Finite topologies and T-partitions

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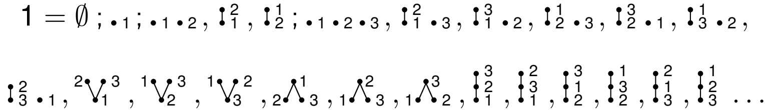
Introduction

- In his thesis, Stanley introduced a quasi-symmetric function attached to any special poset, the generating series of P-partitions.
- These quasi-symmetric functions can be decomposed according to linear extensions of the considered special poset.
- Hopf algebraic version of these results, and generalizations?

Special poset

A special poset (or (P, ω)-poset) is a partial order $\leq_P$ defined on a set $[n] = \{1, \ldots, n\}$, for $n \geq 0$.

We represent special posets by their Hasse graph.



Number of special posets of degree n: A001035 in OEIS.

n	1	2	3	4	5	6	7	8
	1	3	19	219	4 231	130 023	6 129 859	431 723 379

P-partitions

Let P be a special poset of degree n. A P-partition of P is a sequence of nonnegative integers $f = (f(1), \ldots, f(n))$ such that:

1. If $i \leq_P j$, then $f(i) \leq f(j)$.
2. If $i \leq_P j$ and $i > j$, then $f(i) < f(j)$.

The set of P-partitions of P is denoted by $Part(P)$.

- If $P = \begin{smallmatrix}3\\2\\1\end{smallmatrix}$, a P-partition of P is an increasing sequence $(f(1), f(2), f(3))$, that is to say a partition of length 3.
- If $P = \begin{smallmatrix}1\\2\\3\end{smallmatrix}$, a P-partition of P is an strictly decreasing sequence $(f(1), f(2), f(3))$, that is to say a strict partition of length 3.

- If $P =$ , a P-partition of P is a sequence $(f(1), f(2), f(3))$ such that $f(2) \leq f(3) < f(1)$. If $a < b < c$:

$$(baa), \qquad (cab).$$

- If $P =$ , a P-partition of P is a sequence $(f(1), f(2), f(3))$ such that $f(3) < f(1), f(2)$. If $a < b < c$:

$$(bba), \qquad (bca), \qquad (cba).$$

- If $P =$ , a P-partition of P is a sequence $(f(1), f(2), f(3))$ such that $f(1) \leq f(2), f(3)$. If $a < b < c$:

$$(aaa), \quad (aab), \quad (aba), \quad (abb), \quad (abc), \quad (acb).$$

Quasi-symmetric functions

Let $F \in K[[X_1, X_2, \ldots]]$. The series f is quasi-symmetric if for any increasing map $f : \mathbb{N}_{>0} \longrightarrow \mathbb{N}_{>0}$, the coefficients in F of $X_1^{a_1} \ldots X_n^{a_n}$ and of $X_{f(1)}^{a_1} \ldots X_{f(n)}^{a_n}$ are the same.

The algebra of quasi-symmetric functions is denoted by **QSym**. Basis of **QSym**: if $a_1, \ldots, a_n$ are nonnegative integers,

$$M_{(a_1,\ldots,a_n)} = \sum_{i_1 < \ldots < i_n} X_{i_1}^{a_1} \ldots X_{i_n}^{a_n}.$$

Product: quasi-shuffle.

$$\begin{aligned} M_{(a)} M_{(b)} &= M_{(a,b)} + M_{(b,a)} + M_{(a+b)}, \\ M_{(a,b)} M_{(c)} &= M_{(a,b,c)} + M_{(a,c,b)} + M_{(c,a,b)} + M_{(a,b+c)} + M_{(a+c,b)}. \end{aligned}$$

Generating series of P-partitions

Let P a special poset of degree n.

$$\gamma(P) = \sum_{f\ P\text{-partition of } P} X_{f(1)} \ldots X_{f(n)}.$$

It is a quasi-symmetric function.

$$\begin{aligned}
\gamma\left(\begin{smallmatrix}3\\2\\1\end{smallmatrix}\right) &= \sum_{a} X_a^3 + \sum_{a<b} X_a X_b^2 + \sum_{a<b} X_a^2 X_b + \sum_{a<b<c} X_a X_b X_c \\
&= M_{(3)} + M_{(1,2)} + M_{(2,1)} + M_{(1,1,1)} \\
\gamma\left(\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right) &= \sum_{a<b<c} X_a X_b X_c \\
&= M_{(1,1,1)}
\end{aligned}$$

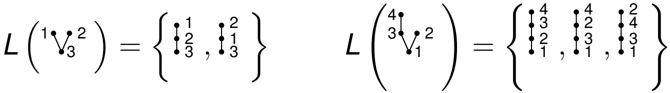
$$\begin{aligned}
\gamma\left(\begin{smallmatrix}1\\3\\2\end{smallmatrix}\right) &= \sum_{a<b} X_b X_a^2 + \sum_{a<b<c} X_c X_a X_b \\
&= M_{(2,1)} + M_{(1,1,1)} \\
\gamma\left({}^{1}\mathtt{V}_{3}^{2}\right) &= \sum_{a<b} X_b^2 X_a + \sum_{a<b<c} X_b X_c X_a + \sum_{a<b<c} X_c X_b X_a \\
&= M_{(1,2)} + 2M_{(1,1,1)} \\
\gamma\left({}^{2}\mathtt{V}_{1}^{3}\right) &= \sum_{a} X_a^3 + \sum_{a<b} X_a^2 X_b + \sum_{a<b} X_a X_b X_a + \sum_{a<b} X_a X_b^2 \\
&\quad + \sum_{a<b<c} X_a X_b X_c + \sum_{a<b<c} X_a X_c X_b \\
&= M_{(3)} + 2M_{(2,1)} + M_{(1,2)} + 2M_{(1,1,1)}.
\end{aligned}$$

Linear extension of a special poset

Let P be a special poset of degree n. A linear extension of P is a total order $\leq_{tot}$ on $[n]$ such that:

$$(i \leq_P j) \Longrightarrow (i \leq_{tot} j).$$

The set of linear extensions of P is denoted by $L(P)$.



Stanley's theorem

Let P be a special poset.

1. $$Part(P) = \bigsqcup_{Q \in L(P)} Part(Q).$$
2. $$\gamma(P) = \sum_{Q \in L(P)} \gamma(Q).$$

Let $A = (A, m, 1_A)$ be an algebra.

What is the structure on the dual of A?

- We want to dualize the product. We linearize it and see it as a linear map m from $A \otimes A$ to A instead of a bilinear map from $A \times A$ to A.
- We want to dualize the unit. We linearize it and see it as the linear map η from K to A sending 1_K to 1_A.

Axioms of algebras: two commutative diagrams of linear maps

- Associativity:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m \otimes Id"] \arrow[d, "Id \otimes m"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"'] & A \end{tikzcd}$$

- Unit:

$$\begin{tikzcd} K \otimes A \arrow[r, "\eta \otimes Id"] \arrow[dr, "\approx"'] & A \otimes A \arrow[d, "m"] & A \otimes K \arrow[l, "Id \otimes \eta"'] \arrow[dl, "\approx"] \\ & A & \end{tikzcd}$$

Axioms of coalgebras

$C = (C, \Delta, \varepsilon)$, with $\Delta : C \longrightarrow C \otimes C$ and $\varepsilon : C \otimes K$, with:

- Coassociativity:

$$\begin{tikzcd}
C \otimes C \otimes C & & C \otimes C \arrow[ll, "Id\otimes\Delta"'] \\
C \otimes C \arrow[u, "\Delta\otimes Id"] & & C \arrow[ll, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

- Counit:

$$\begin{tikzcd}
K \otimes C \arrow[dr, "\approx"'] & C \arrow[l, "\varepsilon\otimes Id"'] \arrow[r, "Id\otimes\varepsilon"] & C \otimes K \arrow[dl, "\approx"] \\
& C \arrow[u, "\Delta"] &
\end{tikzcd}$$

Axioms of bialgebras

$B=(B,m,1_B,\Delta,\varepsilon)$, such that:

- $(B,m,1_B)$ is an algebra.
- (B,Δ,ε) is a coalgebra.
- Compatibilities:

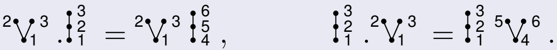
$$\begin{array}{rcl}\Delta(ab)&=&\Delta(a)\Delta(b),\\ \Delta(1_B)&=&1_B\otimes 1_B;\\ \varepsilon(ab)&=&\varepsilon(a)\varepsilon(b),\\ \varepsilon(1_B)&=&1.\end{array}$$

Examples

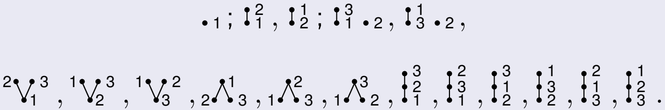
- Let G be a group. The group algebra KG is a bialgebra, with:
$$\Delta(x) = x \otimes x \text{ for any } x \in G.$$
- Let $\mathfrak{g}$ be a Lie algebra. The enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a bialgebra, with:
$$\Delta(x) = x \otimes 1 + 1 \otimes x \text{ for any } x \in \mathfrak{g}.$$
- Let G be an algebraic group. Let $K[G]$ be the algebra of polynomial functions over G. It is a bialgebra, with:
$$\begin{array}{rcl} \Delta(f)(x \otimes y) & = & f(xy), \\ \varepsilon(f) & = & f(1_G). \end{array}$$

Hopf algebra of special posets $\mathcal{H}_{SP}$ (Malvenuto-Reutenauer)

- Basis: the set of special posets.
- Product: shifted concatenation.



- $\mathcal{H}_{SP}$ is freely generated by the set of indecomposable special posets:



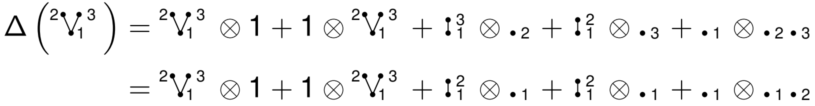
Hopf algebra of special posets $\mathcal{H}_{SP}$ (Malvenuto-Reutenauer)

- Let P be a special poset of degree n. An ideal I of P is a part $I \subseteq [n]$ such that:

$$(x \in I, y \in P, x \leq_P y) \Longrightarrow (y \in I).$$

- Coproduct:

$$\Delta(P) = \sum_{I \text{ ideal of } P} Std(P \setminus I) \otimes Std(I).$$



Hopf algebra **QSym**

$$\Delta(M_{(a_1,\ldots,a_n)}) = \sum_{i=0}^{n} M_{(a_1,\ldots,a_i)} \otimes M_{(a_{i+1},\ldots,a_n)}.$$

Theorem

The map $\gamma : \mathcal{H}_{SP} \longrightarrow$ **QSym** is a surjective Hopf algebra morphism.

Special poset associated to a permutation

Let $\sigma \in \mathfrak{S}_n$. We define a total order $\leq_{tot}$ on $[n]$ by:

$$(i \leq_{tot} j) \Longleftrightarrow (\sigma(i) \leq \sigma(j)).$$

We obtain in this way all special posets such that $\leq_P$ is total.

$$(123) \longleftrightarrow \begin{smallmatrix}3\\2\\1\end{smallmatrix} \qquad (132) \longleftrightarrow \begin{smallmatrix}2\\3\\1\end{smallmatrix}$$

$$(213) \longleftrightarrow \begin{smallmatrix}3\\1\\2\end{smallmatrix} \qquad (231) \longleftrightarrow \begin{smallmatrix}2\\1\\3\end{smallmatrix}$$

$$(312) \longleftrightarrow \begin{smallmatrix}1\\3\\2\end{smallmatrix} \qquad (321) \longleftrightarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix}$$

Hence, linear extensions of any special posets can be seen as permutations.

Hopf algebra of permutations **FQSym** (Malvenuto-Reutenauer)

- Basis: the set of all permutations.
- Product: if $\sigma \in \mathfrak{S}_k$ and $\tau \in \mathfrak{S}_l$,

$$\sigma.\tau = \sum_{\alpha \in Sh(k,l)} \alpha \circ (\sigma \otimes \tau).$$

$$\begin{aligned}(132).(21) &= (13254) + (14253) + (15243) + (14352)\\ &+ (15342) + (15432) + (24351) + (25341)\\ &+ (25431) + (35421).\end{aligned}$$

Hopf algebra of permutations **FQSym** (Malvenuto-Reutenauer)

- Coproduct: if $\sigma \in \mathfrak{S}_n$,

$$\Delta(\sigma) = \sum_{k=0}^{n} \sigma_{|[k]} \otimes Std(\sigma_{|\mathbb{N}_{>0} \setminus [k]}).$$

$$\begin{aligned}\Delta((51423)) &= 1 \otimes (51423) + (1) \otimes (5423) + (12) \otimes (543) \\ &+ (123) \otimes (54) + (1423) \otimes (5) + (51423) \otimes 1 \\ \\ &= 1 \otimes (51423) + (1) \otimes (4312) + (12) \otimes (321) \\ &+ (123) \otimes (21) + (1423) \otimes (1) + (51423) \otimes 1.\end{aligned}$$

Theorem

- The following map is a surjective Hopf algebra morphism:

$$L : \left\{ \begin{array}{rcl} \mathcal{H}_{SP} & \longrightarrow & \mathbf{FQSym} \\ P & \longrightarrow & \displaystyle\sum_{\sigma \in L(P)} \sigma \end{array} \right.$$

- There exists a commutative diagram of Hopf algebra morphisms:

$$\begin{tikzcd} \mathcal{H}_{SP} \arrow[r, two heads, "L"] \arrow[dr, two heads, "\gamma"'] & \mathbf{FQSym} \arrow[d, two heads, "\phi"] \\ & \mathbf{QSym} \end{tikzcd}$$

Let P a special poset of degree n. We put:

$$\Gamma(P) = \sum_{f\ P\text{-partition of } P} X_{f(1)} \ldots X_{f(n)} \in K\langle\langle X_1, X_2, \ldots \rangle\rangle.$$

Hopf algebra of packed words **WQSym** (Thibon-Novelli)

- A packed word is a word $f(1)\ldots f(n)$ such that $f([n]) = [k]$ for a certain k.

$$1 = \emptyset; (1); (12), (21), (11);$$

$$(123), (132), (213), (231), (312), (321),$$

$$(122), (212), (221), (112), (121), (211), (111) \ldots$$

Hopf algebra of packed words **WQSym** (Thibon-Novelli)

- For any packed word $f(1)\ldots f(n)$, we put:

$$M_{f(1)\ldots f(n)} = \sum_{Pack(g(1)\ldots g(n))=f(1)\ldots f(n)} X_{g(1)}\ldots X_{g(n)}.$$

  These elements are a basis of a subalgebra **WQSym** of $K\langle\langle X_1, X_2, \ldots\rangle\rangle$.

- The canonical projection $K\langle\langle X_1, X_2, \ldots\rangle\rangle \longrightarrow K[[X_1, X_2, \ldots,]]$ induces a surjective algebra π morphism **WQSym** $\longrightarrow$ **QSym**.

$$\pi(M_{f(1)\ldots f(n)}) = M_{(\sharp f^{-1}(1),\ldots,\sharp f^{-1}(max(f)))}.$$

Hopf algebra of packed words **WQSym** (Thibon-Novelli)

- Product: If $\max(f) = k$ and $\max(g) = l$,

$$f.g = \sum_{\alpha \in QSh(k,l)} \alpha \circ (f \otimes g).$$

$$\begin{aligned}(132).(21) &= (13254) + (14253) + (15243) + (14352)\\ &+ (15342) + (15432) + (24351) + (25341)\\ &+ (25431) + (35421) + (13221) + (13231)\\ &+ (13241) + (13232) + (13242) + (14231)\\ &+ (14232) + (14243) + (14321) + (14332)\\ &+ (14342) + (24321) + (24331) + (24341)\end{aligned}$$

Hopf algebra of packed words **WQSym**

- Coproduct: if $max(f) = n$,

$$\Delta(f) = \sum_{k=0}^{n} f_{|[k]} \otimes Std(f_{|\mathbb{N}_{>0}\setminus[k]}).$$

$$\begin{aligned}\Delta((511423)) &= 1 \otimes (511423) + (511423) \otimes 1 + (11) \otimes (5423) \\ &+ (112) \otimes (543) + (1123) \otimes (54) + (11423) \otimes (5) \\ &= 1 \otimes (511423) + (511423) \otimes 1 + (11) \otimes (4312) \\ &+ (112) \otimes (321) + (1123) \otimes (21) + (11423) \otimes (1).\end{aligned}$$

Theorem

- The following map is a surjective Hopf algebra morphism:

$$\Gamma : \left\{ \begin{array}{rcl} \mathcal{H}_{SP} & \longrightarrow & \mathbf{WQSym} \\ P & \longrightarrow & \displaystyle\sum_{f \text{ } P\text{-partition of } P} X_{f(1)} \ldots X_{f(n)}. \end{array} \right.$$

- There exists a commutative diagram of Hopf algebra morphisms:

$$\begin{tikzcd} \mathcal{H}_{SP} \arrow[r, two heads, "L"] \arrow[dr, two heads, "\Gamma"'] & \mathbf{FQSym} \arrow[d, "\varphi"] \arrow[dr, two heads, "\phi"] & \\ & \mathbf{WQSym} \arrow[r, two heads, "\pi"'] & \mathbf{QSym} \end{tikzcd}$$

$$\Gamma\left(\substack{1\\3\\2}\right) = \sum_{a<b} X_b X_a^2 + \sum_{a<b<c} X_c X_a X_b$$

$$= (211) + \color{red}{(312)}$$

$$\Gamma\left({}^{1}\mathrm{V}^{2}_{3}\right) = \sum_{a<b} X_b^2 X_a + \sum_{a<b<c} X_b X_c X_a + \sum_{a<b<c} X_c X_b X_a$$

$$= (221) + \color{red}{(231)} + \color{red}{(321)}$$

$$\Gamma\left({}^{2}\mathrm{V}^{3}_{1}\right) = \sum_{a} X_a^3 + \sum_{a<b} X_a^2 X_b + \sum_{a<b} X_a X_b X_a + \sum_{a<b} X_a X_b^2$$

$$+ \sum_{a<b<c} X_a X_b X_c + \sum_{a<b<c} X_a X_c X_b$$

$$= (111) + (112) + (121) + (122) + \color{red}{(123)} + \color{red}{(132)}$$

$$\begin{aligned}
\varphi((123)) &= (123) + (122) + (112) + (111),\\
\varphi((132)) &= (132) + (121),\\
\varphi((213)) &= (213) + (212),\\
\varphi((231)) &= (231) + (221),\\
\varphi((312)) &= (312) + (211),\\
\varphi((321)) &= (321).
\end{aligned}$$

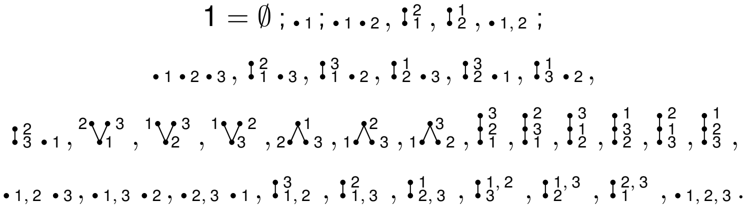
Corollary

φ is injective.

Quasi-orders

- A quasi-order on a set X is a transitive and reflexive relation $\leq_T$.
- If $\leq_T$ is a quasi-order on X, then:
  1. The relation $\sim_T$ defined by:
  $$(x \sim_T y) \iff (x \leq_T y \text{ and } y \leq_T x)$$
  is an equivalence on X.
  2. $X/\sim_T$ is partially ordered by:
  $$(\overline{x} \leq_T \overline{y}) \iff (x \leq_T y).$$

We represent quasi-orders on $[n]$ by the Hasse diagram of $X/\sim$.



n	1	2	3	4	5	6	7	8
	1	4	29	355	6 942	209 527	9 535 241	642 779 354

Sequence A000798 in the OEIS.

- If $\leq$ is a quasi-order on X, the set of ideals of $\leq$ defines a topology on X.
- If T is a topology on X, the following relation is a quasi-order on X:

$$(x \leq y) \Longleftrightarrow (\text{any open set of } T \text{ containing } x \text{ contains } y).$$

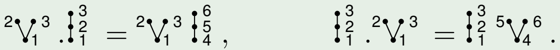
If X is finite, these two correspondences are bijective.

Alexandrov's theorem

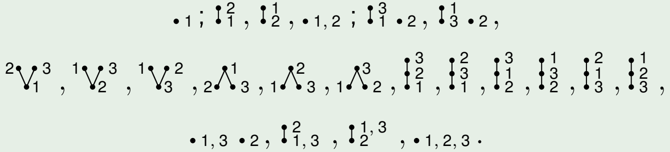
For all $n \geq 0$, there exists a bijection between the set of quasi-orders on $[n]$ and the set of topologies on $[n]$.

Hopf algebra of topologies $\mathcal{H}_T$

- Basis: the set of all finite topologies on sets $[n]$.
- Product: shifted concatenation.



- $\mathcal{H}_T$ is freely generated by the set of indecomposable topologies:



Number of topologies and indecomposable topologies of degree n:

n	1	2	3	4	5	6	7	8
	1	4	29	355	6 942	209 527	9 535 241	642 779 354
	1	3	22	292	6 120	193 594	9 070 536	622 336 756

Hopf algebra of topologies $\mathcal{H}_T$

Coproduct:

$$\Delta(P) = \sum_{I \text{ open set of } P} Std(P \setminus I) \otimes Std(I).$$

$$\begin{aligned}
\Delta\left({}^{2}\mathrm{V}_{1}^{3}\right) &= {}^{2}\mathrm{V}_{1}^{3} \otimes 1 + 1 \otimes {}^{2}\mathrm{V}_{1}^{3} + \bullet_2 \otimes \mathfrak{l}_{1}^{3} + \bullet_3 \otimes \mathfrak{l}_{1}^{2} + \bullet_2 \bullet_3 \otimes \bullet_1 \\
&= {}^{2}\mathrm{V}_{1}^{3} \otimes 1 + 1 \otimes {}^{2}\mathrm{V}_{1}^{3} + \bullet_1 \otimes \mathfrak{l}_{1}^{2} + \bullet_1 \otimes \mathfrak{l}_{1}^{2} + \bullet_1 \bullet_2 \otimes \bullet_1
\end{aligned}$$

T-partitions

Let T be a topology on $[n]$. The associated quasi-order is denoted by $\leq_T$. A T-partition of T is a sequence $(f(1), \ldots, f(n))$ such that:

- $(i \leq_T j) \Longrightarrow (f(i) \leq f(j))$.
- $(i \leq_T j$, not $i \geq_T j$ and $i > j) \Longrightarrow (f(i) < f(j))$.
- $(i < j < k$, $i \sim_T k$ and $f(i) = f(j)) \Longrightarrow (i \sim_T j)$.

T-partitions of ${}^{1}\mathbb{V}^{5}_{2,4}\ \bullet_3$ are sequences $(f(1), \ldots, f(5))$ such that:

- $f(2) = f(4) \leq f(1), f(5)$
- $f(2) = f(4) < f(1)$
- $f(3) \neq f(2), f(4)$

If $a < b < c < d$:

$$(babaa),\quad (babab),\quad (babac),\quad (bacaa),\quad (bacab),\quad (bacac),$$
$$(bacad),\quad (badac),\quad (cabaa),\quad (cabab),\quad (cabac),\quad (cabad),$$
$$(cacab),\quad (cadab),\quad (cbabb),\quad (cbabc),\quad (cbabd),\quad (dabac),$$
$$(dacab),\quad (dbabc).$$

Theorem

The following map is a surjective Hopf algebra morphism:

$$\Gamma : \left\{ \begin{array}{rcl} \mathcal{H}_T & \longrightarrow & \mathbf{WQSym} \\ T & \longrightarrow & \displaystyle\sum_{f\ T\text{-partition of } T} X_{f(1)} \ldots X_{f(n)}. \end{array} \right.$$

$$\begin{aligned}\Gamma\left({}^{1}\mathbb{V}^{5}_{2,4}\quad \bullet_3\right) = {} & (21211)+(21212)+(21213)+(21311)\\ & +(21312)+(21313)+(21314)+(21413)\\ & +(31211)+(31212)+(31213)+(31214)\\ & +(31312)+(31412)+(32122)+(32123)\\ & +(32124)+(41213)+(41312)+(42123)\end{aligned}$$

Theorem

The following map is a surjective Hopf algebra morphism:

$$\Gamma : \left\{ \begin{array}{rcl} \mathcal{H}_T & \longrightarrow & \textbf{WQSym} \\ T & \longrightarrow & \displaystyle\sum_{f\ T\text{-partition of } T} X_{f(1)} \ldots X_{f(n)}. \end{array} \right.$$

$$\begin{aligned} \Gamma\left({}^{1}\!\bigvee^{5}_{2,4}\ \ \cdot_3\right) = {} & (21211) + (21212) + (21213) + (21311) \\ & + (21312) + (21313) + (21314) + (21413) \\ & + (31211) + (31212) + (31213) + (31214) \\ & + (31312) + (31412) + (32122) + (32123) \\ & + (32124) + (41213) + (41312) + (42123) \end{aligned}$$

Linear extensions

Let T be a topology on n. A linear extension of T is an ordered partition $A = (A_1, \ldots, A_k)$ of $[n]$ such that:

- the equivalence classes of $\sim_T$ are $A_1, \ldots, A_k$;
- if, in the poset $[n]/\sim_T$, $A_i \leq_T A_j$, then $i \leq j$.

Linear extensions are represented by packed words $(f(1), \ldots, f(n))$:

$$(f(i) = j) \Longleftrightarrow (i \in A_j)$$

Linear extensions of ${}^{1}\!\bigvee^{5}_{2,4}\ \bullet_3$: packed words $(f(1),\ldots,f(5))$ such that:

- $f(2)=f(4)$, and $f(1),f(2),f(3),f(5)$ are all distinct.
- $f(2)<f(1),f(5)$.

$$\begin{array}{llll}
(21314), & (21413), & (31214), & (31412),\\
(32124), & (41213), & (41312), & (42123).
\end{array}$$

Another product on **WQSym**

- Product: If $\max(f) = k$ and $\max(g) = l$,

$$f.g = \sum_{\alpha \in QSh(k,l)} \alpha \circ (f \otimes g).$$

$$\begin{aligned}(132).(21) = {} & (13254) + (14253) + (15243) \\ & + (14352) + (15342) + (15432) + (24351) \\ & + (25341) + (25431) + (35421) \\ & + (13221) + (13231) + (13241) + (13232) \\ & + (13242) + (14231) + (14232) + (14243) \\ & + (14321) + (14332) + (14342) \\ & + (24321) + (24231) + (24341)\end{aligned}$$

Another product on **WQSym**

- Second product: If $\max(f) = k$ and $\max(g) = l$,

$$f \sqcup\!\sqcup g = \sum_{\alpha \in Sh(k,l)} \alpha \circ (f \otimes g).$$

$$\begin{aligned}(132) \sqcup\!\sqcup (21) = {} & (13254) + (14253) + (15243) \\ & + (14352) + (15342) + (15432) + (24351) \\ & + (25341) + (25431) + (35421)\end{aligned}$$

Theorem

- The following map is a surjective Hopf algebra morphism:

$$L : \left\{ \begin{array}{rcl} \mathcal{H}_T & \longrightarrow & (\mathbf{WQSym}, \sqcup\!\sqcup) \\ T & \longrightarrow & \displaystyle\sum_{f \in L(T)} f \end{array} \right.$$

- There exists a commutative diagram of Hopf algebra morphisms:

$$\begin{tikzcd} \mathcal{H}_T \arrow[r, two heads, "L"] \arrow[dr, "\Gamma"'] & (\mathbf{WQSym}, \sqcup\!\sqcup) \arrow[d, "\varphi"] \\ & \mathbf{WQSym} \end{tikzcd}$$

$$\varphi((132)) = (132) + (121), \qquad \varphi((213)) = (213) + (212),$$
$$\varphi((231)) = (231) + (221), \qquad \varphi((312)) = (312) + (211),$$
$$\varphi((321)) = (321), \qquad \varphi((112)) = (112) + (111),$$
$$\varphi((121)) = (121), \qquad \varphi((211)) = (211),$$
$$\varphi((122)) = (122) + (111), \qquad \varphi((212)) = (212),$$
$$\varphi((221)) = (221), \qquad \varphi((111)) = (111).$$

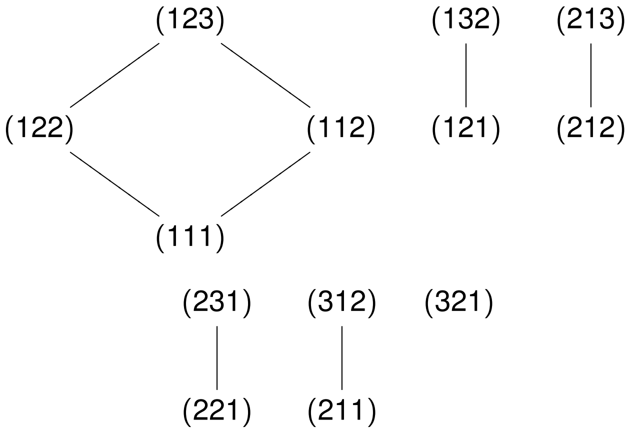
Partial order on packed words

There exists a partial order on the set of packed words such that for any packed word f:

$$\varphi(f) = \sum_{g \leq f} g.$$

Corollary

φ is an isomorphism.


(123)
(132)
(213)
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(121)
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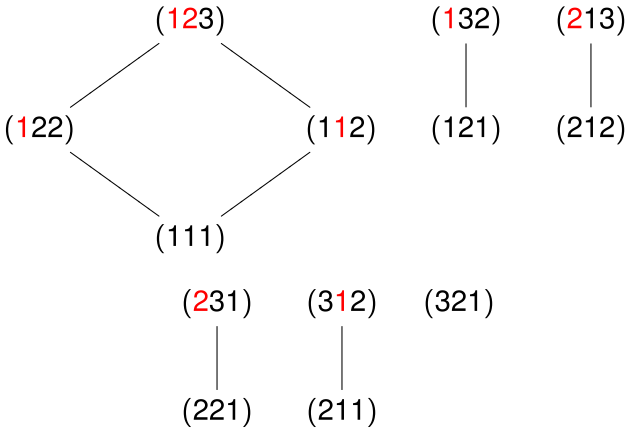
(123)
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Property of the partial order

Let f be a packed word of length n. If $1 \leq i \leq n$, $i \in M(f)$ if:

- $f(i) < \max(f)$
- For all j, $(f(i) = f(j)) \Longrightarrow (j \leq i)$.
- For all j, $(f(j) = f(i) + 1) \Longrightarrow j > i$.

If f, g are two packed words:

$$(f \leq g) \Longleftrightarrow (Std(f) = Std(g) \text{ and } M(f) \subseteq M(g)).$$

If $\sigma \in \mathfrak{S}_n$, $i \in M(\sigma)$ if, and only if, $\sigma(i)$ is an ascent of σ^{-1}.

Corollary

For all $n \geq 1$, for all $\sigma \in \mathfrak{S}_n$:

$$\sharp\{w \text{ packed word of length } n \mid Std(w) = \sigma\} = 2^{\sharp M(\sigma)}.$$

For all $n \geq 1$:

$$\sharp\{\text{packed words of length } n\} = \sum_{\sigma \in \mathfrak{S}_n} 2^{\sharp M(\sigma)}.$$