The Z-invariant massive Laplacian on isoradial graphs

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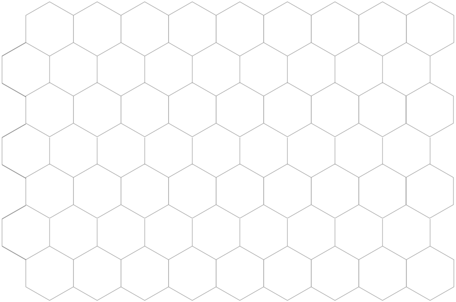
Isoradial graphs

- A graph G is isoradial if it can be embedded in the plane in such a way that all (inner) faces are inscribed in a circle of radius 1, and such that the center of the circles are in the interior of the faces (Duffin-Mercat-Kenyon).



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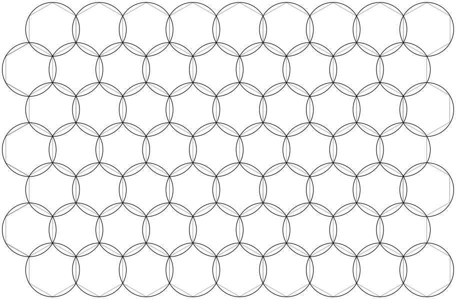

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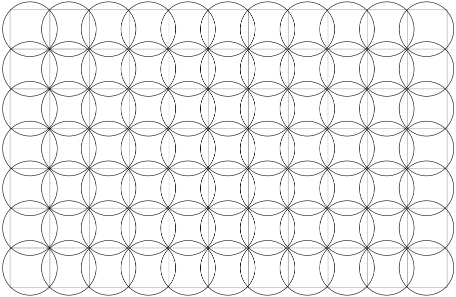

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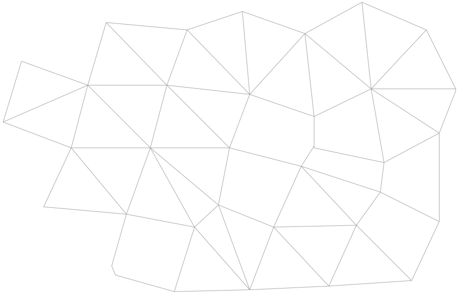

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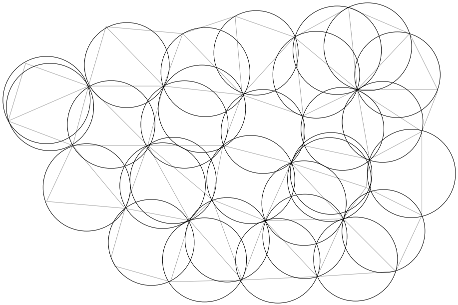
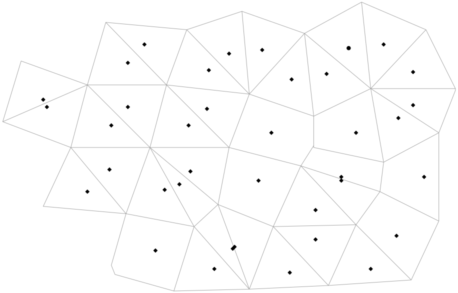


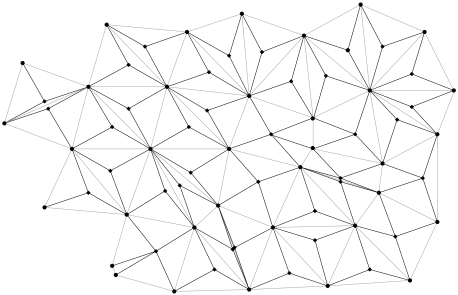
Corresponding diamond graph, angles

- Take the centers of the circumcircles (embedded dual vertices)



Corresponding diamond graph, angles

- Join them to the vertices of G of the face they correspond to.
  $\Rightarrow$ Corresponding rhombus graph $\mathsf{G}^{\diamond}$.



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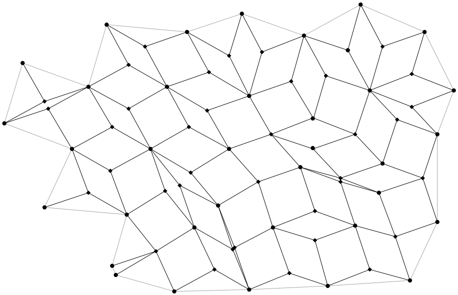
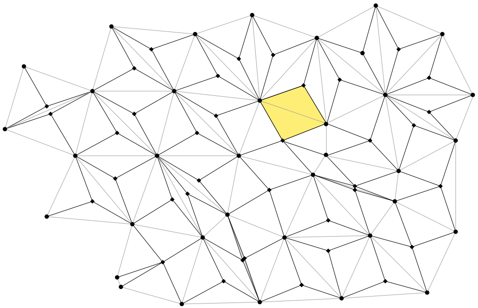
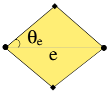


Corresponding diamond graph, angles

- To every edge e corresponds a rhombus and a half-angle θ_e.

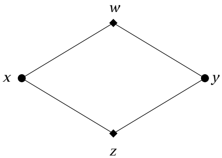




Discrete complex analysis

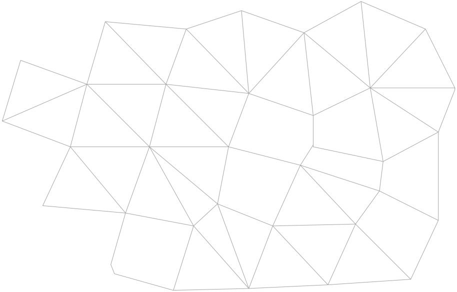
- Let f be a function defined on vertices of G and G^*.
- It is discrete holomorphic if, for every rhombus $xwyz$,

$$\frac{f(y)-f(x)}{y-x} = \frac{f(w)-f(z)}{w-z}.$$



Z-invariant models on isoradial graphs

- Finite isoradial graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$.



- Set of configurations on G: $\mathcal{C}(\mathsf{G})$.

- Parameters: positive weight function on edges/vertices

$$\boxed{w \text{ depends on angles } (\theta_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}}}$$

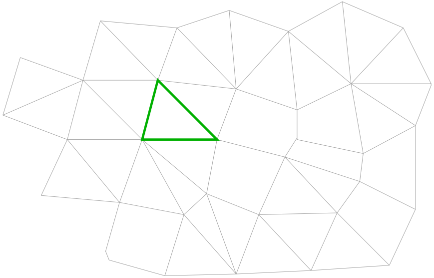
- Boltzmann probability measure on configurations:

$$\forall\, \mathsf{C} \in \mathcal{C}(\mathsf{G}), \quad \mathbb{P}(\mathsf{C}) = \frac{e^{-\mathcal{E}_w(\mathsf{C})}}{Z(\mathsf{G}, w)},$$

where $Z(\mathsf{G}, w) = \sum_{\mathsf{C}\in\mathcal{C}(\mathsf{G})} e^{-\mathcal{E}_w(\mathsf{C})}$ is the partition function.

Z-invariant models on isoradial graphs

- Star-triangle transformation preserves isoradiality.



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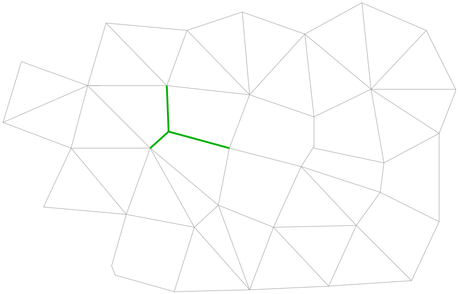

Z-INVARIANT MODELS ON ISORADIAL GRAPHS

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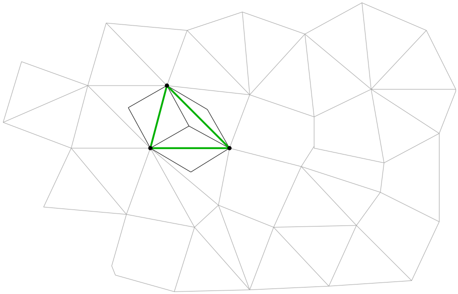

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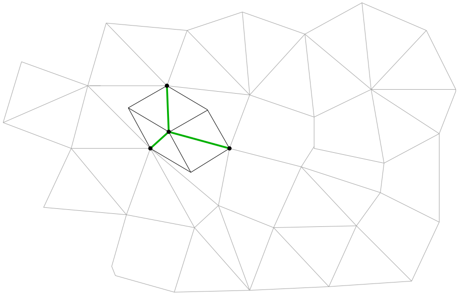

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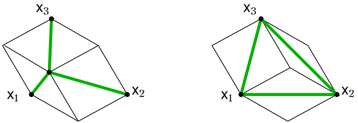


- Decompose the partition function according to the possible configurations outside of the star/triangle.
- The model is Z-invariant (Baxter) if $\exists$ constant $\mathcal{C}$, s.t. for all outer configuration $\mathsf{C}(\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3)$:

$$Z(\mathsf{G}_{\mathsf{Y}}, w, \mathsf{C}(\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3)) = \mathcal{C}\, Z(\mathsf{G}_{\Delta}, w, \mathsf{C}(\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3)).$$

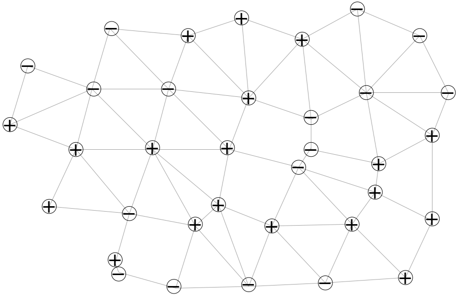
(Yang-Baxter equations)

$\Rightarrow$ Transfer matrices commute (Onsager, 1944).

$\Rightarrow$ Probabilities are not affected by $\mathsf{Y} - \Delta$ transformations.

Probabilities should only depend on the *local* geometry of the graph

Example: the Z-invariant Ising model (Baxter)



$$\forall\, \sigma \in \{-1,1\}^{\mathsf{V}}, \quad \mathbb{P}_{\mathrm{Ising}}(\sigma) = \frac{\exp\left(\sum_{e=\mathsf{xy}\in\mathsf{E}} J(\theta_{\mathsf{e}})\sigma_{\mathsf{x}}\sigma_{\mathsf{y}}.\right)}{Z_{\mathrm{Ising}}(\mathsf{G},J)},$$

THEOREM (BAXTER)

The Ising model is Z-invariant if

$$\forall\, \mathrm{e} \in \mathrm{E},\ J(\theta_{\mathrm{e}}) = \frac{1}{2}\log\left(\frac{1+\mathrm{sn}\left(\frac{2K}{\pi}\theta_{\mathrm{e}}|k\right)}{\mathrm{cn}\left(\frac{2K}{\pi}\theta_{\mathrm{e}}|k\right)}\right),\ k \in [0,1).$$

- $K = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin\tau}}\mathrm{d}\tau$: *Complete elliptic integral of the first kind.*
- sn, cn: *Jacobi elliptic functions.*

- If $k=0$: $\forall\, \mathrm{e} \in \mathrm{E},\ J(\theta_{\mathrm{e}}) = \frac{1}{2}\log\left(\frac{1+\sin\theta_{\mathrm{e}}}{\cos\theta_{\mathrm{e}}}\right)$.
  - The model is critical (Li, Duminil-Copin-Cimasoni, Lis), conformally invariant (Chelkak - Smirnov).
  - Local expressions for probabilities of the corresponding dimer model (Boutillier-dT).
- $k \neq 0$:  (Boutillier-dT-Raschel).

The Laplacian [...] on critical planar graphs (Kenyon)

- Infinite isoradial graph G.
- Conductances: $\rho = (\tan(\theta_{\mathsf{e}}))_{\mathsf{e}\in\mathsf{E}}$.
- Let Δ be the discrete Laplacian on G represented by the matrix Δ:

$$\forall\, \mathsf{x},\mathsf{y} \in \mathsf{V}, \quad \Delta(\mathsf{x},\mathsf{y}) = \begin{cases} \rho(\theta_{\mathsf{xy}}) & \text{if } \mathsf{x} \sim \mathsf{y} \\ -\sum_{\mathsf{y}\sim\mathsf{x}} \rho(\theta_{\mathsf{xy}}) & \text{if } \mathsf{x} = \mathsf{y} \\ 0 \text{ otherwise.} \end{cases}$$

- The Laplacian Δ is an operator from $\mathbb{C}^{\mathsf{V}}$ to $\mathbb{C}^{\mathsf{V}}$

$$\forall f \in \mathbb{C}^{\mathsf{V}}, \quad (\Delta f)(\mathsf{x}) = \sum_{\mathsf{y}\in\mathsf{V}} \Delta(\mathsf{x},\mathsf{y}) f(\mathsf{y}) = \sum_{\mathsf{y}\sim\mathsf{x}} \rho(\theta_{\mathsf{xy}})(f(\mathsf{y}) - f(\mathsf{x})).$$

- The restriction to G of a discrete holomorphic function is discrete harmonic.

The Laplacian [...] on critical planar graphs (Kenyon)

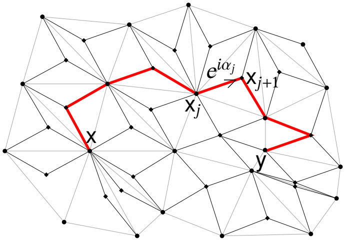
- The Green function G is the inverse of the Laplacian: $\Delta G = \mathrm{Id}$.
- Discrete exponential function (Mercat):

$\mathrm{Exp} : \mathsf{V}^{\diamond} \times \mathsf{V}^{\diamond} \times \mathbb{C} \to \mathbb{C}$. Let $\mathsf{x}, \mathsf{y} \in \mathsf{V}^{\diamond}$.

Path in $\mathsf{E}^{\diamond}$: $\mathsf{x} = \mathsf{x}_1, \ldots, \mathsf{x}_n = \mathsf{y}$,

$$\mathrm{Exp}_{\mathsf{x}_j, \mathsf{x}_{j+1}}(\lambda) = \frac{(\lambda + e^{i\alpha_j})}{(\lambda - e^{i\alpha_j})}$$

$$\mathrm{Exp}_{\mathsf{x},\mathsf{y}}(\lambda) = \prod_{j=1}^{n-1} \mathrm{Exp}_{\mathsf{x}_j, \mathsf{x}_{j+1}}(\lambda).$$



Theorem (Kenyon)

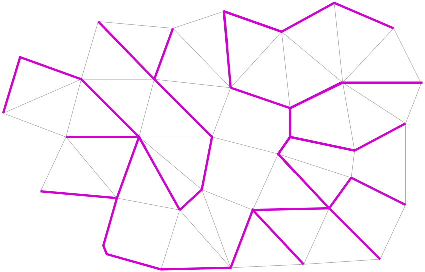
The Green function has the following explicit expression:

$$\forall\, \mathsf{x}, \mathsf{y} \in \mathsf{V}, \quad G(\mathsf{x}, \mathsf{y}) = -\frac{1}{8\pi^2 i} \oint_{\gamma} \mathrm{Exp}_{\mathsf{x},\mathsf{y}}(\lambda) \log(\lambda)\, \mathrm{d}\lambda,$$

where γ is a contour in $\mathbb{C}$ containing all the poles of $\mathrm{Exp}_{\mathsf{x},\mathsf{y}}$.

Relation to statistical mechanics

- Spanning trees of G



- Boltmann probability measure:

$$\forall\, \mathsf{T} \in \mathcal{T}(\mathsf{G}), \quad \mathbb{P}_{\mathrm{tree}}(\mathsf{T}) = \frac{\prod_{\mathsf{e}\in\mathsf{T}} \rho(\theta_{\mathsf{e}})}{Z_{\mathrm{tree}}(\mathsf{G},\rho)}.$$

Relation to statistical mechanics

Theorem (Kirchhoff)

$$Z_{\text{tree}}(\mathsf{G},\rho) = \det \Delta^{(\mathsf{r})},$$

where $\Delta^{(r)}$ is the matrix Δ from which the line and column corresponding to the vertex r are removed.

Theorem (Burton - Pemantle)

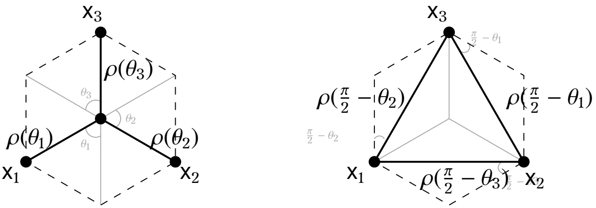
For every subset of edges $\{\mathsf{e}_1,\dots,\mathsf{e}_k\}$ of G:

$$\mathbb{P}_{\text{tree}}(\mathsf{e}_1,\dots,\mathsf{e}_k) = \det[(H(\mathsf{e}_i,\mathsf{e}_j))_{1\le i,j\le k}],$$

where H is the transfer impedance matrix. Coefficients are differences of Green functions.

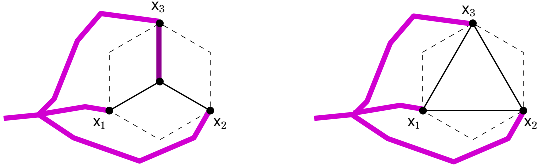
- Kenyon's results yield local formulas for $\mathbb{P}_{\text{tree}}$ and for the free energy when the graph is infinite.

Z-INVARIANCE FOR SPANNING TREES



Decompose $Z_{\text{tree}}(\mathsf{G}, \rho)$ according to the possible configurations outside of the $\mathsf{Y} - \Delta$.

Z-INVARIANCE FOR SPANNING TREES



Example: x_1, x_2, x_3 are connected to r

	C_Y	C_Δ
$\{x_1, x_2, x_3\}$	$\sum_{\ell=1}^{3} \rho(\theta_\ell)$	1
$\{x_i, x_j\}$	$\rho(\theta_k)(\sum_{\ell\neq k} \rho(\theta_\ell))$	$\sum_{\ell\neq k} \rho(\frac{\pi}{2} - \theta_\ell)$
$\{x_i\}$	$\prod_{\ell=1}^{3} \rho(\theta_\ell)$	$\sum_{\ell=1}^{3} \prod_{\ell'\neq\ell} \rho(\frac{\pi}{2} - \theta_{\ell'})$
$\{\emptyset\}$	0	0

REMARK

The spannig tree model with conductances $\rho = (\tan(\theta_e))_{e\in E}$ is Z-invariant [Kenelly].

Away from the critical point ? Massive Laplacian

- Let $k \in [0,1)$ (the elliptic modulus), $k' = \sqrt{1-k^2}$, $\bar{\theta}_{\mathsf{e}} = \frac{2K}{\pi}\theta_{\mathsf{e}}$.
- Define conductances and masses on G:

$$\forall\, \mathsf{e} \in \mathsf{E},\ \rho(\theta_{\mathsf{e}}) = \mathrm{sc}(\bar{\theta}_{\mathsf{e}} \,|\, k)$$

$$\forall\, \mathsf{x} \in \mathsf{V},\ m^2(\mathsf{x}) = \sum_{j=1}^{n} \mathrm{A}(\bar{\theta}_j|k) - \frac{2}{k'}(K-E) - \sum_{j=1}^{n} \rho(\bar{\theta}_j|k).$$

  - $E = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin\tau}\,\mathrm{d}\tau$: complete elliptic int. of the second kind.
  - $\mathrm{E}(u|k) = \int_0^u \mathrm{dn}^2(v|k)\mathrm{d}v$: Jacobi epsilon function.
  - $\mathrm{A}(u|k) = -\frac{i}{k'}\mathrm{E}(iu|k')$.

Family of massive Laplacians

- The massive Laplacian $\Delta^{m(k)}$ on G is represented by the matrix :

$$\forall\, \mathsf{x},\mathsf{y} \in \mathsf{V}, \quad \Delta^{m(k)}(\mathsf{x},\mathsf{y}) = \begin{cases} \rho(\theta_{\mathsf{xy}}) & \text{if } \mathsf{x} \sim \mathsf{y} \\ -m^2(x) - \sum_{\mathsf{y}\sim\mathsf{x}} \rho(\theta_{\mathsf{xy}}) & \text{if } \mathsf{x} = \mathsf{y} \\ 0 & \text{otherwise.} \end{cases}$$

- The massive Laplacian $\Delta^{m(k)}$ is the operator:

$$\forall f \in \mathbb{C}^{\mathsf{V}}, \quad (\Delta^{m(k)} f)(\mathsf{x}) = \sum_{\mathsf{y}\sim\mathsf{x}} \rho(\theta_{\mathsf{xy}})(f(\mathsf{y}) - f(\mathsf{x})) - m^2(x) f(x).$$

- The massive Green function $G^{m(k)}$ is the inverse of the massive Laplacian: $\Delta^{m(k)} G^{m(k)} = \mathrm{Id}$.

The discrete massive exponential function

- Let $\mathbb{T}(k) = \mathbb{C}/(4K\mathbb{Z} + i4K'\mathbb{Z})$.

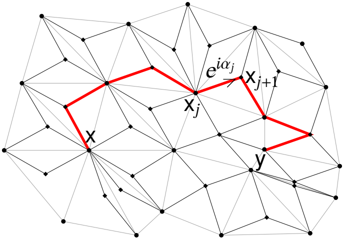
$$\mathrm{Exp}(\cdot|k) : \mathsf{V}^{\diamond} \times \mathsf{V}^{\diamond} \times \mathbb{T}(k) \to \mathbb{C}.$$

Let $\mathsf{x}, \mathsf{y} \in \mathsf{V}^{\diamond}$.

Path in $\mathsf{E}^{\diamond}$: $\mathsf{x} = \mathsf{x}_1, \ldots, \mathsf{x}_n = \mathsf{y}$,

$$\mathrm{Exp}_{\mathsf{x}_j,\mathsf{x}_{j+1}}(u|k) = -i\sqrt{k'}\,\mathrm{sc}(u_{\bar{\alpha}_j}),\ \ u_{\bar{\alpha}_j} = \frac{u - \bar{\alpha}_j}{2}.$$

$$\mathrm{Exp}_{\mathsf{x},\mathsf{y}}(u|k) = \prod_{j=1}^{n-1} \mathrm{Exp}_{\mathsf{x}_j,\mathsf{x}_{j+1}}(u|k).$$



Lemma

The discrete massive exponential function is well defined, i.e., independent of the choice of the path from x *to* y.

Proposition

For every $u \in \mathbb{T}(k)$, *for every* $\mathsf{y} \in \mathsf{V}$, *the function* $\mathrm{Exp}_{(\cdot,y)}(u|k) \in \mathbb{C}^{\mathsf{V}}$ *is massive harmonic:* $\Delta^m \mathrm{Exp}_{(\cdot,y)}(u|k) = 0$.

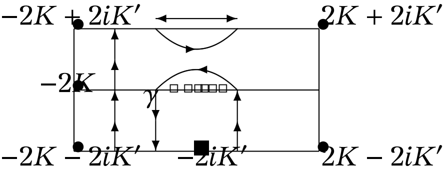
Local expression for the massive Green function

Theorem

For every pair of vertices x, y *of* G,

$$G^{m(k)}(\mathsf{x},\mathsf{y}) = -\frac{k'}{4i\pi}\oint_{\gamma_{x,y}} H(u|k)\,\mathrm{Exp}_{\mathsf{x},\mathsf{y}}(u|k)\mathrm{d}u,$$

where $\gamma_{x,y}$ *is the following contour,* $H(u|k) = \frac{u}{4K} + \frac{K'}{\pi}Z(u/2|k)$ *and* Z *is Jacobi zeta function.*



Torus $\mathbb{T}(k)$, contour of $\gamma_{x,y}$. White squares are poles of $\mathrm{Exp}_{\mathsf{x},\mathsf{y}}(\cdot|k)$, the black square is the pole of H.

Idea of the proof, consequences

Idea of the proof (Kenyon)

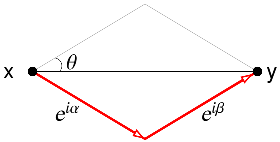
- Show that $\forall\, \mathsf{x},\mathsf{y} \in \mathsf{V},\ \Delta^{m(k)} G^{m(k)}(\mathsf{x},\mathsf{y}) = \delta(\mathsf{x},\mathsf{y})$.
- If $\mathsf{x} \neq \mathsf{y}$, deform the contours into a common contour and use the fact that massive exponential functions are massive harmonic.
- If $\mathsf{x} = \mathsf{y}$, explicit residue computation. Use the jump of the function H on the torus $\mathbb{T}(k)$.

Consequences

- Locality of the formula.
- Asymptotics of $G^{m(k)}(\mathsf{x},\mathsf{y})$, when $|\mathsf{x}-\mathsf{y}| \to \infty$.
- Explicit computations.

Example of computation

If x ∼ y in G, then

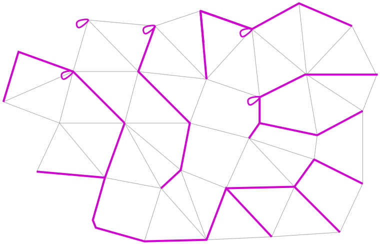


$$\mathrm{Exp}_{\mathsf{x},\mathsf{y}}(u) = -(k')^2\,\mathrm{sc}(u_{\bar\alpha})\,\mathrm{sc}(u_{\bar\beta}).$$

$$\begin{aligned}
G^{m(k)}(\mathsf{x},\mathsf{y}) &= \frac{(k')^2}{4i\pi}\oint_\gamma H(u)\,\mathrm{sc}(u_{\bar\alpha})\,\mathrm{sc}(u_{\bar\beta})\mathrm{d}u\\
&= \frac{(k')^2}{4i\pi}\oint_\gamma H(u)\,\mathrm{sc}\left(\frac{u}{2}\right)\mathrm{sc}\left(\frac{u-2\bar\theta}{2}\right)\mathrm{d}u,\ \text{(change of variable)}\\
&= \frac{H(2K+2\bar\theta)-H(2K)}{\mathrm{sc}(\bar\theta)} - \frac{K'k'}{\pi\,\mathrm{dn}(\bar\theta)},\ \text{(residues } 2K, 2K+2\bar\theta, 2iK'\text{)}\\
&= \frac{H(2\bar\theta)}{\mathrm{sc}(\bar\theta)} - \frac{K'\,\mathrm{dn}(\bar\theta)}{\pi},\ \text{(addition formula for } H\text{)}.
\end{aligned}$$

Local formula for rooted spanning forests

- Rooted spanning forests



- Boltmann probability measure:

$$\forall\, \mathsf{F} \in \mathcal{F}(\mathsf{G}), \quad \mathbb{P}_{\text{forest}}(\mathsf{F}) = \frac{\prod_{\mathsf{T} \in \mathsf{F}, \mathsf{T} \text{ rooted in } \mathsf{x}} \left(\prod_{\mathsf{e} \in \mathsf{T}} \rho(\theta_{\mathsf{e}})\right) m^2(\mathsf{x})}{Z_{\text{forest}}(\mathsf{G}, \rho, m^2)}.$$

- Explicit expression for probability measure on spanning forests of an infinite isoradial graph, periodic or not.

Z-INVARIANCE FOR ROOTED SPANNING FORESTS

	C_{Y}	C_{Δ}
$\{x_1,x_2,x_3\}$	$m^2(x_0)+\sum_{\ell=1}^{3}\rho(\theta_\ell)$	1
$\{x_i,x_j\}$	$\rho(\theta_k)\left[\sum_{\ell\neq k}\rho(\theta_\ell)\right]+m^2(x_0)\rho(\theta_k)+$ $m^2(x_k)\left[\sum_{\ell=1}^{3}\rho(\theta_\ell)+m^2(x_0)\right]$	$\sum_{\ell\neq k}\rho(K-\theta_\ell)+m'^2(x_k)$
$\{x_i\}$	$\prod_{\ell=1}^{3}\rho(\theta_\ell)+m^2(x_0)\prod_{\ell\neq i}\rho(\theta_\ell)+$ $\sum_{\ell\neq i}m^2(x_\ell)\rho(\theta_{\overline{\{i,\ell\}}})\left[\sum_{\ell'\in\{i,\ell\}}\rho(\theta_{\ell'})\right]+$ $m^2(x_0)[m^2(x_k)\rho(\theta_j)+m^2(x_j)\rho(\theta_k)]+$ $\left[\prod_{\ell\neq i}m^2(x_\ell)\right]\left[\sum_{\ell=1}^{3}\rho(\theta_\ell)+m^2(x_0)\right]$	$\sum_{\ell=1}^{3}\prod_{\ell'\neq\ell}\rho(K-\theta_{\ell'})+$ $\sum_{\ell\neq i}m'^2(x_\ell)\left[\sum_{\ell'\in\{i,\ell\}}\rho(K-\theta_{\ell'})\right]+\prod_{\ell\neq i}m'^2(x_\ell)$
$\{\emptyset\}$	$\left[\sum_{i=0}^{3}m^2(x_i)\right]\left[\prod_{i=1}^{3}\rho(\theta_i)\right]+m^2(x_0)\sum_{i=1}^{3}m^2(x_i)\prod_{\ell\neq i}\rho(\theta_\ell)+$ $\sum_{i=1}^{3}\left[\prod_{\ell\neq i}m^2(x_\ell)\right]\rho(\theta_i)\left[\sum_{\ell\neq i}\rho(\theta_i)\right]+$ $m^2(x_0)\sum_{i=1}^{3}\left[\prod_{\ell\neq i}m^2(x_\ell)\right]\rho(\theta_i)+$ $\left[\prod_{i=1}^{3}m^2(x_k)\right]\left[\sum_{i=1}^{3}\rho(\theta_i)+m^2(x_0)\right]$	$\left[\sum_{i=1}^{3}m'^2(x_i)\right]\left[\sum_{i=1}^{3}\prod_{\ell\neq i}\rho(K-\theta_\ell)\right]+$ $\sum_{i=1}^{3}\left[\prod_{\ell\neq i}m'^2(x_\ell)\right]\left[\sum_{i\neq\ell}\rho(K-\theta_\ell)\right]+$ $\prod_{i=1}^{3}m'^2(x_k)$

Z-invariance for rooted spanning forests

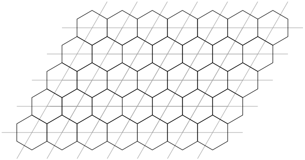
Theorem

For every $k \in [0,1)$, the rooted spanning forest model with weights ρ, m^2, is Z-invariant.

- When $k = 0$, $\rho(\theta_{\mathsf{e}}) = \tan(\theta_{\mathsf{e}}), m^2(\mathsf{x}) = 0$: one recovers the "critical" case.

When the graph G is $\mathbb{Z}^2$-periodic

Exhaustion by toroidal graphs G: $\mathsf{G}_n = \mathsf{G}/n\mathbb{Z}^2$.



The free energy is:

$$f(k) = -\lim_{n\to\infty} \frac{1}{n^2} \log Z_{\text{forest}}(\mathsf{G}_n, \rho, m^2).$$

Theorem

The free energy is equal to

$$f(k) = |\mathsf{V}_1| \int_0^K 4H'(2\theta) \log \mathrm{sc}(\theta) \mathrm{d}\theta + \sum_{e\in\mathsf{E}_1} \int_0^{\theta_e} \frac{2H(2\theta)\mathrm{sc}'(\theta)}{\mathrm{sc}(\theta)} \mathrm{d}\theta,$$

When $k = 0$, one recovers Kenyon'result.

Second order phase transition

Proposition

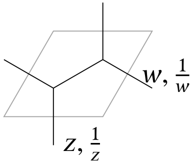
When $k \to 0$,

$$f(k) = f(0) - k^2 \log(k)|\mathsf{V}_1| + O(k^2).$$

where $f(0)$ is the free energy of spanning trees.

Spectral curve

- Fundamental domain: G_1.



- $\Delta^{m(k)}(z,w)$: massive Laplacian matrix of G_1, with weights $z, \frac{1}{z}, w, \frac{1}{w}$.
- Characteristic polynomial: $P_{\Delta^{m(k)}}(z,w) = \det \Delta^{m(k)}(z,w)$.
- Spectral curve of the massive Laplacian:

$$C_{\Delta^{m(k)}} = \{(z,w) \in \mathbb{C}^2 : P_{\Delta^{m(k)}}(z,w) = 0\}$$

Theorem

- *For every $k \in (0,1)$, $C_{\Delta^{m(k)}}$ is a Harnack curve of genus 1.*
- *Every Harnack curve of genus 1 with $(z,w) \leftrightarrow (z^{-1}, w^{-1})$ symmetry arises for such a massive Laplacian.*