Coloring graphs and other metric spaces

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Question (Hadwiger 1945, Nelson 1950, Gardner 1960)

What is the fewest number of colors needed to color the points of $\mathbb{R}^2$, such that every two points at unit distance have different colors?

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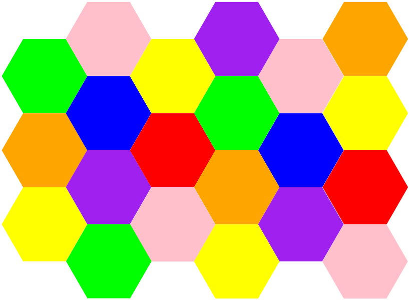

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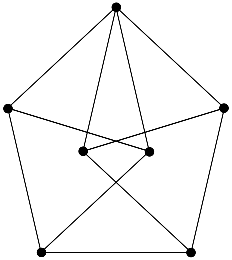

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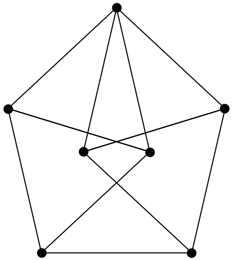
In ZFC, a theorem of de Bruijn and Erdős (1951) tells you that the chromatic number of a graph is the supremum of the chromatic numbers of its finite subgraphs.

Unit-distance graphs

A unit-distance graph is a graph whose vertices can be mapped to $\mathbb{R}^2$ such that any two vertices are adjacent if and only if their images are at distance 1.



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Deciding whether a graph is a unit-distance graph is complete for the Existential Theory of the Reals, and in particular NP-hard.

Unit-distance graphs

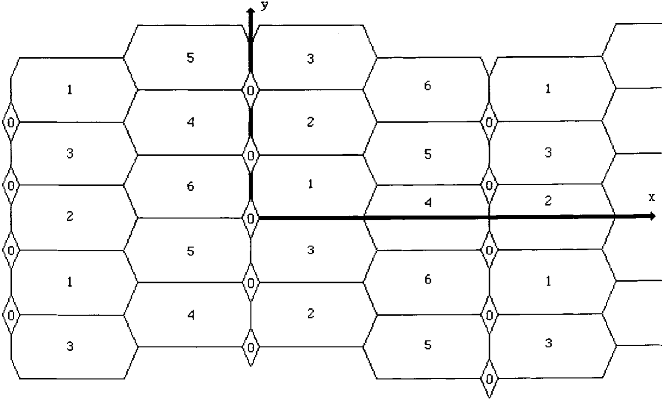

FIG. 3. A good 7-coloring of $(\mathbb{R}^2, 1)$.

THEOREM 3. *Every unit-distance graph on* 6197 *or fewer vertices is* 6-*colorable.*

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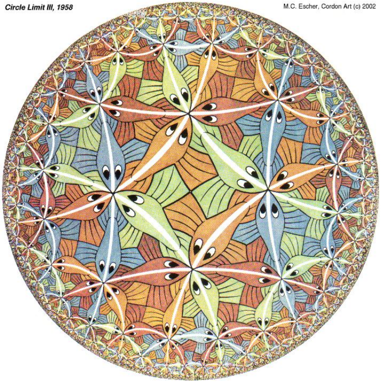

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Is $\chi(\mathbb{H}^2, d)$ bounded by a universal constant (independent of d)?

Infinite q-ary trees

Let T_q be the infinite q-ary tree.

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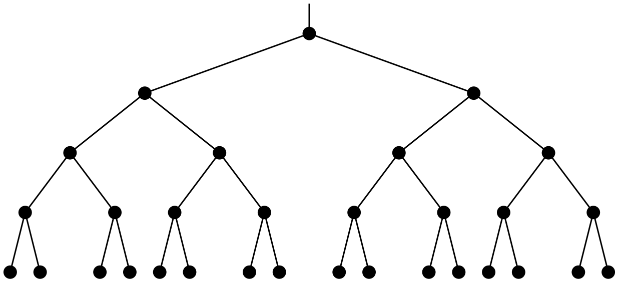

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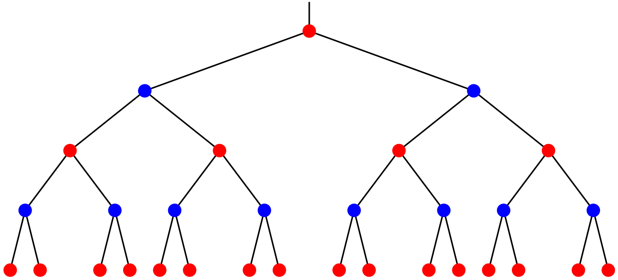

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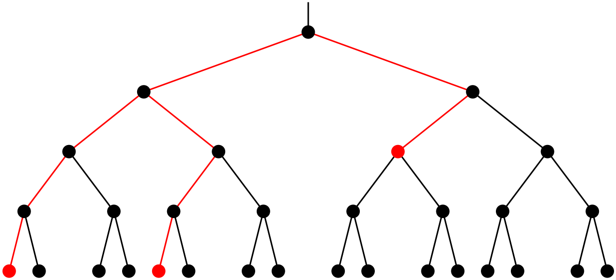

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Theorem (Bousquet, E., Harutyunyan, de Joannis de Verclos 2017)

If d is even, then

$$\left(\tfrac{1}{4} - o(1)\right)\frac{d\log(q-1)}{\log d} \leq \chi(T_q, d) \leq (2 + o(1))\frac{d\log(q-1)}{\log d}$$

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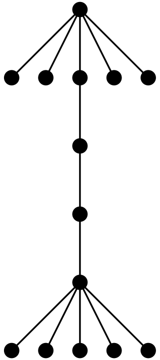

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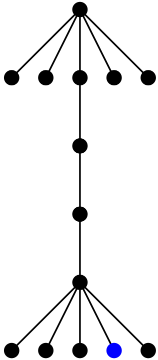

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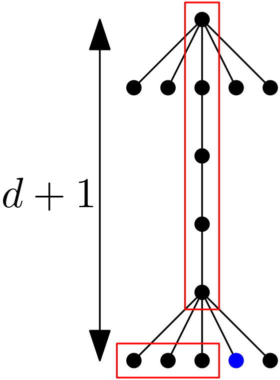

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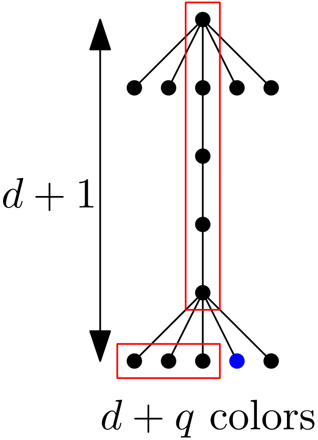

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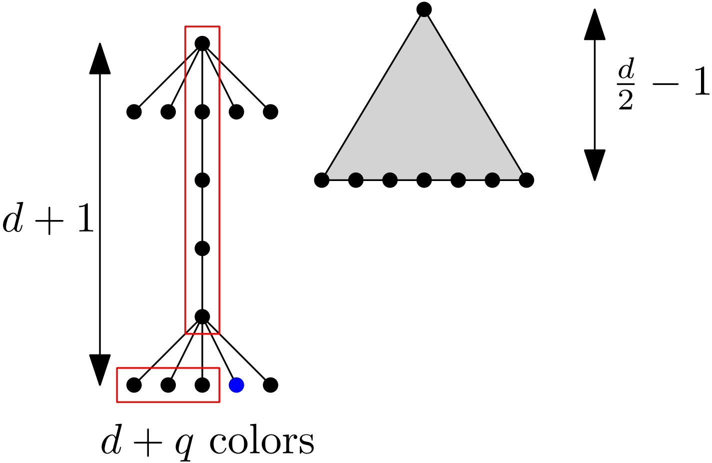

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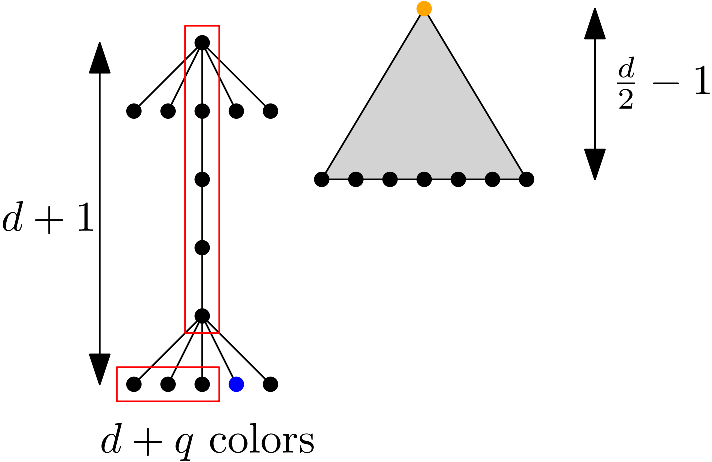

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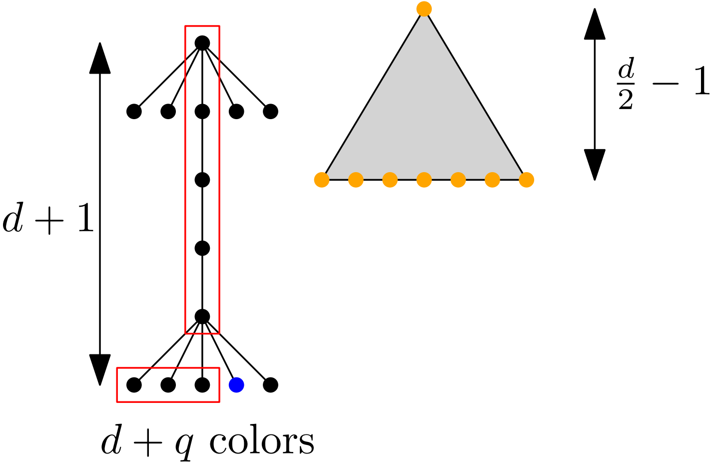

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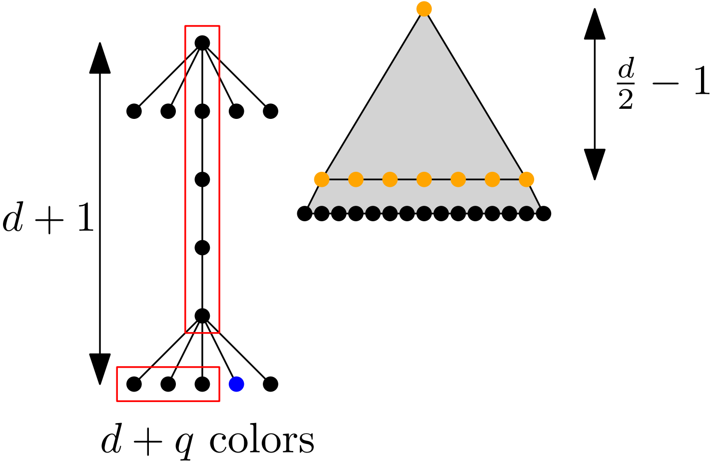

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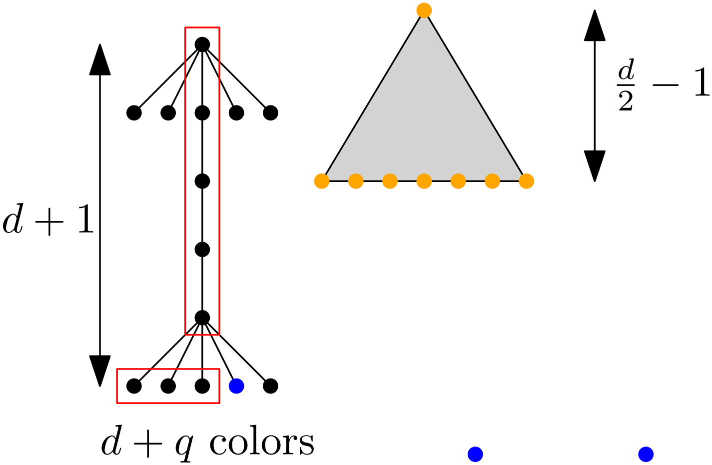

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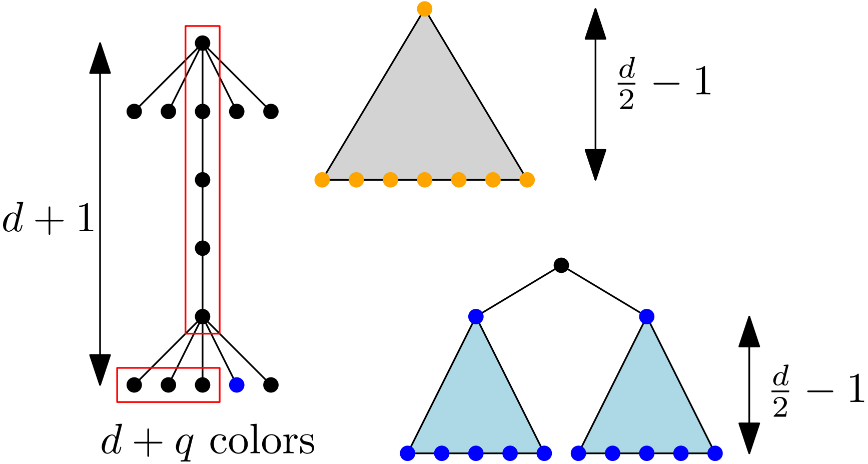

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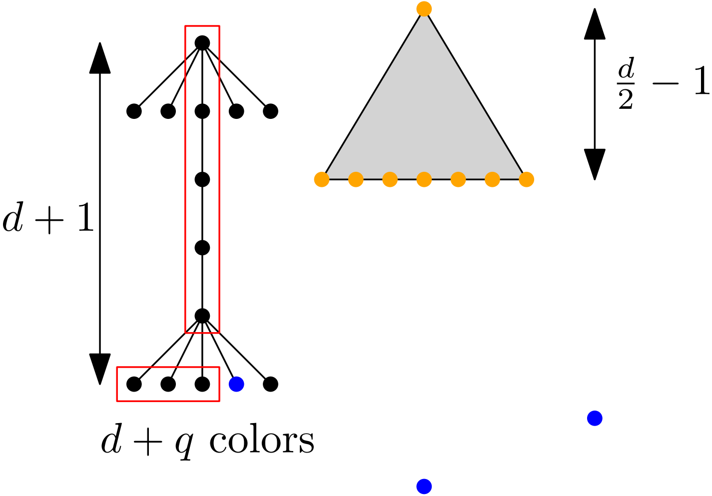

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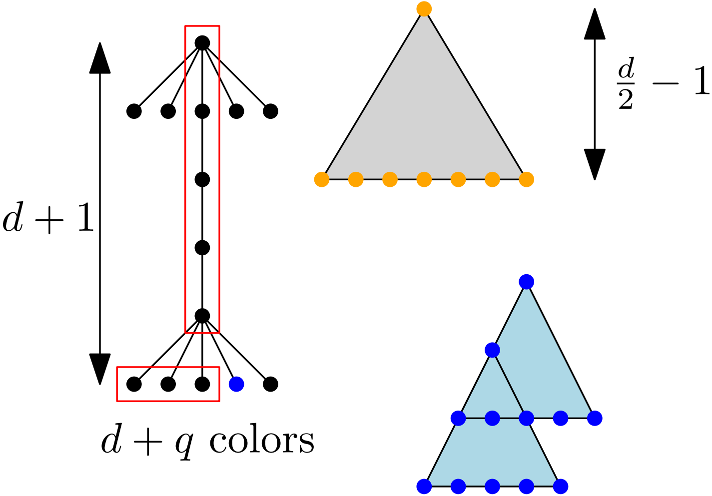

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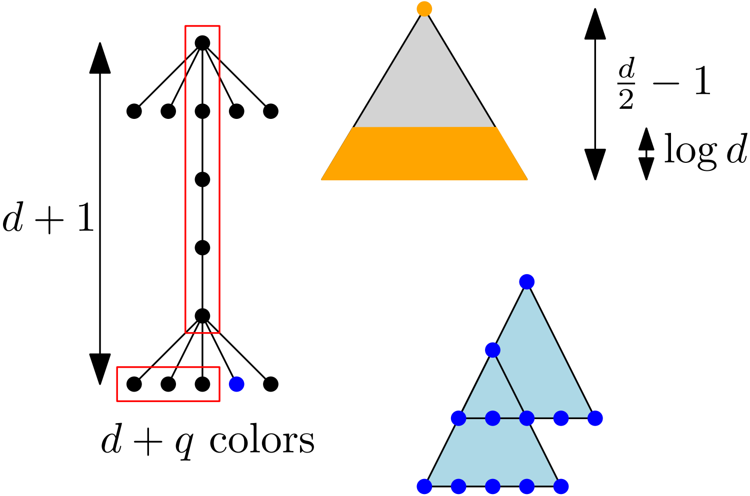

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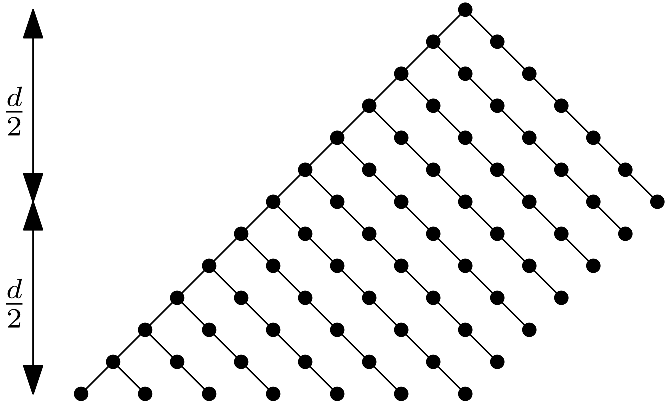

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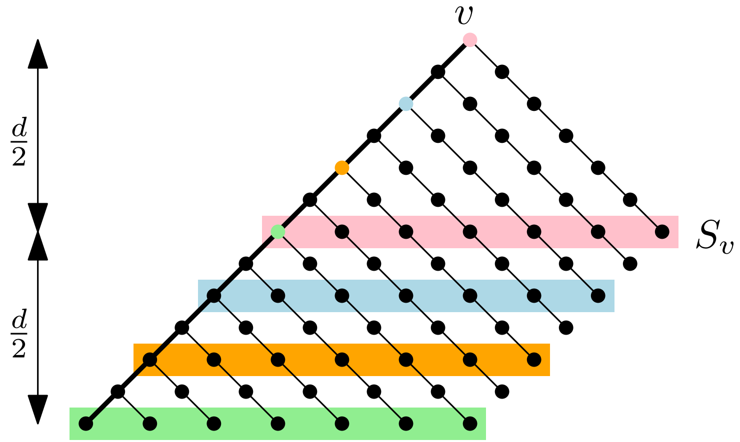

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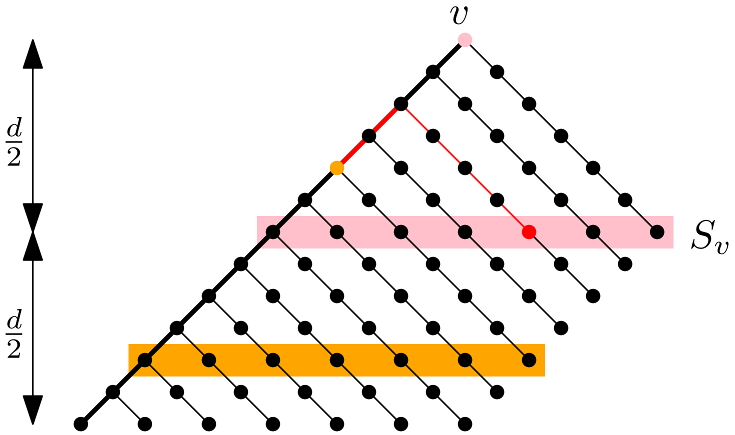

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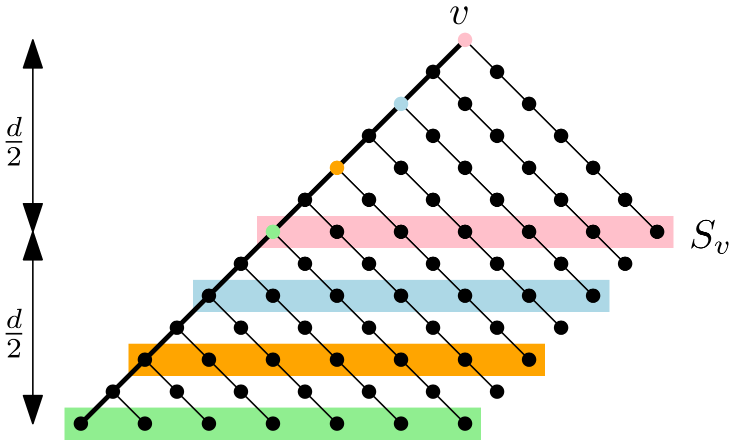

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Theorem (van den Heuvel, Kierstead, and Quiroz 2016)

Let G be a planar graph.

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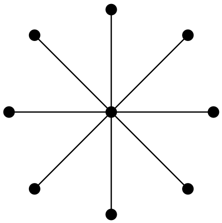

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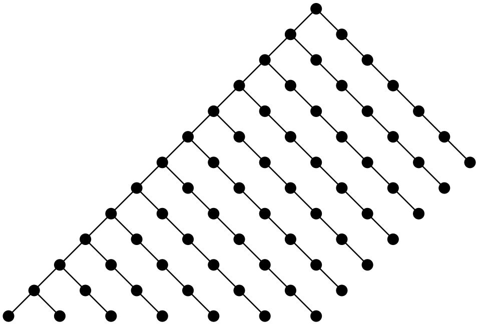

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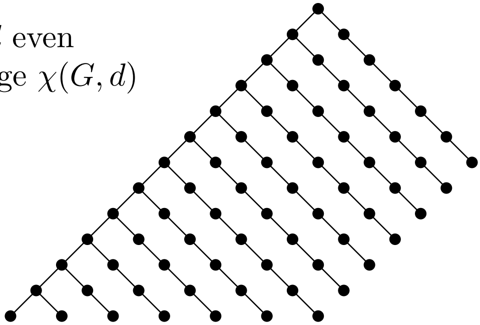

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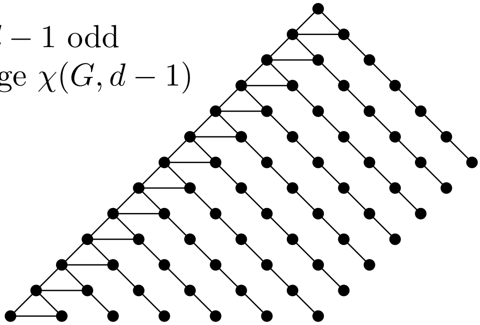

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- Is $\chi(\mathbb{H}^2, d)$ bounded by a constant that does not depend on d?
- It is known that $\chi(\mathbb{Q}^2) = 2$ (Woodall 1973). What about $\chi(\mathbb{Q} \times \mathbb{R})$? (Axenovich et al. 2012).