The two-variable circle method

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Outline

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Motivation: Integer partitions

Definition

A *partition* of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \dots, \lambda_m$ such that $\lambda_1 + \cdots + \lambda_m = n$. The integers $\lambda_1, \dots, \lambda_m$ are called the *parts* of the partition.

Example

There are 5 partitions of 4:

$$4, 3+1, 2+2, 2+1+1 \text{ and } 1+1+1+1.$$

Let $p(n)$ denote the number of partitions of n.

Natural questions

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Let n, be positive integers and let $Q(n,k)$ denote the number of partitions of n into k distinct parts. Then

$$1+\sum_{n\geq 1}\sum_{k\geq 1} Q(n,k) z^k q^n = (1+zq)(1+zq^2)(1+zq^3)(1+zq^4)\cdots$$
$$= \prod_{n\geq 1}(1+zq^n).$$

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Recurrence relation: $Q(n,k) = Q(n-k,k) + Q(n-k,k-1)$.

$\Rightarrow$ There are 522 partitions of 50 into 7 distinct parts.

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Let $p(n,k)$ denote the number of partitions of n into k parts. Then, by the same principle:

$$1+\sum_{n\geq 1}\sum_{k\geq 1} p(n,k) z^k q^n = \prod_{n\geq 1} \frac{1}{(1-zq^n)}.$$

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Recurrence relation: $p(n,k) = p(n-1,k-1) + p(n-k,k)$.
By Euler's pentagonal number theorem

$$\left(\sum_{n\geq 0} p(n)q^n\right)\left(\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n(3n-1)}{2}}\right) = 1,$$

we have

$$p(n) = p(n-1)-p(n-2)+p(n-5)+p(n-7)-p(n-12)-p(n-15)+\cdots.$$

$\rightarrow$ Allows to compute $p(1),\ldots,p(n)$ in time $O\left(n^{\frac{3}{2}}\right)$.

Natural questions

Using the previous algorithm, one can compute the first values of $p(n)$:

$$p(10) = 42, p(20) = 627, p(50) = 204226,$$
$$p(100) = 190569292, p(200) = 3972999029388.$$

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Answer:

Theorem (Hardy-Ramanujan 1918)

As $n \to \infty$,

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

Proof: circle method

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where

$$A_k(n) = \sum_{\substack{0 \le h < k \\ (h,k)=1}} \omega_{h,k} e^{\frac{-2\pi i n h}{k}},$$

and $\omega_{h,k}$ is a (particular) 24-th root of unity.

Proof: slightly modified version of the circle method

Outline

A modular form

The generating function for partitions is

$$P(q) := \sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1} \frac{1}{(1-q^n)} = \mathrm{e}^{\frac{2i\pi\tau}{24}} \frac{1}{\eta(\tau)},$$

where $q = \mathrm{e}^{2i\pi\tau}$ and $\eta(\tau) := \mathrm{e}^{i\pi\tau/12} \prod_{k=1}^{\infty} \left(1 - \mathrm{e}^{2i\pi k\tau}\right)$.

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The function η is a **modular form**:

- some holomorphicity conditions
- $\forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \eta\left(\frac{a\tau+b}{c\tau+d}\right) = \nu(A)(c\tau+d)^{\frac{1}{2}}\eta(\tau).$

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Example

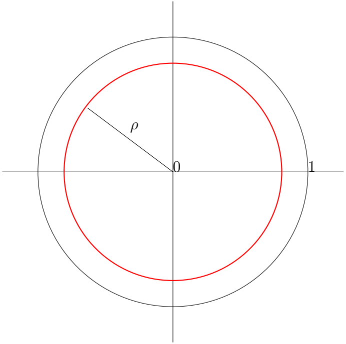
$$\eta\left(\frac{-1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau).$$

An integral on a circle

By Cauchy's theorem, we have:
For all $n \in \mathbb{N}$,

$$p(n) = \frac{1}{2i\pi} \oint_{\gamma} \frac{P(q)}{q^{n+1}} \, dq,$$

where γ is any circle centered at the origin with radius $\rho < 1$.



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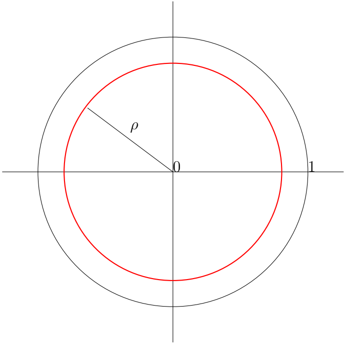
$\prod_{k=1}^{N} \frac{1}{1-q^k}$ has a pole of order $\lfloor \frac{N}{k} \rfloor$ at every point $q = \mathrm{e}^{\frac{2i\pi h}{k}}$ with $(h,k) = 1$.



Cutting the circle

By the transformation formula for η, we can evaluate $P(q)$ close to every singularity $\exp(2i\pi h/k)$.

Method:

- Choose a correct value for the radius (tending to 1 as N tends to ∞)
- Cut the circle into N small arcs (according to which singularity is the closest)
- Give an asymptotic estimation of $P(q)$ on each of these arcs
- Integrate each of them and add them
- Let N tend to infinity

The final result

Theorem (Hardy-Ramanujan-Rademacher 1937)

For every positive integer n,

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) k^{1/2} \left[\frac{\mathrm{d}}{\mathrm{d}x} \frac{\sinh\left(\frac{\pi}{k}\left(\frac{2}{3}\left(x - \frac{1}{24}\right)\right)^{1/2}\right)}{\left(x - \frac{1}{24}\right)^{1/2}} \right]_{x=n},$$

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Corollary:

$$p(n) \underset{n\to\infty}{\sim} \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}.$$

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General principle

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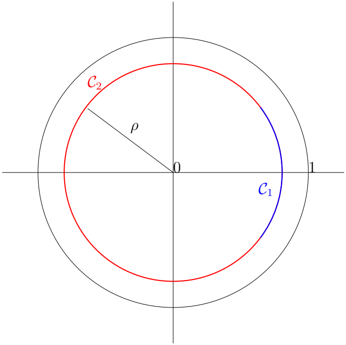
- Cut the circle into a major arc $\mathcal{C}_1$ and a minor arc $\mathcal{C}_2$,
- Give an asymptotic estimate of the integral on $\mathcal{C}_1$,
- Show that the integral on $\mathcal{C}_2$ is negligible compared to the integral on $\mathcal{C}_1$.



Asymptotics for $p(n)$

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We write $p(n) = M + E$, where

$$M := \frac{1}{2i\pi} \int_{\mathcal{C}_1} \frac{P(q)}{q^{n+1}} \, dq,$$
$$E := \frac{1}{2i\pi} \int_{\mathcal{C}_2} \frac{P(q)}{q^{n+1}} \, dq.$$

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The correct radius for the upcoming calculations is $e^{\frac{-\pi}{\sqrt{6n}}}$.

Writing $q = e^{-z} = e^{\frac{-\pi}{\sqrt{6n}}(1+ix)}$, we choose $\mathcal{C}_1$ to be the portion of the circle where $|x| \leq 1$ and $\mathcal{C}_2$ the one where $1 \leq |x| \leq \sqrt{6n}$.

Asymptotic behaviour of $P(q)$ close to $q = 1$

Theorem

Assume that $|x| \leq 1$. As n tends to infinity,

$$P(q) = \sqrt{\frac{z}{2\pi}} e^{\frac{\pi^2}{6z}} + O\left(n^{\frac{-3}{4}} e^{\pi\sqrt{\frac{n}{6}}}\right).$$

Beginning of the proof:

$$\begin{aligned} P(q) &= \frac{q^{\frac{1}{24}}}{\eta(\tau)} \\ &= \sqrt{-i\tau} \frac{q^{\frac{1}{24}}}{\eta\left(\frac{-1}{\tau}\right)} \\ &= \sqrt{-i\tau} \frac{e^{\frac{2\pi i\tau}{24}}}{e^{\frac{-2i\pi}{24\tau}} \prod_{k\geq 1}\left(1 - e^{\frac{-2k\pi i}{\tau}}\right)} = \cdots \end{aligned}$$

Asymptotic behaviour of $P(q)$ far from $q = 1$

Lemma

Let $P(q) = \frac{q^{\frac{1}{24}}}{\eta(\tau)}$ *be the generating function for partitions. Assume that* $\tau = u + iv \in \mathbb{H}$*. For* $Mv \leq |u| \leq \frac{1}{2}$ *and* $v \to 0$*, we have that*

$$|P(q)| \ll \sqrt{v} \exp\left[\frac{1}{v}\left(\frac{\pi}{12} - \frac{1}{2\pi}\left(1 - \frac{1}{\sqrt{1+M^2}}\right)\right)\right].$$

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The previous lemma with $M = 1$, $u = \frac{-x}{2\sqrt{6n}}$ and $v = \frac{1}{2\sqrt{6n}}$ gives the following.

Theorem

Assume that $1 \leq |x| \leq \sqrt{6n}$*. As* n *tends to infinity,*

$$|P(q)| \ll n^{\frac{-1}{4}} e^{\pi\sqrt{\frac{n}{6}} - \frac{1}{\pi}\sqrt{\frac{3n}{2}}}.$$

The integral on $\mathcal{C}_1$

After changes of variable ($v = 1 + ix$) and some calculation, we obtain

$$M = \frac{1}{i2^{\frac{3}{2}}(6n)^{\frac{3}{4}}} \int_{1-i}^{1+i} \sqrt{v} e^{\pi \sqrt{\frac{n}{6}}\left(\frac{1}{v}+v\right)} dv + O\left(n^{\frac{-5}{4}} e^{\pi\sqrt{\frac{2n}{3}}}\right)$$

$$= \frac{\pi}{\sqrt{2}(6n)^{\frac{3}{4}}} \left(I_{\frac{-3}{2}}\left(\pi\sqrt{\frac{2n}{3}}\right) + O\left(e^{\frac{\pi}{2}\sqrt{\frac{3n}{2}}}\right)\right) + O\left(n^{\frac{-5}{4}} e^{\pi\sqrt{\frac{2n}{3}}}\right),$$

where $I_{\frac{-3}{2}}$ is the Bessel function defined as

$$I_{-s-1}(2u) := \frac{1}{2\pi i} \int_{\Gamma} t^s e^{\pi u \left(t + \frac{1}{t}\right)} dt.$$

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$$\Rightarrow M \underset{n\to\infty}{=} \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} + O\left(n^{\frac{-5}{4}} e^{\pi\sqrt{\frac{2n}{3}}}\right)$$

The integral on $\mathcal{C}_2$

By the estimate for $P(q)$ far from the dominant pole, we have

Theorem

As $n \to \infty$,

$$E \ll n^{\frac{1}{4}} e^{\pi\sqrt{\frac{2n}{3}} - \frac{1}{\pi}\frac{\sqrt{3n}}{2}}.$$

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Theorem

As $n \to \infty$,

$$E \ll n^{\frac{1}{4}} e^{\pi \sqrt{\frac{2n}{3}} - \frac{1}{\pi} \frac{\sqrt{3n}}{2}}.$$

This is exponentially small compared to

$$M = \frac{1}{4n\sqrt{3}} e^{\pi \sqrt{\frac{2n}{3}}} + O\left(n^{\frac{-5}{4}} e^{\pi \sqrt{\frac{2n}{3}}} \right).$$

Thus

$$p(n) = M + E \underset{n \to \infty}{\sim} \frac{1}{4n\sqrt{3}} e^{\pi \sqrt{\frac{2n}{3}}}.$$

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Ramanujan's congruences

Ramanujan's congruences (1919)

For every non-negative integer n,

$$\begin{aligned} p(5n+4) &\equiv 0 \mod 5, \\ p(7n+5) &\equiv 0 \mod 7, \\ p(11n+6) &\equiv 0 \mod 11. \end{aligned}$$

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Original proof using q-series identities

Is there a combinatorial explanation?

The rank

In 1944, Dyson defines the rank to explain the congruences mod 5 and 7.

Definition

The *rank* of a partition is defined as its largest part minus its number of parts.

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Theorem

For all n,

$$\sum_{m \equiv 0 \mod 5} N(m, 5n+4) = \cdots = \sum_{m \equiv 4 \mod 5} N(m, 5n+4).$$

$$\sum_{m \equiv 0 \mod 7} N(m, 7n+5) = \cdots = \sum_{m \equiv 6 \mod 7} N(m, 7n+5).$$

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The rank fails to explain the congruences modulo 11.

The crank

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Definition (Andrews-Garvan 1988)

If for a partition λ, $o(\lambda)$ denotes the number of ones in λ, and $\mu(\lambda)$ is the number of parts strictly larger than $o(\lambda)$, then the *crank* of λ is defined by

$$\mathrm{crank}(\lambda) := \begin{cases} \text{largest part of } \lambda & \text{if } o(\lambda) = 0, \\ \mu(\lambda) - o(\lambda) & \text{if } o(\lambda) > 0. \end{cases}$$

Let $M(m,n)$ denote the number of partitions of n with crank m.

Theorem

The crank explains the three congruences. In particular

$$\sum_{m \equiv 0 \mod 11} M(m, 11n+6) = \cdots = \sum_{m \equiv 10 \mod 11} M(m, 11n+6).$$

Dyson's conjecture

Conjecture (Dyson 1989)

As n and m tend to infinity, we have

$$M(m,n) \sim \frac{1}{4}\beta \operatorname{sech}^2\left(\frac{1}{2}\beta m\right) p(n),$$

with $\beta := \frac{\pi}{\sqrt{6n}}$.

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- What is the error term?
- Is there also a two-variable asymptotic formula for the rank?

Outline

The solution

Theorem (Bringmann-D. 2014)

Dyson's conjecture is true. Precisely, if $|m| \leq \frac{1}{\pi\sqrt{6}}\sqrt{n}\log n$*, we have as* $n \to \infty$,

$$M(m,n) = \frac{\beta}{4}\operatorname{sech}^2\left(\frac{\beta m}{2}\right) p(n)\left(1 + O\left(\beta^{\frac{1}{2}}|m|^{\frac{1}{3}}\right)\right),$$

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Proof with the *two-variable circle method*

A Jacobi form

The generating function for $M(m,n)$ is the following (except for $M(m,0)$ and $M(m,1)$):

$$C(\zeta;q) := \sum_{\substack{m\in\mathbb{Z}\\ n\in\mathbb{N}}} M(m,n)\zeta^m q^n$$

$$= \frac{i\left(\zeta^{\frac{1}{2}}-\zeta^{-\frac{1}{2}}\right)q^{\frac{1}{24}}\eta^2(\tau)}{\theta\left(w;\tau\right)},$$

where $q := e^{2\pi i\tau}$, $\zeta := e^{2\pi iw}$, and

$$\theta\left(w;\tau\right) := i\zeta^{\frac{1}{2}}q^{\frac{1}{8}}\prod_{n=1}^{\infty}\left(1-q^n\right)\left(1-\zeta q^n\right)\left(1-\zeta^{-1}q^{n-1}\right).$$

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For $w \in \mathbb{C}$ and $\tau \in \mathbb{H}$, we have

$$\theta\left(\frac{w}{\tau}; -\frac{1}{\tau}\right) = -i\sqrt{-i\tau}e^{\frac{\pi i w^2}{\tau}}\theta(w;\tau).$$

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$\Rightarrow$ The two-variable generating function for the crank has modular transformation properties.

The two-variable circle method

- By Cauchy's theorem, define

$$C_m(q) := \sum_{n=0}^{\infty} M(m,n)\, q^n = \int_{-\frac{1}{2}}^{\frac{1}{2}} C\left(e^{2\pi i w}; q\right) e^{-2\pi i m w}\, dw$$

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$$M(m,n) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{C_m(q)}{q^{n+1}} dq,$$

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- With the transformation formulas, we estimate $C_m(q)$ close to and far from the dominant pole $q = 1$ and we cut the circle $\mathcal{C}$ into a major arc around 1 and a minor arc. Again, the integral on the minor arc is asymptotically negligible.

Estimates of $\mathcal{C}_m(q)$

Close to $q = 1$

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- We obtain that for $|x| \leq 1$ ($q = e^{-\beta(1+ixm^{-\frac{1}{3}})}$), as $n \to \infty$,

$$\mathcal{C}_m\left(q\right) = \frac{z^{\frac{3}{2}}}{4(2\pi)^{\frac{1}{2}}} \operatorname{sech}^2\left(\frac{\beta m}{2}\right) e^{\frac{\pi^2}{6z}} + O\left(\beta^{\frac{5}{2}} m^{\frac{2}{3}} \operatorname{sech}^2\left(\frac{\beta m}{2}\right) e^{\pi\sqrt{\frac{n}{6}}}\right).$$

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Far from $q = 1$

Assume that $1 \leq |x| \leq \frac{\pi m^{\frac{1}{3}}}{\beta}$. Then we have, as $n \to \infty$,

$$|\mathcal{C}_m(q)| \ll n^{\frac{1}{2}} \exp\left(\pi\sqrt{\frac{n}{6}} - \frac{\sqrt{6n}}{8\pi} m^{-\frac{2}{3}}\right).$$

Integral on the second circle

Define

$$M := \frac{\beta}{2\pi m^{\frac{1}{3}}} \int_{|x|\leq 1} \mathcal{C}_m \left(e^{-\beta\left(1+ixm^{-\frac{1}{3}}\right)} \right) e^{\beta n\left(1+ixm^{-\frac{1}{3}}\right)} dx,$$

$$E := \frac{\beta}{2\pi m^{\frac{1}{3}}} \int_{1\leq|x|\leq \frac{\pi m^{\frac{1}{3}}}{\beta}} \mathcal{C}_m \left(e^{-\beta\left(1+ixm^{-\frac{1}{3}}\right)} \right) e^{\beta n\left(1+ixm^{-\frac{1}{3}}\right)} dx.$$

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We have as $n \to \infty$

$$M = \frac{\beta}{4}\operatorname{sech}^2\left(\frac{\beta m}{2}\right) p(n) \left(1 + O\left(\frac{m^{\frac{1}{3}}}{n^{\frac{1}{4}}}\right)\right).$$

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As $n \to \infty$

$$E \ll n^{\frac{1}{2}} \exp\left(\pi\sqrt{\frac{2n}{3}} - \frac{\sqrt{6n}}{8\pi} m^{-\frac{2}{3}}\right) \ll M.$$

Outline

The theorem

The rank has the same asymptotic formula as the crank.

Theorem (D.-Mertens 2014)

If $|m| \leq \frac{1}{\pi\sqrt{6}}\sqrt{n}\log n$, *we have as* $n \to \infty$,

$$N(m,n) = \frac{\beta}{4}\mathrm{sech}^2\left(\frac{\beta m}{2}\right)p(n)\left(1 + O\left(\beta^{\frac{1}{2}}|m|^{\frac{1}{3}}\right)\right),$$

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Proof: again with the two-variable circle method, but more technical difficulties.

A mock Jacobi form

We always assume $\tau \in \mathbb{H}$, $w \in \mathbb{R}$, $q := e^{2\pi i\tau}$, and $\zeta := e^{2\pi i w}$. The generating function for the rank is the following.

$$\begin{aligned}
R(\zeta; q) : &= \sum_{n=0}^{\infty} \sum_{m\in\mathbb{Z}} N(m,n)\zeta^m q^n \\
&= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta q)_n(\zeta^{-1}q)_n} \\
&= \frac{q^{\frac{1}{24}}}{\eta(\tau)} \left[\frac{i\left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right)\eta^3(3\tau)}{\theta(3w;3\tau)} - \zeta^{-1}\left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right) A_1(3w,-\tau;3\tau) \right. \\
&\qquad \left. -\zeta\left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right) A_1(3w,\tau;3\tau) \right],
\end{aligned}$$

where A_1 is an *Appell-Lerch sum*.

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The Appell-Lerch sum

$$A_1(u, v; \tau) := e^{\pi i u} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\frac{n^2+n}{2}} e^{2\pi i n v}}{1 - e^{2\pi i u} q^n}$$

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Example

$$-\frac{1}{\tau} e^{\frac{\pi i (u^2-2uv)}{\tau}} A_1\left(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\right) + A_1(u,v;\tau) = \frac{1}{2i} h(u-v;\tau)\theta(v;\tau),$$

where

$$h(z;\tau) := \int_{-\infty}^{\infty} \frac{e^{\pi i \tau w^2 - 2\pi z w}}{\cosh(\pi w)} dw.$$

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- As for the crank, by Cauchy's theorem, define

$$R_m(q) := \sum_{n=0}^{\infty} N(m,n)\, q^n = \int_{-\frac{1}{2}}^{\frac{1}{2}} R\left(e^{2\pi i w}; q\right) e^{-2\pi i m w} dw$$

and write

$$N(m,n) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{R_m(q)}{q^{n+1}} dq,$$

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Perspectives

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- Is it possible to find a circle method for Jacobi forms with more than two variables?