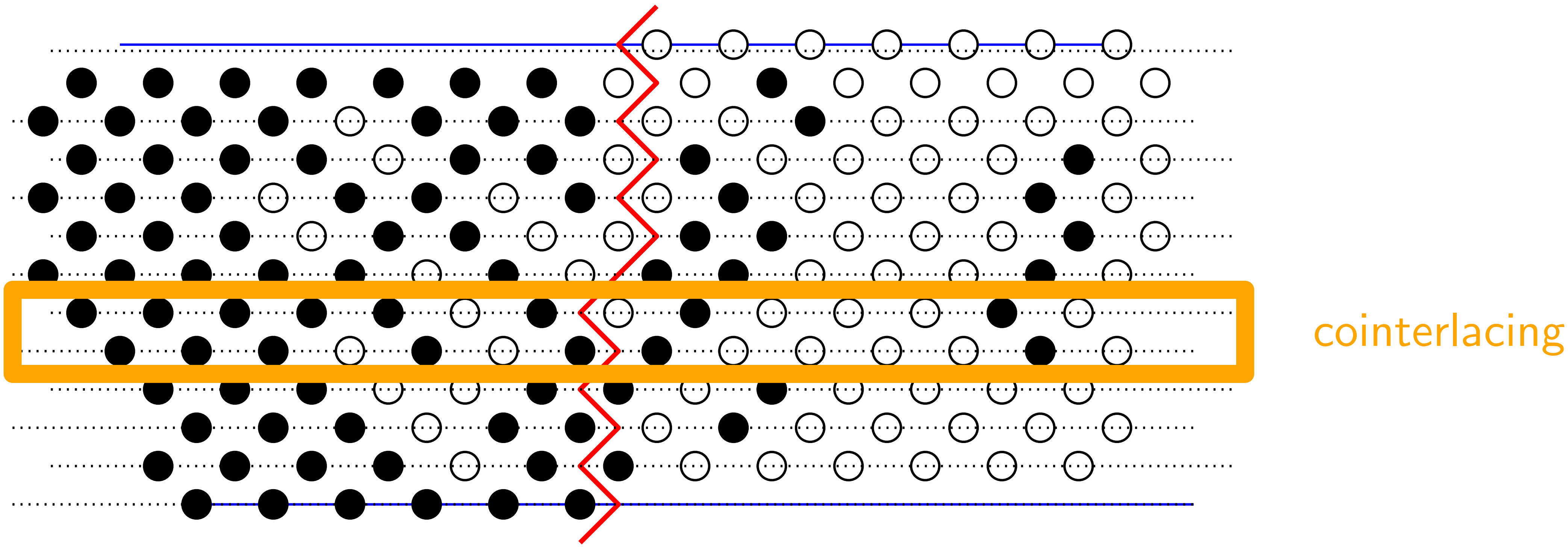
λ

μ



Particles



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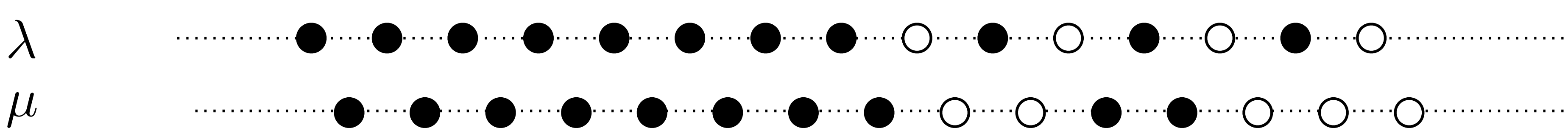

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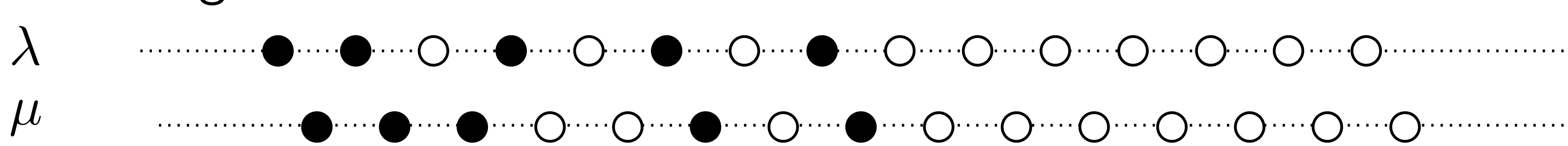

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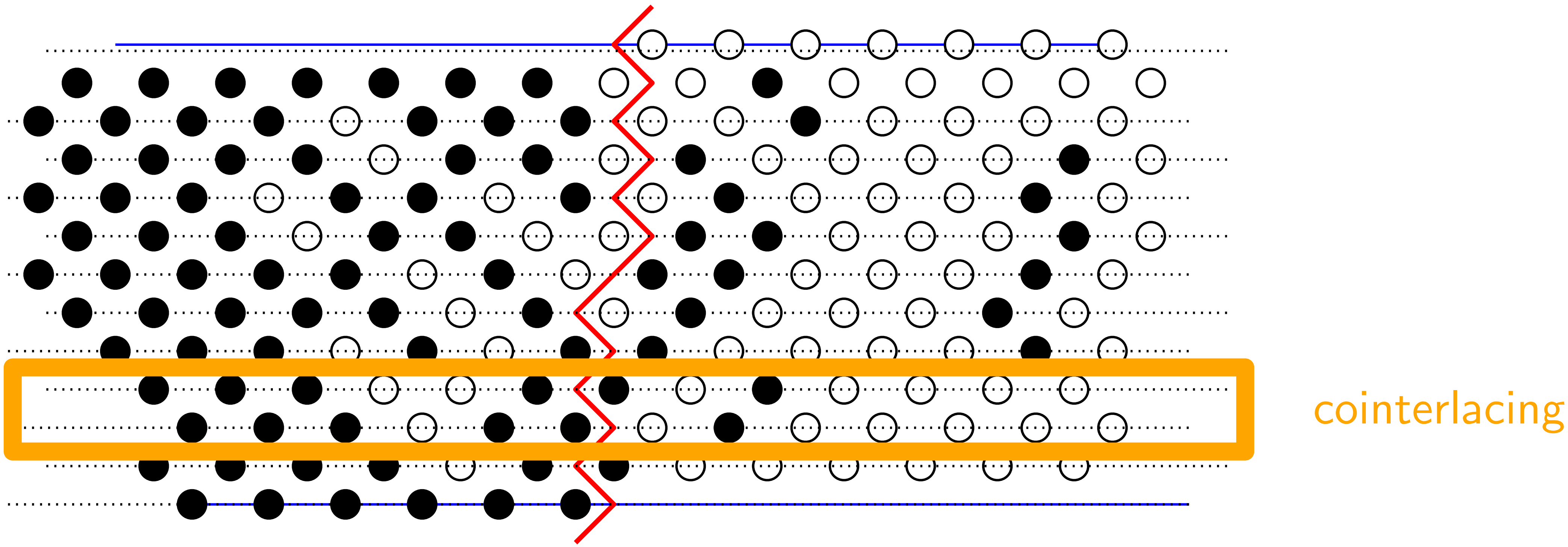

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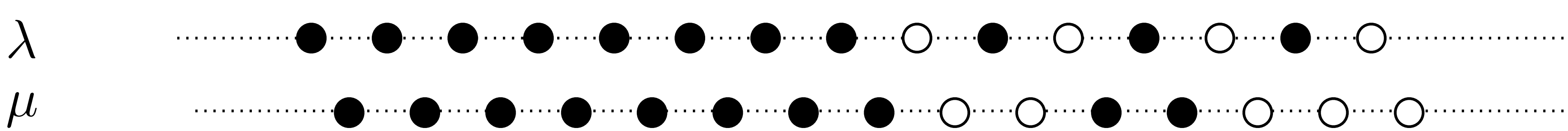

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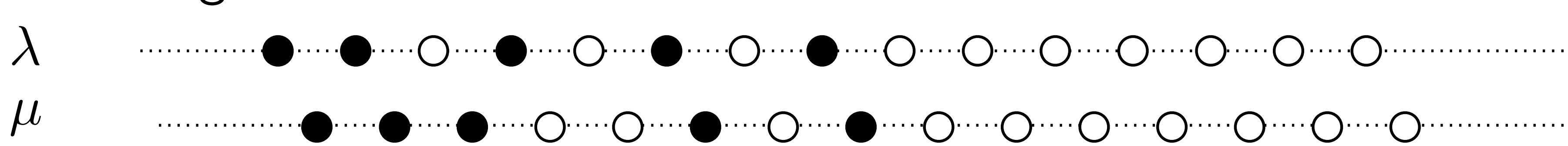

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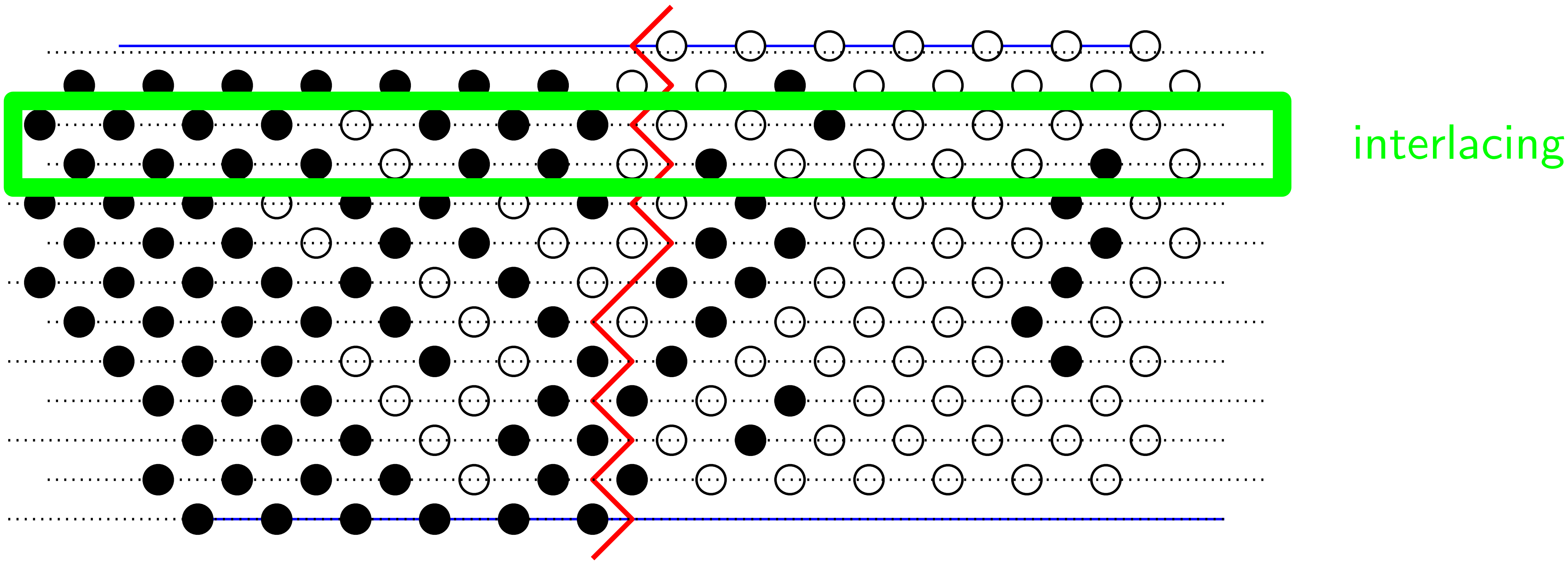

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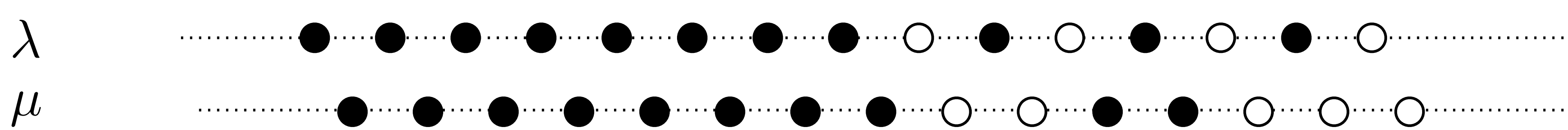

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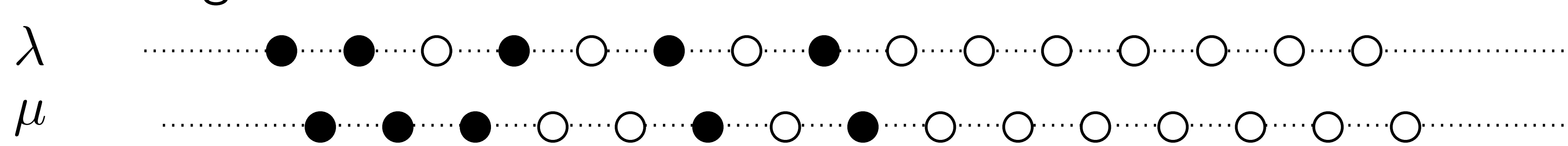
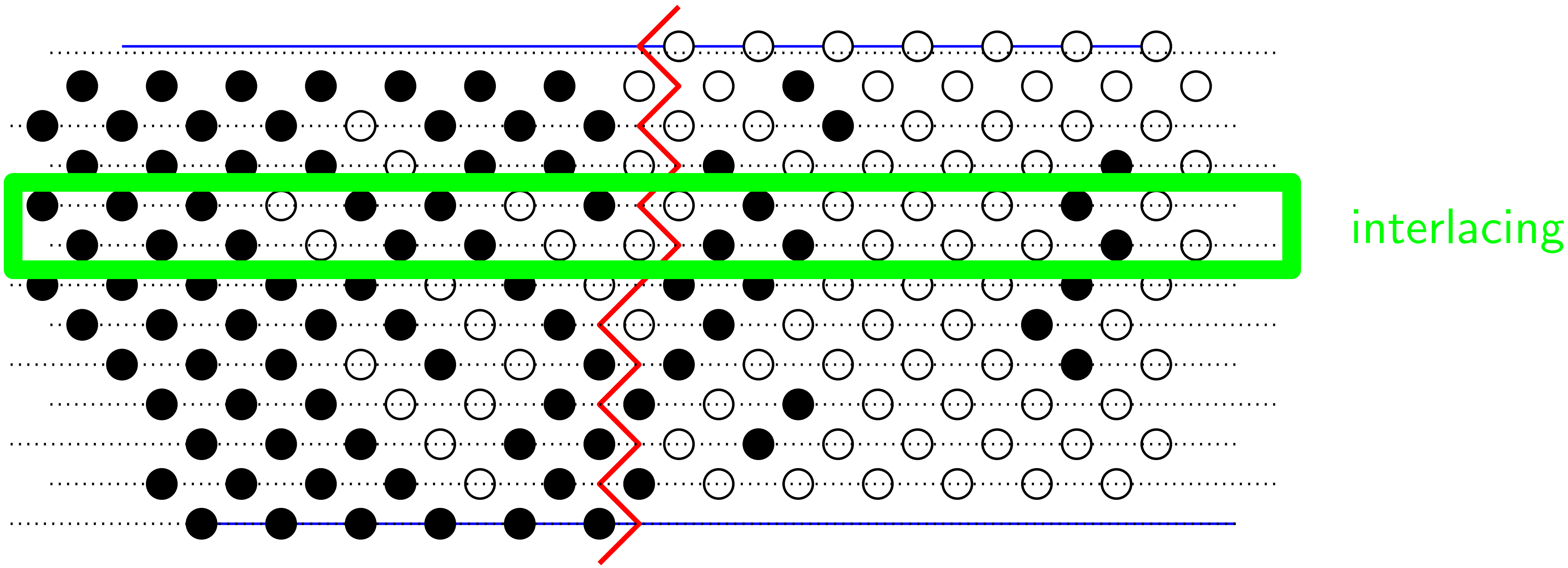


Particles



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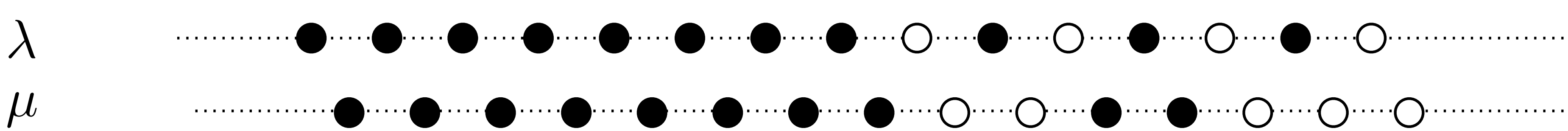

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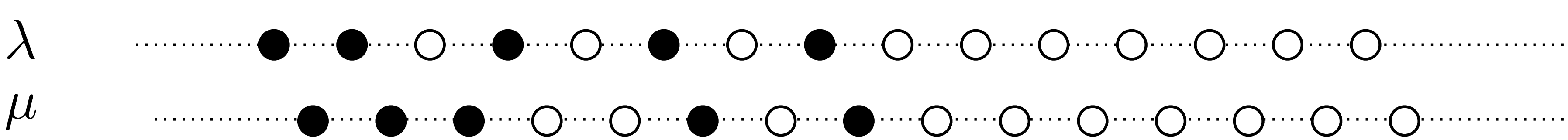
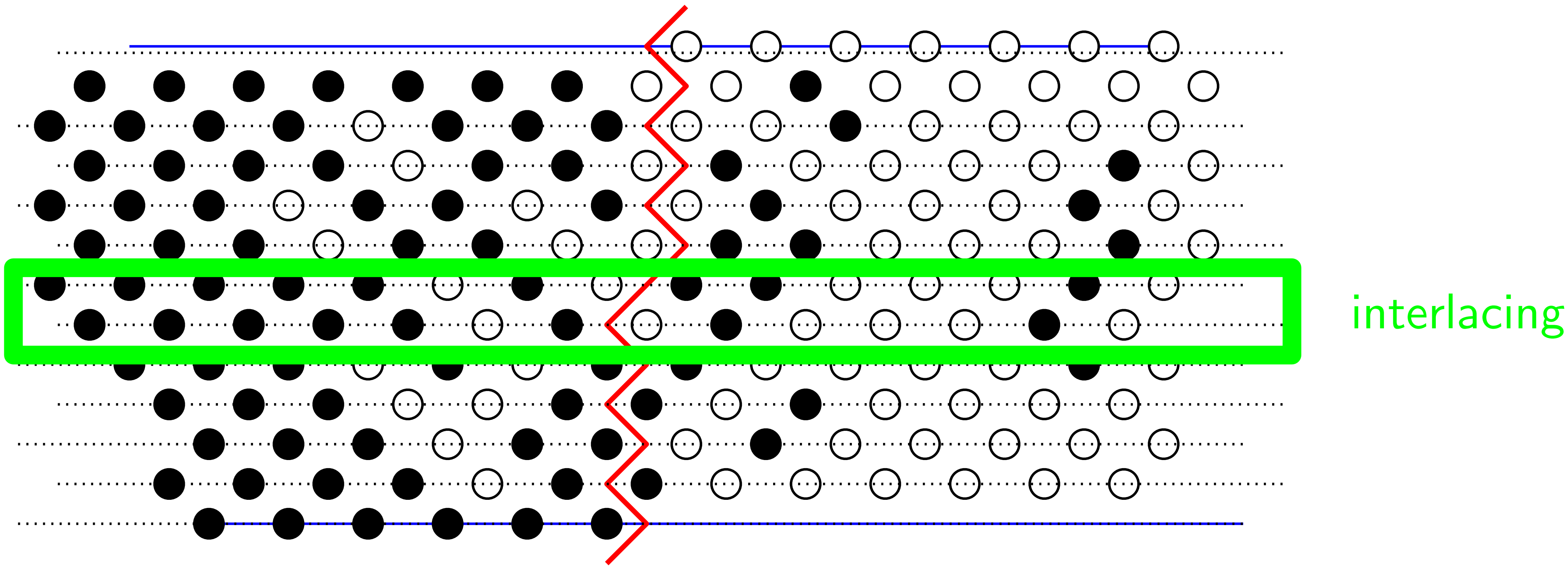


Particles



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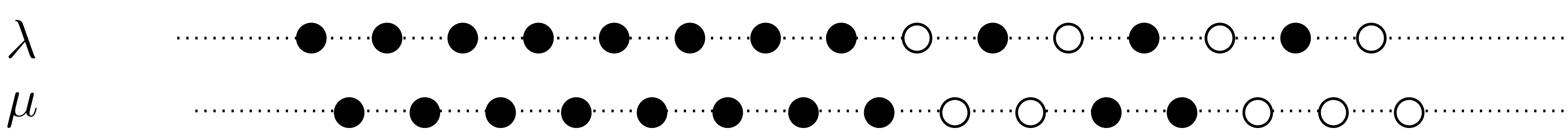

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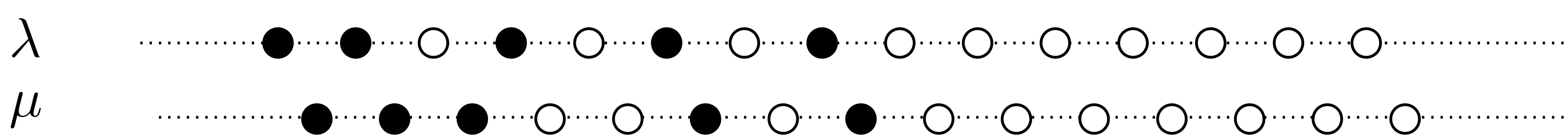
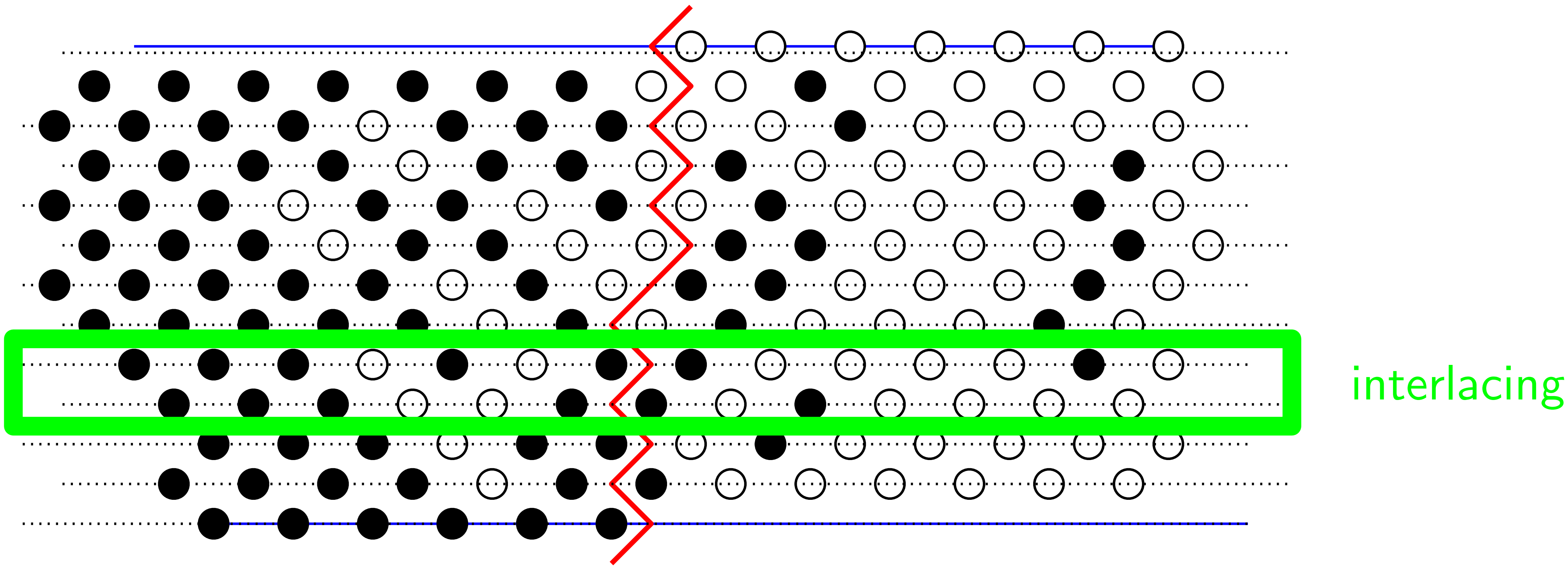


Particles



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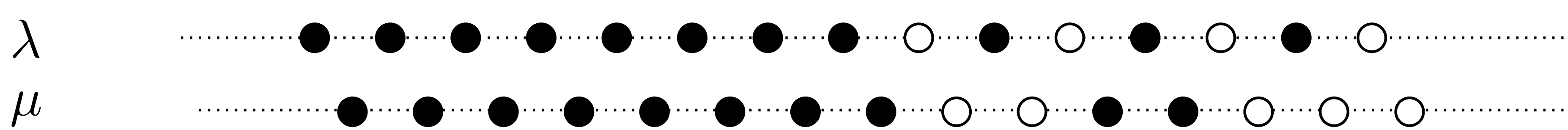

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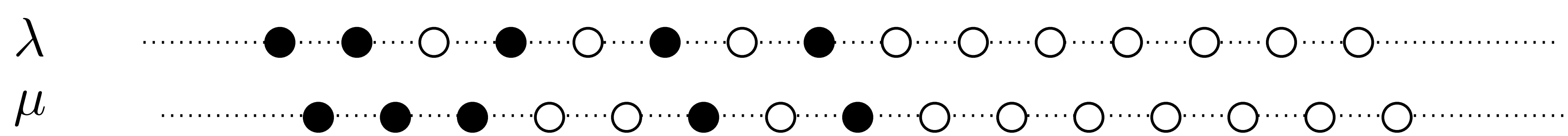
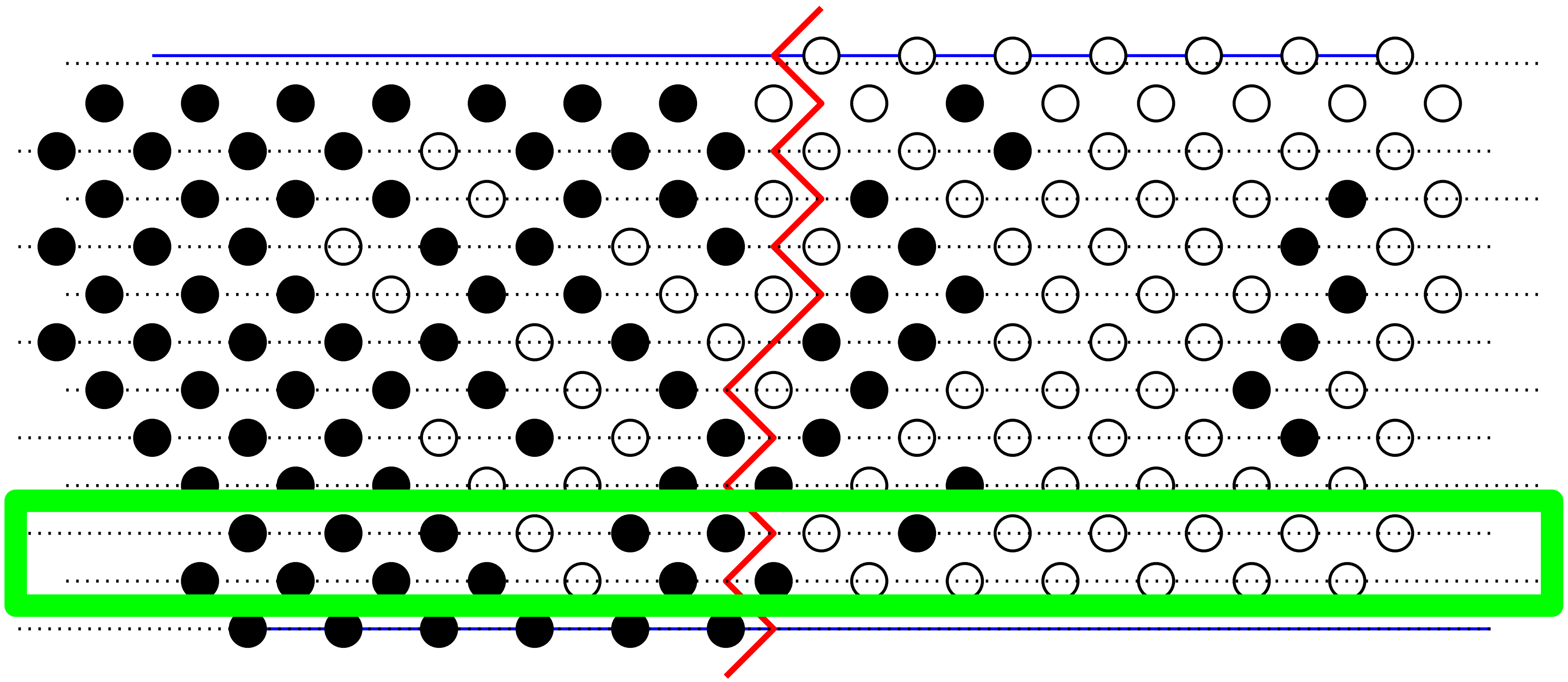


Particles



interlacing

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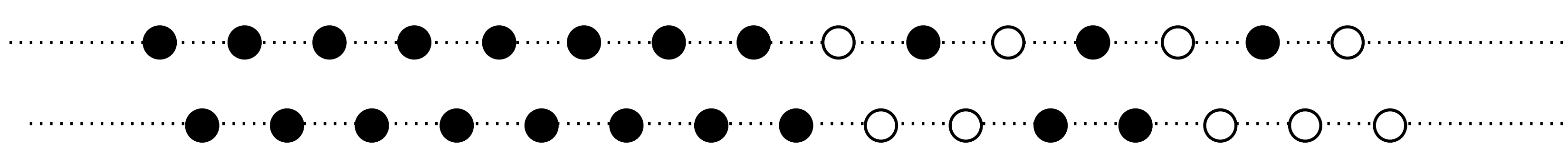
λ

μ



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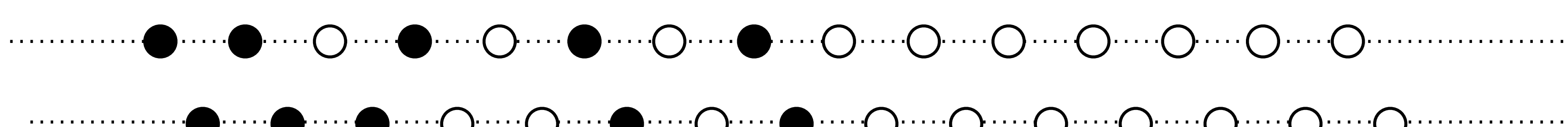
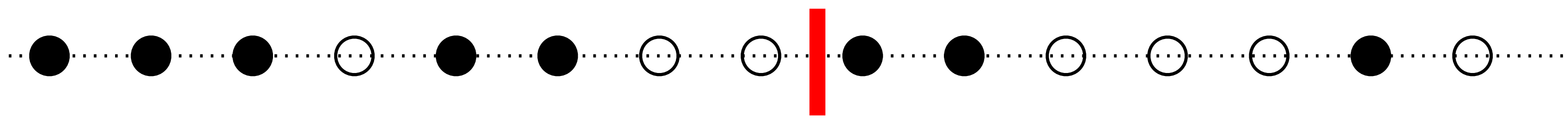
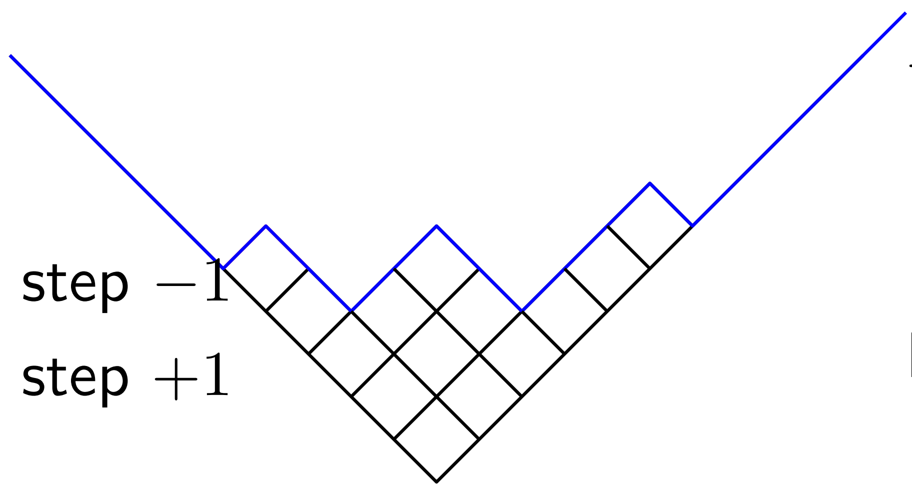


Integer partitions

From particles to integer partitions





● $=$ step -1

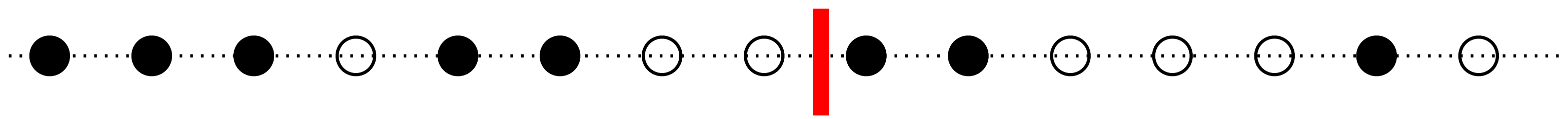
○ $=$ step $+1$

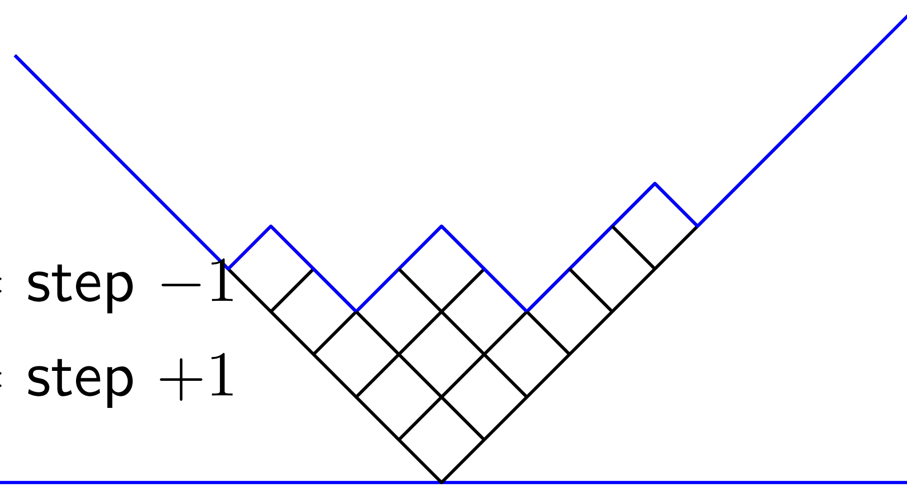
$\rightarrow$ partition $\lambda = (6, 3, 3, 1, 1)$.

Integer partition of n: non increasing sequence $(\lambda_1, \lambda_2, \dots)$ with $\sum_i \lambda_i = n$

Integer partitions

From particles to integer partitions





$\rightarrow$ partition $\lambda = (6,3,3,1,1)$.

● $=$ step -1

○ $=$ step $+1$

Integer partition of n: non increasing sequence $(\lambda_1, \lambda_2, \dots)$ with $\sum_i \lambda_i = n$

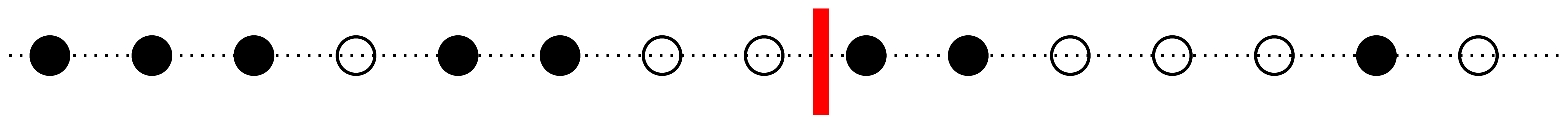
$\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ and $\mu = (\mu_1, \mu_2, \mu_3, \dots)$

Interlacing: $\mu \prec \lambda$ iff $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots$

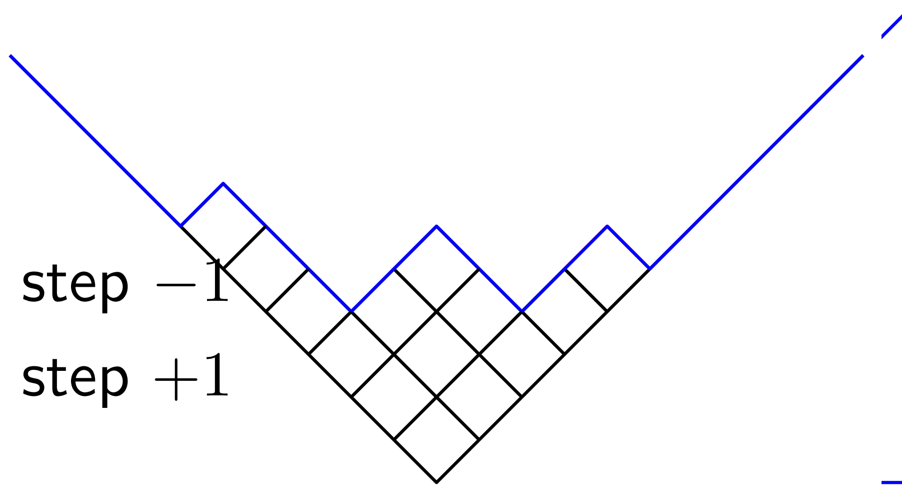
Cointerlacing $\mu \prec' \lambda$ iff $\mu' \prec \lambda'$

Integer partitions

From particles to integer partitions



$\lambda' = (5, 3, 3, 1, 1, 1)$



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Integer partition of n: non increasing sequence $(\lambda_1, \lambda_2, \dots)$ with $\sum_i \lambda_i = n$

$\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ and $\mu = (\mu_1, \mu_2, \mu_3, \dots)$

Horizontal strip

Interlacing: $\mu \prec \lambda$ iff $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots$

Cointerlacing $\mu \prec' \lambda$ iff $\mu' \prec \lambda'$ **Vertical strip**

Sequences of partitions

- **Thm:**[B-C-C] Steep tilings with period w are in bijection with sequences of partitions $(\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(2\ell)})$ such that $\lambda^{(0)} = \lambda^{(2\ell)} = \emptyset$ and:

$$\begin{array}{ll} w_{2k} = + & \lambda^{(2k)} \succ' \lambda^{(2k-1)} \\ w_{2k} = - & \lambda^{(2k)} \prec' \lambda^{(2k-1)} \end{array}$$

$$\begin{array}{ll} w_{2k-1} = + & \lambda^{(2k-1)} \succ \lambda^{(2k-2)} \\ w_{2k-1} = - & \lambda^{(2k-1)} \prec \lambda^{(2k-2)} \end{array}$$

The number of flips corresponds to the sum of the sizes of all the partitions

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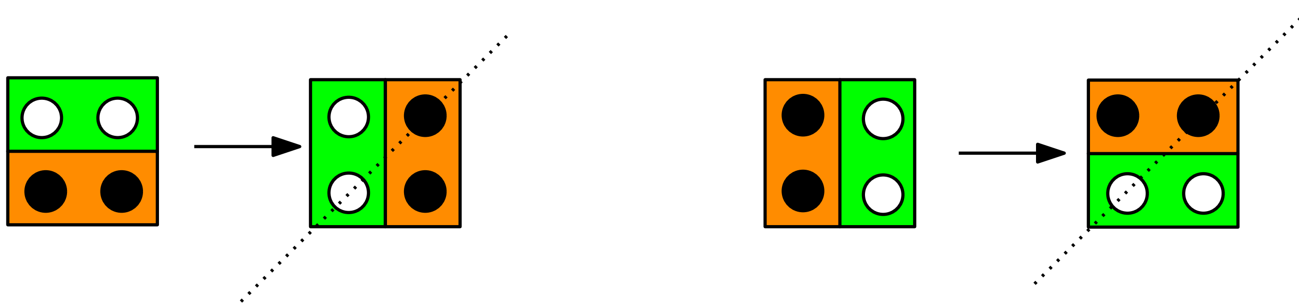
The number of flips corresponds to the sum of the sizes of all the partitions

Idea: True for the minimal tiling and stays true after any flip



even diag.

odd diag.

Interlacing= horizontal strip

Co-interlacing=vertical strip

Super-Schur functions/ Hook Schur functions

Vertex operators

Interlacing= horizontal strip

Co-interlacing=vertical strip

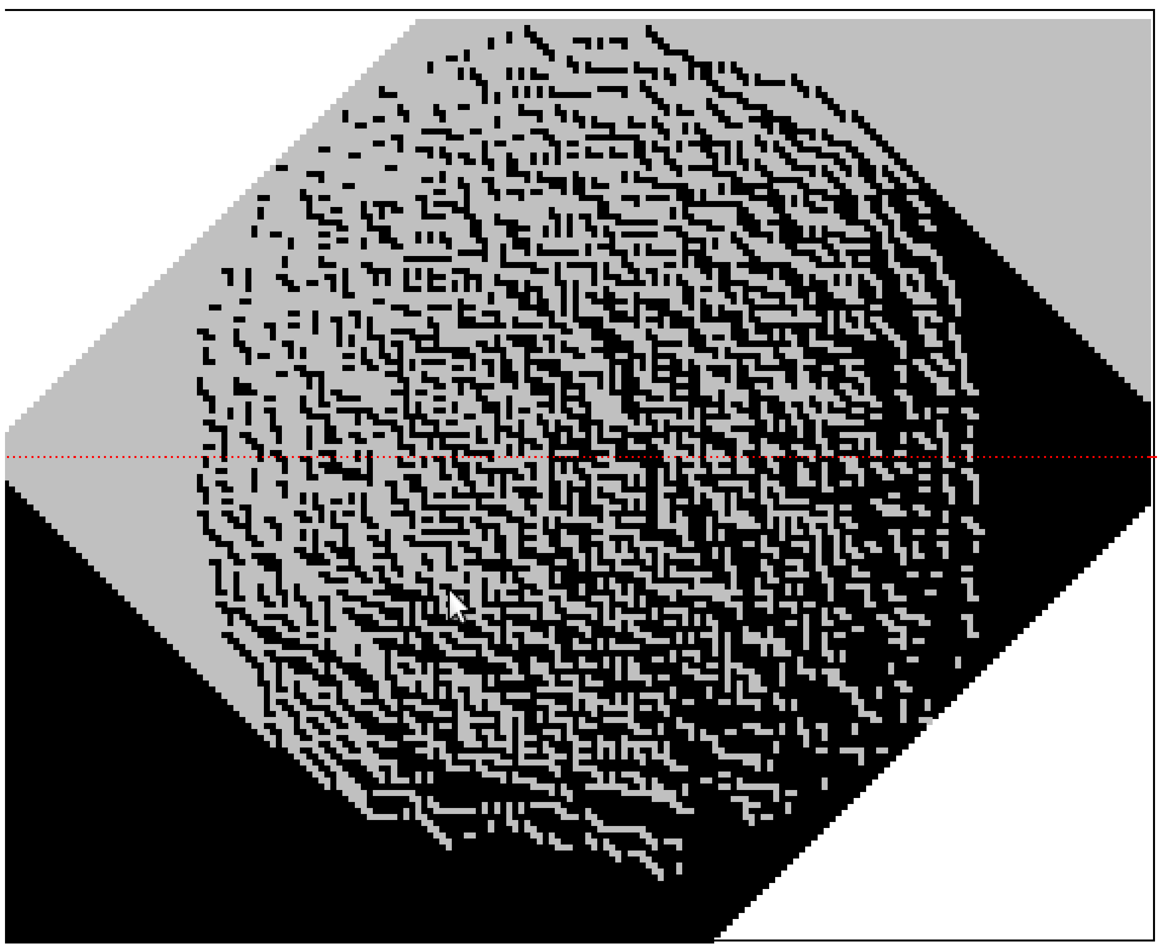
Shake!

$$\boxed{\begin{array}{l}\text{Super-Schur functions/ Hook Schur functions} \\ \\ \qquad\qquad\qquad\qquad\text{Vertex operators}\end{array}}$$

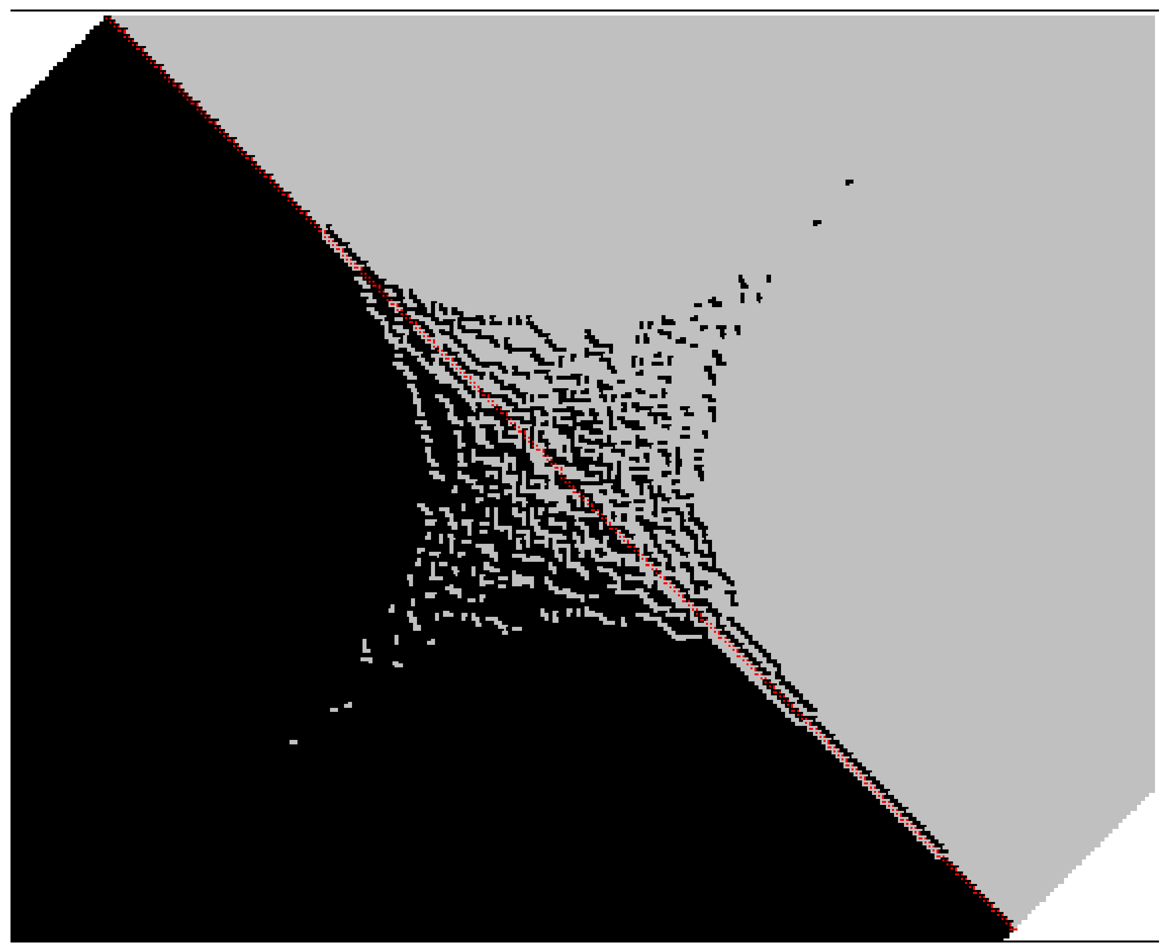
$$T_w(q) = \prod_{\substack{i<j \\ w_i=+,\ w_j=-}} (1 + \epsilon q^{j-i})^{\epsilon}$$

Bijective proofs and Random generation

(with Dan Betea (Paris 6) and Mirjana Vuletić (UMass))

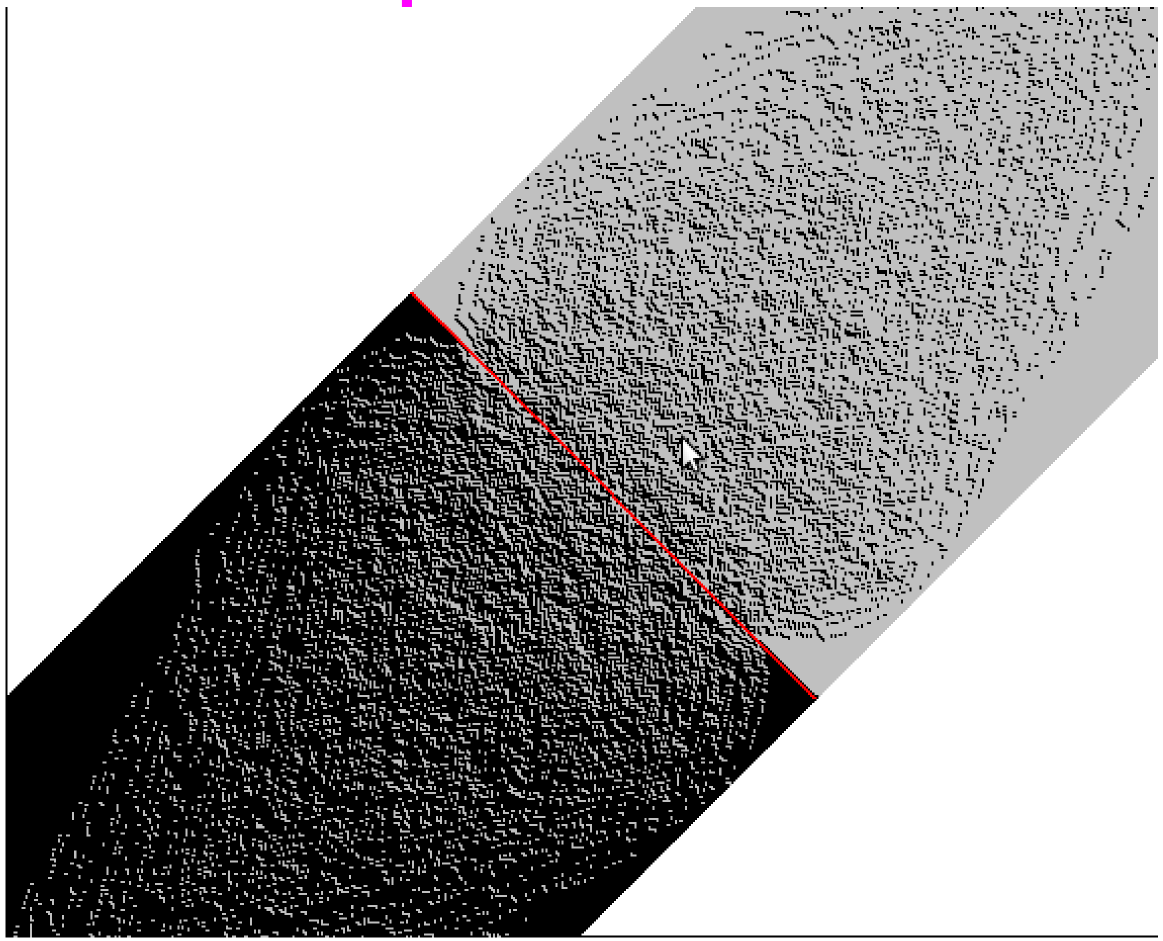


aztec diamond



pyramid partitions

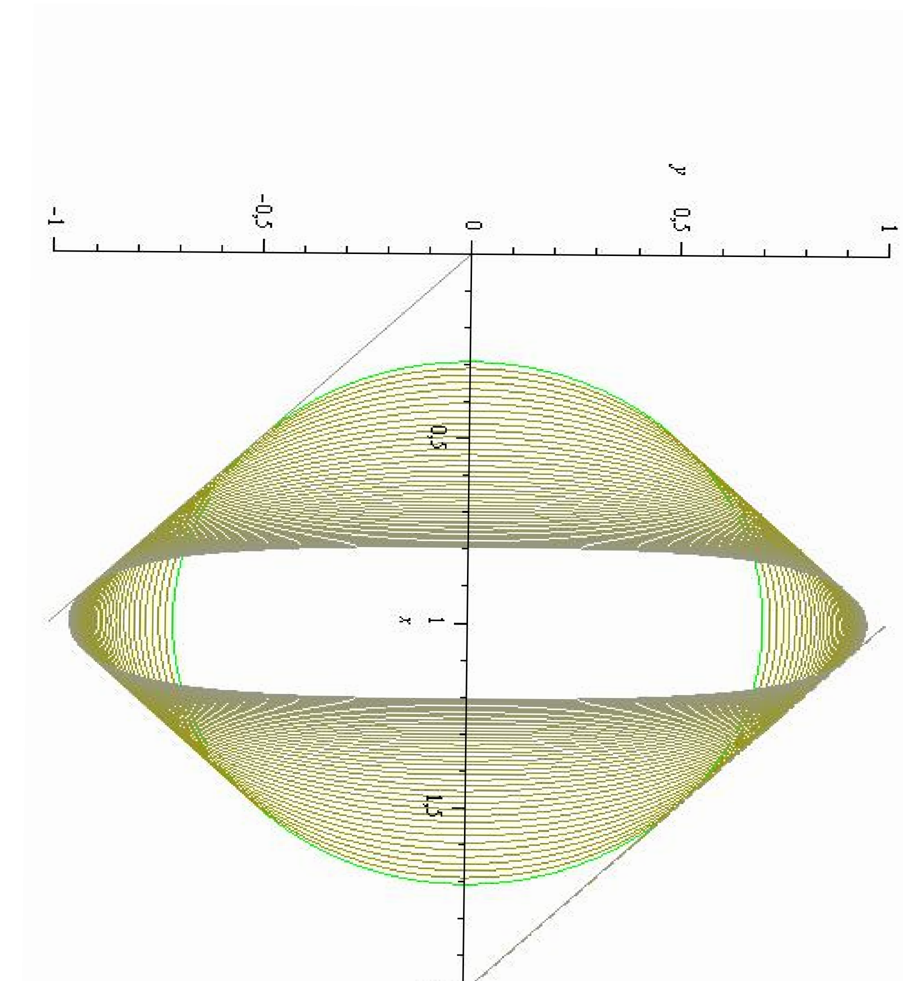
Limit shapes



New phenomenon:

limit shapes of ℓ-pyramid partitions

with Mirjana Vuletic (UMASS)



Aztec diamond with $1/4 \leq q \leq 1$

Movie (©Dan Betea 2013)

Generalizations

Steep tilings on a cylinder

- **Thm:**[B-C-C] Cylindric steep tilings with period w are in bijection with sequences of partitions $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(2\ell)})$ such that $\lambda^{(0)} = \lambda^{(2\ell)}$

$$T_w^{(c)}(q) = T_w(q) \times \prod_{k\geq 1} \frac{1}{1-q^{2\ell k}} \prod_{w_i=+,\ w_j=-} (1+\epsilon q^{2\ell k+j-i})^{\epsilon}$$

Free steep tilings : allow flips on the boundaries $y = x$ and $y = x - 2\ell$

- **Thm:**[B-C-C] Free steep tilings with period w are in bijection with sequences of partitions $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(2\ell)})$

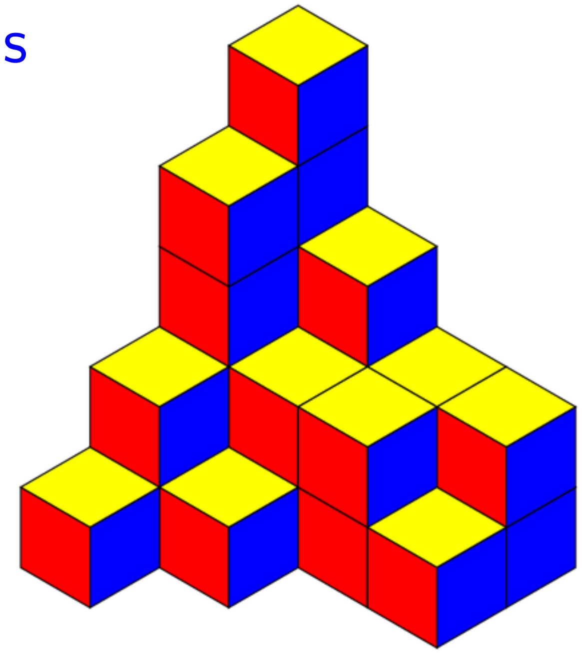
$$T_w^{(f)}(u,v,q) = \sum q^{\#\text{flips}} u^{|\lambda^{(0)}|} v^{|\lambda^{(2\ell)}|}$$

$$T_w^{(f)}(q,u,v) = T_w(q) \times \prod_{k\geq 1} \frac{1}{1-(uv)^k} \prod_{w_i\neq w_j} (1+\epsilon u^{2k+1} v^{2k+1} q_i q_j)^{\epsilon}$$
$$\times \prod_{w_i=w_j} (1+\epsilon u^{2k} v^{2k} q_i q_j)^{\epsilon} \prod_i \frac{1}{1-u^k v^k q^i}$$

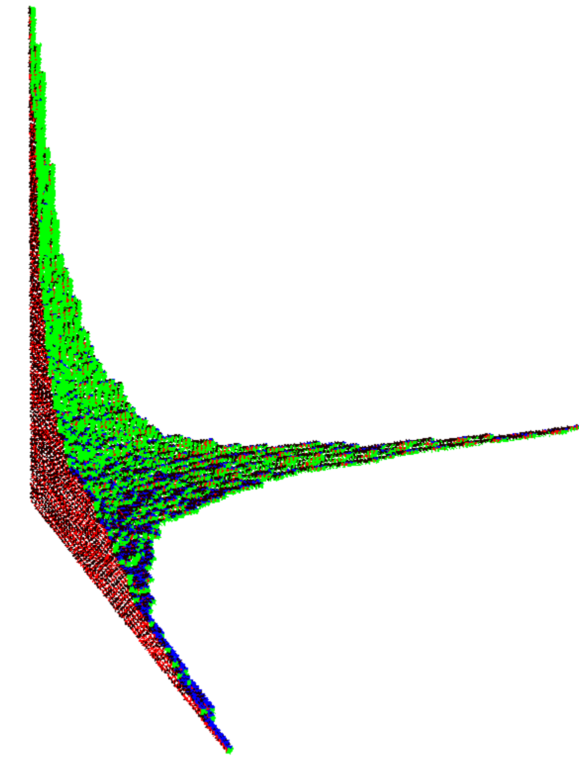
with $u \to q^{2\ell+1}$, $q_i = q^i/u$ if $w_i = +$ and q^{-i}/v if $w_i = -$

Some results who look alike: plane partitions

- Plane partitions

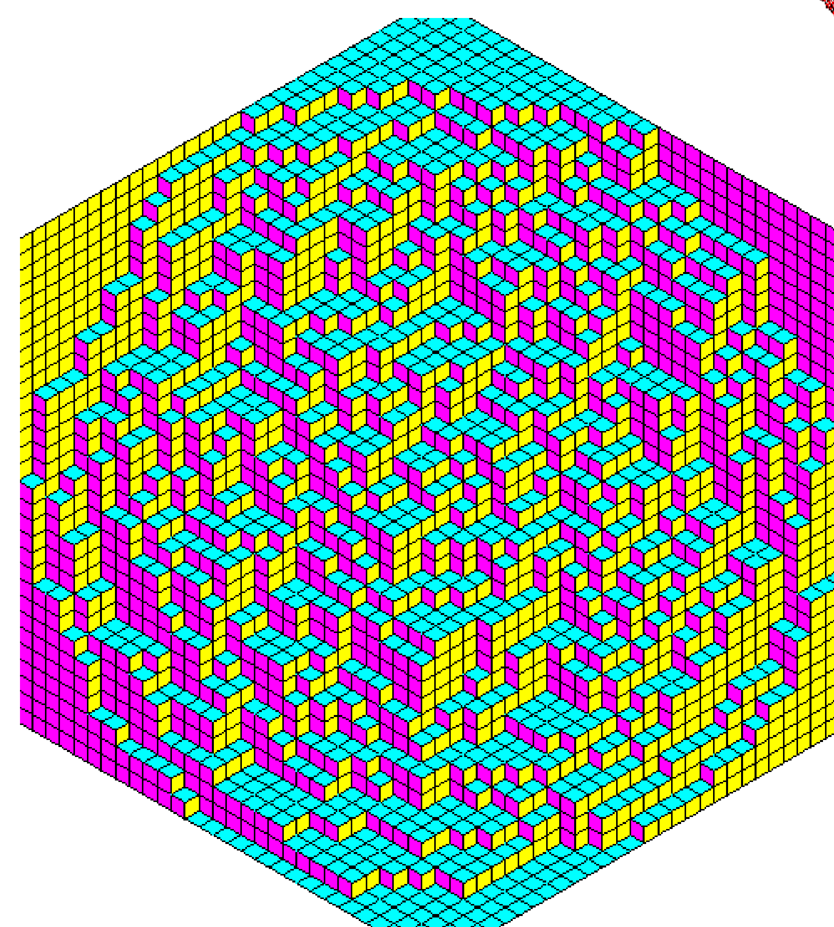


- Limite shape [Okounkov-Reshetikhin 05]



- Generating function [MacMahon 1916]

$$\sum_{P} q^{\#\text{cubes}(P)} = \prod_{i\geq 1} \frac{1}{(1-q^i)^i}$$



- box $n\times n\times n$ arctic circle [Cohn-Larsen-Propp 98]

Some results who look alike: plane partitions

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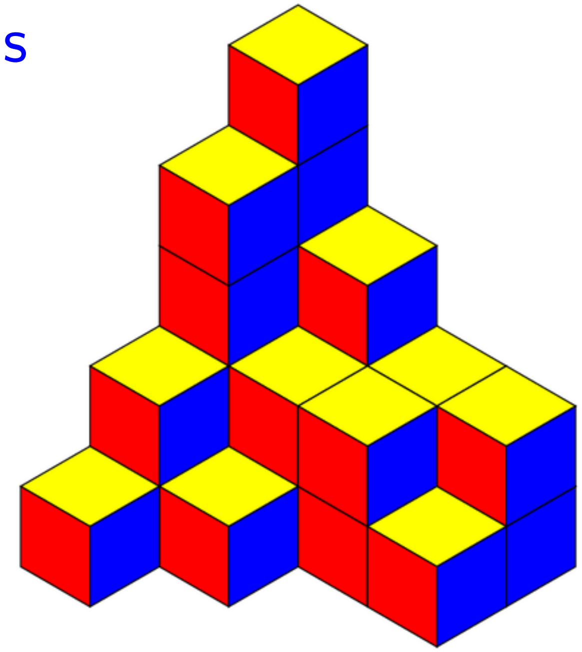

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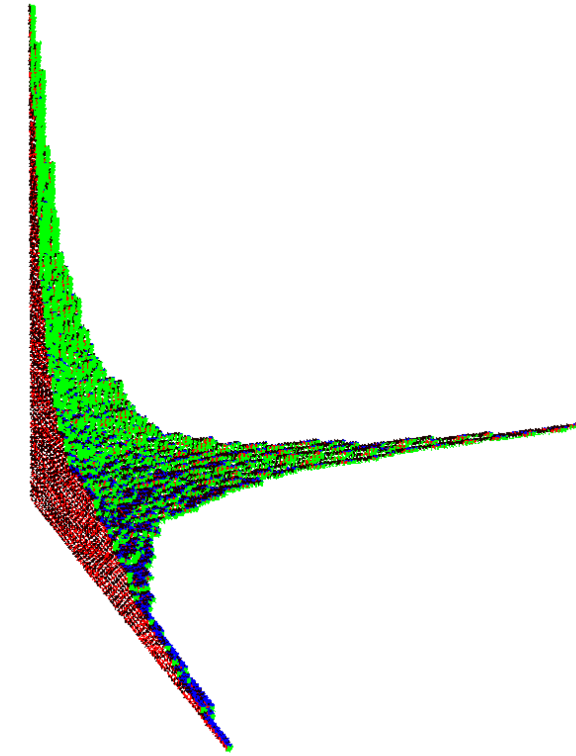

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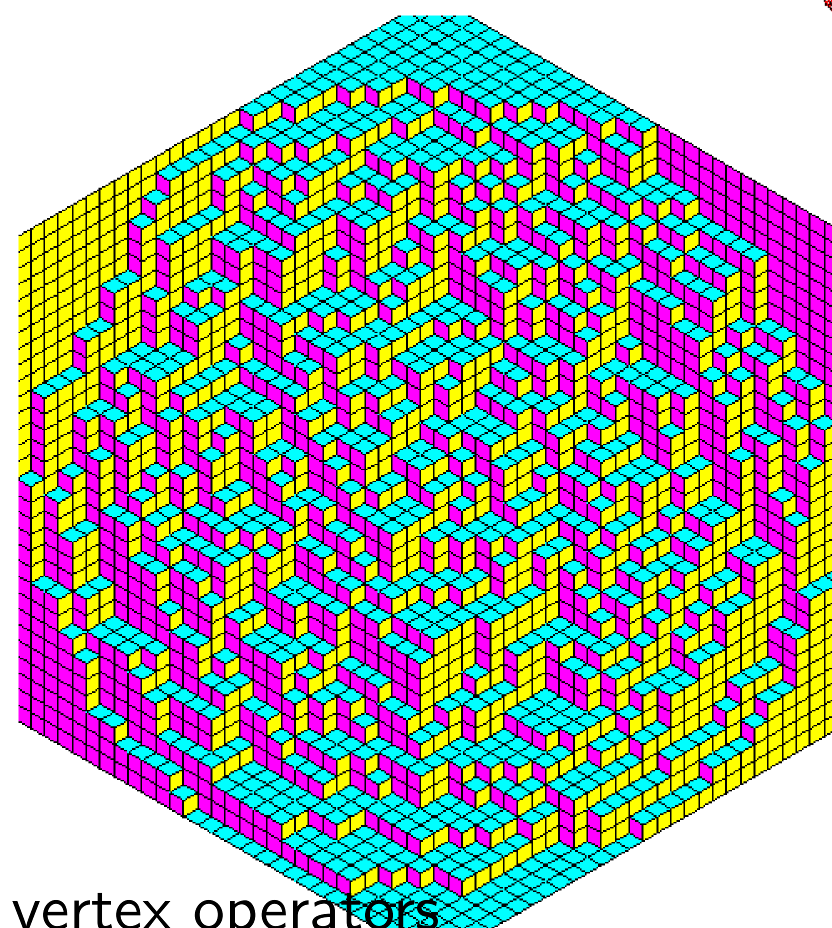


- box $n \times n \times n$ arctic circle [Cohn-Larsen-Propp 98]

- Reverse partitions in λ [Stanley 70s]

$$\prod_{c\in\lambda} \frac{1}{(1-q^{\text{hook}(c)})}$$

[Okounkov Reshetikin 00s] Schur functions, vertex operators

Steep tilings

Plane partitions

Steep tilings

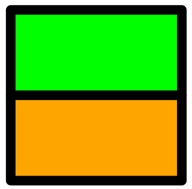
Add/remove vertical strip going from even to odd diagonals

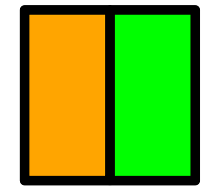
Add/remove horizontal strip going from odd to even diagonals

Plane partitions

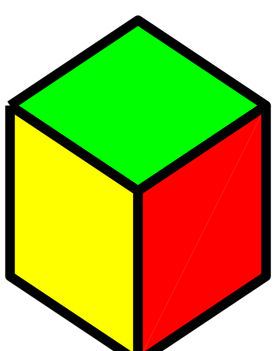
Add/remove horizontal strip

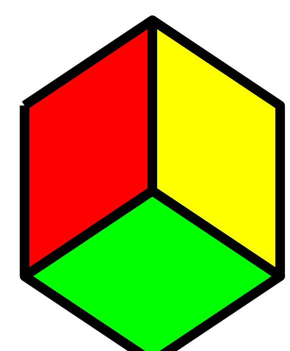
Steep tilings $\leftrightarrow$ **Domino tilings**





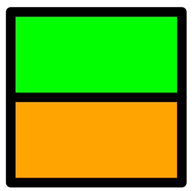
Plane partitions $\leftrightarrow$ **Lozenge tilings**

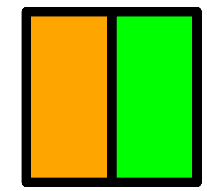




Steep tilings $\leftrightarrow$ **Domino tilings**

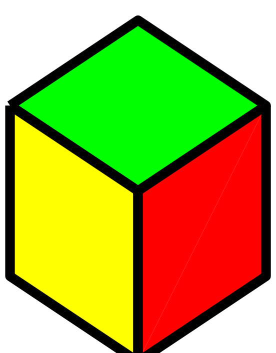
$\leftrightarrow$ Dimers on the square lattice

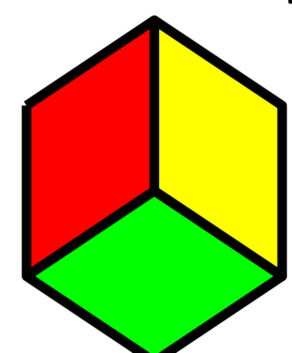




Plane partitions $\leftrightarrow$ **Lozenge tilings**

$\leftrightarrow$ Dimers on the honeycomb lattice





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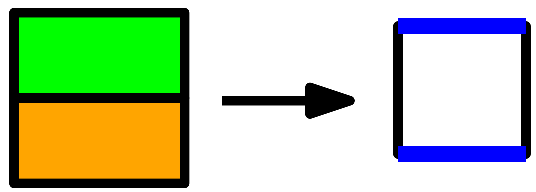

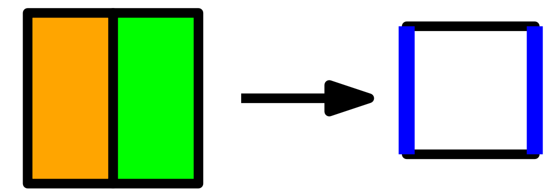

Plane partitions $\leftrightarrow$ **Lozenge tilings**

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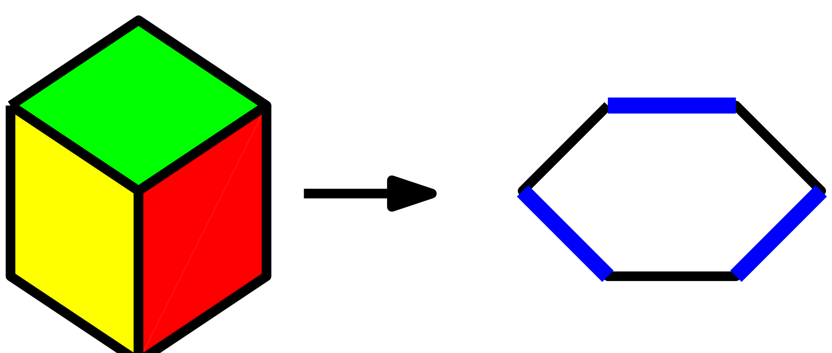
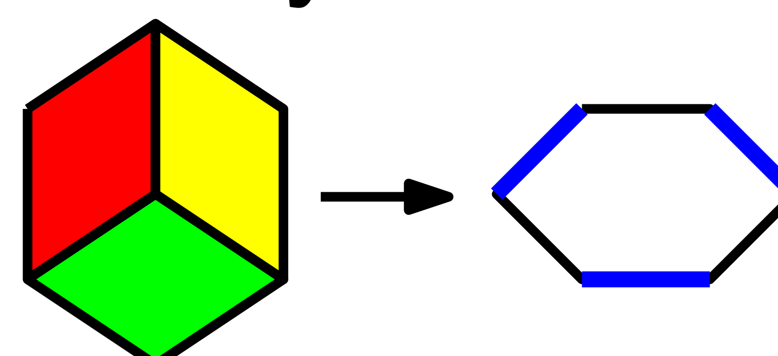
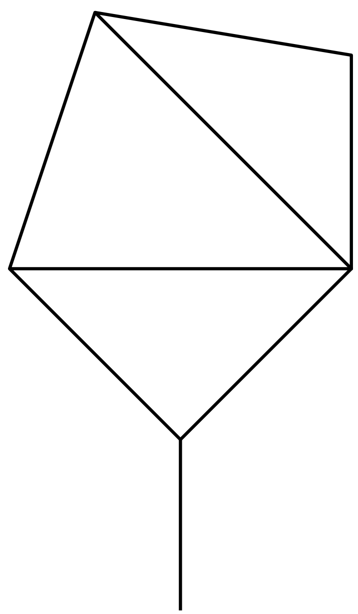




3. Dimers on rail yard graphs

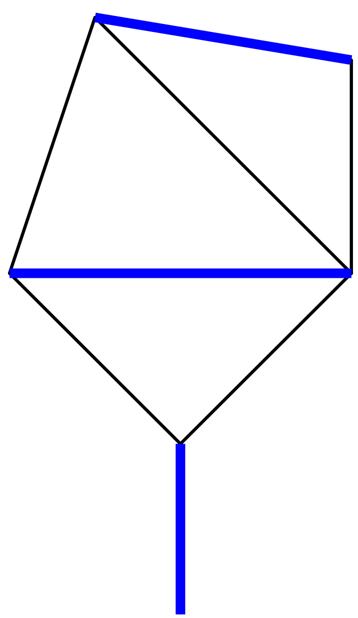
Dimer model

A Graph



Dimer model

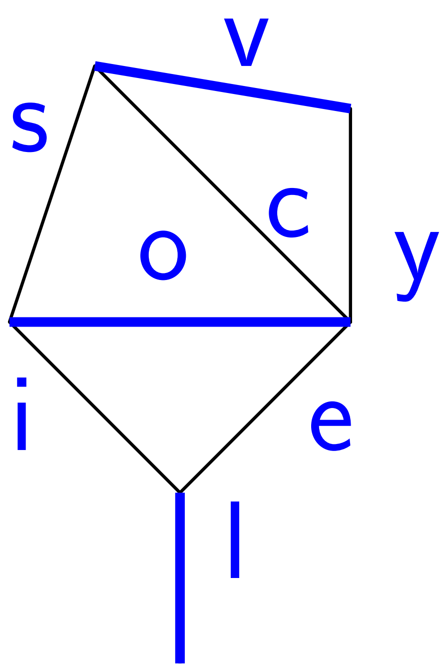
A Graph



A subset of edges such that each vertex belongs to one edge

Dimer model

A Graph



Weight= product of the weights of the chosen edges

Weight of the example = lov

A subset of edges such that each vertex belongs to one edge

Rail Yard Graphs

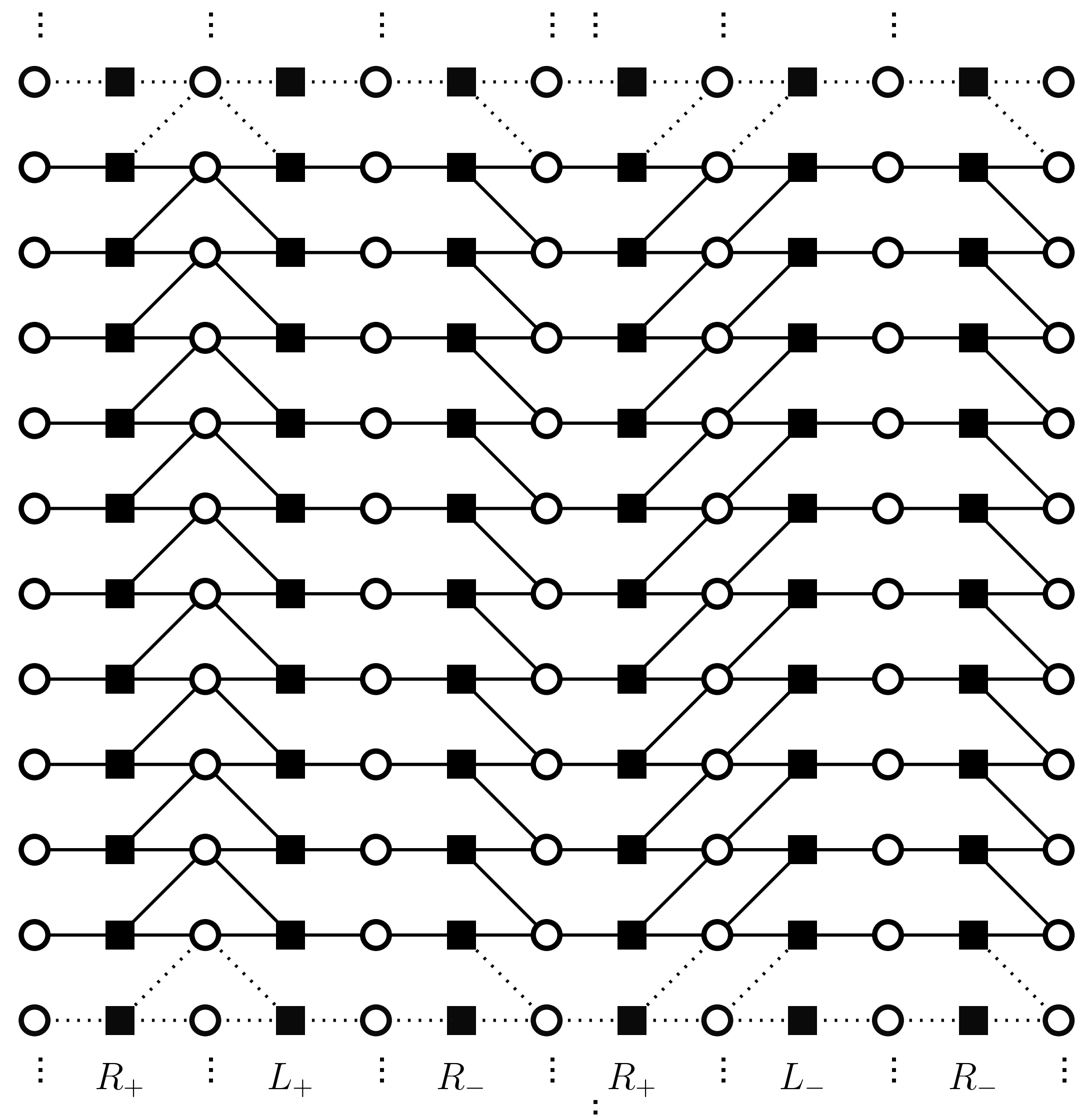
○ odd vertex

■ even vertex

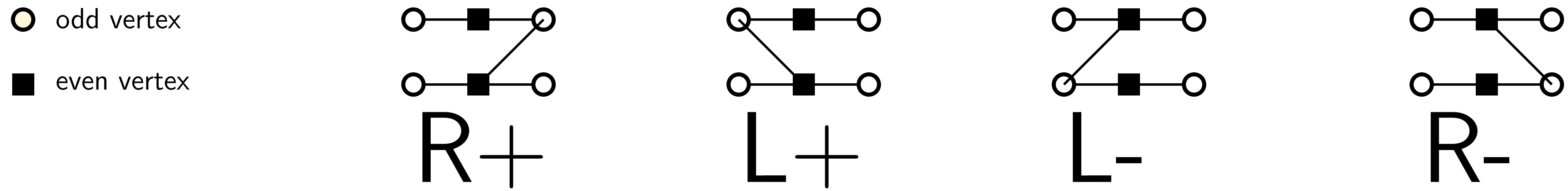
Rail Yard Graphs

○ odd vertex

■ even vertex



Rail Yard Graphs

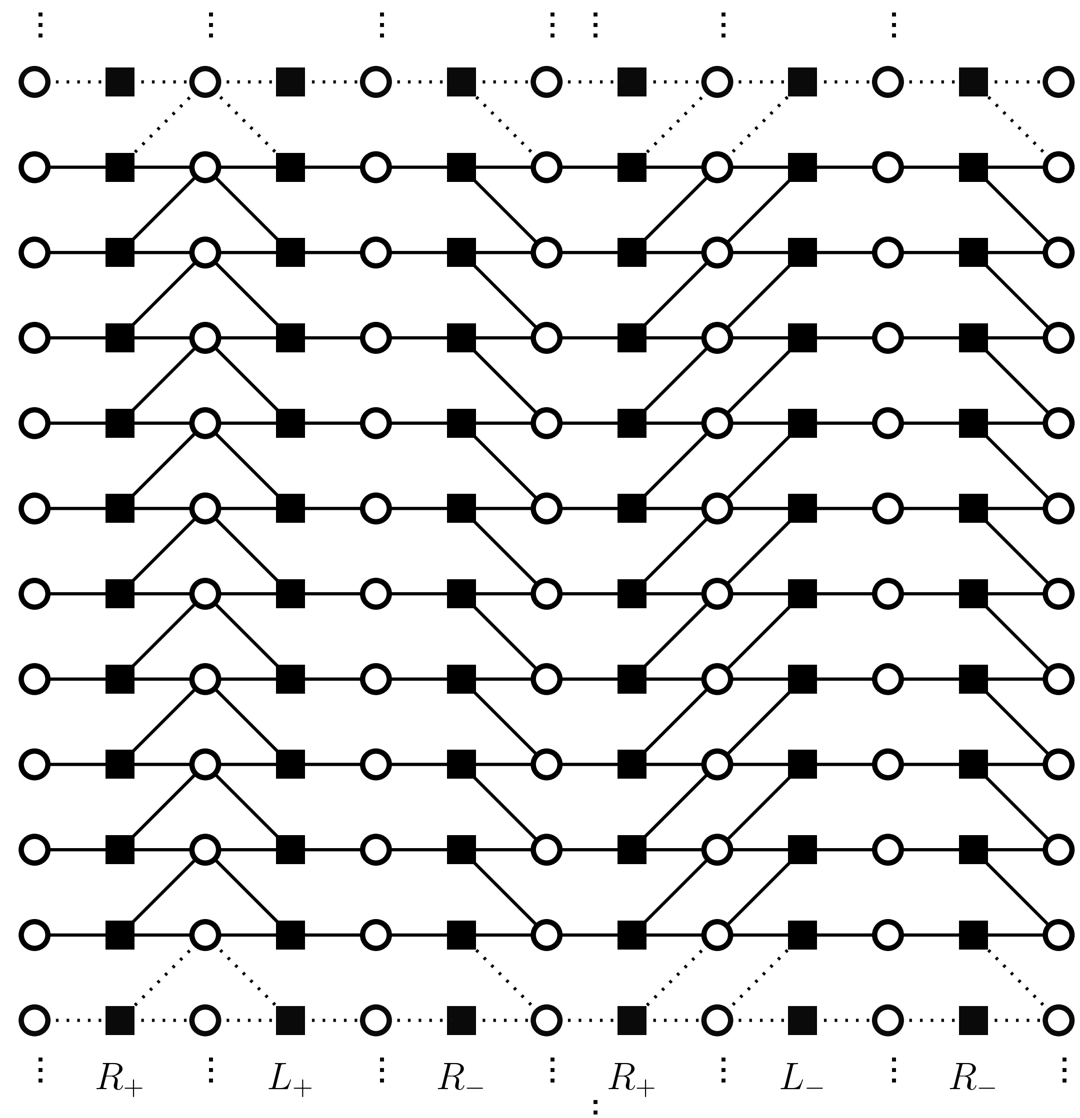


Finite width

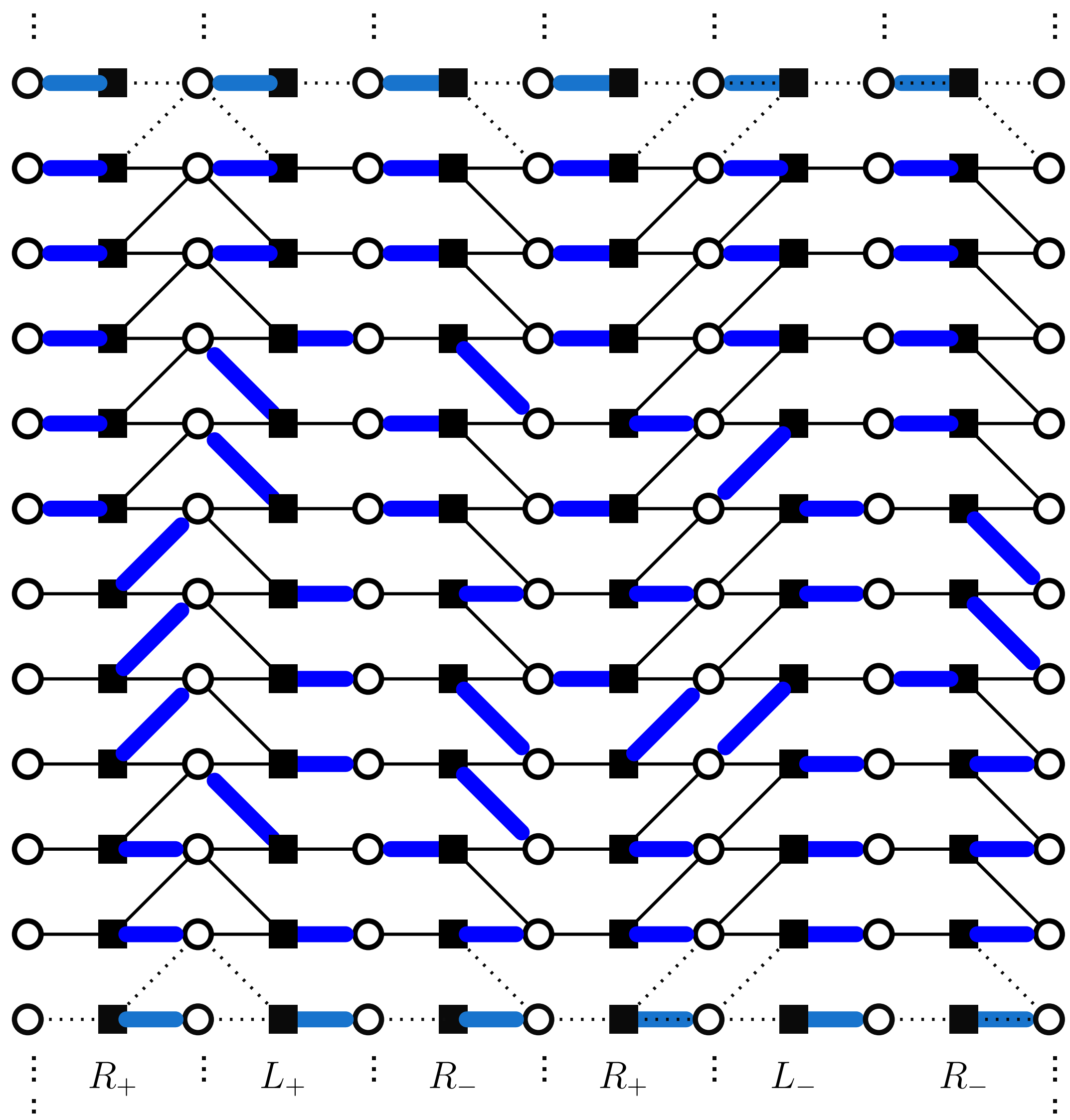
Infinite height

One pattern per column

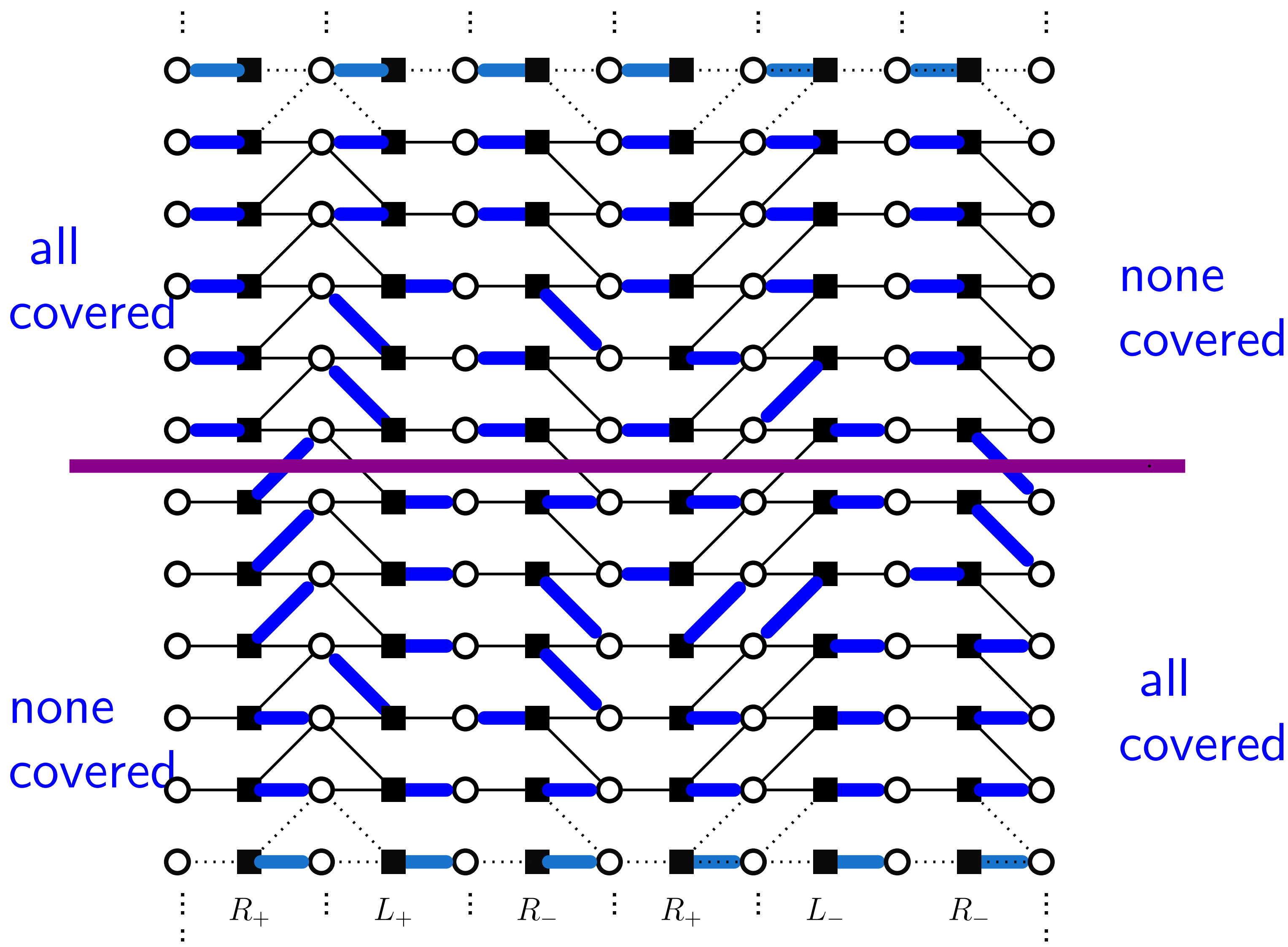
$\Rightarrow$ faces of degree 4,6,8



Dimer coverings on RYG



Dimer coverings on RYG



Generating function

$a_i \in \{L, R\}$, $b_i \in \{+, -\}$ $\quad a_i b_i$ pattern in column i

x_i follows the number of **diagonal edges** in column i

Theorem [BBCCR 14]

$$\boxed{Z(\ell, \underline{a}, \underline{b}; \underline{x}) = \prod_{\substack{1\le i<j\le \ell \\ b_i=+, b_j=-}} z_{ij}}$$

$$a_i \neq a_j\colon\ z_{ij} = (1 + x_i x_j)$$

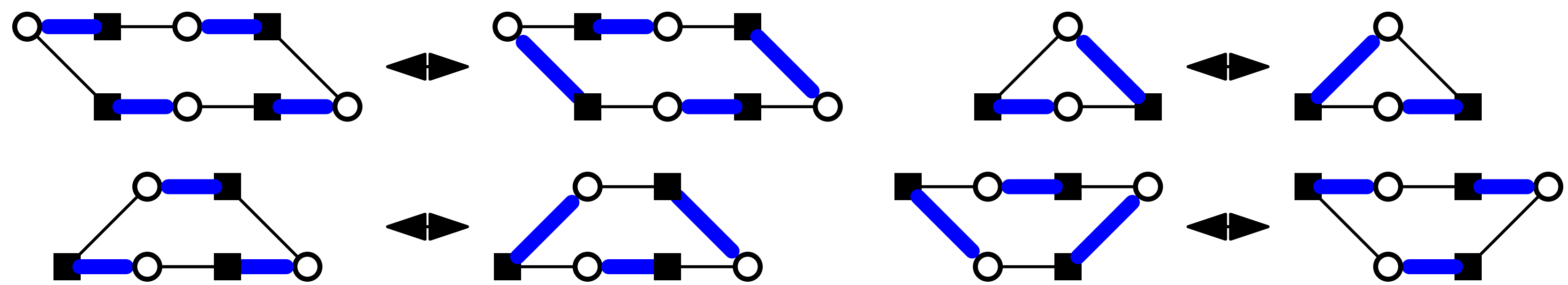
$$a_i = a_j\colon\ z_{ij} = \frac{1}{1 - x_i x_j}$$

Flip generating function

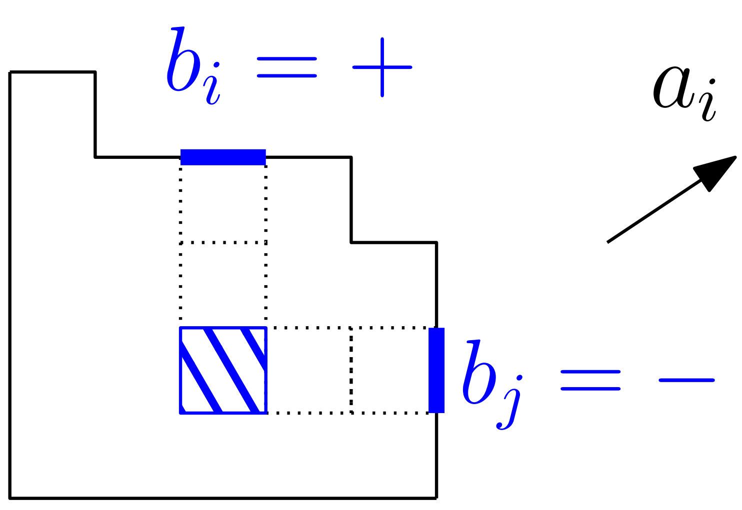
$$Z(\ell,\underline{a},\underline{b};\underline{x}) = \prod_{\substack{\ell\le i<j\le r\\ b_i=+,b_j=-}} z_{ij}$$

Setting $x_i = q^i$, if $b_i = +$ and $1/q^i$, if $b_i = -$

Flips



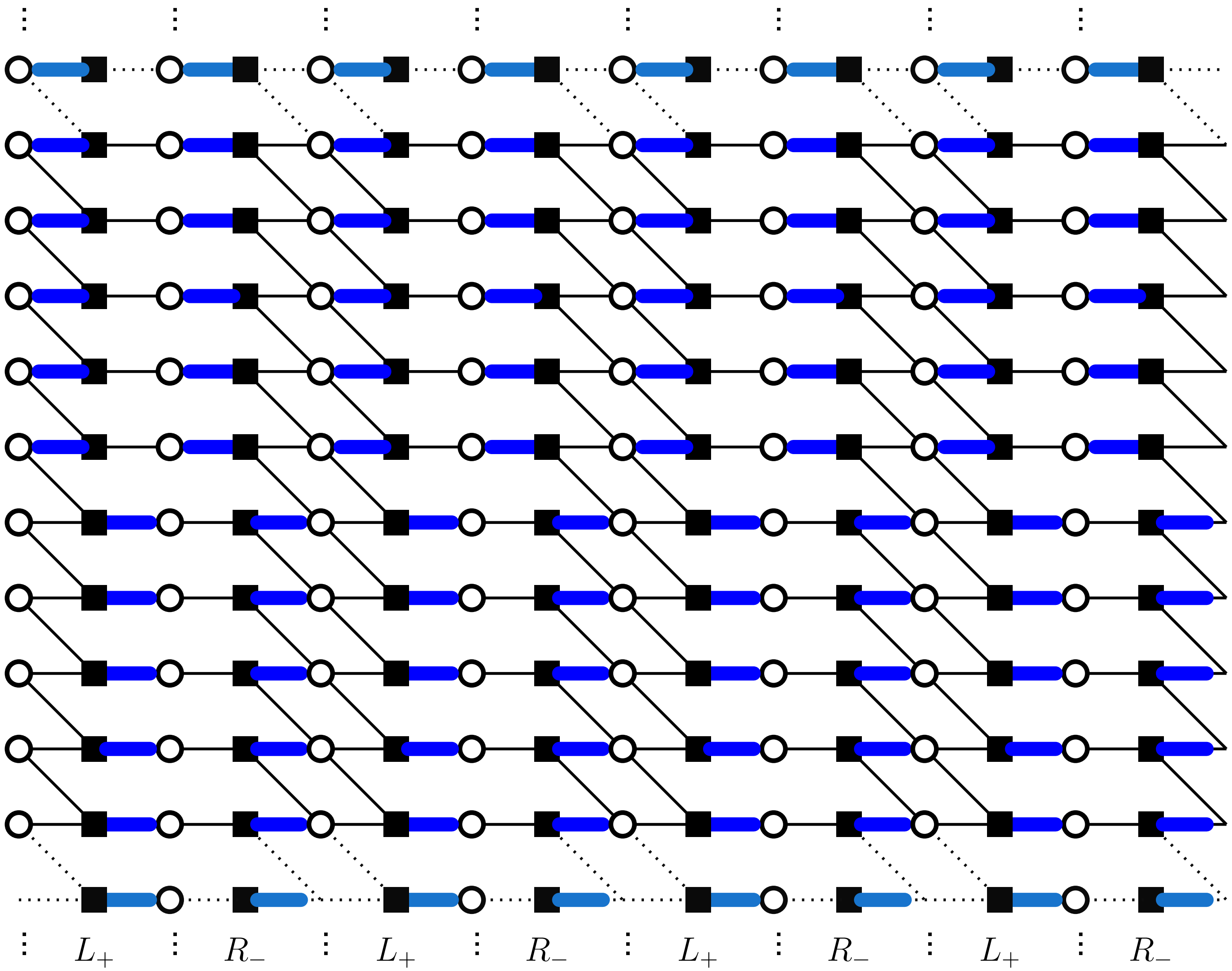
$$\boxed{Z(q) = \sum\nolimits_{\text{coverings}} q^{\#\text{flips}}}$$



$a_i \neq a_j$: $z_{ij} = (1 + q^{\text{hook}})$

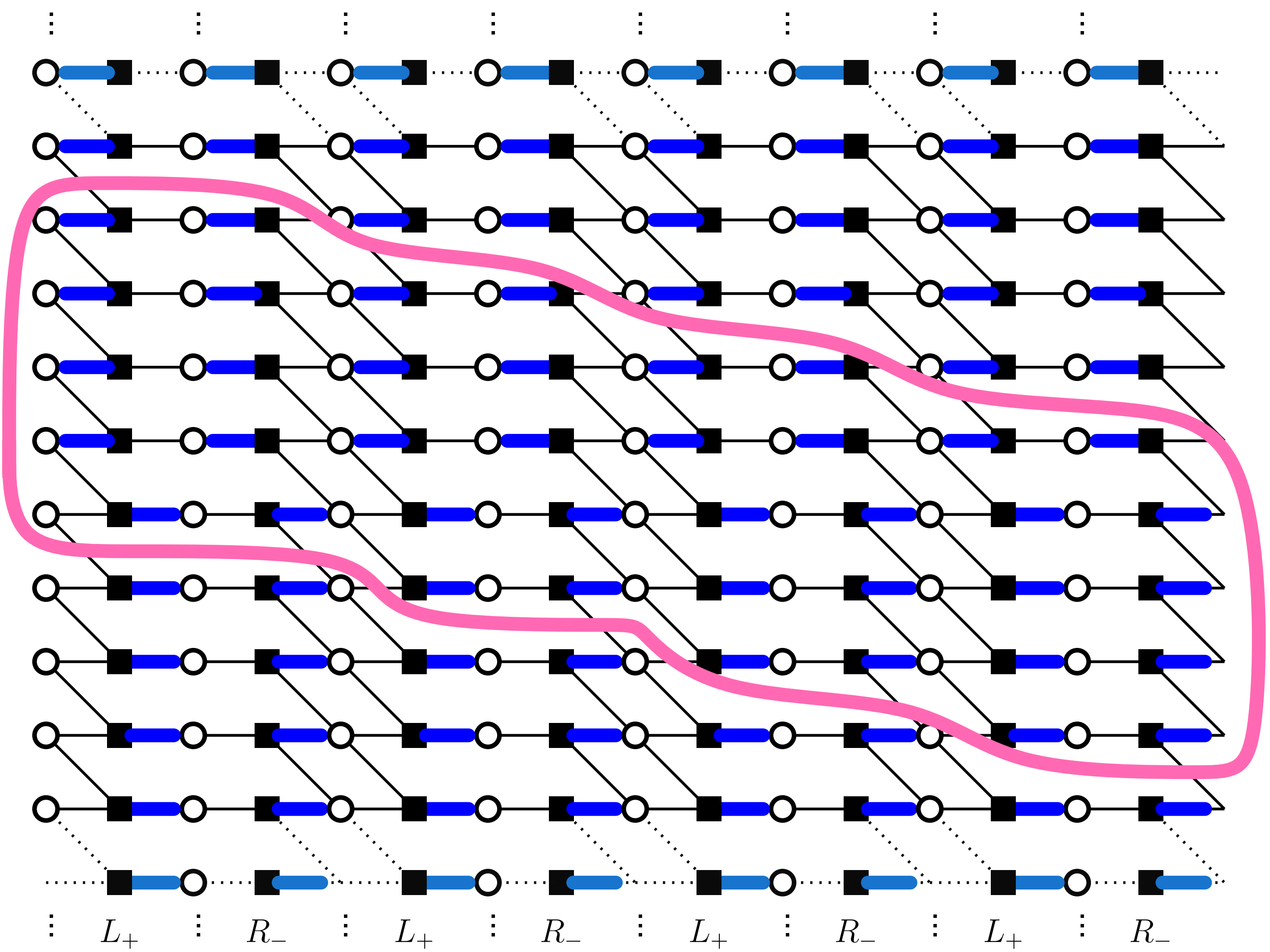
$a_i = a_j$: $z_{ij} = \dfrac{1}{1 - q^{\text{hook}}}$

Back to the aztec diamond



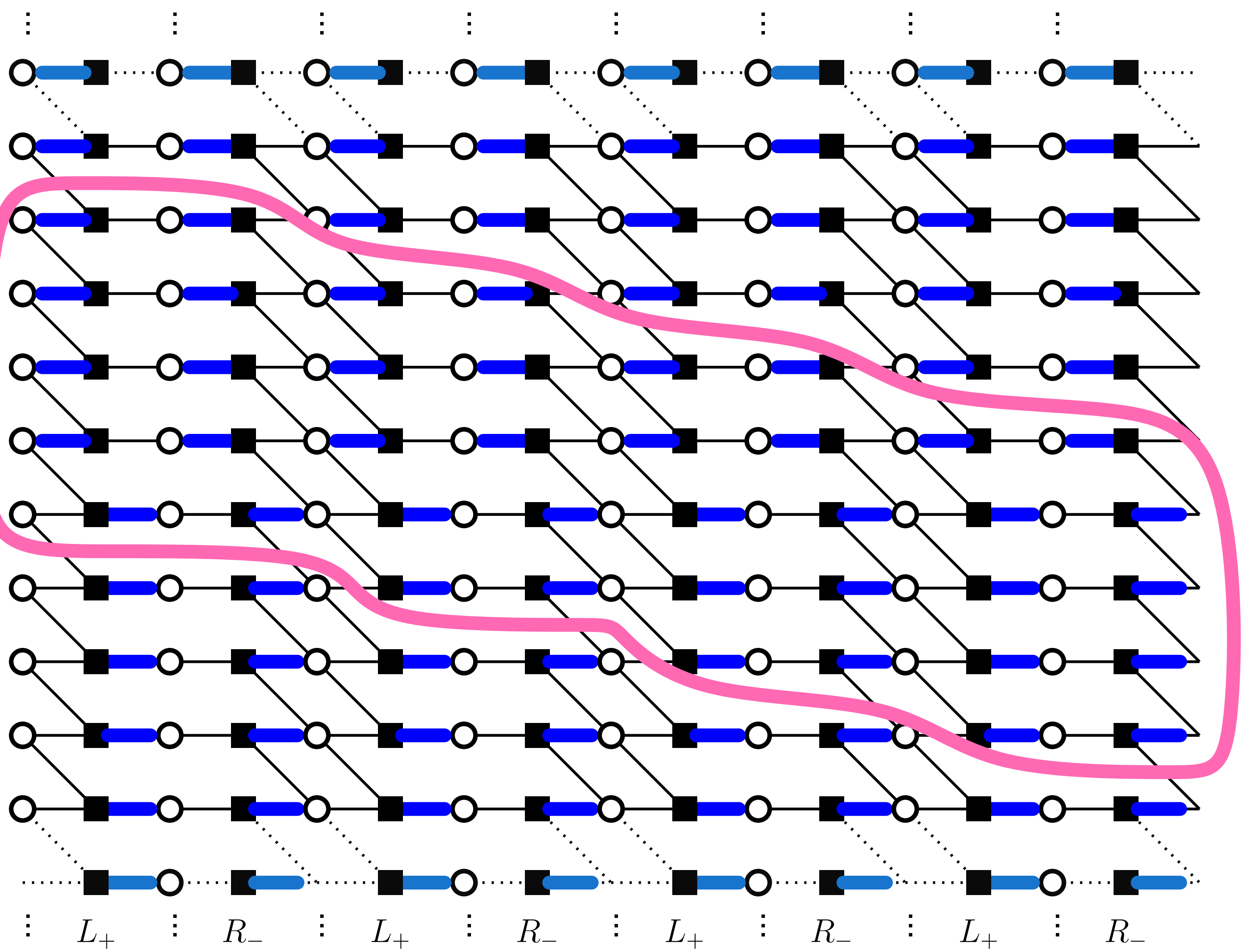
aztec diamond $(L_+R_-)^\ell$

Back to the aztec diamond

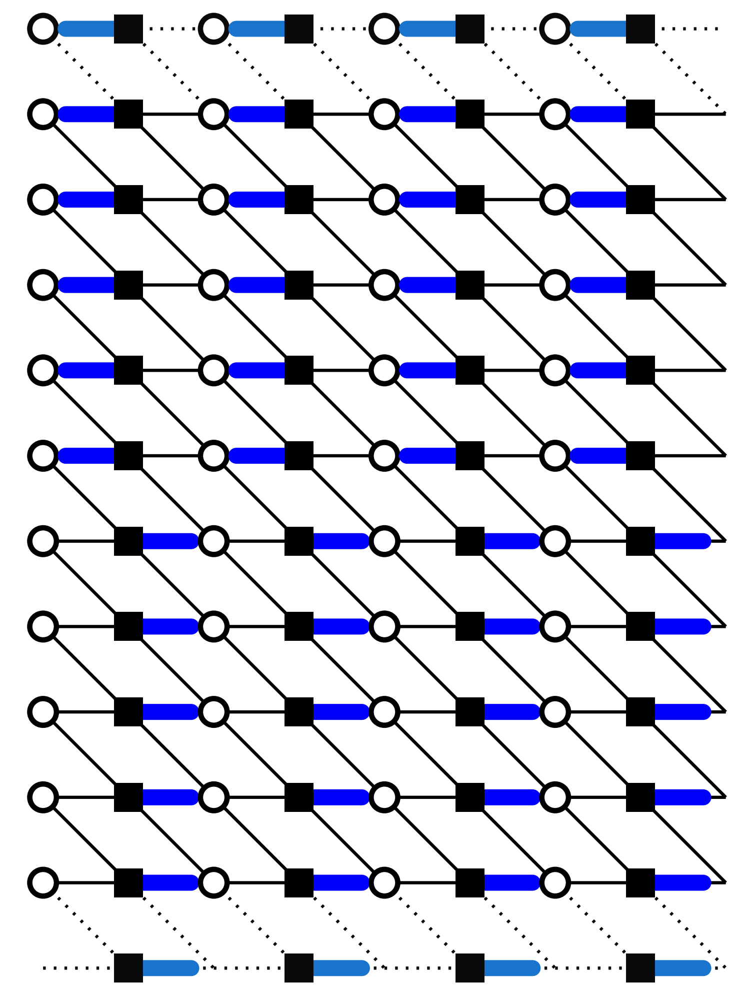


aztec diamond $(L_+R_-)^\ell$

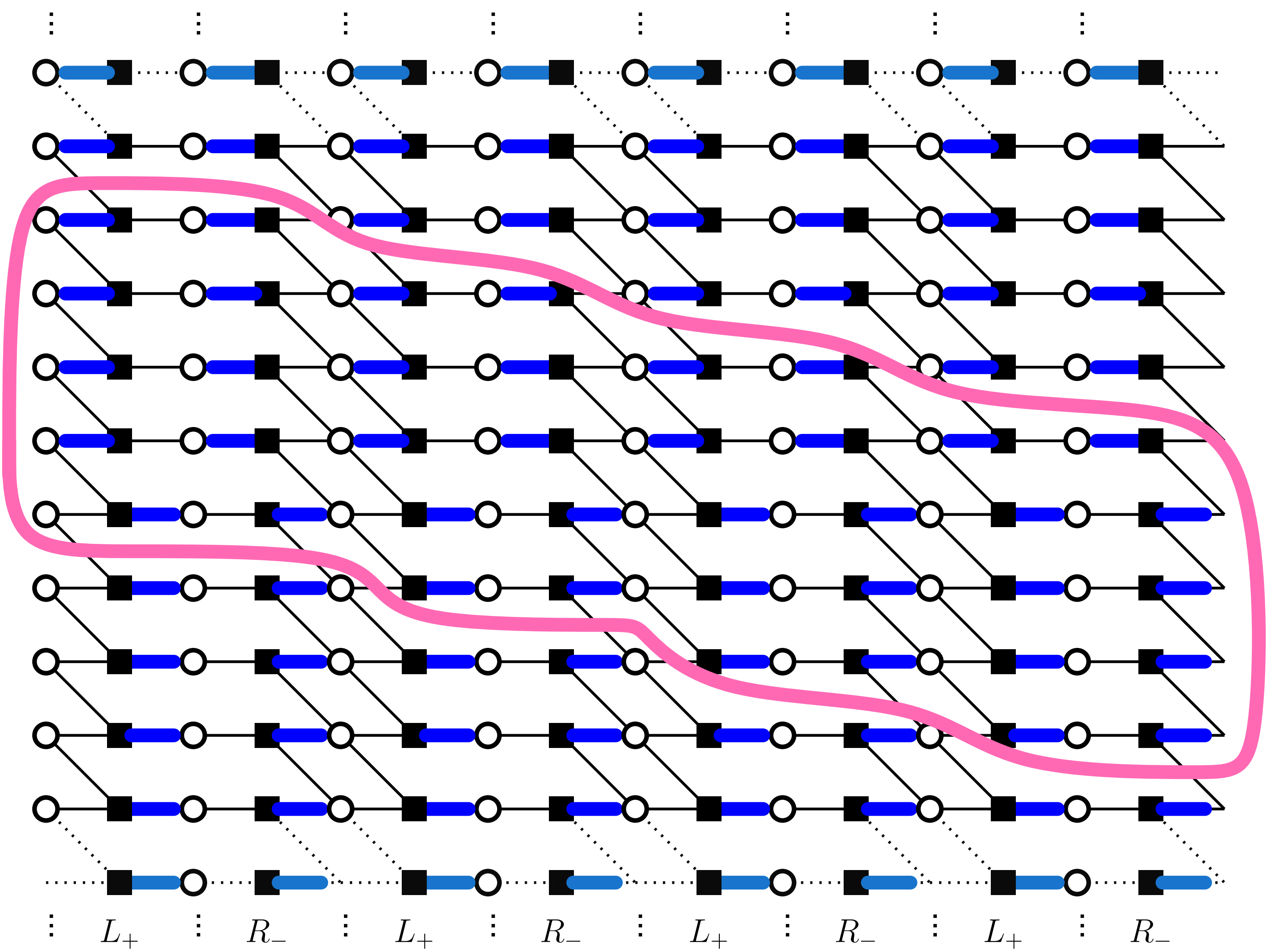
Back to the aztec diamond



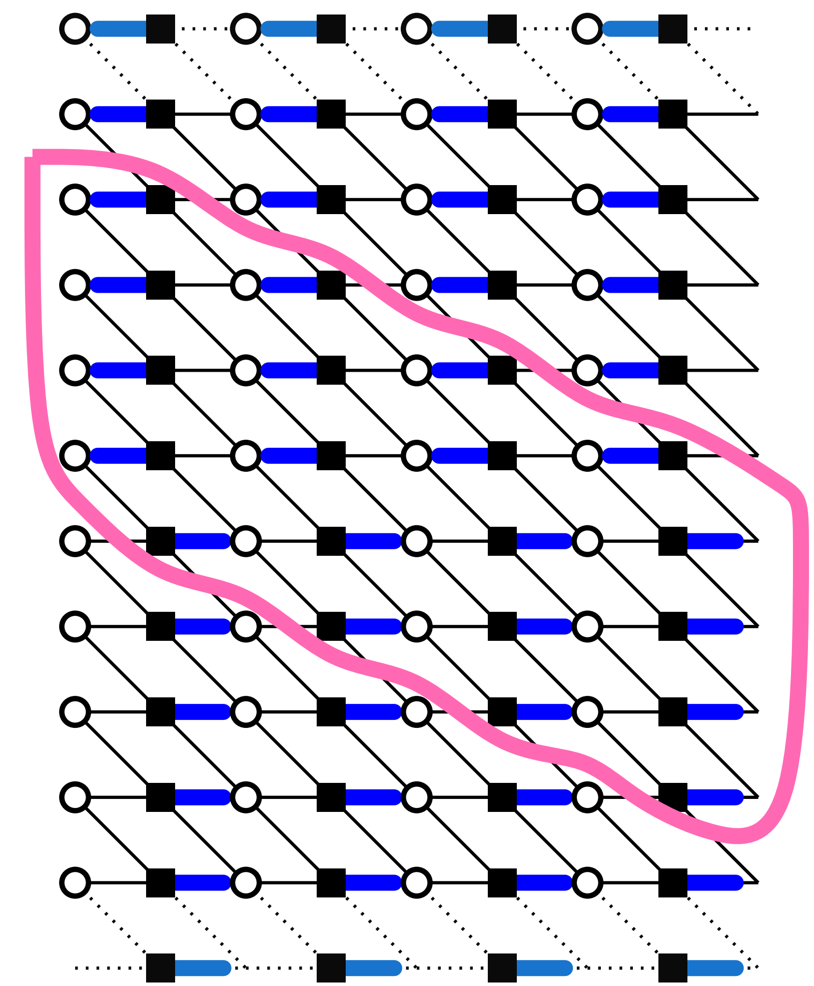
aztec diamond $(L_+R_-)^\ell$



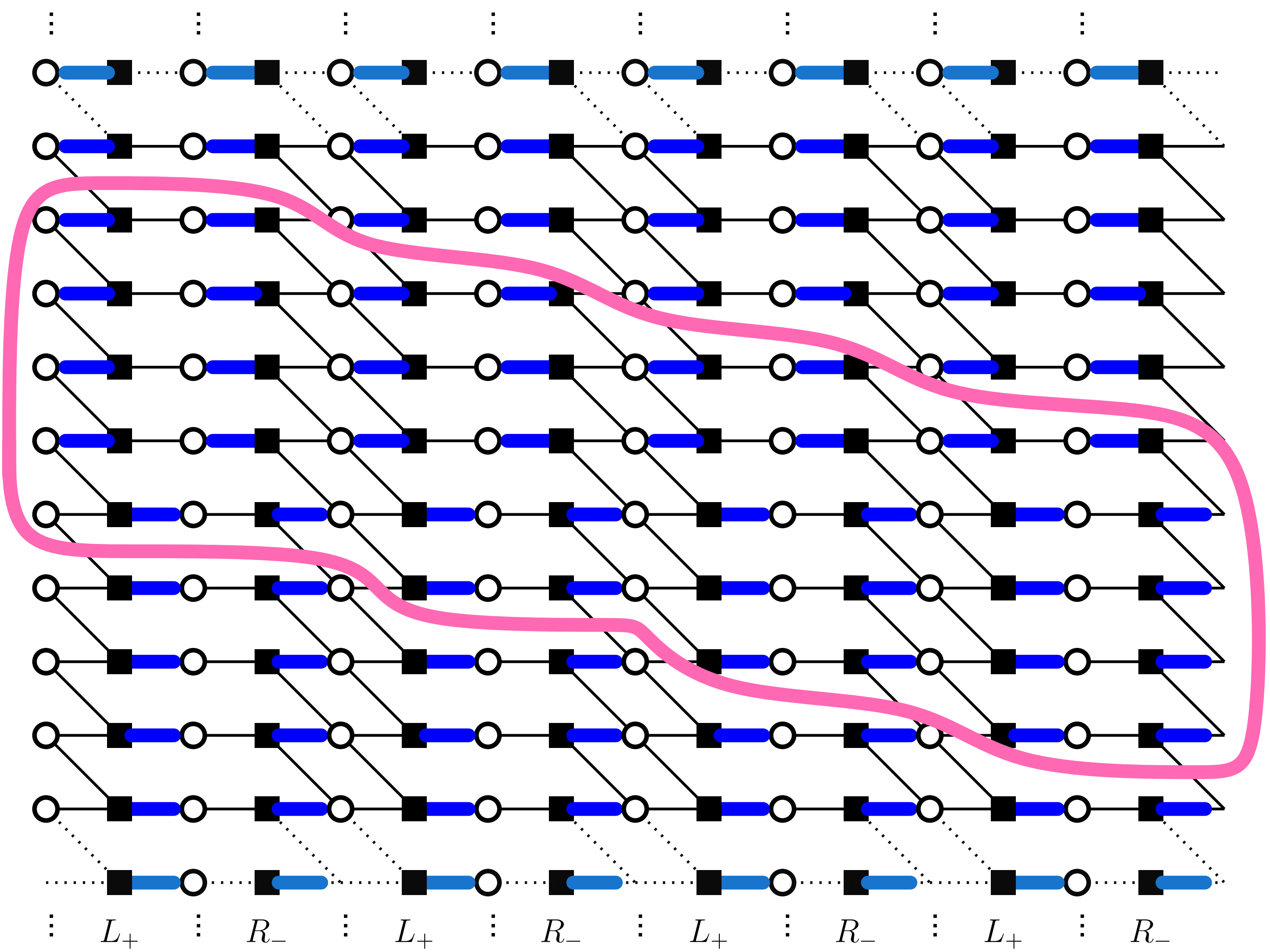
Back to the aztec diamond



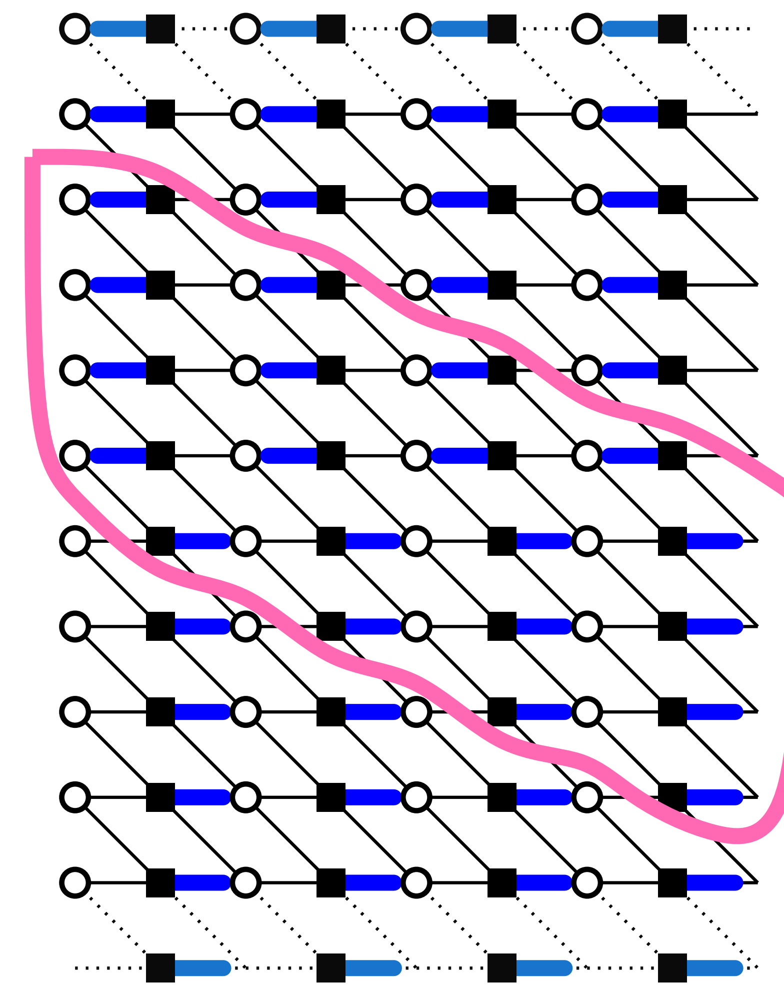
aztec diamond $(L_+R_-)^\ell$



Back to the aztec diamond



aztec diamond $(L_+R_-)^\ell$



Pyramid partitions $(L_-R_-)^\ell(L_+R_+)^\ell$

Plane partitions $L_-^\ell L_+^\ell$

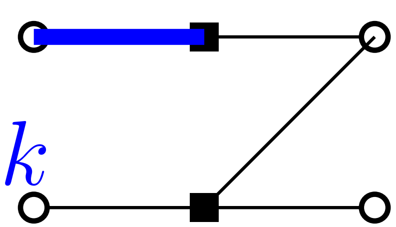
Dimer correlations

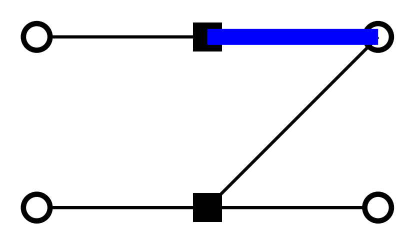
ψ_k creation operator ψ_k^* annihilation operator

Vertex operators

Γ_{R+} Γ_{L+} Γ_{L-} Γ_{R-}

$\psi_k^* \psi_k$





$\psi_k \Gamma_{R+} \psi_k^*$

Correlation kernel

$$F_i(z) = \frac{\prod_{\substack{m:2m<i \\ (a_m,b_m)=(R,+)}} (1+x_m z) \prod_{\substack{m:2m>i \\ (a_m,b_m)=(L,-)}} (1-\frac{x_m}{z})}{\prod_{\substack{m:2m<i \\ (a_m,b_m)=(L,+)}} (1-x_m z) \prod_{\substack{m:2m>i \\ (a_m,b_m)=(R,-)}} (1+\frac{x_m}{z})}$$

$$C_{\alpha\beta} = \left[z^{k_\alpha} w^{-k'_\beta} \right] \frac{F_{i_\alpha}(z)}{F_{i'_\beta}(w)} \frac{\sqrt{zw}}{z-w}$$

- Thm [BCCRR]: The probability that dimers $(e_1, \ldots, e_s)$ with $e_i = (w_i, b_i)$ belong to a random configuration is

$$\mathrm{cste} \times \det(C_{w_i b_j})$$

Correlation kernel

$$F_i(z) = \frac{\prod_{\substack{m:2m<i \\ (a_m,b_m)=(R,+)}} (1+x_m z) \prod_{\substack{m:2m>i \\ (a_m,b_m)=(L,-)}} (1-\frac{x_m}{z})}{\prod_{\substack{m:2m<i \\ (a_m,b_m)=(L,+)}} (1-x_m z) \prod_{\substack{m:2m>i \\ (a_m,b_m)=(R,-)}} (1+\frac{x_m}{z})}$$

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$$\text{cste} \times \det(C_{w_i b_j})$$

C is the inverse of the Kasteleyn matrix

Time to wrap it up

Fermionic and bosonic operators $\Rightarrow$ Correlation kernel

We can compute the gf of coverings with a fixed collection of dimers

Free dimer configurations, cylindric dimer configurations

Asymptotics...

Open questions:

Can we treat those with cluster algebras?

Can we allow the pattern to change in a column?

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Merci!

Merci!

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