Counting lattice walks by winding angle

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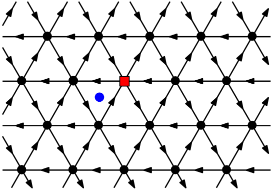
October 2020

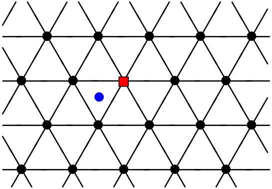
LATTICE WALKS BY WINDING ANGLE

The model: count walks starting at ■ by end point and winding angle around ●.

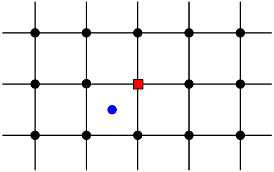
Cell-centred lattices:



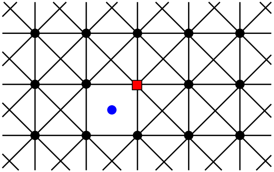
Kreweras lattice



Triangular Lattice



Square Lattice



King Lattice

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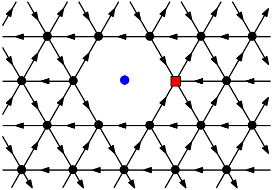
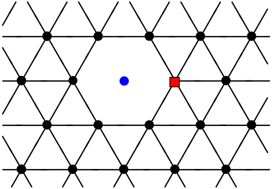
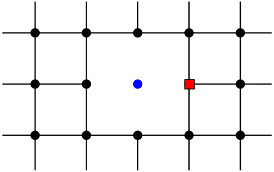
Vertex-centred lattices:



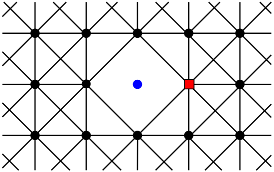
Kreweras lattice



Triangular Lattice



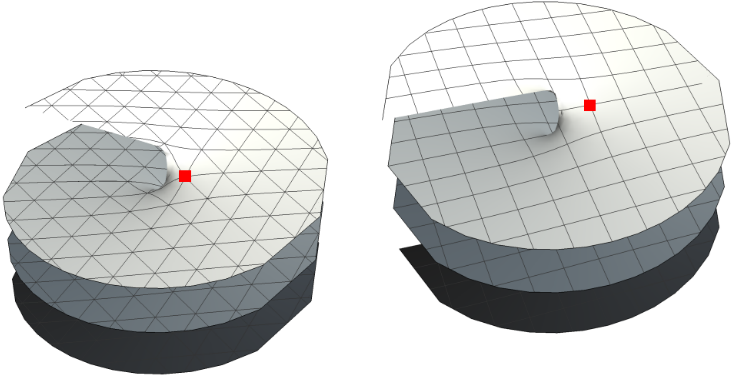
Square Lattice



King Lattice

Lattice walks by winding angle

The model: count walks starting at ■ (by end point).



Left: Cell-centred triangular lattice
Right: Vertex-centred square lattice

Why study walks by winding angle?

Physics motivation: Models a long-chain polymer growing in the vicinity of a rod

Bélisle, Berger, Brereton, Butler, Duplantier, Durrett, Faraway, Fisher, Frish, Grosberg, Hu, Le Gall, Privman, Redner, Roberts, Rudnick, Saluer, Shi, Spitzer, . . .

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More real world applications:

Square lattice walks by winding angle

[Timothy Budd, 2017]: enumeration of square lattice walks (starting and ending on an axis or diagonal) by winding angle

- **Method:** Matrices counting paths, eigenvalue decomposition etc.
- **Solution:** Jacobi theta function expressions
- **Corollaries:**
    - Square lattice walks in cones (eg. Gessel walks)
    - Loops around the origin (without a fixed starting point)
    - Algebraicity results, asymptotic results, etc.

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This work:

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- Slightly different set of results
- Extension to three other lattices

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This talk: Kreweras lattice (mostly)

JACOBI THETA FUNCTION

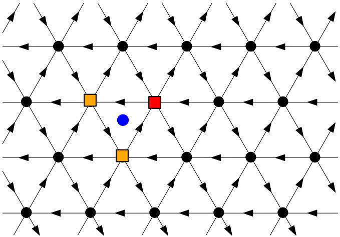
All results are in terms of the series:

$$T_k(u,q) = \sum_{n=0}^{\infty} (-1)^n (2n+1)^k q^{n(n+1)/2} (u^{n+1} - (-1)^k u^{-n})$$
$$= (u \pm 1) - 3^k q(u^2 \pm u^{-1}) + 5^k q^3 (u^3 \pm u^{-2}) + O(q^6).$$

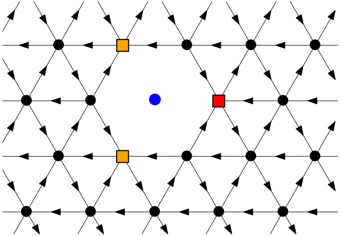
Related to Jacobi Theta function $\vartheta(z,\tau) \equiv \vartheta_{11}(z,\tau)$ by

$$\vartheta^{(k)}(z,\tau) \equiv \left(\frac{\partial}{\partial z}\right)^k \vartheta(z,\tau) = e^{\frac{(\pi\tau - 2z)i}{2}} i^k T_k(e^{2iz}, e^{2i\pi\tau}).$$

Preview: Kreweras almost-excursions



Cell-centred Kreweras lattice



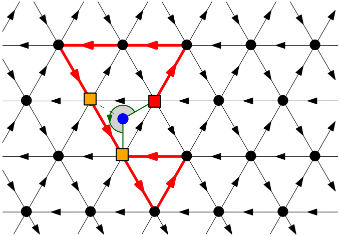
Vertex-centred Kreweras lattice

On each lattice: count walks ■ $\rightarrow$ (■ or ■). Walks with length n and winding angle $\frac{2\pi k}{3}$ contribute $t^n s^k$.

Cell-centred: $E(t,s) = 1 + st + \left(s^2 + s^{-1}\right) t^2 + \ldots$

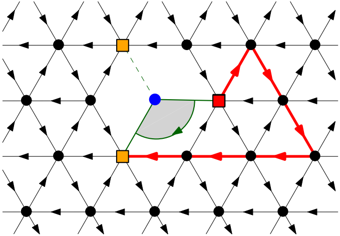
Vertex-centred: $\tilde{E}(t,s) = 1 + \left(s^{-1} + 4 + s\right) t^3 + \ldots$

Preview: Kreweras almost-excursions



Cell-centred Kreweras lattice

Contributes s^2t^8 to $E(t,s)$



Vertex-centred Kreweras lattice

Contributes $s^{-1}t^6$ to $\tilde{E}(t,s)$

On each lattice: count walks ■ → (■ or ■). Walks with length n and winding angle $\frac{2\pi k}{3}$ contribute $t^n s^k$.

Cell-centred: $E(t,s) = 1 + st + \left(s^2 + s^{-1}\right) t^2 + \ldots$

Vertex-centred: $\tilde{E}(t,s) = 1 + \left(s^{-1} + 4 + s\right) t^3 + \ldots$

Preview: Kreweras almost-excursions

Define $T_k(u,q) = \sum_{n=0}^{\infty} (-1)^n (2n+1)^k q^{n(n+1)/2} (u^{n+1} - (-1)^k u^{-n})$

$$= (u \pm 1) - 3^k q(u^2 \pm u^{-1}) + 5^k q^3 (u^3 \pm u^{-2}) + O(q^6).$$

Let $q(t) \equiv q = t^3 + 15t^6 + 279t^9 + \cdots$ satisfy

$$t = q^{1/3} \frac{T_1(1,q^3)}{4T_0(q,q^3) + 6T_1(q,q^3)}.$$

The gf for cell-centred Kreweras-lattice almost-excursions is:

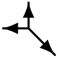
$$E(t,s) = \frac{s}{(1-s^3)t} \left(s - q^{-1/3} \frac{T_1(q^2,q^3)}{T_1(1,q^3)} - q^{-1/3} \frac{T_0(q,q^3) T_1(sq^{-2/3},q)}{T_1(1,q^3) T_0(sq^{-2/3},q)} \right).$$

The gf for vertex-centred Kreweras-lattice almost-excursions is:

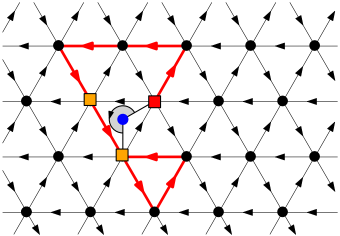
$$\tilde{E}(t,s) = \frac{s(1-s)q^{-\frac{2}{3}}}{t(1-s^3)} \frac{T_0(q,q^3)^2}{T_1(1,q^3)^2} \left(\frac{T_1(q,q^3)^2}{T_0(q,q^3)^2} - \frac{T_2(q,q^3)}{T_0(q,q^3)} - \frac{T_2(s,q)}{2T_0(s,q)} + \frac{T_3(1,q)}{6T_1(1,q)} + \frac{T_3(1,q^3)}{3T_1(1,q^3)} \right).$$

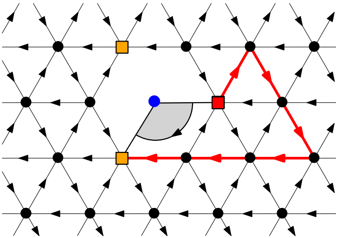
Talk outline

Focus: Kreweras lattice (for parts 1 to 4).

- **Part 1:** Decomposition of lattice $\rightarrow$ functional equations
- **Part 2:** Solving the functional equations (with theta functions!)
- **Part 3:** Corollaries: walks restricted to cones
  - **New result:** Excursions with step set  avoiding a quadrant
- **Part 4:** Analysing the solution
  - Algebraicity results using modular forms
  - Asymptotic results
- **Part 5:** Square, triangular and king lattices
- **Part 6:** Final comments and open problems

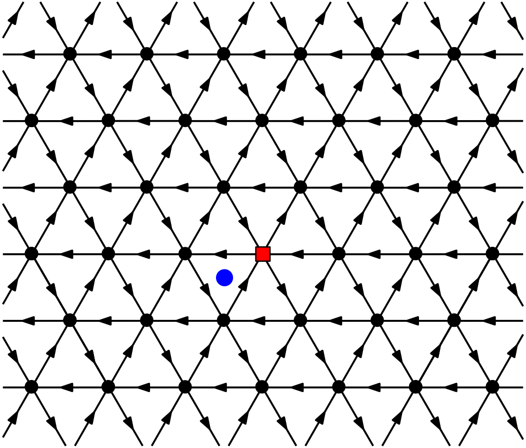
Part 1: Functional equations for Kreweras walks by winding angle


Cell-centred Kreweras lattice


Vertex-centred Kreweras lattice

KREWERAS WALKS BY WINDING NUMBER

The model: Count walks starting at ■ by end point and winding around ●.



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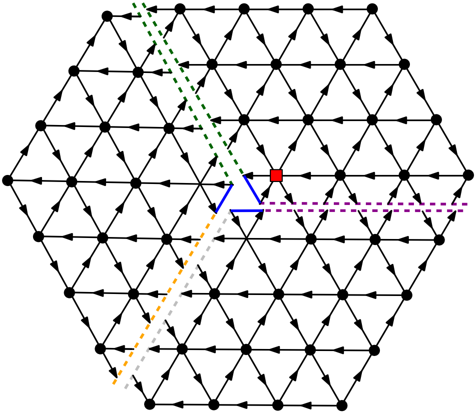

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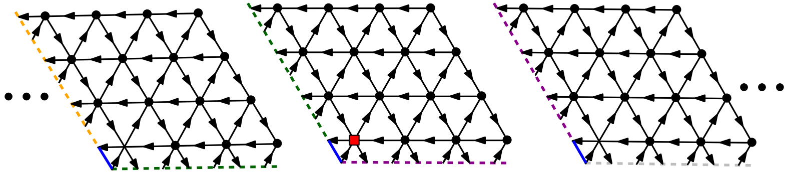

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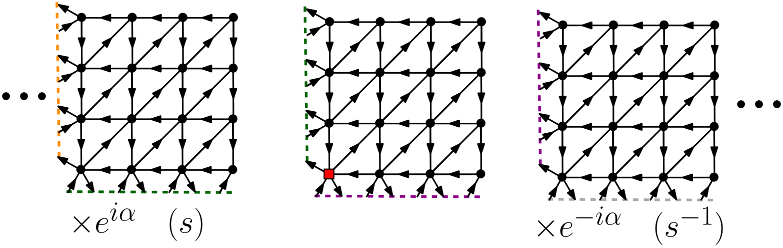

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

Note: $Q(0,0) = E(t, e^{i\alpha})$

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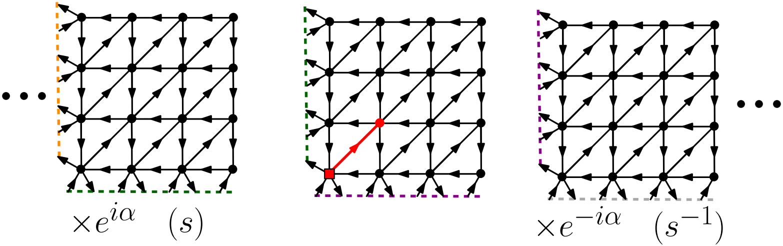

This example contributes txy.

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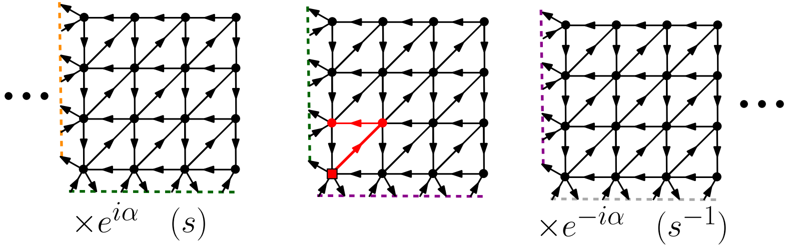

This example contributes $t^2 y$.

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

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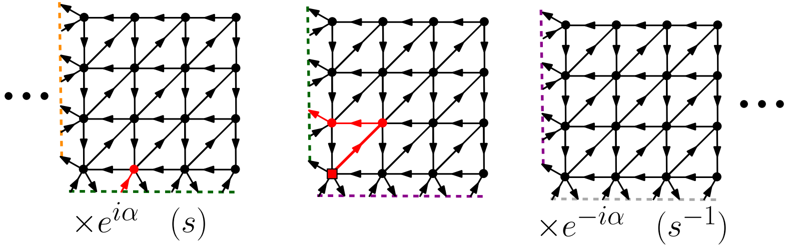

This example contributes $t^3 x e^{i\alpha}$.

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

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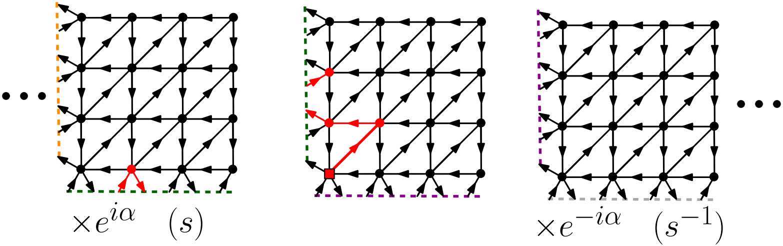

This example contributes $t^4 y^2$.

Definition: $Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

Note: $Q(0,0) = E(t, e^{i\alpha})$

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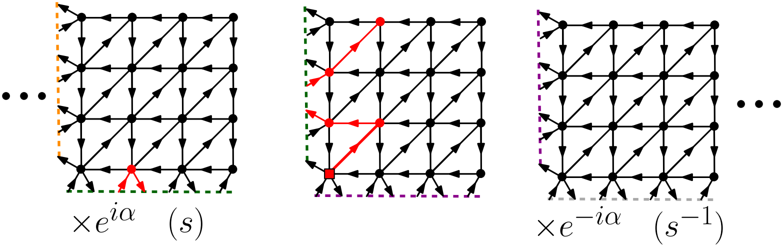

This example contributes t^5xy^3.

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|}x^{x(p)}y^{y(p)}e^{i\alpha n(p)}$

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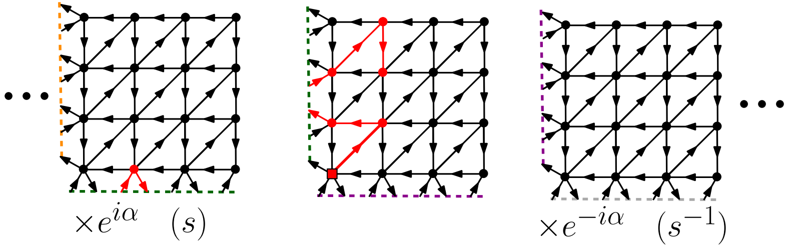

This example contributes $t^6 x y^2$.

Definition: $Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

Note: $Q(0, 0) = E(t, e^{i\alpha})$

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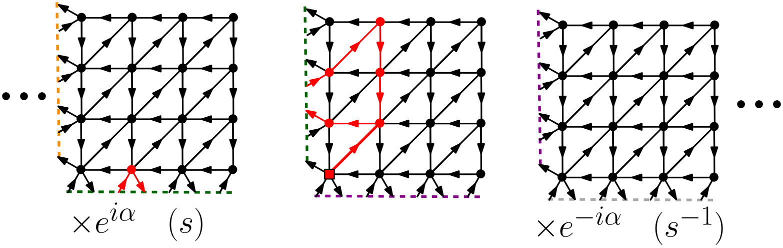

This example contributes $t^7 xy$.

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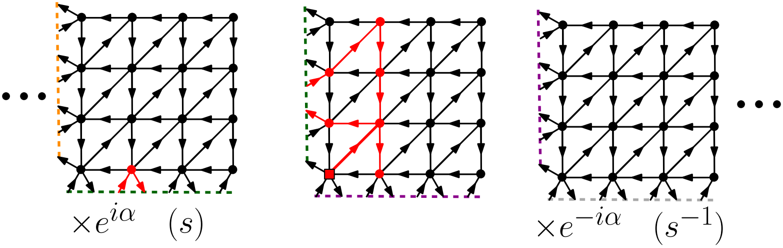

This example contributes $t^8 x$.

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

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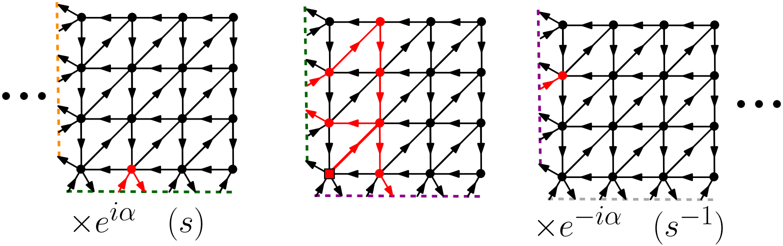

This example contributes $t^9 y^2 e^{-i\alpha}$.

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KREWERAS WALKS BY WINDING NUMBER

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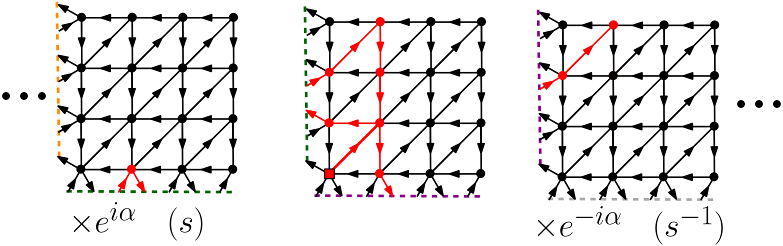

This example contributes $t^{10}xy^3e^{-i\alpha}$.

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|}x^{x(p)}y^{y(p)}e^{i\alpha n(p)}$

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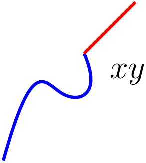
FUNCTIONAL EQUATION

Recursion → functional equation: separate by *type* of final step.

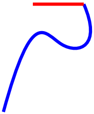
$$Q(x,y) = \quad 1$$

$$+$$



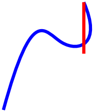
$$xytQ(x,y)$$

$$+$$



$$\frac{t}{x}(Q(x,y) - Q(0,y))$$

$$+$$



$$\frac{t}{y}(Q(x,y) - Q(x,0))$$

$$+\, e^{i\alpha}tQ(0,x)$$

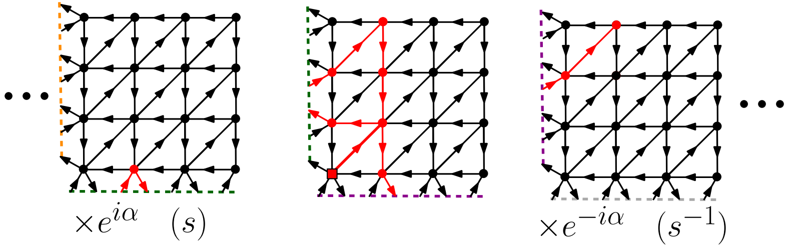
(Final step goes through left wall)

$$+\, e^{-i\alpha}tyQ(y,0)$$

(Final step goes through bottom wall)

KREWERAS WALKS BY WINDING NUMBER

The model: Count walks starting at the red point by end point.

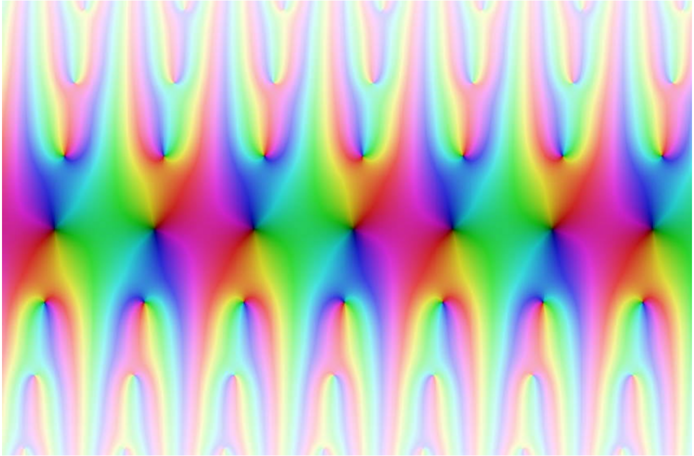


Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$.

Characterised by:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y)-Q(0,y)}{x} + t\frac{Q(x,y)-Q(x,0)}{y}$$
$$+e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

Part 2: Solution (using theta functions)



Solution to Kreweras walks by winding number

Equation to solve:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y) - Q(0,y)}{x} + t\frac{Q(x,y) - Q(x,0)}{y}$$
$$+ e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

SOLUTION TO KREWERAS WALKS BY WINDING NUMBER

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$$+ e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

Solution:

Step 1: Fix $t \in [0, 1/3), \alpha \in \mathbb{R}$. All series converge for $|x|, |y| < 1$.

Solution to Kreweras walks by winding number

Equation to solve:

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Solution:

Step 1: Fix $t \in [0, 1/3), \alpha \in \mathbb{R}$. All series converge for $|x|, |y| < 1$.

Step 2: Write equation as $K(x,y)Q(x,y) = R(x,y)$, where

$$K(x,y) = 1 - txy - t/y - t/x$$

$$R(x,y) = 1 - \frac{t}{x}Q(0,y) - \frac{t}{y}Q(x,0) + e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

Solution to Kreweras walks by winding number

Equation to solve:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y) - Q(0,y)}{x} + t\frac{Q(x,y) - Q(x,0)}{y} + e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

Solution:

Step 1: Fix $t \in [0, 1/3), \alpha \in \mathbb{R}$. All series converge for $|x|, |y| < 1$.

Step 2: Write equation as $K(x,y)Q(x,y) = R(x,y)$, where

$$K(x,y) = 1 - txy - t/y - t/x$$

$$R(x,y) = 1 - \frac{t}{x}Q(0,y) - \frac{t}{y}Q(x,0) + e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

Step 3: Consider the curve $K(x,y) = 0$ (Then $R(x,y) = 0$).

SOLUTION TO KREWERAS WALKS BY WINDING NUMBER

Equation to solve:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y)-Q(0,y)}{x} + t\frac{Q(x,y)-Q(x,0)}{y} + e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

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Step 1: Fix $t \in [0,1/3), \alpha \in \mathbb{R}$. All series converge for $|x|, |y| < 1$.

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Step 3: Consider the curve $K(x,y) = 0$ (Then $R(x,y) = 0$).

Parameterisation involves the Jacobi theta function $\vartheta(z,\tau)$.

So far: Similar to elliptic approaches to quadrant models [Bernardi, Bousquet-Mélou, Fayolle, Iasnogorodski, Kurkova, Malyshev, Raschel, Trotignon]

JACOBI THETA FUNCTION $\vartheta(z,\tau)$

Definition: For $\tau, z \in \mathbb{C}$, $\mathrm{im}(\tau) > 0$,

$$\vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\left(\frac{2n+1}{2}\right)^2 i\pi\tau + (2n+1)iz}$$

Useful facts (for fixed τ):

- $\vartheta(z+\pi,\tau) = -\vartheta(z,\tau)$
- $\vartheta(z+\pi\tau,\tau) = -e^{-2iz-i\pi\tau}\vartheta(z,\tau)$

PARAMETERISATION OF $K(x,y)=0$ USING $\vartheta(z,\tau)$

Definition: For $\tau, z \in \mathbb{C}$, $\mathrm{im}(\tau) > 0$,

$$\vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\left(\frac{2n+1}{2}\right)^2 i\pi\tau + (2n+1)iz}$$

Useful facts (for fixed τ):

- $\vartheta(z+\pi,\tau) = -\vartheta(z,\tau)$
- $\vartheta(z+\pi\tau,\tau) = -e^{-2iz-i\pi\tau}\vartheta(z,\tau)$

Parameterisation: The curve

$$K(x,y) := 1 - txy - t/y - t/x = 0$$

is parameterised by

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta\left(z-\pi\tau,3\tau\right)}{\vartheta\left(z+\pi\tau,3\tau\right)\vartheta\left(z-2\pi\tau,3\tau\right)} \quad \text{and} \quad Y(z) = X(z+\pi\tau),$$

where τ is determined by $t = e^{-\frac{\pi\tau i}{3}} \dfrac{\vartheta'(0,3\tau)}{4i\vartheta(\pi\tau,3\tau) + 6\vartheta'(\pi\tau,3\tau)}$.

SOLUTION TO KREWERAS WALKS BY WINDING NUMBER

Equation to solve:

$$K(x,y)Q(x,y) = R(x,y),$$

where

$$K(x,y) = 1 - txy - t/y - t/x,$$
$$R(x,y) = 1 - \frac{t}{x}Q(0,y) - \frac{t}{y}Q(x,0) + e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

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Define

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta(z-\pi\tau,3\tau)}{\vartheta(z+\pi\tau,3\tau)\vartheta(z-2\pi\tau,3\tau)}.$$

Then $K(X(z),X(z+\pi\tau)) = 0$.

Solution to Kreweras walks by winding number

Equation to solve:

$$K(x,y)Q(x,y) = R(x,y),$$

where

$$K(x,y) = 1 - txy - t/y - t/x,$$
$$R(x,y) = 1 - \frac{t}{x}Q(0,y) - \frac{t}{y}Q(x,0) + e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

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$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta(z-\pi\tau,3\tau)}{\vartheta(z+\pi\tau,3\tau)\vartheta(z-2\pi\tau,3\tau)}.$$

Then $K(X(z),X(z+\pi\tau)) = 0$. Hence $R(X(z),X(z+\pi\tau)) = 0$ (assuming $|X(z)| \leq 1$ and $|X(z+\pi\tau)| \leq 1$).

Equation to solve:

$$K(x,y)Q(x,y) = R(x,y),$$

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Define

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta\left(z-\pi\tau,3\tau\right)}{\vartheta\left(z+\pi\tau,3\tau\right)\vartheta\left(z-2\pi\tau,3\tau\right)}.$$

Then $K(X(z),X(z+\pi\tau)) = 0$. Hence $R(X(z),X(z+\pi\tau)) = 0$ (assuming $|X(z)| \leq 1$ and $|X(z+\pi\tau)| \leq 1$).

New equation to solve:

$$R(X(z),X(z+\pi\tau)) = 0,$$

Equation to solve:

$$K(x,y)Q(x,y) = R(x,y),$$

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$$K(x,y) = 1 - txy - t/y - t/x,$$
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Define

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta\left(z-\pi\tau,3\tau\right)}{\vartheta\left(z+\pi\tau,3\tau\right)\vartheta\left(z-2\pi\tau,3\tau\right)}.$$

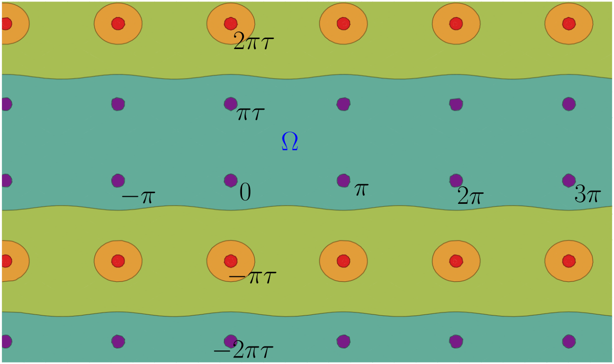
Then $K(X(z),X(z+\pi\tau)) = 0$. Hence $R(X(z),X(z+\pi\tau)) = 0$ (assuming $|X(z)| \leq 1$ and $|X(z+\pi\tau)| \leq 1$).

New equation to solve:

$$R(X(z),X(z+\pi\tau)) = 0,$$

Solution to Kreweras walks by winding number

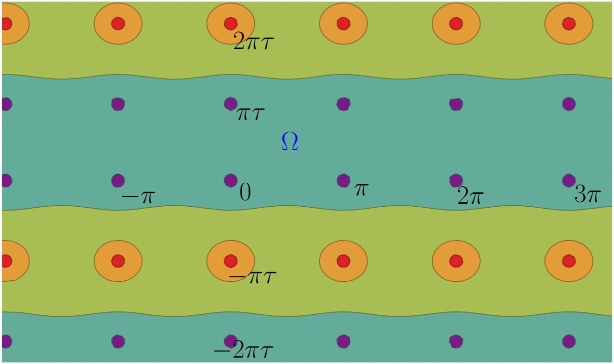
Plot of $\left\{z : |X(z)| \in \left[0, \frac{1}{3}\right), \left(\frac{1}{3}, 1\right), (1, 3), (3, 9), (9, \infty]\right\}$.



For $z \in \Omega$, $|X(z)| < 1 \Rightarrow Q(X(z), 0)$ and $Q(0, X(z))$ are well defined.

Solution to Kreweras walks by winding number

Plot of $\left\{z : |X(z)| \in \left[0, \frac{1}{3}\right), \left(\frac{1}{3}, 1\right), (1,3), (3,9), (9,\infty]\right\}$.



For $z \in \Omega$, $\quad |X(z)| < 1 \Rightarrow Q(X(z), 0)$ and $Q(0, X(z))$ are well defined.
Near $\mathrm{Re}(z) = 0$, we have $z \in \Omega$ and $z + \pi\tau \in \Omega$.

Solution to Kreweras walks by winding number

Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$R(X(z), X(z+\pi\tau)) = 0$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta\left(z-\pi\tau,3\tau\right)}{\vartheta\left(z+\pi\tau,3\tau\right)\vartheta\left(z-2\pi\tau,3\tau\right)}.$$

$$R(x,y) = 1 - \frac{t}{x}Q(0,y) - \frac{t}{y}Q(x,0) + e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$1 = \frac{t}{X(z)} Q(0, X(z+\pi\tau)) + \frac{t}{X(z+\pi\tau)} Q(X(z), 0) - e^{i\alpha} t Q(0, X(z)) - e^{-i\alpha} t X(z+\pi\tau) Q(X(z+\pi\tau), 0),$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau) \vartheta(z-\pi\tau, 3\tau)}{\vartheta(z+\pi\tau, 3\tau)\, \vartheta(z-2\pi\tau, 3\tau)}.$$

Solution to Kreweras walks by winding number

Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$1 = \frac{t}{X(z)}Q(0, X(z + \pi\tau)) + \frac{t}{X(z + \pi\tau)}Q(X(z), 0) - e^{i\alpha}tQ(0, X(z)) - e^{-i\alpha}tX(z + \pi\tau)Q(X(z + \pi\tau), 0),$$

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$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z, 3\tau)\vartheta(z - \pi\tau, 3\tau)}{\vartheta(z + \pi\tau, 3\tau)\vartheta(z - 2\pi\tau, 3\tau)}.$$

For z near 0, define

$$L(z) = \frac{t}{X(z + \pi\tau)}Q(X(z), 0) - e^{i\alpha}tQ(0, X(z)).$$

Both $L(z)$ and $L(z + \pi\tau)$ converge.

Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$1 = \frac{t}{X(z)}Q(0, X(z + \pi\tau)) + \frac{t}{X(z + \pi\tau)}Q(X(z), 0)$$
$$- e^{i\alpha}tQ(0, X(z)) - e^{-i\alpha}tX(z + \pi\tau)Q(X(z + \pi\tau), 0),$$

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Solution to Kreweras walks by winding number

Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$1 = \frac{t}{X(z)} Q(0, X(z+\pi\tau)) + L(z) - e^{-i\alpha} t X(z+\pi\tau) Q(X(z+\pi\tau), 0),$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau) \vartheta(z-\pi\tau, 3\tau)}{\vartheta(z+\pi\tau, 3\tau) \vartheta(z-2\pi\tau, 3\tau)}.$$

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$$L(z) = \frac{t}{X(z+\pi\tau)} Q(X(z), 0) - e^{i\alpha} t Q(0, X(z)).$$

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Solution to Kreweras walks by winding number

Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$\begin{aligned} 1 =& \frac{t}{X(z)} Q(0, X(z + \pi\tau)) + L(z) \\ &- e^{-i\alpha} t X(z + \pi\tau) Q(X(z + \pi\tau), 0), \end{aligned}$$

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Equation to solve: (near $\mathrm{Re}(z)=0$)

$$1 = \frac{t}{X(z)}Q(0,X(z+\pi\tau)) + L(z) - \frac{e^{-i\alpha}t}{X(z)X(z+2\pi\tau)}Q(X(z+\pi\tau),0),$$

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Both $L(z)$ and $L(z+\pi\tau)$ converge.

Solution to Kreweras walks by winding number

Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$1 = -\frac{e^{-i\alpha}}{X(z)} L(z + \pi\tau) + L(z).$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau)\vartheta\left(z - \pi\tau, 3\tau\right)}{\vartheta\left(z + \pi\tau, 3\tau\right)\vartheta\left(z - 2\pi\tau, 3\tau\right)}.$$

For z near 0, define

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We can solve this exactly:

$$L(z) = -\frac{e^{3i\alpha}}{1-e^{3i\alpha}}\left(1+\frac{e^{-i\alpha}}{X(z)}+e^{-2i\alpha}X(z-\pi\tau)\right)$$
$$-\frac{e^{i\alpha+\frac{5i\pi\tau}{3}}\vartheta(\pi\tau,3\tau)\vartheta'(0,\tau)}{(1-e^{3i\alpha})\vartheta(\frac{\alpha}{2}-\frac{2\pi\tau}{3},\tau)\vartheta'(0,3\tau)}\frac{\vartheta(z-2\pi\tau,3\tau)\vartheta(z-\frac{\alpha}{2}+\frac{2\pi\tau}{3},\tau)}{\vartheta(z,\tau)\vartheta(z,3\tau)}$$

SOLUTION TO KREWERAS WALKS BY WINDING NUMBER

Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$1 = -\frac{e^{-i\alpha}}{X(z)} L(z+\pi\tau) + L(z).$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau) \vartheta\left(z - \pi\tau, 3\tau\right)}{\vartheta\left(z + \pi\tau, 3\tau\right) \vartheta\left(z - 2\pi\tau, 3\tau\right)}.$$

We can solve this exactly:

$$L(z) = -\frac{e^{3i\alpha}}{1 - e^{3i\alpha}} \left(1 + \frac{e^{-i\alpha}}{X(z)} + e^{-2i\alpha} X(z - \pi\tau)\right)$$
$$- \frac{e^{i\alpha + \frac{5i\pi\tau}{3}} \vartheta(\pi\tau, 3\tau) \vartheta'(0, \tau)}{(1 - e^{3i\alpha}) \vartheta(\frac{\alpha}{2} - \frac{2\pi\tau}{3}, \tau) \vartheta'(0, 3\tau)} \frac{\vartheta(z - 2\pi\tau, 3\tau) \vartheta(z - \frac{\alpha}{2} + \frac{2\pi\tau}{3}, \tau)}{\vartheta(z, \tau) \vartheta(z, 3\tau)}$$

We can extract $E(t, e^{i\alpha}) = Q(0, 0)$...

Kreweras walks by winding number: Solution

Recall: τ is determined by

$$t = e^{-\frac{\pi\tau i}{3}} \frac{\vartheta'(0, 3\tau)}{4i\vartheta(\pi\tau, 3\tau) + 6\vartheta'(\pi\tau, 3\tau)}.$$

The gf $E(t, e^{i\alpha}) = Q(0, 0) \equiv Q(t, \alpha, 0, 0)$ is given by:

$$E(t, e^{i\alpha}) = \frac{e^{i\alpha}}{t(1 - e^{3i\alpha})} \left(e^{i\alpha} - e^{\frac{4\pi\tau i}{3}} \frac{\vartheta'(2\pi\tau, 3\tau)}{\vartheta'(0, 3\tau)} - e^{\frac{\pi\tau i}{3}} \frac{\vartheta(\pi\tau, 3\tau)\vartheta'(\frac{\alpha}{2} - \frac{2\pi\tau}{3}, \tau)}{\vartheta'(0, 3\tau)\vartheta(\frac{\alpha}{2} - \frac{2\pi\tau}{3}, \tau)} \right).$$

KREWERAS WALKS BY WINDING NUMBER: SOLUTION

Recall: τ is determined by

$$t = e^{-\frac{\pi\tau i}{3}} \frac{\vartheta'(0,3\tau)}{4i\vartheta(\pi\tau,3\tau)+6\vartheta'(\pi\tau,3\tau)}.$$

The gf $E(t,e^{i\alpha}) = Q(0,0) \equiv Q(t,\alpha,0,0)$ is given by:

$$E(t,e^{i\alpha}) = \frac{e^{i\alpha}}{t(1-e^{3i\alpha})}\left(e^{i\alpha} - e^{\frac{4\pi\tau i}{3}}\frac{\vartheta'(2\pi\tau,3\tau)}{\vartheta'(0,3\tau)} - e^{\frac{\pi\tau i}{3}}\frac{\vartheta(\pi\tau,3\tau)\vartheta'(\frac{\alpha}{2}-\frac{2\pi\tau}{3},\tau)}{\vartheta'(0,3\tau)\vartheta(\frac{\alpha}{2}-\frac{2\pi\tau}{3},\tau)}\right).$$

Equivalently:

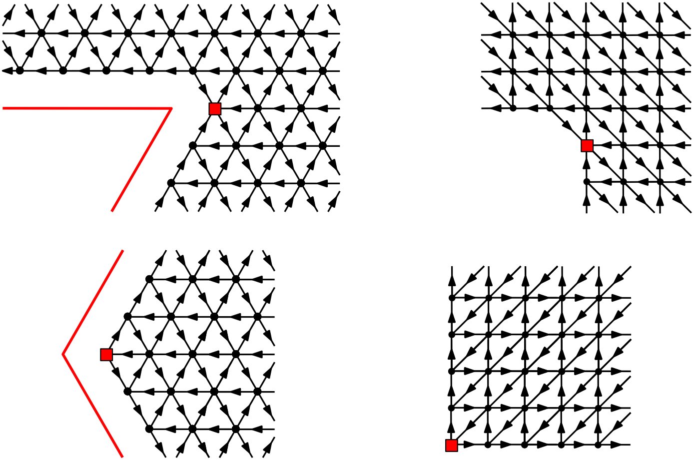
Let $q(t) \equiv q = t^3 + 15t^6 + 279t^9 + \cdots$ satisfy

$$t = q^{1/3}\frac{T_1(1,q^3)}{4T_0(q,q^3)+6T_1(q,q^3)}.$$

The gf for cell-centred Kreweras-lattice almost-excursions is:

$$E(t,s) = \frac{s}{(1-s^3)t}\left(s - q^{-1/3}\frac{T_1(q^2,q^3)}{T_1(1,q^3)} - q^{-1/3}\frac{T_0(q,q^3)T_1(sq^{-2/3},q)}{T_1(1,q^3)T_0(sq^{-2/3},q)}\right).$$

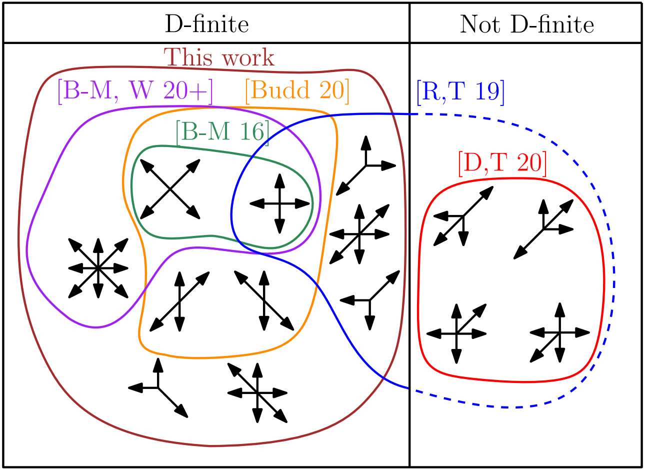
Part 3: Walks in cones



Walks in cones with small steps

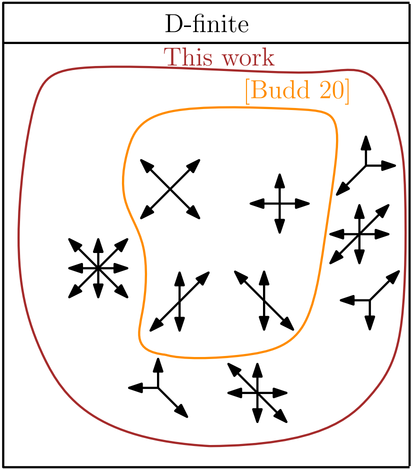
- **Quarter plane walks:** Completely classified into rational, algebraic, D-finite, D-algebraic cases.
  [Mishna, Rechnitzer 09], [Bousquet-Mélou, Mishna 10], [Bostan, Kauers 10], [Fayolle, Raschel 10], [Kurkova, Raschel 12], [Melczer, Mishna 13], [Bostan, Raschel, Salvy 14], [Bernardi, Bousquet-Mélou, Raschel 17], [Dreyfus, Hardouin, Roques, Singer 18]
- **Half plane walks:** Easy
- **Three quarter plane walks:** Active area of research
  (Previously) solved in 6-12 of the 74 non-trivial cases
  [Bousquet-Mélou 16], [Raschel-Trotignon 19], [Budd 20], [Bousquet-Mélou, Wallner 20+]
- **Walks on the slit plane $\mathbb{C} \setminus \mathbb{R}_{<0}$:** solved in all cases
  [Bousquet-Mélou, 01], [Bousquet-Mélou, Schaeffer, 02], [Rubey 05]

WALKS IN THE 3/4-PLANE: SOLVED CASES



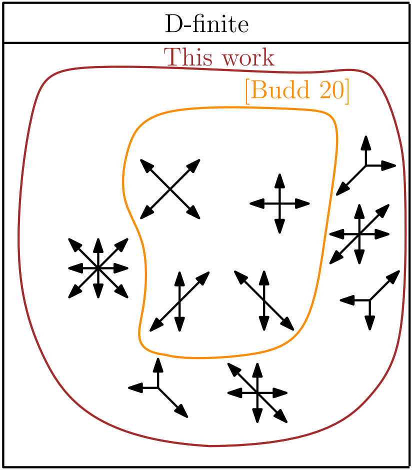
[Bousquet-Mélou 16],[Raschel, Trotignon 19], [Budd 20], [Bousquet-Mélou, Wallner 20+]

WALKS IN THE 5/4-PLANE: SOLVED CASES



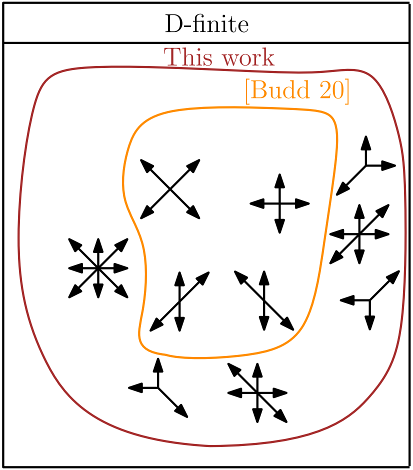
[Budd 20]

Walks in the 6/4-plane: solved cases

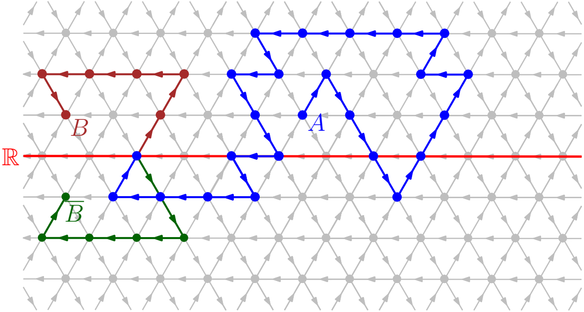


[Budd 20]

Walks in the 7/4-plane: solved cases



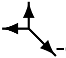
[Budd 20]

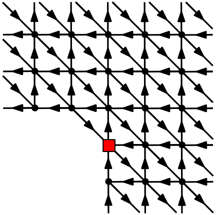


In the upper half plane: Use reflection principle

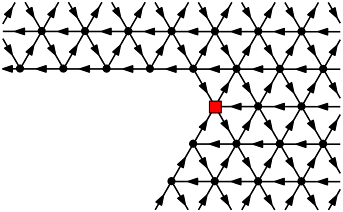
$$\begin{aligned}
&\#(\text{Walks from } A \text{ to } B \text{ above } \mathbb{R}) \\
= &\#(\text{Walks from } A \text{ to } B) - \#(\text{Walks from } A \text{ to } B \text{ through } \mathbb{R}) \\
= &\#(\text{Walks from } A \text{ to } B) - \#(\text{Walks from } A \text{ to } \overline{B})
\end{aligned}$$

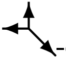
Counting Kreweras excursions in 5/6-plane

New result: -excursions avoiding a quadrant.

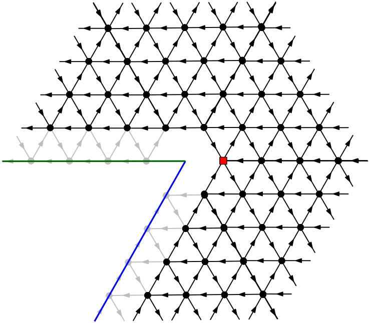


$\equiv$

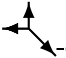


New result: -excursions avoiding a quadrant.

Equivalently: Walks avoiding the blue and green lines



COUNTING KREWERAS EXCURSIONS IN 5/6-PLANE

New result: -excursions avoiding a quadrant.
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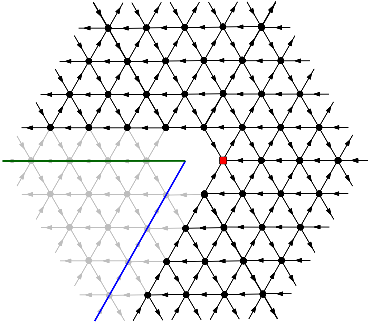

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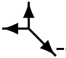
New result: -excursions avoiding a quadrant.
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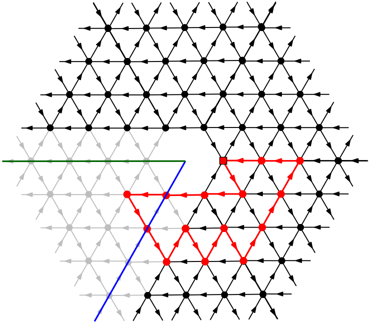

Reflection principle: For walks passing through at least one such line: reflect walk after first intersection.
Winding angle $\alpha \to -\frac{4\pi}{3} - \alpha$ or $2\pi - \alpha$.

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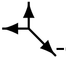
New result: -excursions avoiding a quadrant.
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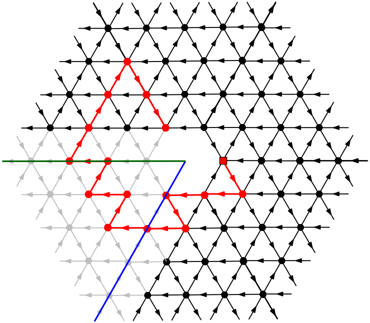

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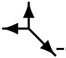
New result: -excursions avoiding a quadrant.
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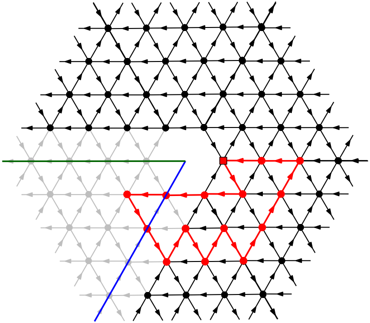

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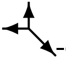
Winding angle $\frac{10\pi k}{3} \to -\frac{4\pi}{3} + \frac{10\pi j}{3}$.

$$\#(\text{Walks } \blacksquare \to \blacksquare \text{ avoiding lines})$$

$$= \left(\sum_{k\in\mathbb{Z}} [s^{5k}]\tilde{E}(t,s)\right) - \left(\sum_{k\in\mathbb{Z}} [s^{5k-2}]\tilde{E}(t,s)\right)$$

$$= \frac{1}{5}\sum_{j=1}^{4}\left(1 - e^{\frac{4\pi ij}{5}}\right)\tilde{E}\left(t, e^{\frac{2\pi i}{5}}\right)$$

Counting Kreweras excursions in 5/6-plane

New result: -excursions avoiding a quadrant.
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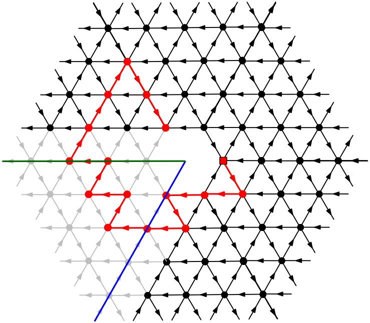

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Counting Kreweras excursions in $k/6$-plane

More generally: Let $C_{k,r}(t)$ count whole-plane Kreweras excursions...

- Starting adjacent to the origin,
- Avoiding the origin,
- Having winding angle 0,
- Having intermediate winding angles restricted to $\left[-\frac{r\pi}{3}, \frac{(k-r)\pi}{3}\right]$

I.e., Kreweras excursions in the $k/6$-plane

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I.e., Kreweras excursions in the $k/6$-plane

Previous slide:

$$C_{5,2}(t) = \frac{1}{5}\sum_{j=1}^{4}\left(1-e^{\frac{4\pi ij}{5}}\right)\tilde{E}\left(t, e^{\frac{2\pi i}{5}}\right).$$

More generally:

$$C_{k,r}(t) = \frac{1}{k}\sum_{j=1}^{k-1}\left(1-e^{\frac{2\pi ijr}{k}}\right)\tilde{E}\left(t, e^{\frac{2\pi ij}{k}}\right).$$

Part 4: Analysis of solutions

ANALYSIS OF SOLUTION

From the exact solution we extract:

- **Asymptotic distribution ([Bélisle, 1989]):** For random excursions of length n, $\frac{\text{winding angle}}{c\log(n)}$ has asymptotic density

$$4\frac{(x-1)e^{x}+(x+1)e^{-x}}{(e^{x}-e^{-x})^{2}}.$$

- **Asymptotics ([Denisov, Wachtel, 2015]):** Let c_n count Kreweras-lattice excursions in a cone of angle $\alpha \in \frac{\pi}{3}\mathbb{N}$.

$$c_n \sim -\frac{2\cdot 3^{5-\frac{6}{k}}\sin^2\left(\frac{\pi}{k}\right)}{\pi k^2\left(1+2\cos\left(\frac{2\pi}{k}\right)\right)\Gamma\left(-\frac{3}{k}\right)}n^{-1-\frac{3}{k}}3^{n}.$$

- **Conditions for algebraicity:** Let $C_\alpha(t)$ count Kreweras-lattice excursions in a cone of angle $\alpha \in \frac{\pi}{3}\mathbb{N}$. This satisfies a non-trivial polynomial equation $P(C_\alpha(t),t)=0$ if and only if $\alpha \notin \pi\mathbb{Z}$. (uses modular forms as in [Zagier, 08] and [E.P., Zinn-Justin, 20])

Analysis of solution: Asymptotics

Fix α.
Writing $\hat{\tau} = -\frac{1}{3\tau}$ and $\hat{q} = e^{2\pi i\hat{\tau}}$, the dominant singularity $t = 1/3$ of $\tilde{E}(t, e^{i\alpha})$ corresponds to $\hat{q} = 0$.

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Series in $\hat{q}$:

$$t = \frac{1}{3} - 3\hat{q} + 18\hat{q}^2 + O(\hat{q}^3)$$

$$t\tilde{E}(t, e^{i\alpha}) = a_0 + a_1\hat{q} - \frac{27\alpha e^{i\alpha}}{2\pi(1 + e^{i\alpha} + e^{2i\alpha})}\hat{q}^{\frac{3\alpha}{2\pi}} + o\left(\hat{q}^{\frac{3\alpha}{2\pi}}\right),$$

$\rightarrow \tilde{E}(t, e^{i\alpha})$ as a series in $(1 - 3t)$,

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$$[t^n]\tilde{E}(t, e^{i\alpha}) \sim -\frac{3^{5-\frac{3\alpha}{\pi}} e^{\alpha i}\alpha}{2\pi(1 + e^{\alpha i} + e^{2\alpha i})\Gamma\left(-\frac{3\alpha}{2\pi}\right)} n^{-\frac{3\alpha}{2\pi}-1}3^n,$$

$$[t^n]C_{k,r}(t) \sim -\frac{2 \cdot 3^{5-\frac{6}{k}} \sin^2\left(\frac{r\pi}{k}\right)}{\pi k^2\left(1 + 2\cos\left(\frac{2\pi}{k}\right)\right)\Gamma\left(-\frac{3}{k}\right)} n^{-1-\frac{3}{k}}3^n.$$

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Previously: Terms 3^n and $n^{-1-\frac{3}{k}}$ known [Denisov, Wachtel, 2015].

Analysis of solution: Algebraicity

Recall: $\vartheta(z,\tau)$ is differentially algebraic $\rightarrow$ so are $\tilde{E}(t,s)$ and $Q(t,\alpha,x,y)$.

For $\alpha \in \frac{\pi}{3}(\mathbb{Q}\setminus\mathbb{Z})$ **we get algebraicity** (Ideas from [Zagier, 08] and [E.P., Zinn-Justin, 20+]):

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Recall: The gf for excursions in the $k/6$-plane is

$$C_{k,r}(t) = \frac{1}{k}\sum_{j=1}^{k-1}\left(1-e^{\frac{2\pi ijr}{k}}\right)\tilde{E}\left(t,e^{\frac{2\pi ij}{k}}\right).$$

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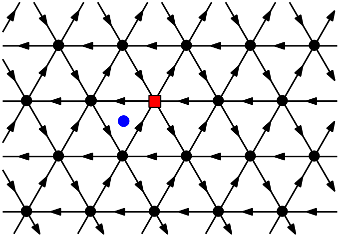
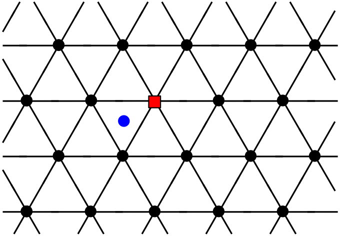
Algebraic iff $3 \nmid k$. (always D-finite).

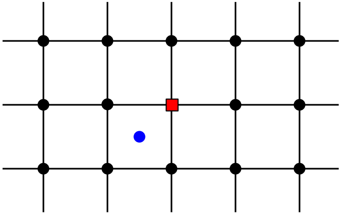
Part 5: Other lattices



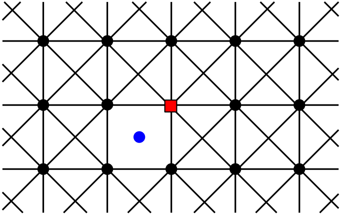
Kreweras lattice



Triangular Lattice

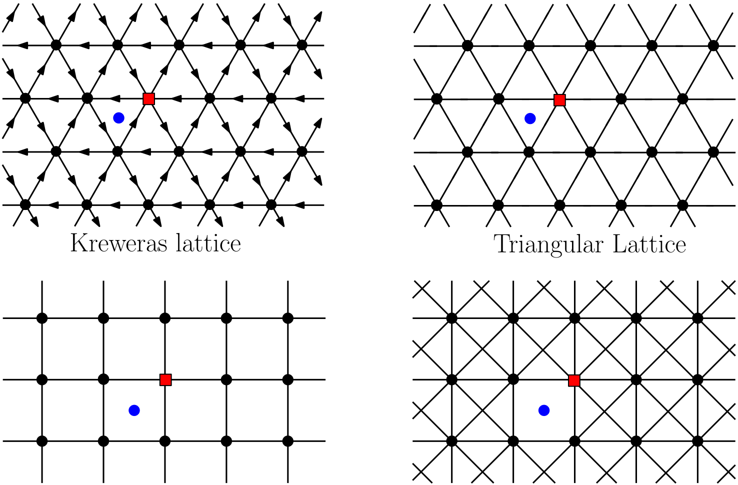


Square Lattice



King Lattice

Important property: Decomposable into congruent sectors



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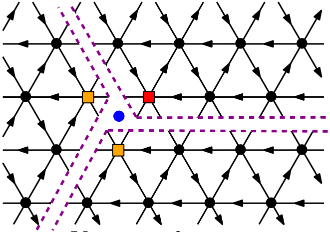

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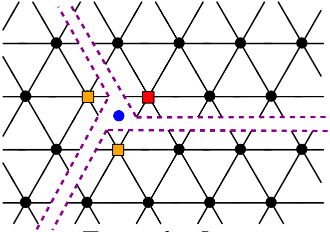

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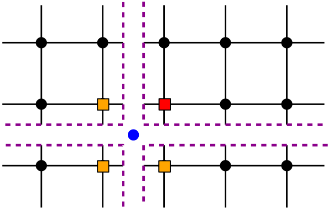

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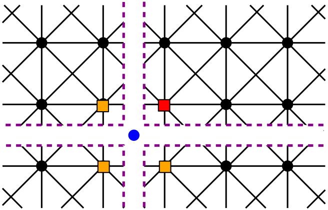
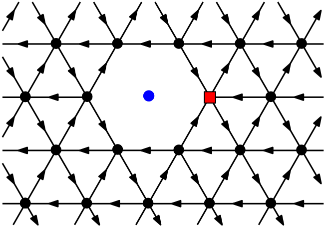
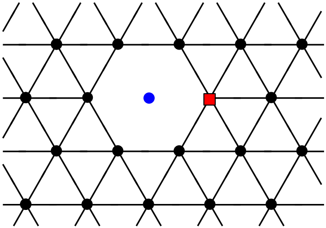


King Lattice

Decompose into rotationally congruent sectors



Kreweras lattice



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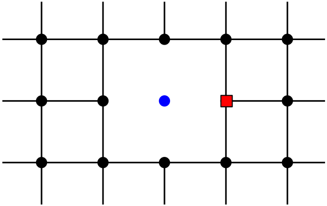

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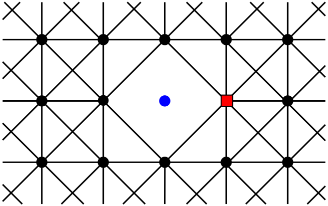

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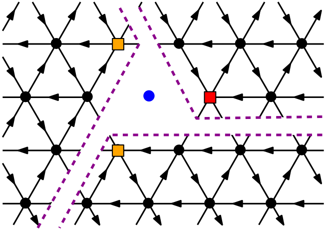

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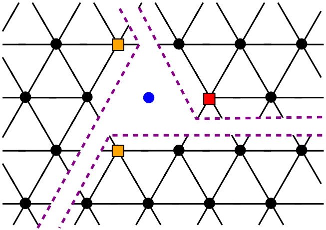

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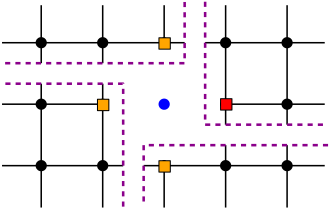

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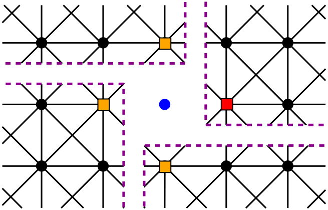

King Lattice

Recall: Kreweras almost-excursions

Define $T_k(u,q) = \sum_{n=0}^{\infty} (-1)^n (2n+1)^k q^{n(n+1)/2} (u^{n+1} - (-1)^k u^{-n})$

$$= (u \pm 1) - 3^k q(u^2 \pm u^{-1}) + 5^k q^3 (u^3 \pm u^{-2}) + O(q^6).$$

Let $q(t) \equiv q = t^3 + 15t^6 + 279t^9 + \cdots$ satisfy

$$t = q^{1/3} \frac{T_1(1,q^3)}{4T_0(q,q^3) + 6T_1(q,q^3)}.$$

The gf for cell-centred Kreweras-lattice almost-excursions is:

$$E(t,s) = \frac{s}{(1-s^3)t} \left(s - q^{-1/3} \frac{T_1(q^2,q^3)}{T_1(1,q^3)} - q^{-1/3} \frac{T_0(q,q^3) T_1(sq^{-2/3},q)}{T_1(1,q^3) T_0(sq^{-2/3},q)} \right).$$

The gf for vertex-centred Kreweras-lattice almost-excursions is:

$$\tilde{E}(t,s) = \frac{s(1-s)q^{-\frac{2}{3}}}{t(1-s^3)} \frac{T_0(q,q^3)^2}{T_1(1,q^3)^2} \left(\frac{T_1(q,q^3)^2}{T_0(q,q^3)^2} - \frac{T_2(q,q^3)}{T_0(q,q^3)} - \frac{T_2(s,q)}{2T_0(s,q)} + \frac{T_3(1,q)}{6T_1(1,q)} + \frac{T_3(1,q^3)}{3T_1(1,q^3)} \right).$$

Square lattice almost-excursions

Define $T_k(u,q) = \sum_{n=0}^{\infty}(-1)^n(2n+1)^k q^{n(n+1)/2}(u^{n+1}-(-1)^k u^{-n})$

$$= (u \pm 1) - 3^k q(u^2 \pm u^{-1}) + 5^k q^3(u^3 \pm u^{-2}) + O(q^6).$$

Let $q(t) \equiv q = t + 4t^3 + 34t^5 + 360t^7 + \cdots$ satisfy

$$t = \frac{qT_0(q^2,q^8)T_1(1,q^8)}{2T_0(q^4,q^8)(T_0(q^2,q^8)+2T_1(q^2,q^8))}.$$

The gf for cell-centred Square-lattice almost-excursions is:

$$\frac{s^2}{(1-s^4)t}\left(s - s^{-1} + \frac{T_0(q^4,q^8)}{qT_1(1,q^8)} - \frac{T_0(q^4,q^8)T_1(s^{-1}q,q^2)}{qT_1(1,q^8)T_0(s^{-1}q,q^2)}\right).$$

The gf for vertex-centred Square-lattice almost-excursions is:

$$\frac{sT_0(q^4,q^8)}{qt(1+s^2)T_1(1,q^8)}\left(1 + \frac{2T_1(q^2,q^8)}{T_0(q^2,q^8)} + \frac{(1-s)T_1(s^{-1},q^2)}{(1+s)T_0(s^{-1},q^2)}\right).$$

Part 6: Final comments

Jacobi theta function/ Weierstrass function parameterisation combinatorial functional equation solution method

JACOBI THETA FUNCTION/ WEIERSTRASS FUNCTION PARAMETERISATION COMBINATORIAL FUNCTIONAL EQUATION SOLUTION METHOD

This method...

- Sometimes works on equations with two catalytic variables
- Successful on
  - Various 2 dimensional lattice walk models [Bernardi, Bousquet-Mélou, E.P., Fayolle, Kurkova, Raschel, Trotignon]
  - Some planar map models [Bousquet Mélou, E.P., Kostov, Zinn-Justin].

Questions for the audience:

- Does anyone have a nice equation to try?
- Can anyone suggest a better name for the method?

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Thank you!

Thank you!

BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$

Write $K(x,y)=A(x)y^2+B(x)y+C(x)$, then

$$Y(x)=\frac{-B(x)\pm\sqrt{B(x)^2-4A(x)C(x)}}{2A(x)}$$

parameterizes $K(x,Y(x))=0$. Typically, $Y_+(x)$ is meromorphic on:

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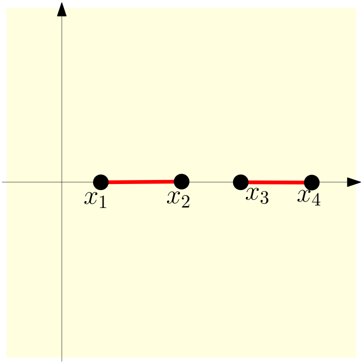

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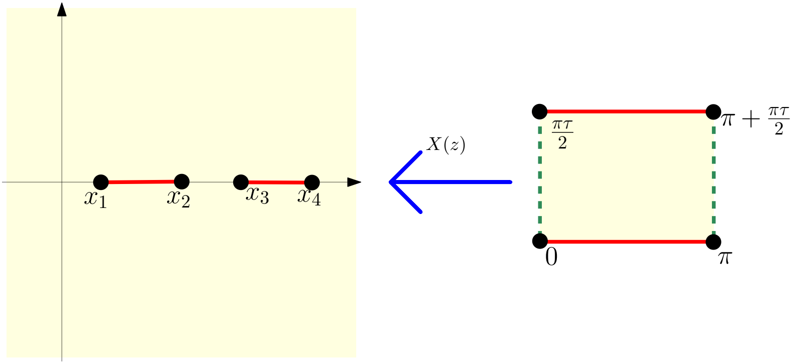

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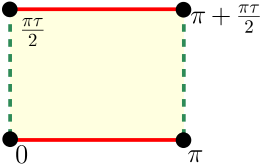

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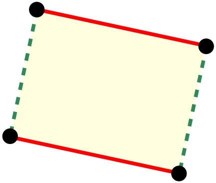

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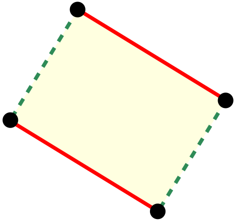

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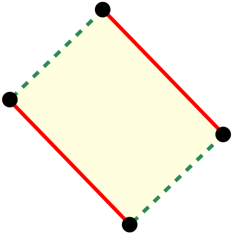

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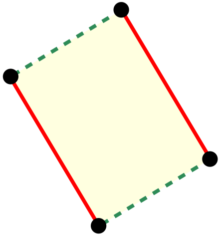

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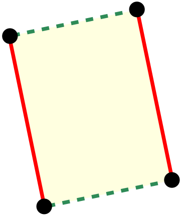

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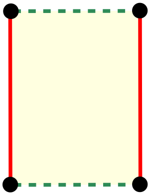

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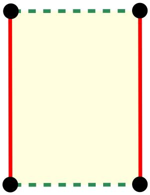

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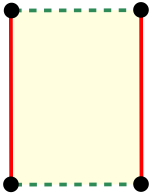

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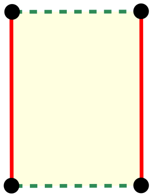

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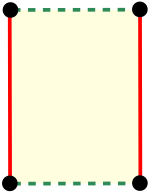

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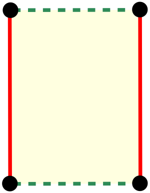

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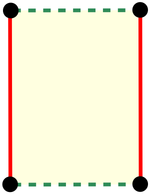

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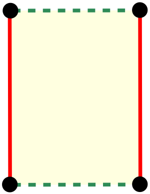

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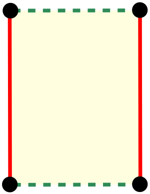

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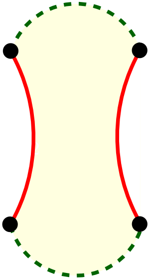

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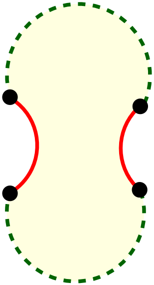

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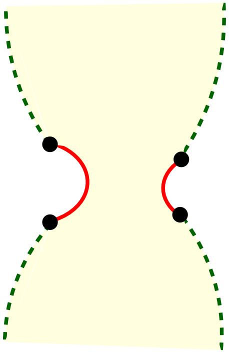

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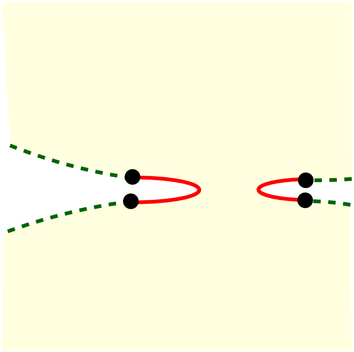

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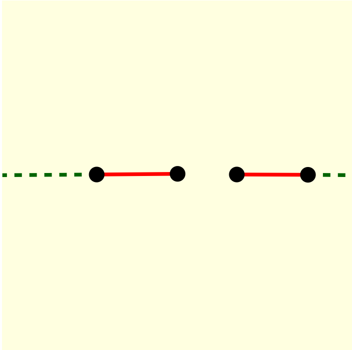

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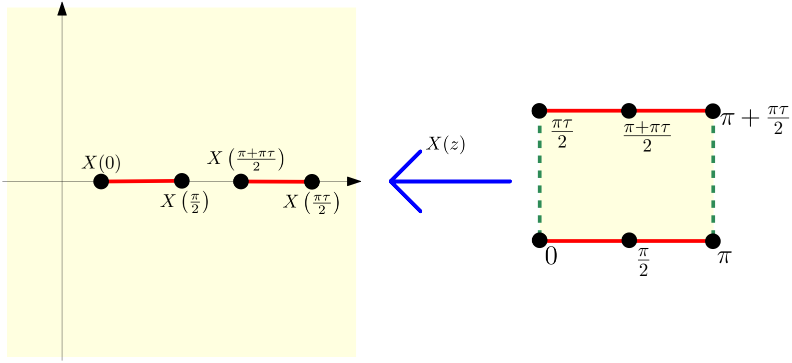

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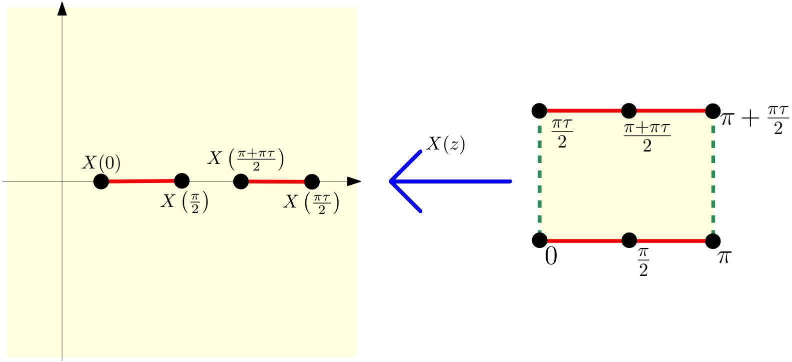
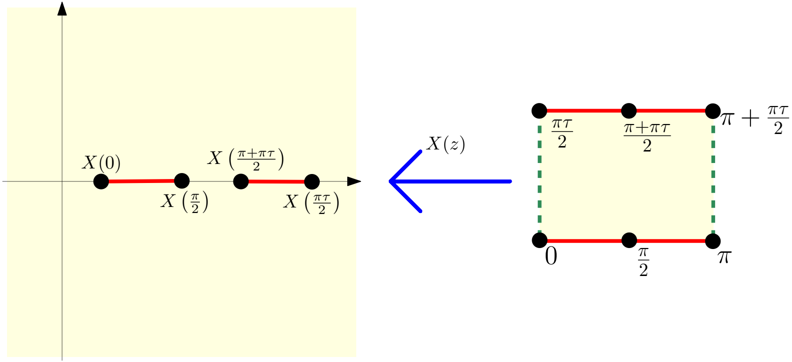


By symmetry, for $r \in \mathbb{R}$:

- $X(r) = X(\pi - r) = X(-r)$
- $X(\frac{\pi\tau}{2} + r) = X(\frac{\pi\tau}{2} - r)$

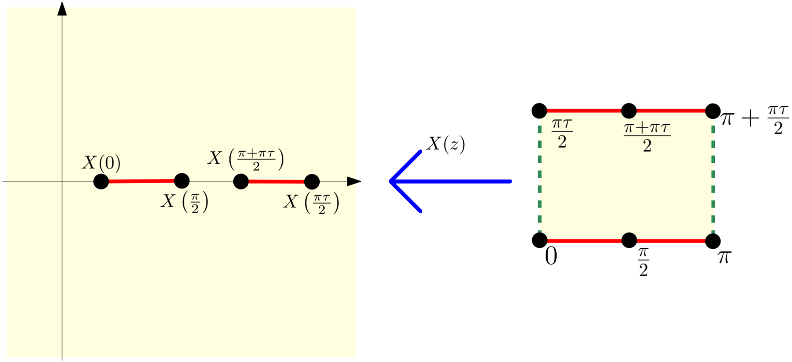
BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$



For $z \in \mathbb{C}$:

- $X(z) = X(\pi - z) = X(-z)$
- $X(z) = X(\pi\tau - z)$

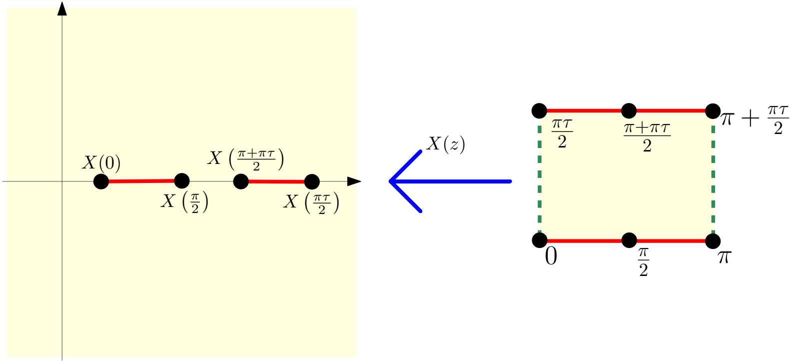
BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$



For $z \in \mathbb{C}$:

- $X(z) = X(\pi - z) = X(-z) = X(\pi\tau + z)$
- $X(z) = X(\pi\tau - z)$

BONUS SLIDE: PARAMETERIZATION OF $K(x, y) = 0$

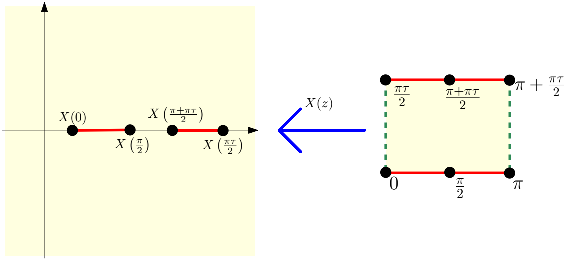


For $z \in \mathbb{C}$:

- $X(z) = X(\pi - z) = X(-z) = X(\pi\tau + z)$

$$X(z) = c\frac{\vartheta(z-\alpha)\vartheta(z+\alpha)}{\vartheta(z-\beta)\vartheta(z+\beta)}$$

BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$



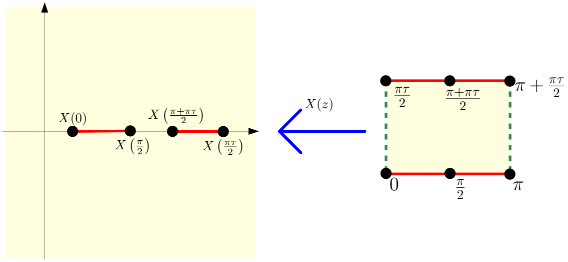
Recall:

$$y(x)=\frac{-B(x)\pm\sqrt{B(x)^2-4A(x)C(x)}}{2A(x)}.$$

Consider $Y(z)=y(X(z))$. By symmetry, for $r\in\mathbb{R}$:

- $X(r)=X(-r)$, so $Y(r)+Y(-r)=-\frac{B(X(r))}{A(X(r))}$.
- Similarly, $Y\left(\frac{\pi\tau}{2}+r\right)+Y\left(\frac{\pi\tau}{2}-r\right)=-\frac{B\left(X\left(\frac{\pi\tau}{2}+r\right)\right)}{A\left(X\left(\frac{\pi\tau}{2}+r\right)\right)}$.

Bonus slide: Parameterization of $K(x, y) = 0$



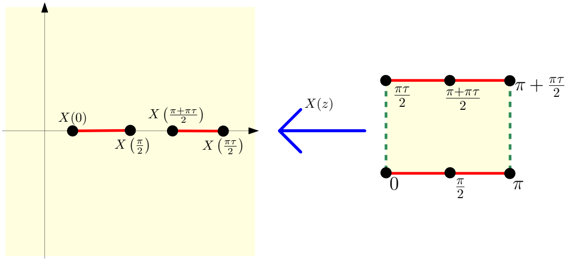
Recall:

$$y(x) = \frac{-B(x) \pm \sqrt{B(x)^2 - 4A(x)C(x)}}{2A(x)}.$$

Consider $Y(z) = y(X(z))$. For $z \in \mathbb{C}$:

- $Y(z) + Y(-z) = -\frac{B(X(z))}{A(X(z))}$.
- $Y(z) + Y(\pi\tau - z) = -\frac{B(X(z))}{A(X(z))}$.

BONUS SLIDE: PARAMETERIZATION OF $K(x, y) = 0$



For $z \in \mathbb{C}$:

- $Y(z) + Y(-z) = -\frac{B(X(z))}{A(X(z))}$.
- $Y\left(z\right) + Y\left(\pi\tau - z\right) = -\dfrac{B\left(X\left(z\right)\right)}{A\left(X\left(z\right)\right)}$.

So $Y(z) = Y(z + \pi\tau) = Y(z + \pi)$

$$\Rightarrow Y(z) = c\frac{\vartheta(z-\gamma)\vartheta(z-\delta)}{\vartheta(z-\epsilon)\vartheta(z-\gamma-\delta+\epsilon)}.$$

BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$

Equation characterising $Q(x,y) \equiv Q(t,x,y)$ for quadrant walks:

$$K(x,y)Q(x,y)+R(x,y)=0.$$

$K(x,y)=0$ is parameterised by

$$X(z)=c_1\frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z)=c_2\frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)},$$

where the constants satisfy $\alpha_j+\beta_j=\gamma_j+\delta_j$ for $j=1,2$.
So, $R(X(z),Y(z))=0$.

Bonus slide: Parameterization of $K(x,y)=0$

In general: $K(x,y)=0$ is parameterised by

$$X(z)=c_1\frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z)=c_2\frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)},$$

with $\alpha_j+\beta_j=\gamma_j+\delta_j$ for $j=1,2$.

BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$

For Kreweras paths:

$$Q(x,y) = 1+xytQ(x,y)+\frac{t}{x}\left(Q(x,y)-Q(0,y)\right)+\frac{t}{y}\left(Q(x,y)-Q(x,0)\right).$$

Then $K(x,y) = xy - tx^2y^2 - tx - ty = 0$ is parameterised by

$$X(z) = c_1\frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2\frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)},$$

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Bonus slide: Parameterization of $K(x,y)=0$

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with $\alpha_j + \beta_j = \gamma_j + \delta_j$ for $j = 1, 2$.

- $K(0,0) = 0$, so WLOG $\alpha_1 = \alpha_2 = 0$.

BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$

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$$X(z) = c_1 \frac{\vartheta(z)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)},$$

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with $\alpha_j+\beta_j=\gamma_j+\delta_j$ for $j=1,2$.

- $K(0,0)=0$, so WLOG $\alpha_1=\alpha_2=0$.
- as $x\to 0$, we have $y(x)\sim -x$ or $y(x)\sim -\frac{1}{x^2}$, so $Y(z)$ has a double pole at $z=\beta_1$.

Bonus slide: Parameterization of $K(x, y) = 0$

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$$X(z) = c_1 \frac{\vartheta(z)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z)\vartheta(z-\beta_2)}{\vartheta(z-\beta_1)^2},$$

with $\alpha_j + \beta_j = \gamma_j + \delta_j$ for $j = 1, 2$.

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Then $K(x,y) = xy - tx^2y^2 - tx - ty = 0$ is parameterised by

$$X(z) = c_1 \frac{\vartheta(z)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z)\vartheta(z-2\beta_1)}{\vartheta(z-\beta_1)^2},$$

with $\alpha_j + \beta_j = \gamma_j + \delta_j$ for $j = 1, 2$.

- $K(0,0) = 0$, so WLOG $\alpha_1 = \alpha_2 = 0$.
- as $x \to 0$, we have $y(x) \sim -x$ or $y(x) \sim -\frac{1}{x^2}$, so $Y(z)$ has a double pole at $z = \beta_1$.

Bonus slide: Parameterization of $K(x,y)=0$

For Kreweras paths:

$$Q(x,y) = 1+xytQ(x,y)+\frac{t}{x}\left(Q(x,y)-Q(0,y)\right)+\frac{t}{y}\left(Q(x,y)-Q(x,0)\right).$$

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Then $K(x,y) = xy - tx^2y^2 - tx - ty = 0$ is parameterised by

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{9}}\vartheta(z)\vartheta\left(z - \frac{\pi\tau}{3}\right)}{\vartheta\left(z + \frac{\pi\tau}{3}\right)\vartheta\left(z - \frac{2\pi\tau}{3}\right)} \quad \text{and} \quad Y(z) = \frac{e^{-\frac{4\pi\tau i}{9}}\vartheta(z)\vartheta\left(z + \frac{\pi\tau}{3}\right)}{\vartheta\left(z - \frac{\pi\tau}{3}\right)\left(z + \frac{2\pi\tau}{3}\right)},$$

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Bonus slide: Parameterization of $K(x, y) = 0$

Then $K(x, y) = xy - tx^2y^2 - tx - ty = 0$ is parameterised by

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z, 3\tau)\vartheta\left(z - \pi\tau, 3\tau\right)}{\vartheta\left(z + \pi\tau, 3\tau\right)\vartheta\left(z - 2\pi\tau, 3\tau\right)} \quad \text{and} \quad Y(z) = X(z + \pi\tau),$$

where

$$t = \frac{1}{X(z)Y(z) + X(z)^{-1} + Y(z)^{-1}}.$$

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$$t = e^{-\frac{\pi\tau i}{3}} \frac{\vartheta'(0, 3\tau)}{4i\vartheta(\pi\tau, 3\tau) + 6\vartheta'(\pi\tau, 3\tau)}.$$