Multi-parameter hook formula for labelled trees

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joint work with Ian P. Goulden (Waterloo)
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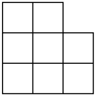
Outline of the talk

1. What is a hook formula?
2. Main result and specializations
3. A combinatorial proof of our hook formula: splicing trees

Frame-Robinson-Thrall formula (1954) for counting tableaux

Fix a Young diagram λ with n boxes.



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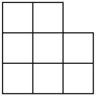

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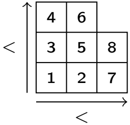

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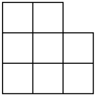

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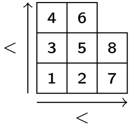


is given by

$$\frac{n!}{\prod_{\square \in \lambda} h_\square}.$$

$h_\square$: hook-length of the box $\square$, *i.e.* number of boxes at its right in the same row or above it in the same column.

In our example: the hook-lengths are

2	1	
4	3	1
5	4	2

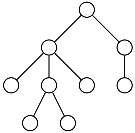
so there are

$8!/(5*4*4*3*2*2) = 42$ standard Young tableaux of shape λ.

Knuth formula for increasing trees (1973)

The same kind of formula holds for trees!

Fix a Tree T with n nodes.



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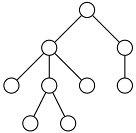

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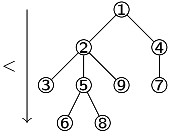

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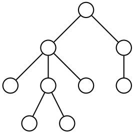

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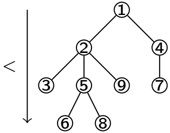

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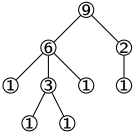

so there are

$9!/(9 * 6 * 3 * 2) = 1120$ increasing labellings of T.

Hook summation formulas

But these objects are in bijection with permutations.

- By Robinson-Schensted algorithm, pairs of standard Young tableaux of the same shape are in bijection with permutations, so

$$\sum_{\lambda \vdash n} \left(\frac{n!}{\prod_{\square \in \lambda} h_\square} \right)^2 = n!.$$

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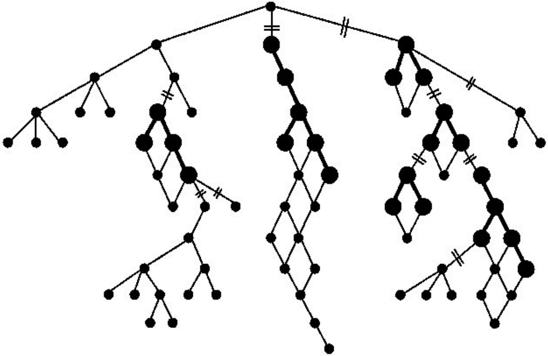
- By binary search tree algorithm, increasing labellings of binary trees are in bijection with permutations, so

$$\sum_{T \text{ binary tree}} \frac{n!}{\prod_{\circ \in V_T} h_\circ} = n!$$

These formulas are called hook summation formulas.

A large amount of work around these hook formulas

- formulas for other objects than trees or Young diagrams: in particular, *d*-complete posets that include both.



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- in summation formulas, one can replace $1/h_\square^2$ or $1/h_\circ$ by more involved expressions such that the sum is still simple.

Example (Postnikov formula)

$$\sum_{\substack{T \text{ binary} \\ \text{tree of size } n}} \prod_{v \in T} \left(x + \frac{1}{h_T(v)} \right) = \frac{1}{(n+1)!} \prod_{i=1}^{n-1} \big((n+1+i)x + n + 1 - i \big).$$

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- formulas for other objects than trees or Young diagrams: in particular, d-complete posets that include both.
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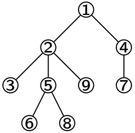
- interpretations in combinatorial Hopf algebra theory, in convex geometry, in commutative algebra.
- ...

Main result

A hook summation formula over labelled increasing tree with n nodes.

A labelled increasing tree T



Children of a given vertex are not ordered. By convention, we draw them in increasing order from left to right.

⚠ in our formula, we sum over labelled trees.

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Theorem (FGL, 2013)

Let $(x_i)_{1\leq i\leq n}$ and $(y_{i,j})_{1\leq i\leq j\leq n}$ be formal parameters.

$$\sum_T \left[\prod_{i=2}^{n} x_{f_i(T)} \left(\sum_{j \in \mathfrak{h}_i(T)} y_{i,j} \right) \right] = x_1 y_{n,n} \prod_{i=2}^{n-1} \left(y_{i,i} \sum_{j=1}^{i} x_j + x_i \sum_{j=i+1}^{n} y_{i,j} \right).$$

$f_i(T)$: parent of i in T;
$\mathfrak{h}_i(T)$: vertex set of the sub-tree of T rooted in i.

Example :

$$\text{weight}\left(\begin{array}{c} ① \\ | \\ ② \\ | \\ ③ \end{array} \right) = x_1(y_{2,2}+y_{2,3})x_2 y_{3,3}$$

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A specialization ($y_{i,j} = x_j + \delta_{i,j} - 1$) appeared in representation theory of symmetric groups.

An interesting specialization

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$$\mathrm{LHS} = \sum_{\substack{T \text{ labelled} \\ \text{tree}}} \mathrm{weight}(T) = \sum_{\substack{U \text{ unlabelled} \\ \text{tree}}} \#\{\text{labellings}\}\, \mathrm{weight}(U)$$

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&= \sum_{\substack{U \text{ unlabelled} \\ \text{tree}}} \frac{n!}{\prod_v |\mathfrak{h}_v(T)|} \prod_{v \text{ non-root}} \big(y + z \cdot |\mathfrak{h}_v(T)|\big) \\
&= (n-1)! \sum_{\substack{U \text{ unlabelled} \\ \text{tree}}} \prod_{v \text{ non-root}} \left(\frac{y}{|\mathfrak{h}_v(T)|} + z\right).
\end{aligned}$$

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Summing over labelled tree is natural to get a multi-parameter generalization!

Question

Is there a formula similar to our main result with a sum over labelled binary tree?

Another interesting specialization: recovering Cayley formula

Set $y_{i,j} = x_j$ for every $i \leq j$. Then

$$\mathrm{RHS} = x_1 \dots x_n \left(\sum_{j=1}^{n} x_j \right)^{n-2}.$$

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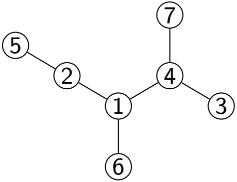
We would like to show that

$$\mathrm{LHS} = \sum_T \left[\prod_{i=2}^{n} x_{f_i(T)} \left(\sum_{j \in \mathfrak{h}_i(T)} x_j \right) \right] = \sum_{T \text{ Cayley tree}} x_1^{\deg_1(T)} \dots x_n^{\deg_n(T)}.$$

Reminder:
A Cayley tree
(no root, no plane embedding)



Another interesting specialization: recovering Cayley formula

Let X a subset of $[n]$. We define:

$$\mathrm{LHS}(X) = \sum_{\substack{T \text{ incresing tree} \\ \text{with label set } X}} \left[\prod_{i \in X \setminus \{\min(X)\}} x_{f_i(T)} \left(\sum_{j \in \mathfrak{h}_i(T)} x_j \right) \right]$$

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Proof that $\mathrm{LHS}(X) = \mathrm{Cay}(X)$

Both satisfy the same induction (and coincide for $|X| = 2$)

$$F(X) = \sum_d x_{\min(X)}^d \sum_{X_1 \sqcup \dots \sqcup X_d = X \setminus \{\min(X)\}} \left(\prod_{i=1}^r F(X_i) \sum_{v \in X_i} x_v \right).$$

Towards a combinatorial formulation

$$\sum_T \left[\prod_{i=2}^{n} x_{f_i(T)} \left(\sum_{j \in \mathfrak{h}_i(T)} y_{i,j} \right) \right] = x_1 y_{n,n} \prod_{i=2}^{n-1} \left(y_{i,i} \sum_{j=1}^{i} x_j + x_i \sum_{j=i+1}^{n} y_{i,j} \right).$$

Reminder: this is our main result.

We would like a combinatorial formulation.

Towards a combinatorial formulation

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  This implies also $2 \leq_T 5$. Because we are using trees!

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In the example, $\pi = \{\{2, 5, 7\}, \{3, 4\}, \{6\}\}$

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- The contribution of a tree T is $\prod_i x_i^{\deg_T(i)}$.

Towards a combinatorial formulation

$$\sum_T \left[\prod_{i=2}^n x_{f_i(T)} \left(\sum_{j \in \mathfrak{h}_i(T)} y_{i,j} \right) \right] = x_1 y_{n,n} \prod_{i=2}^{n-1} \left(y_{i,i} \sum_{j=1}^{i} x_j + x_i \sum_{j=i+1}^{n} y_{i,j} \right).$$

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$$[M_y]\,\mathrm{LHS} = \sum_T \prod_i x_i^{\deg_T(i)},$$

where the sum runs over π-compatible trees.

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$$[M_y]\,\mathrm{RHS} = x_1\, x_2\, x_3 \left(\sum_{j=1}^{4} x_j \right) x_5 \left(\sum_{j=1}^{6} x_j \right).$$

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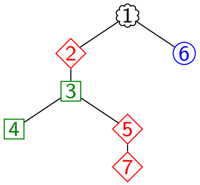
In general,

$$[M_y]\,\mathrm{RHS} = x_1 \cdot \left[\prod_{\substack{i \text{ not max} \\ \text{in its part}}} x_i \right] \cdot \left[\prod_{\substack{i \text{ max} \\ \text{in its part} \\ i \neq n}} \left(\sum_{j=1}^{i} x_j \right) \right].$$

Combinatorial reformulation of the main theorem

Fix a set-partition of $\{2, \dots, n\}$ (in the example $\pi = \{\{2,5,7\},\{3,4\},\{6\}\}$). One has to find a bijection between



increasing trees T such that, for any two elements in the same part, one is the ancestor of the other.

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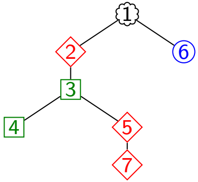

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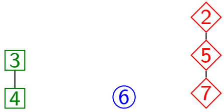


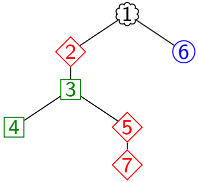
$a_{\square} = 1 \quad a_{\circ} = 3 \quad \emptyset$

a number for each part (except the one containing n) less or equal than the maximum of the part (called *anchor point*)

$$a_{\square} \leq 4, \;\; a_{\circ} \leq 6.$$

Combinatorial reformulation of the main theorem

Fix a set-partition of $\{2, \dots, n\}$ (in the example $\pi = \{\{2,5,7\},\{3,4\},\{6\}\}$). One has to find a bijection between



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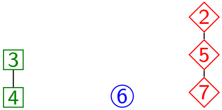

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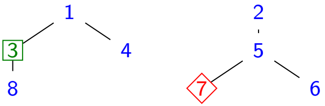
which respects the degree:

$$\deg_{\text{left}}(i) = \deg_{\text{right}}(i) + |a^{-1}(i)| + \delta_{i,1}.$$

Elementary splicing

Let T_1 and T_2 with marked vertices v_1 and v_2. Assume $v_1 < v_2$.



In the example $v_1 = 3$, $v_2 = 7$.

Elementary splicing

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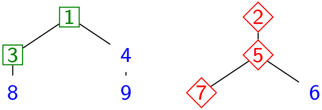

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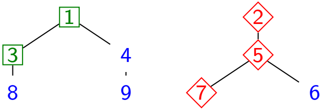

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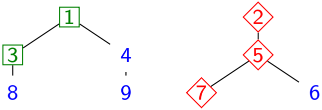

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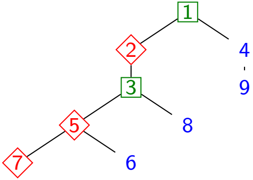
We add other vertices with the same parent than in the original trees:



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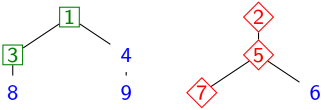

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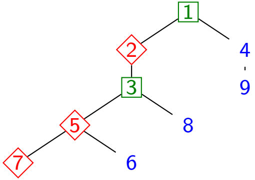
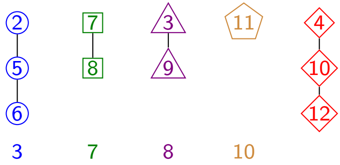


Obs. only the degree of v_1 has increase by 1, other degrees are unchanged.

The bijection on an example



Start with the set of chains above with anchor points.

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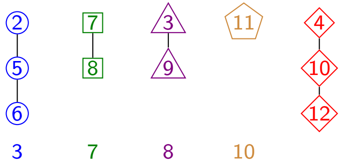

Start with the set of chains above with anchor points.
Step 0: we add a root labeled 1 with a free edge to the list.

The free edge symbolizes that we must increase the degree of the corresponding vertex of 1 during the construction.

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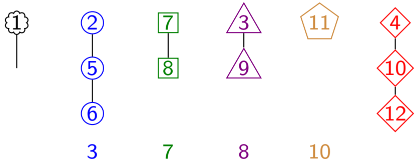

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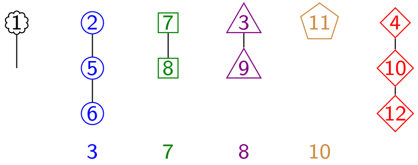

We will splice successively the chains together (always with v_1 a vertex with a free edge, v_2 the max of its tree).
First step: we add a free edge to 3 and splice $2, 5, 6$ with $3, 9$ (*external splice*).

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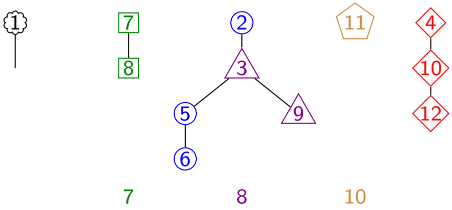

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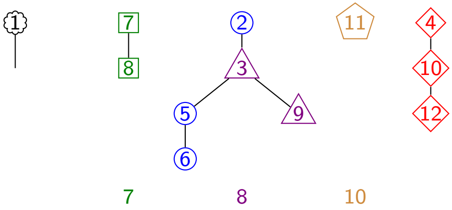

Second step: 7 is in the component we must splice. Thus, we splice 7, 8 on the free edge and add a free edge to 7 (*internal splice*).

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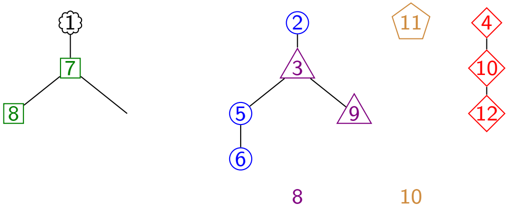

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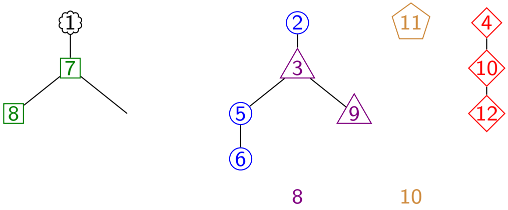

Third step: 8 is in the root component ⇒ again an internal splice. We splice the tree $2, 3, 5, 6, 9$ onto the free edge and add a free edge to 8.

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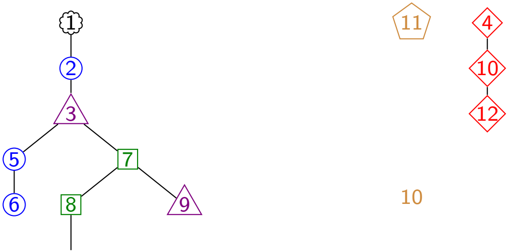

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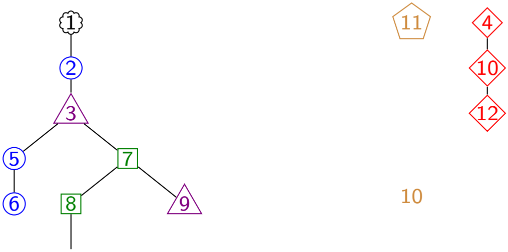

Fourth step: an external splice. We add a free edge to 10 and splice 11 onto it.

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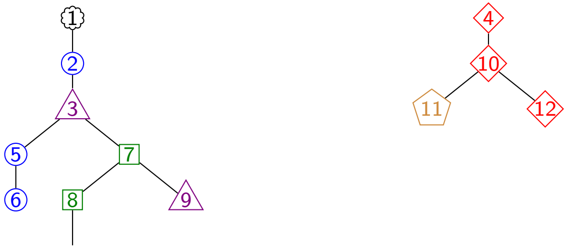

Last step: we splice the tree containing the maximum onto the free edge.

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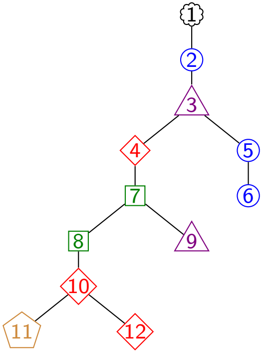

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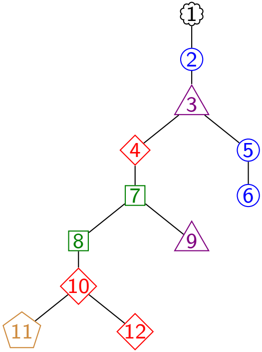

Here is the resulting partitioned tree.
The degree condition is fulfilled by construction.

Summary and conclusion

Construction by successive splicings:

- if the anchor point is in the component we want to splice or in the root component, we splice onto the free edge and add an edge to the anchor point (internal splicing).
- if the anchor point is in another component, we add a free edge to the anchor point and splice the tree on this free edge (external splicing).

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The described procedure defines a bijection.

Corollary (FGL, 2013)

$$\sum_T \left[\prod_{i=1}^{n} x_{f_i(T)} \left(\sum_{j \in \mathfrak{h}_i(T)} y_{i,j} \right) \right] = x_1 y_{n,n} \prod_{i=2}^{n-1} \left(y_{i,i} \sum_{j=1}^{i} x_j + x_i \sum_{j=i+1}^{n} y_{i,j} \right).$$