Bijections autour des bois de Schnyder

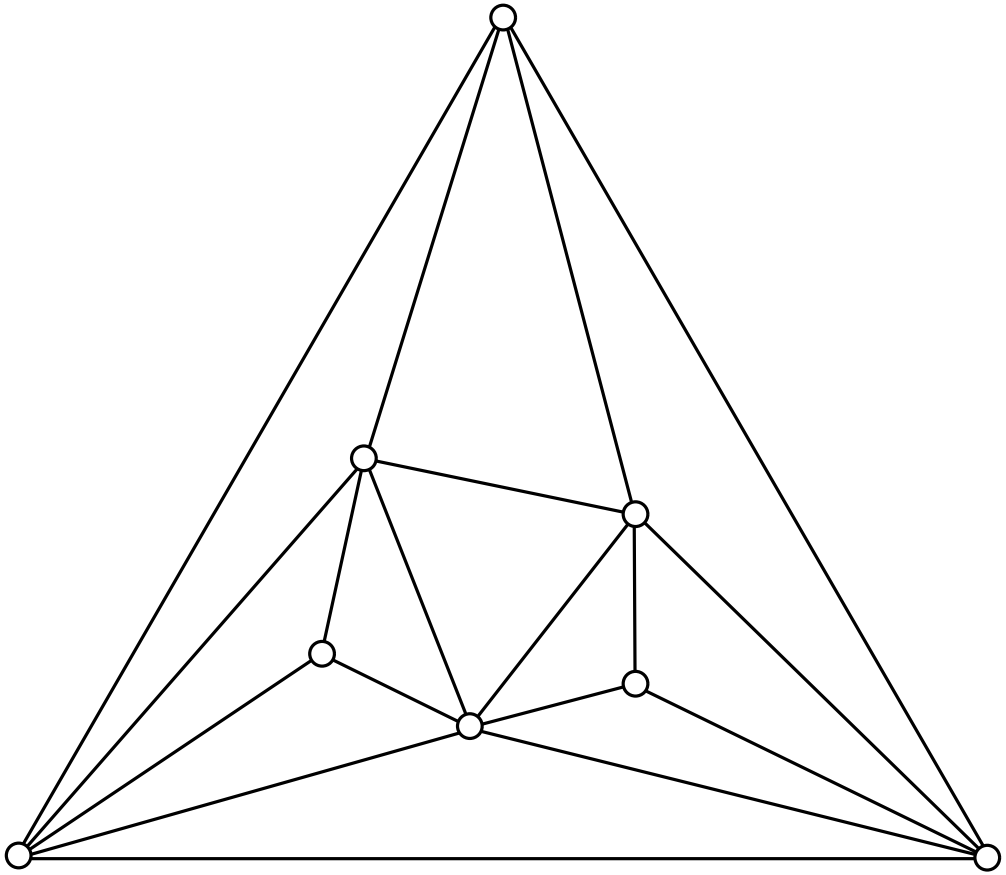
Éric Fusy (LIX, École Polytechnique)

travaux avec Olivier Bernardi, Dominique Poulalhon et Gilles Schaeffer

Séminaire Philippe Flajolet, IHP, 31 janvier 2013

Schnyder structures on simple triangulations

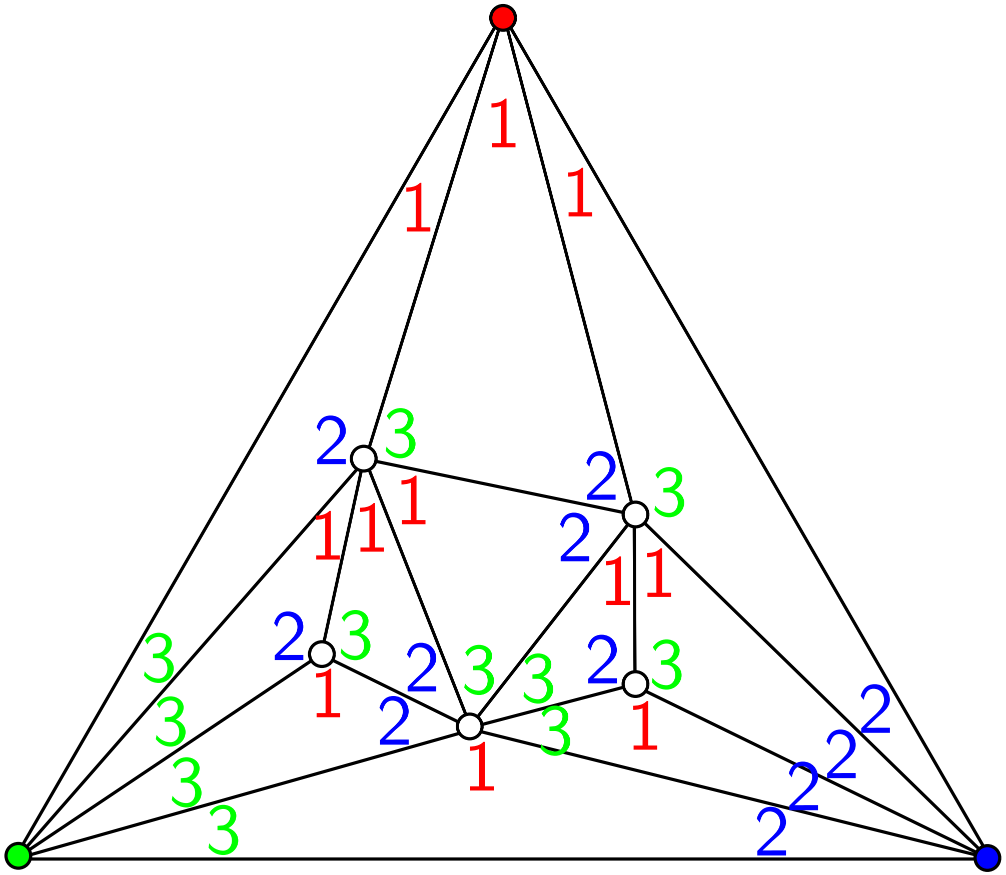
Let T be a **simple triangulation** (topological, **up to isotopy**)



Schnyder structures on simple triangulations

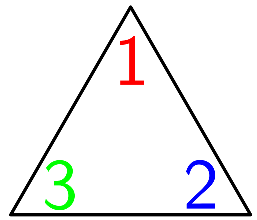
1) Schnyder labellings

[Schnyder'89]

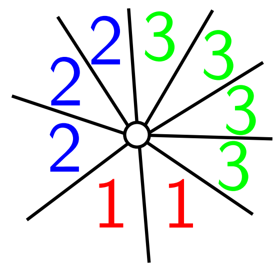


T can be endowed with a **labelling** of the corners by $\{1, 2, 3\}$ such that

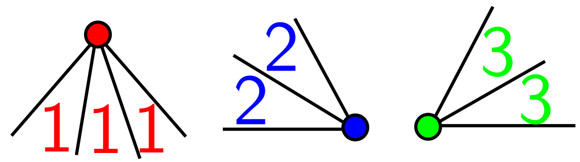
inner faces



inner vertices

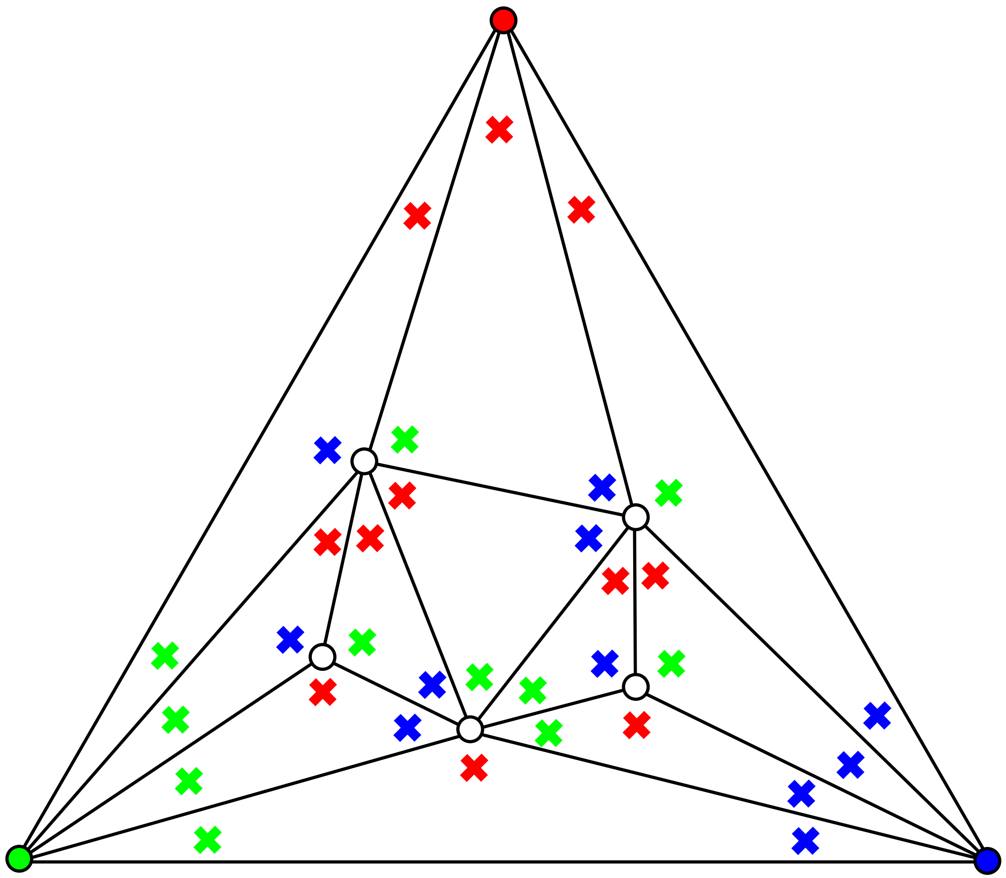


outer vertices



Schnyder structures on simple triangulations

1) Schnyder labellings [Schnyder'89]



T can be endowed with a **labelling** of the corners by $\{\times, \times, \times\}$ such that

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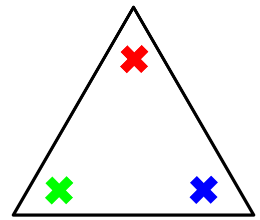
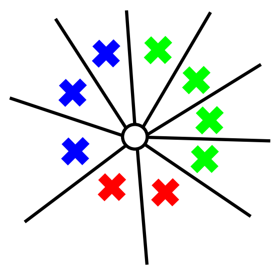
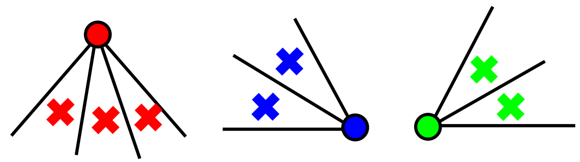


inner vertices

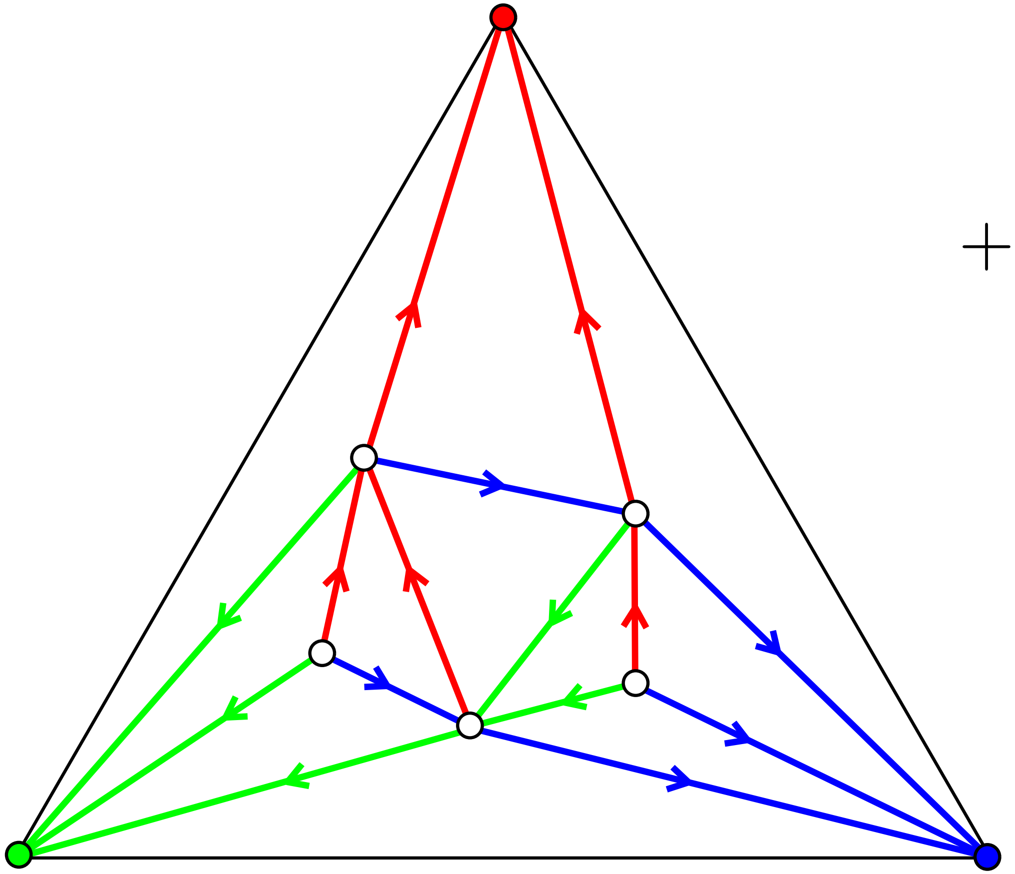


outer vertices



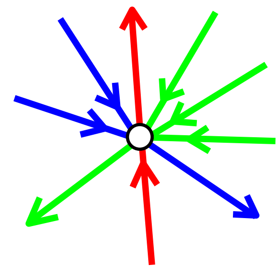
Schnyder structures on simple triangulations

2) Schnyder woods [Schnyder'89]

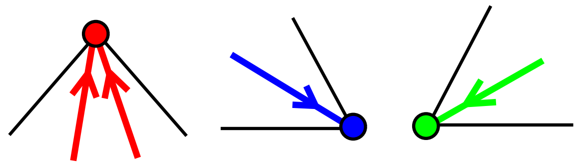


T can be endowed with a **tricoloration** +**orientation** of the inner edges such that

inner vertices



outer vertices



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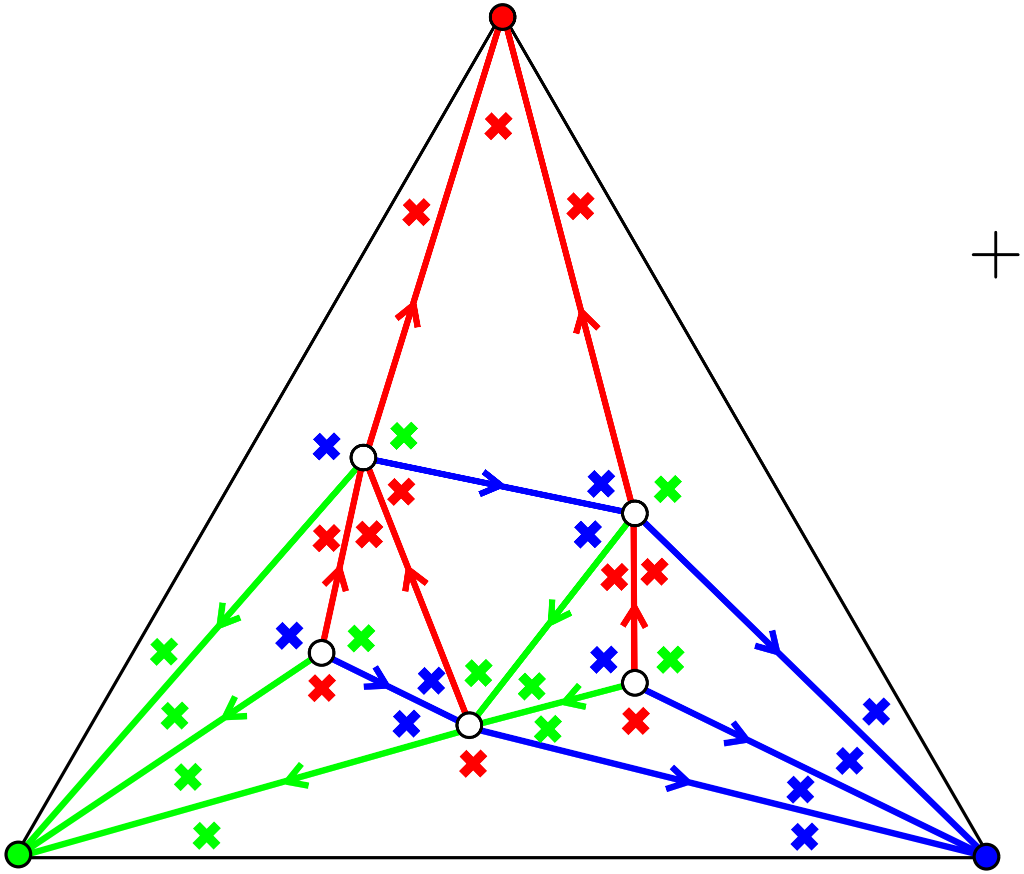

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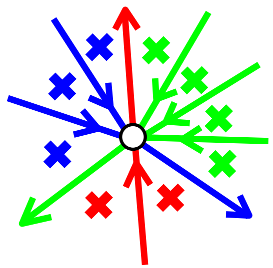
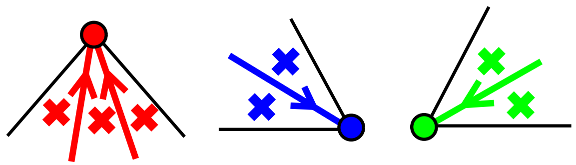


outer vertices



Schnyder woods $\leftrightarrow$ Schnyder labellings

Schnyder structures on simple triangulations

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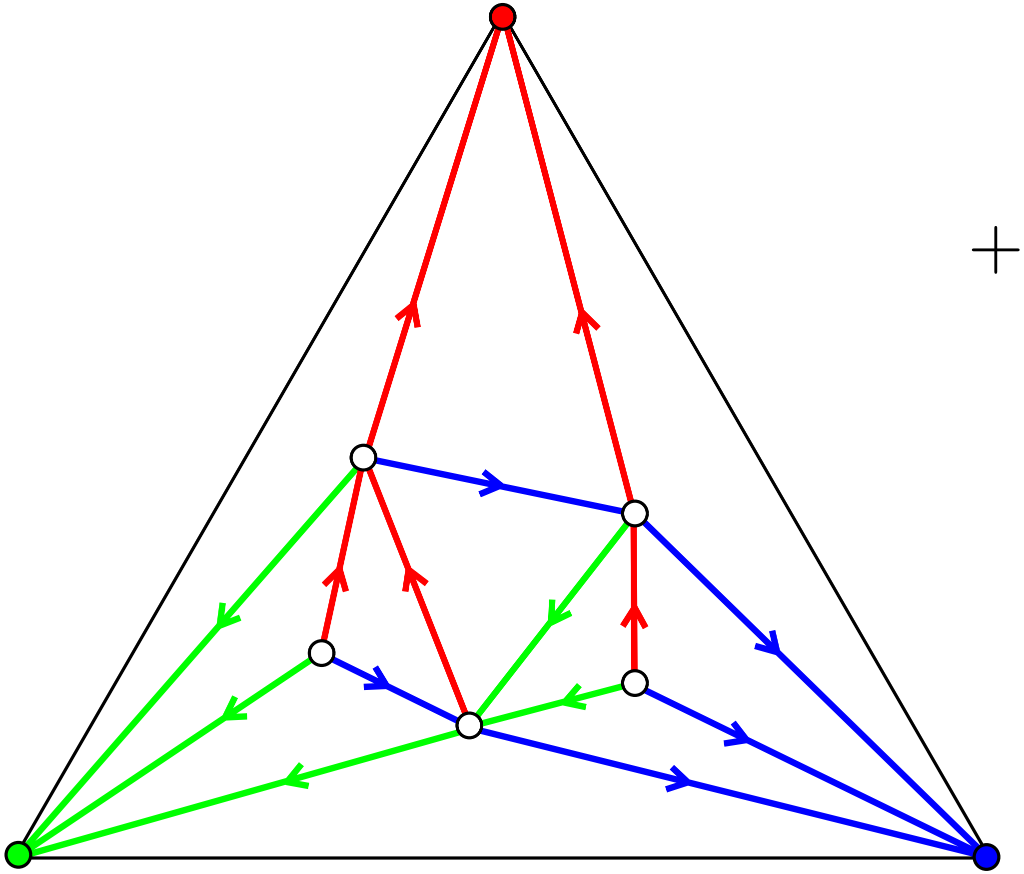

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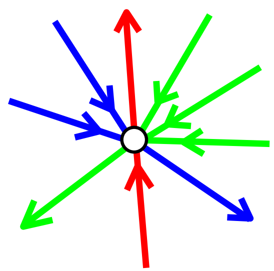
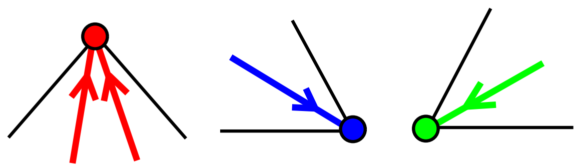


outer vertices



Schnyder structures on simple triangulations

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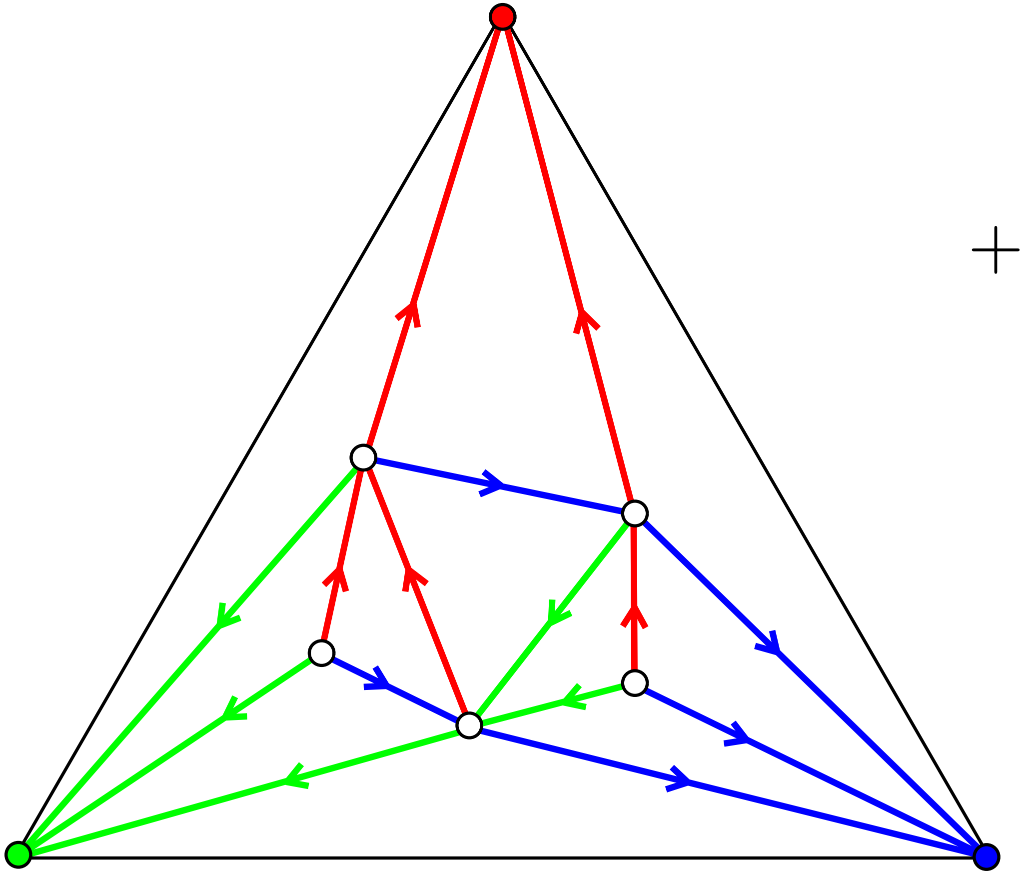

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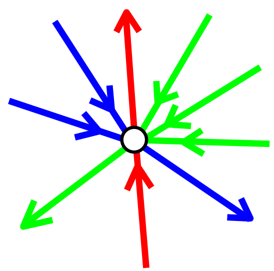

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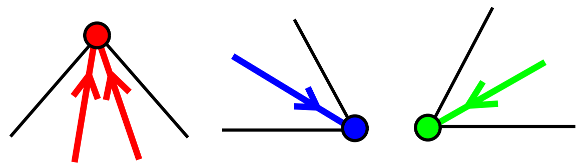
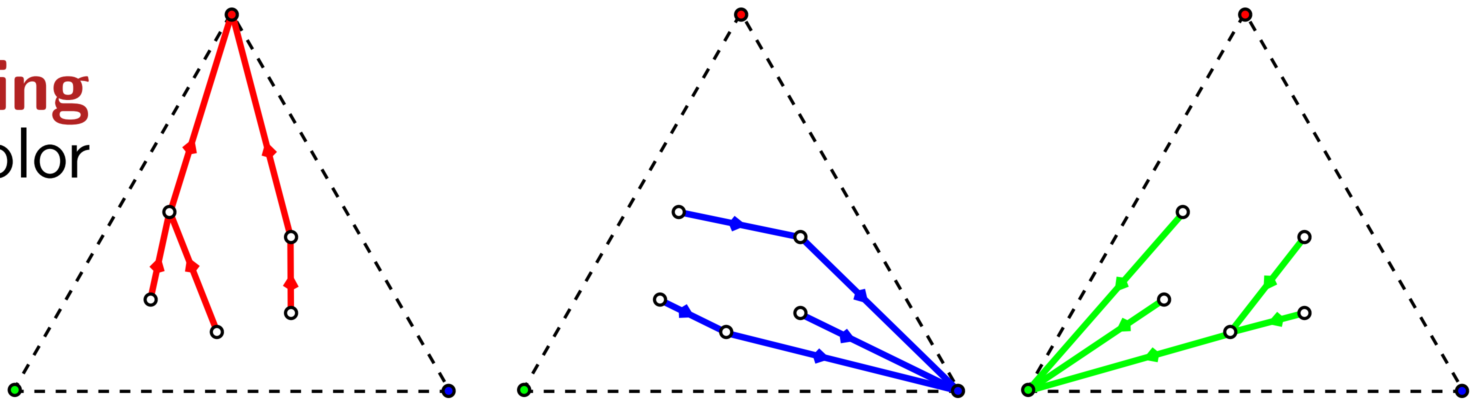


yields a **spanning tree** in each color



Schnyder structures on simple triangulations

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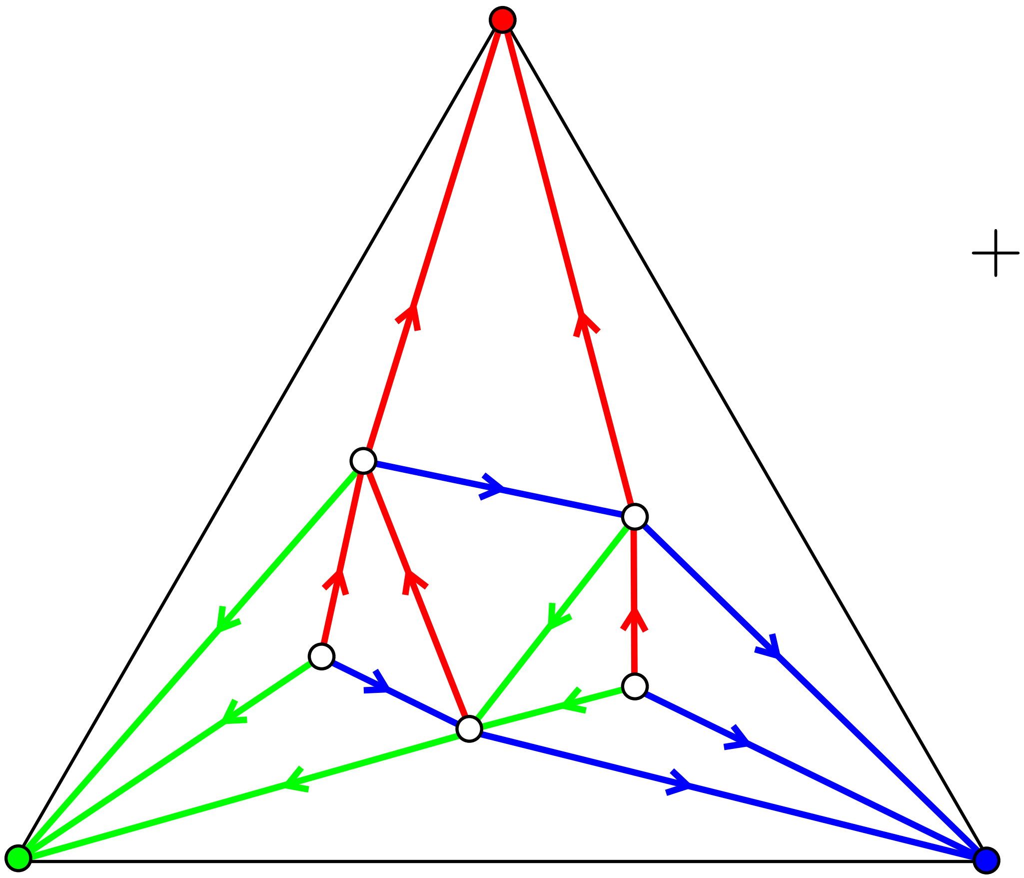
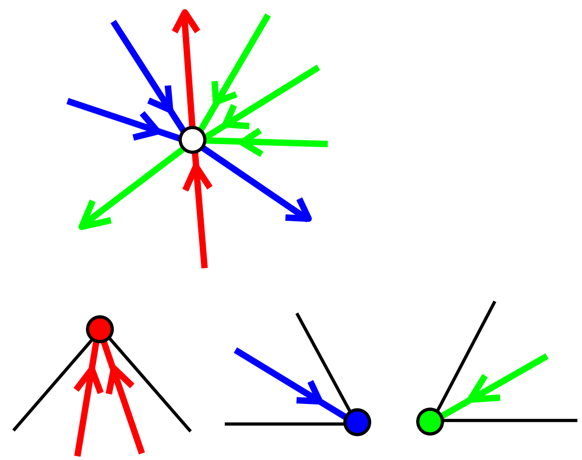


T can be endowed with a **tricoloration** +**orientation** of the inner edges such that

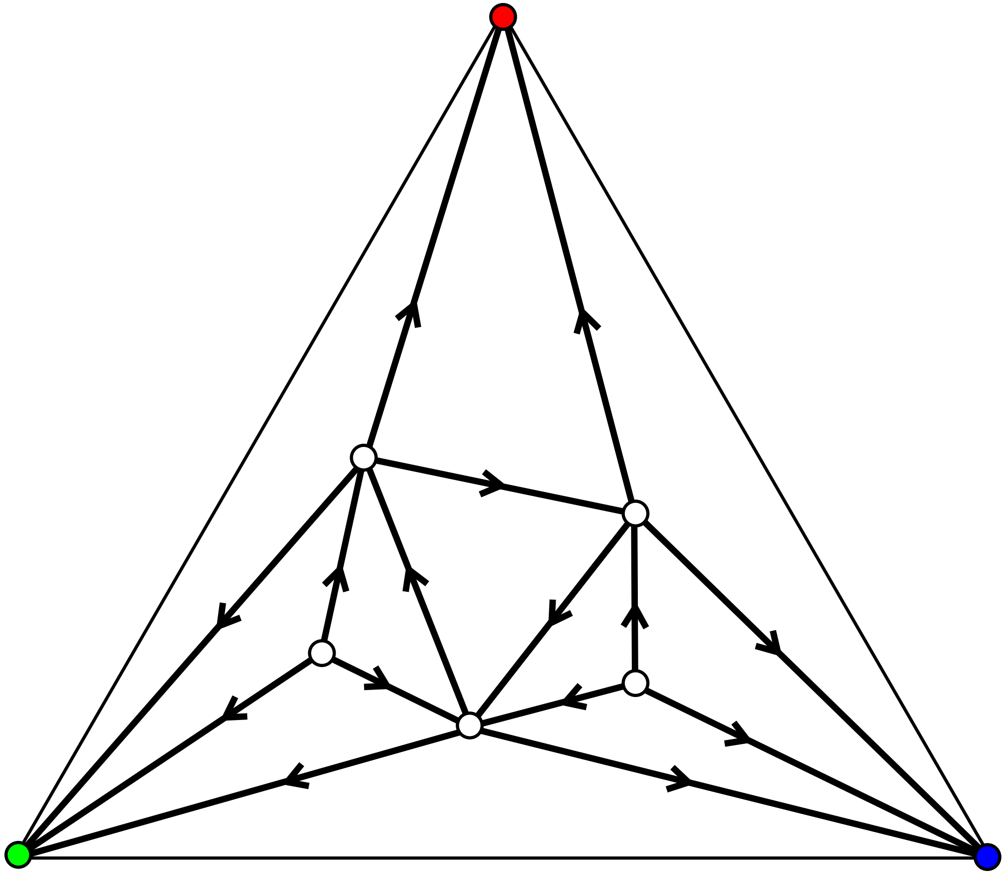
inner vertices

outer vertices

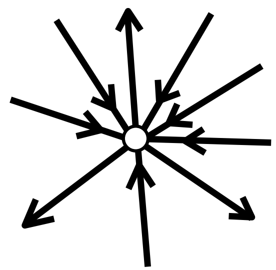


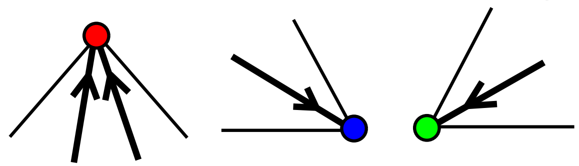
Schnyder structures on simple triangulations

3) 3-orientations [Schnyder'89]



T can be endowed with an **orientation** of its inner edges such that

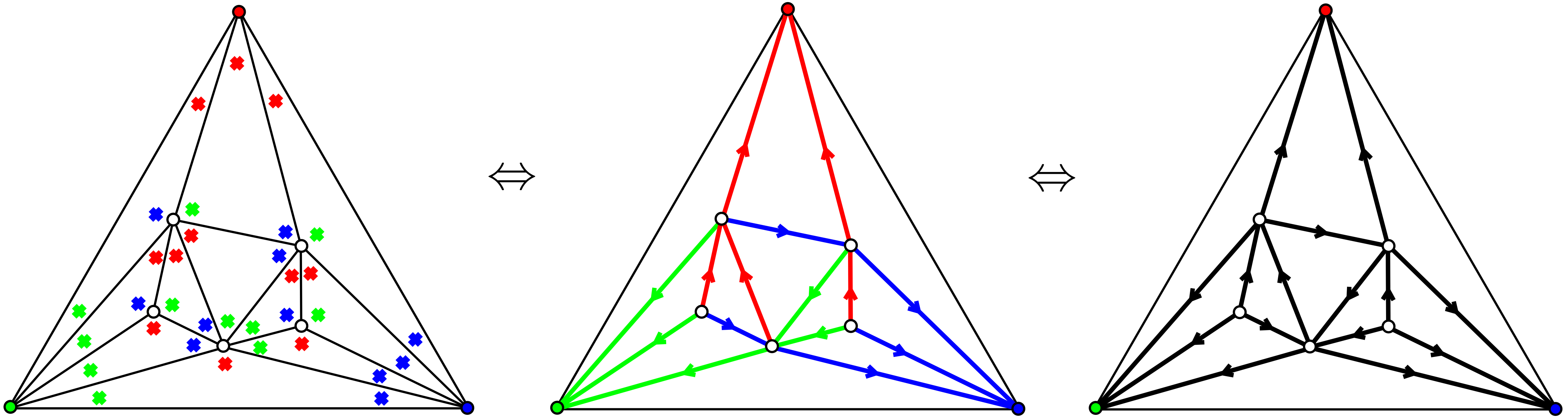
inner vertices  outdeg=3

outer vertices  outdeg=0

Schnyder structures on simple triangulations

The **3 incarnations** of Schnyder structures:

Schnyder labelling | Schnyder wood | 3-orientation



Schnyder structures on simple triangulations

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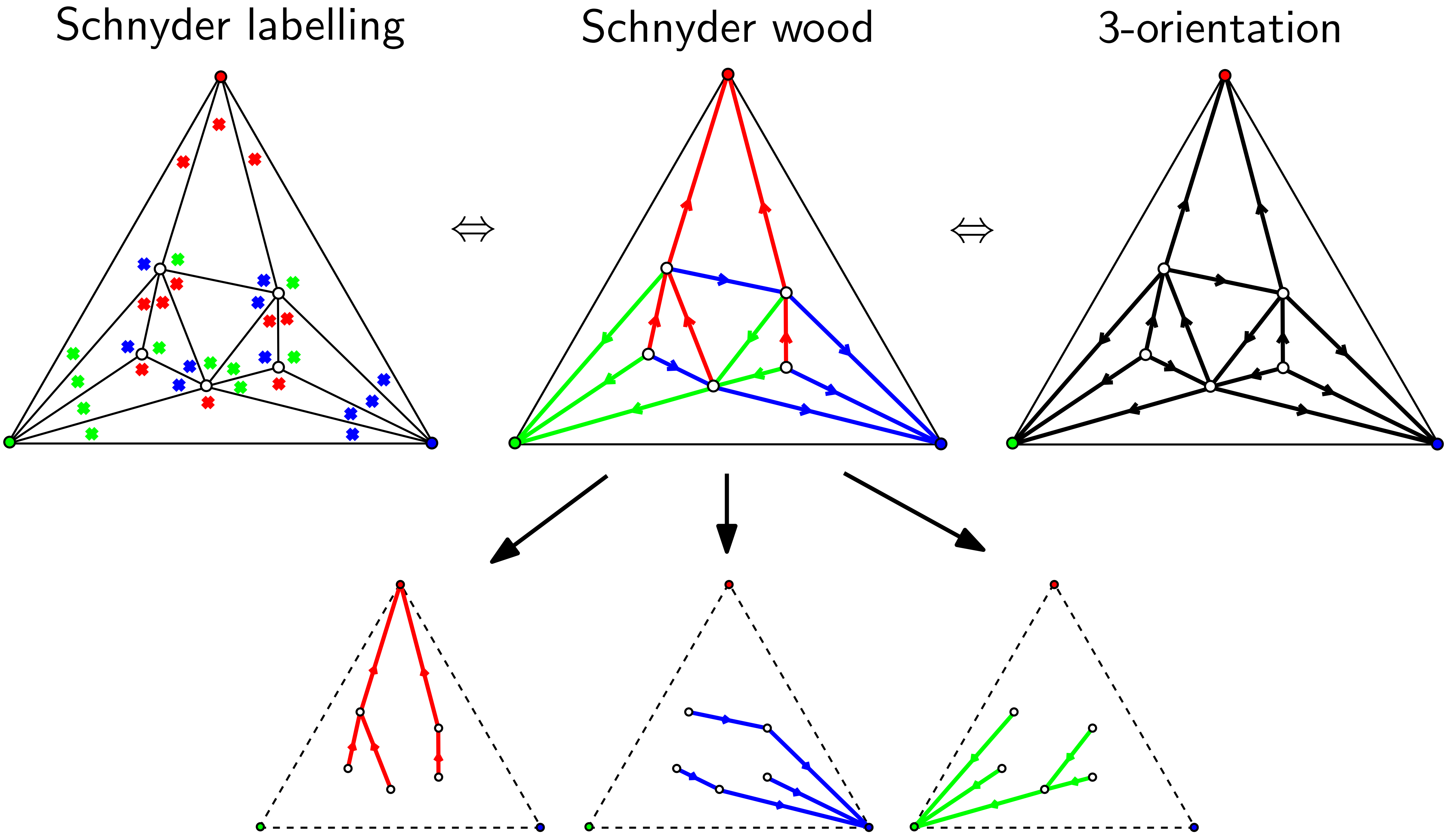
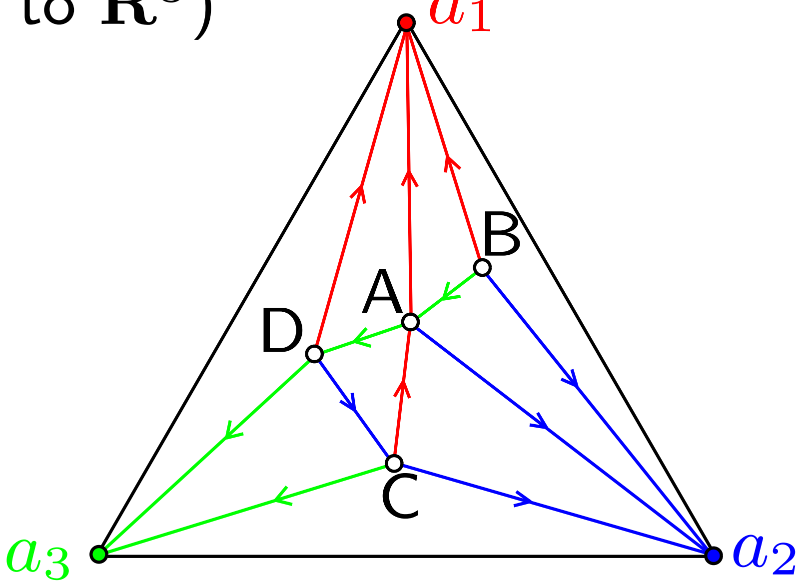


Applications of Schnyder woods [Schnyder'89,90]

Associate 3 **coordinates** to each vertex of T
(mapping from V to $\mathbf{R}^3$)



Applications of Schnyder woods

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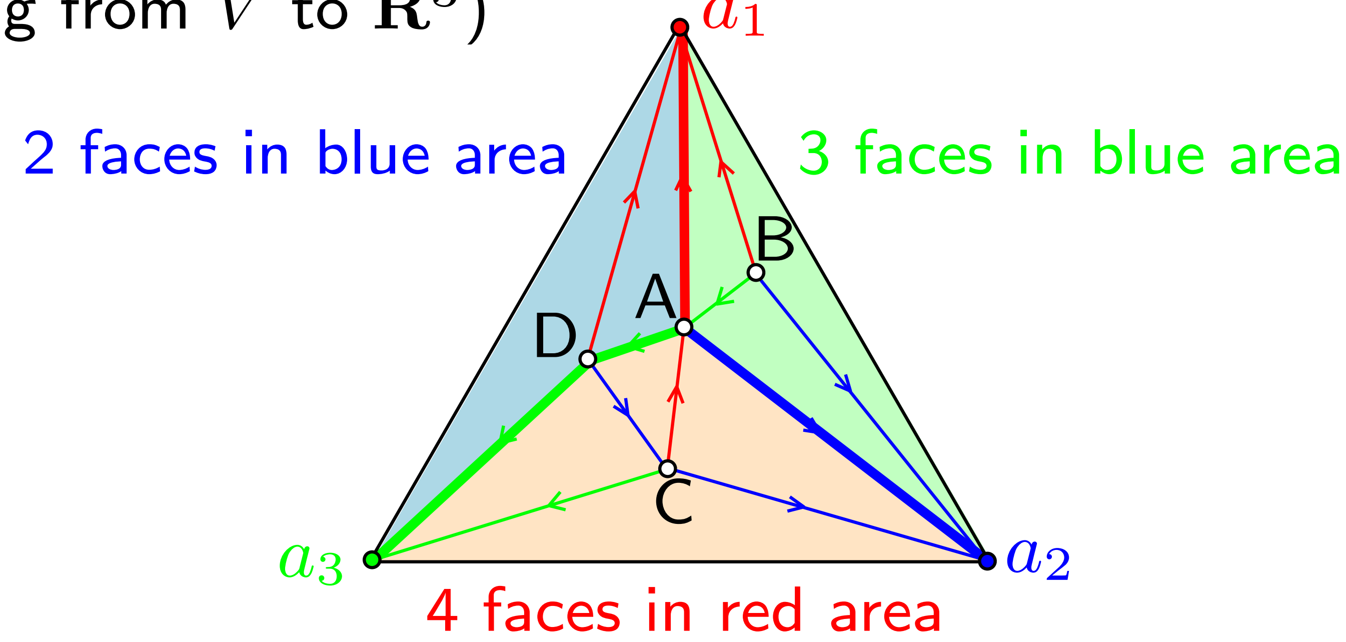

$$A \rightarrow (4, 2, 3)$$

Applications of Schnyder woods

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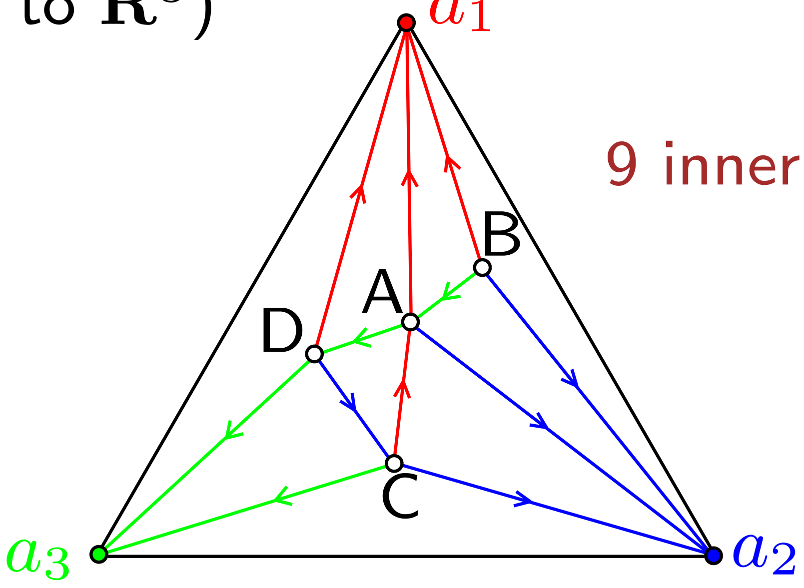

9 inner faces

$a_1 \to (9,0,0)$
$a_2 \to (0,9,0)$
$a_3 \to (0,0,9)$
$A \to (4,2,3)$
$B \to (5,3,1)$
$C \to (1,4,4)$
$D \to (2,1,6)$

all in $x+y+z=9$

Applications of Schnyder woods

[Schnyder'89,90]

Associate 3 **coordinates** to each vertex of T
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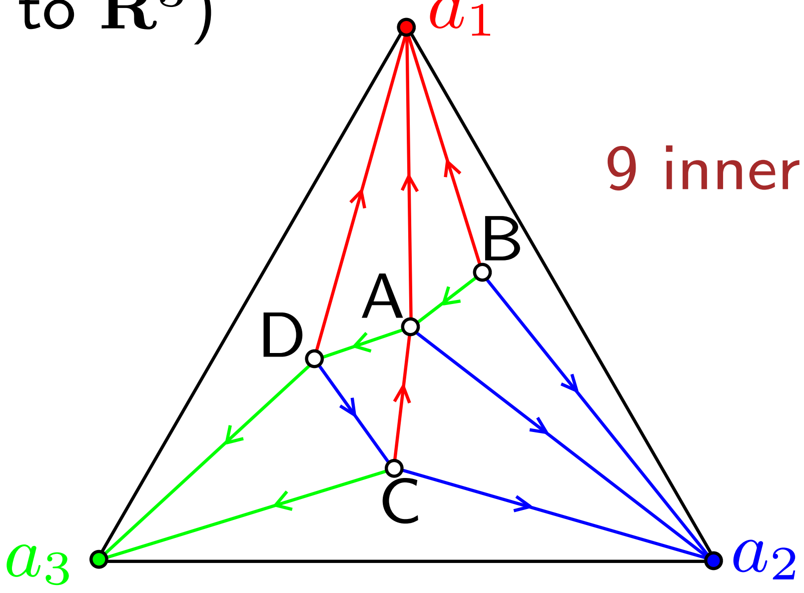
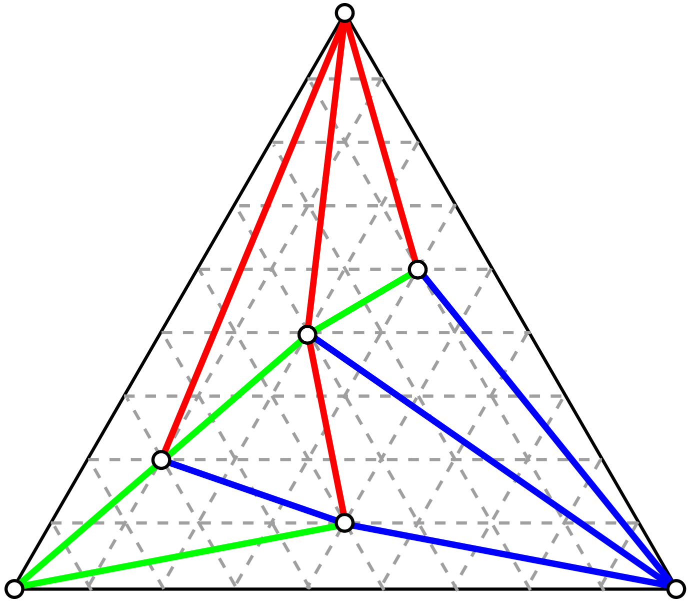


9 inner faces

$$\boxed{\begin{array}{l} a_1 \rightarrow (9,0,0) \\ a_2 \rightarrow (0,9,0) \\ a_3 \rightarrow (0,0,9) \\ A \rightarrow (4,2,3) \\ B \rightarrow (5,3,1) \\ C \rightarrow (1,4,4) \\ D \rightarrow (2,1,6) \end{array}}$$

all in $x+y+z=9$

Straight-line drawing algo



Planarity criterion

$G=(V,E)$ is planar iff

$\exists\ \Phi : V \cup E \rightarrow \mathbf{R}^3$

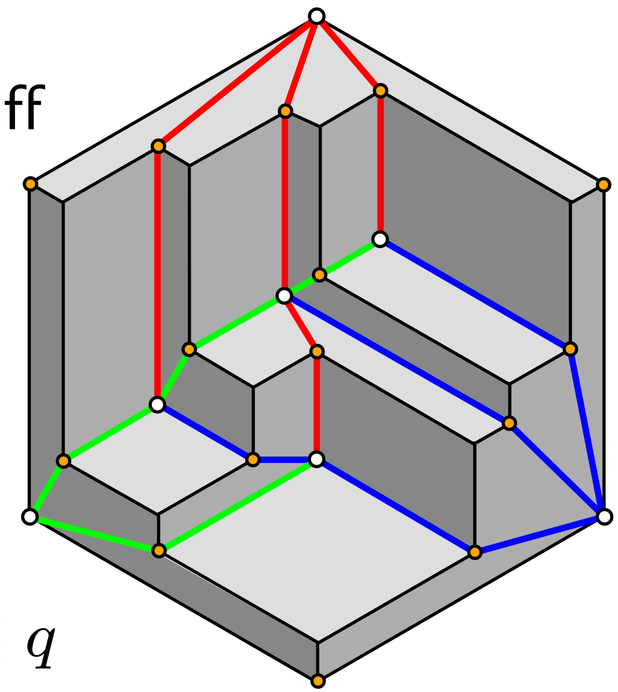
such that

$\forall p \neq q \in (V \cup E)^2$

$$\Phi(p) \leq_{\mathbf{R}^3} \Phi(q)$$

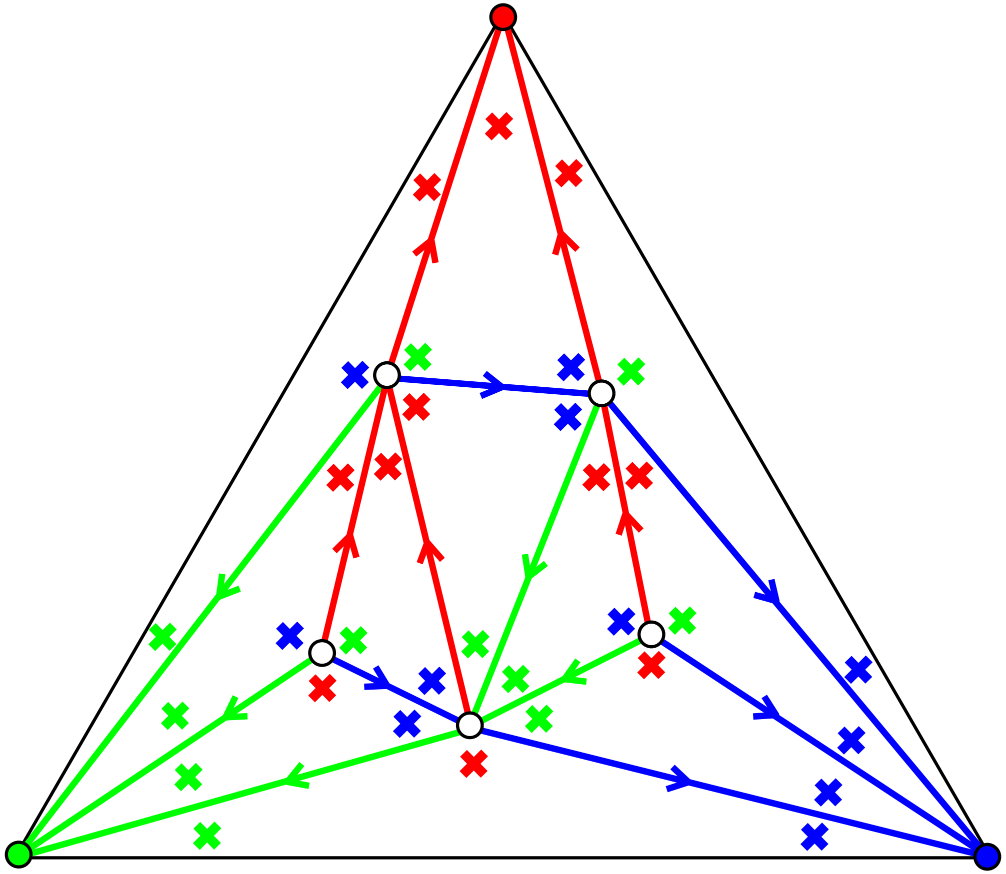
$$\Updownarrow$$

$p \in V, q \in E$ and $p \in q$



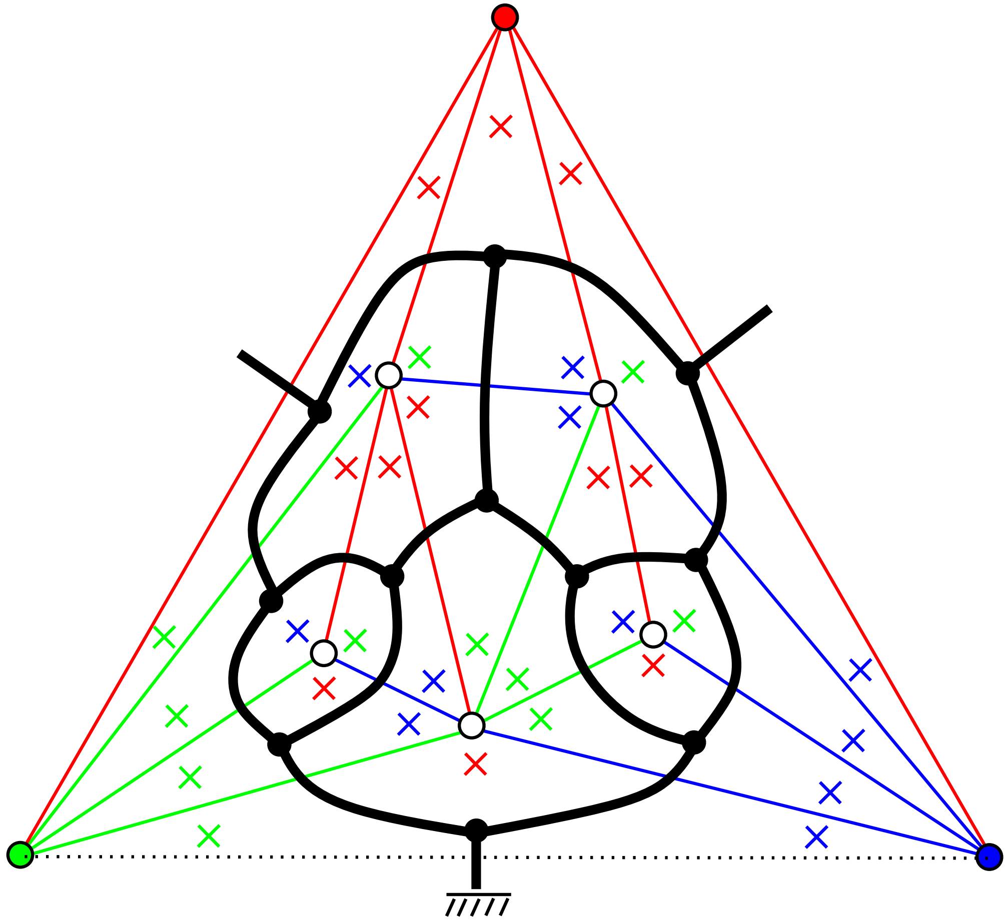
Bijection for Schnyder woods [Bonichon'02]

revisited in the dual setting



Bijection for Schnyder woods [Bonichon'02]

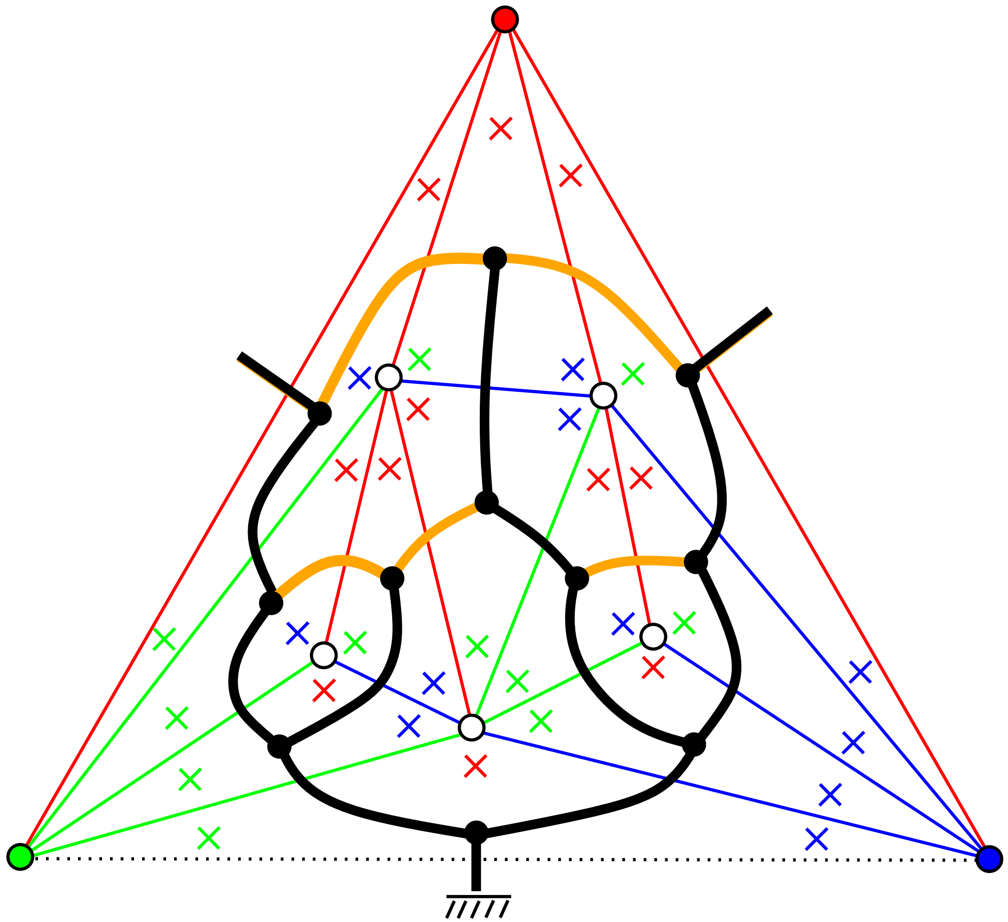
Take the (3-regular) **dual** of the triangulation



Bijection for Schnyder woods [Bonichon'02]

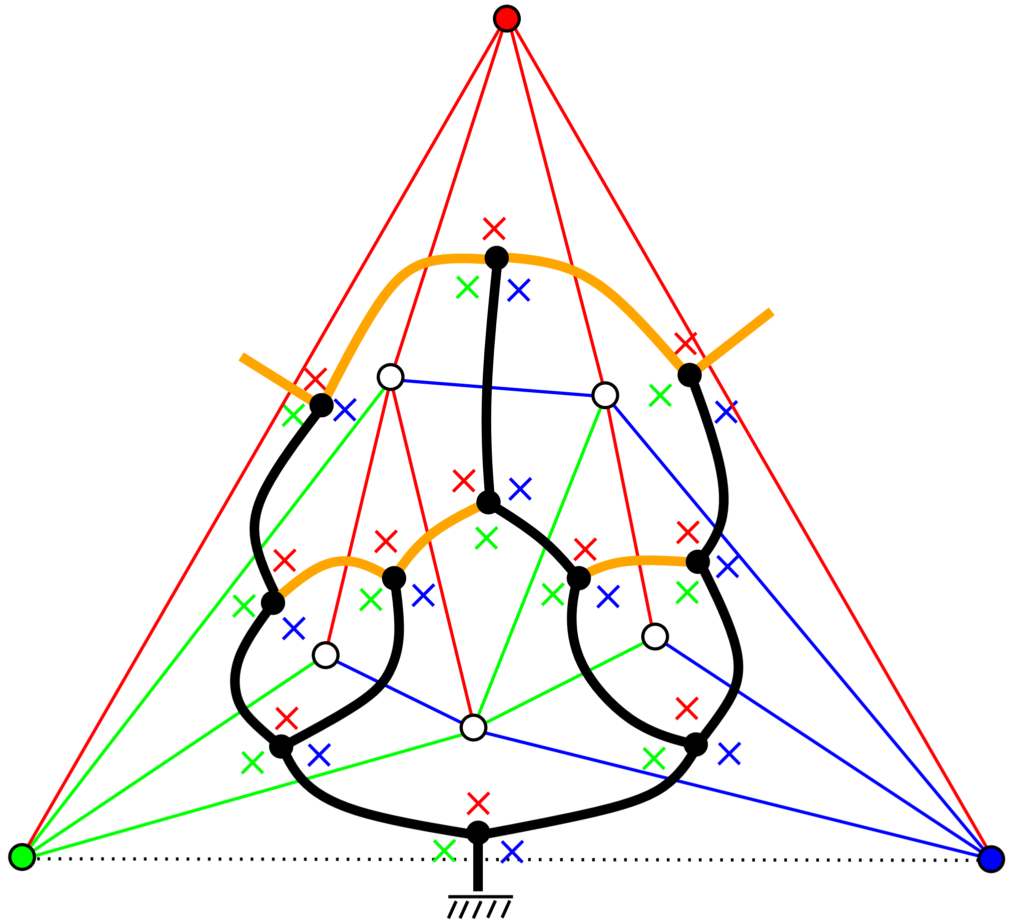
In black the **dual tree** of the red tree

In orange the **dual of the red edges**



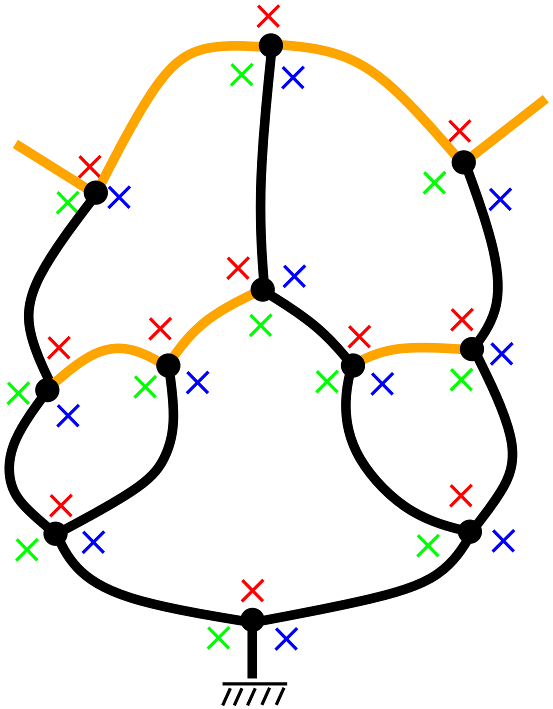
Bijection for Schnyder woods [Bonichon'02]

move corner-labels toward black vertices



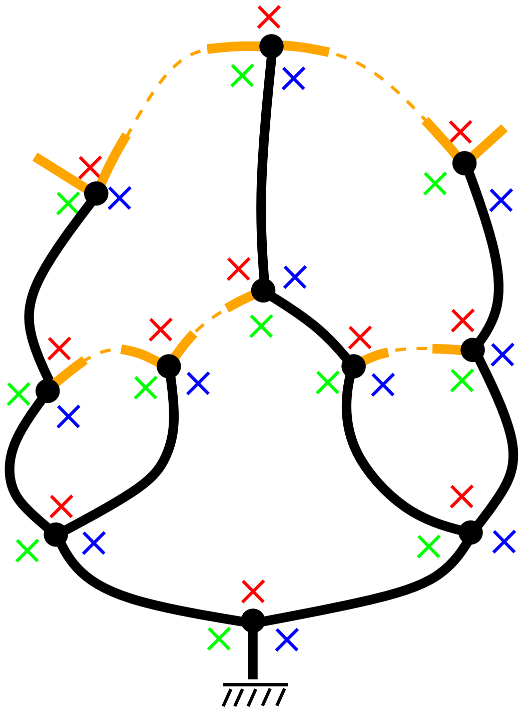
Bijection for Schnyder woods [Bonichon'02]

Erase the triangulation, keep the dual



Bijection for Schnyder woods [Bonichon'02]

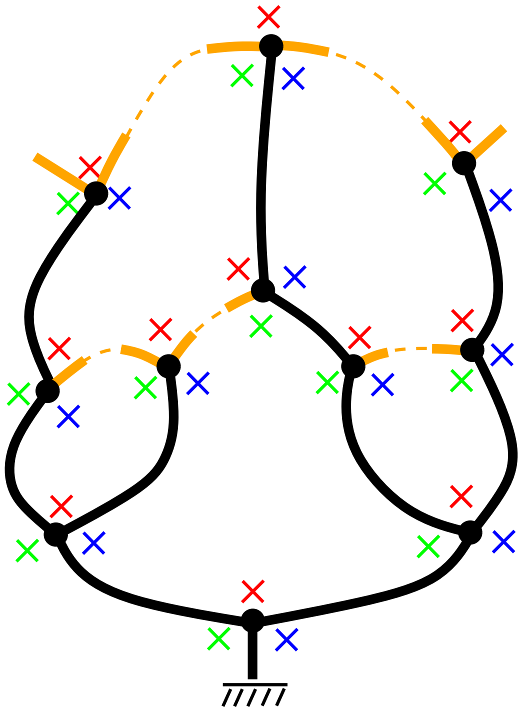
Cut the orange edges at their middle



Bijection for Schnyder woods [Bonichon'02]

Cut the orange edges at their middle

$\Rightarrow$ **binary tree** such that there is a parenthesis **matching of the leaves**



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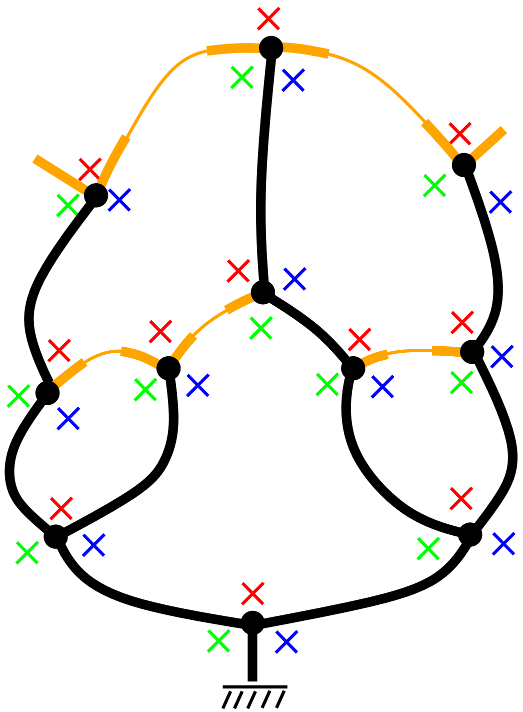
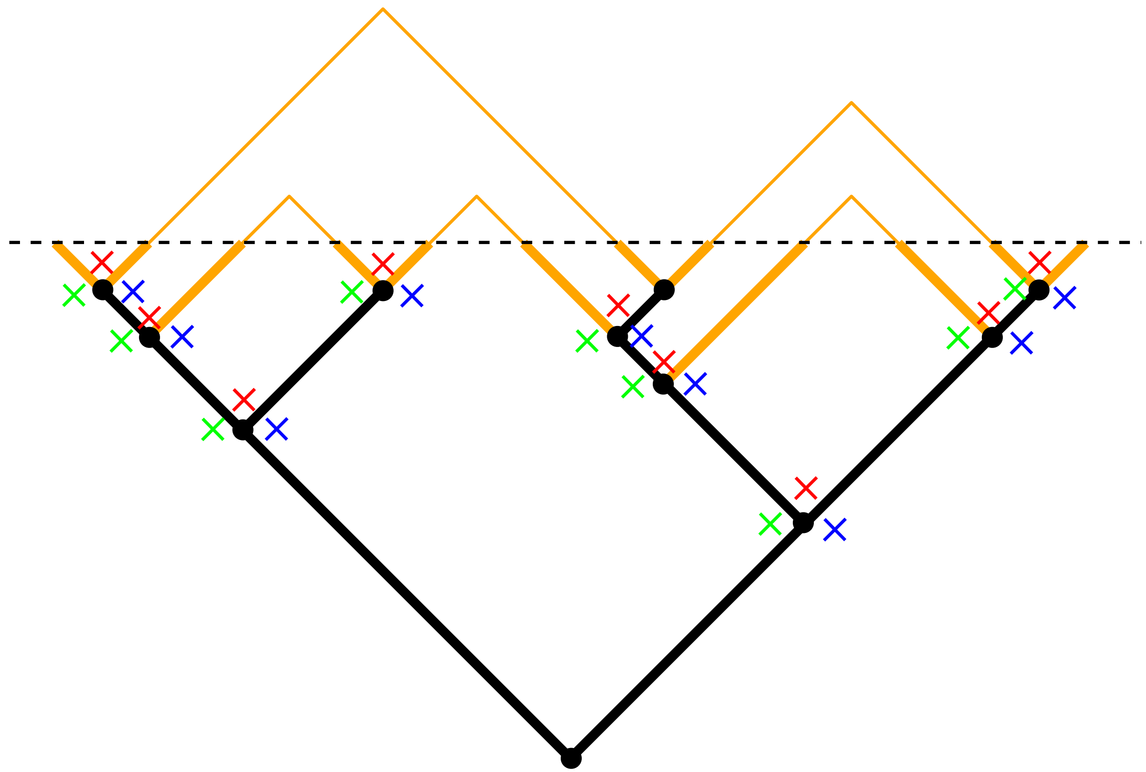


$\Rightarrow$

rectilinear representation



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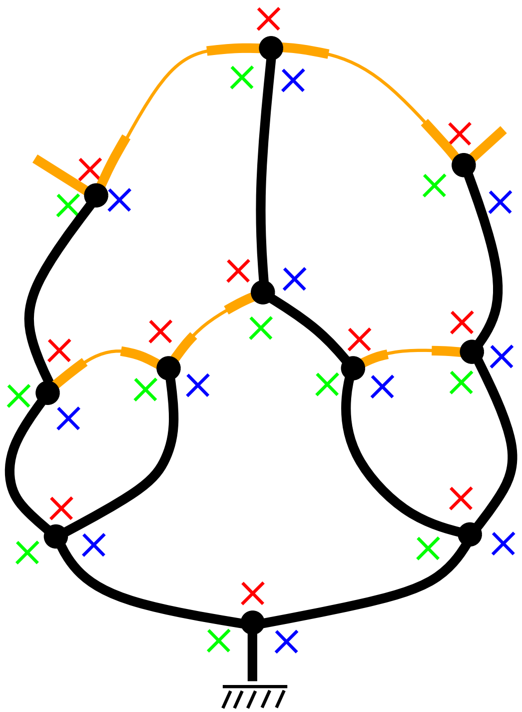

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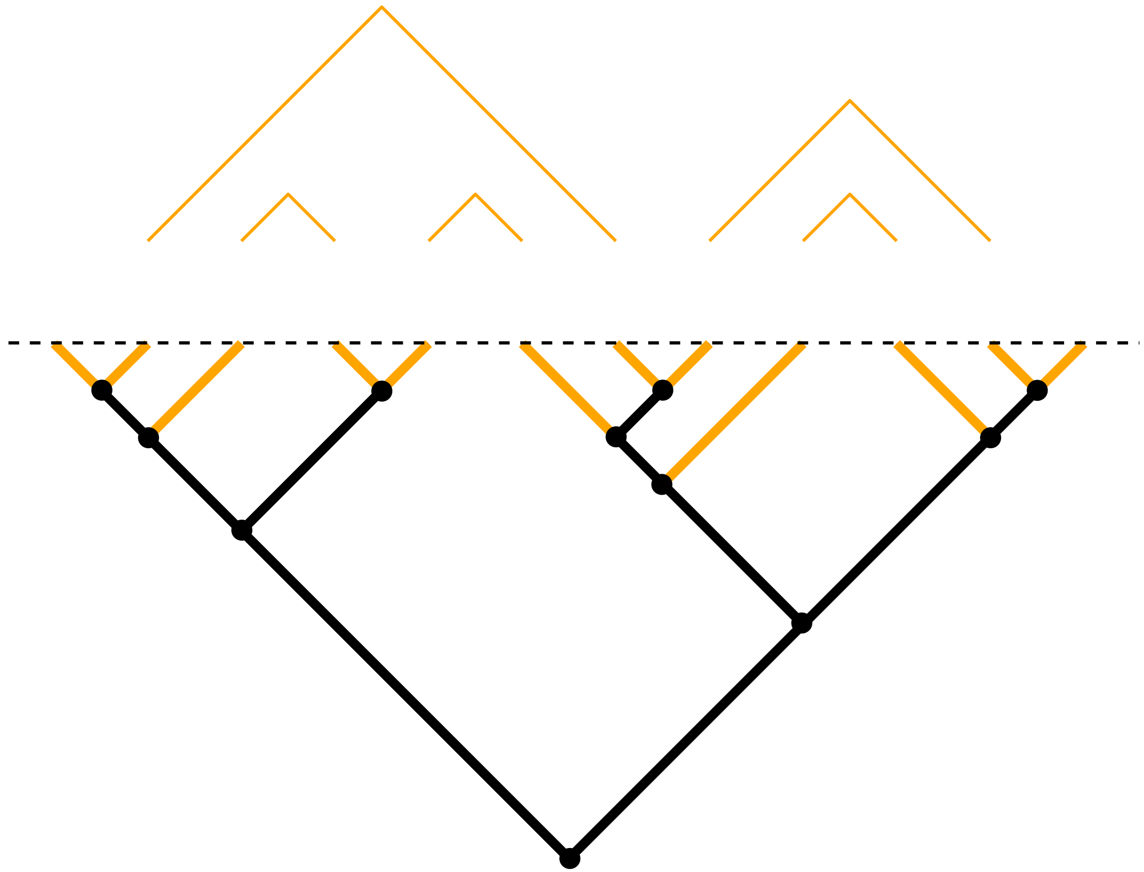
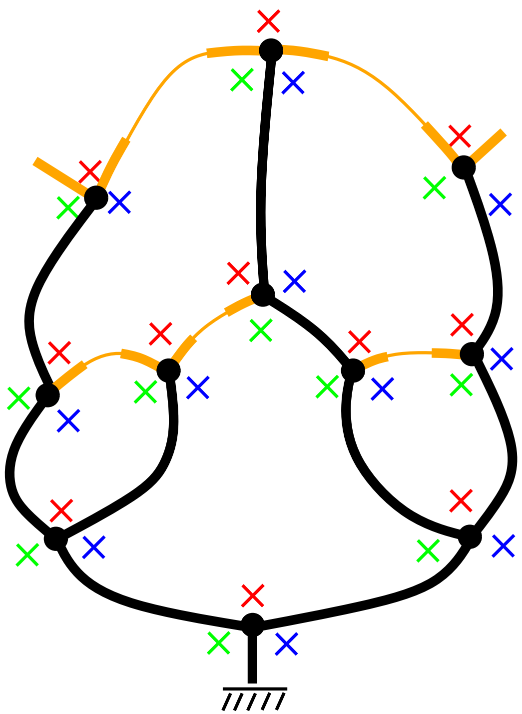


Bijection for Schnyder woods [Bonichon'02]

binary tree such that there is a parenthesis **matching of the leaves**

rectilinear representation (encoded by **two words**)



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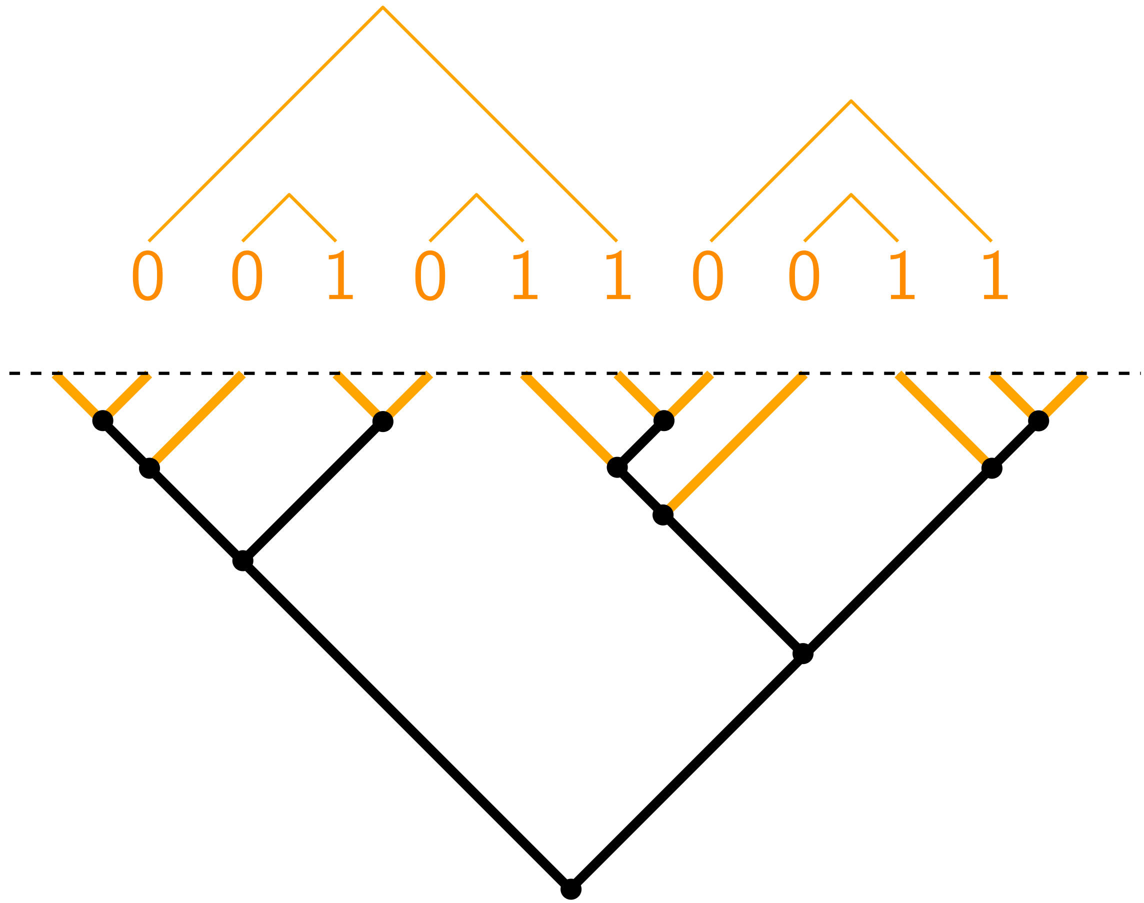

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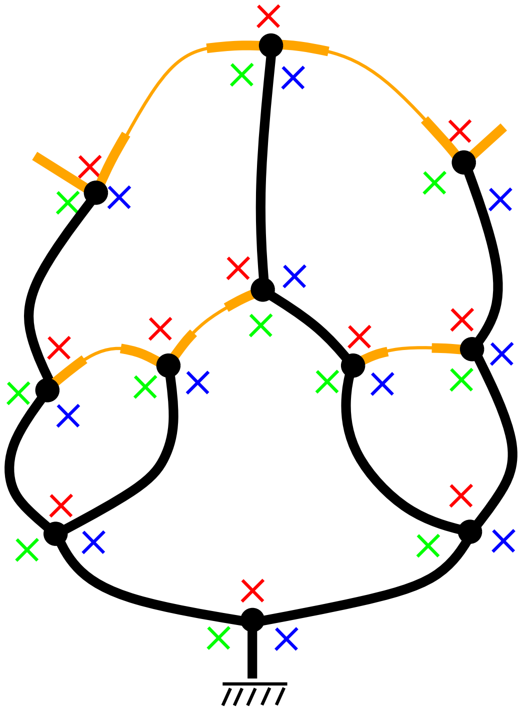

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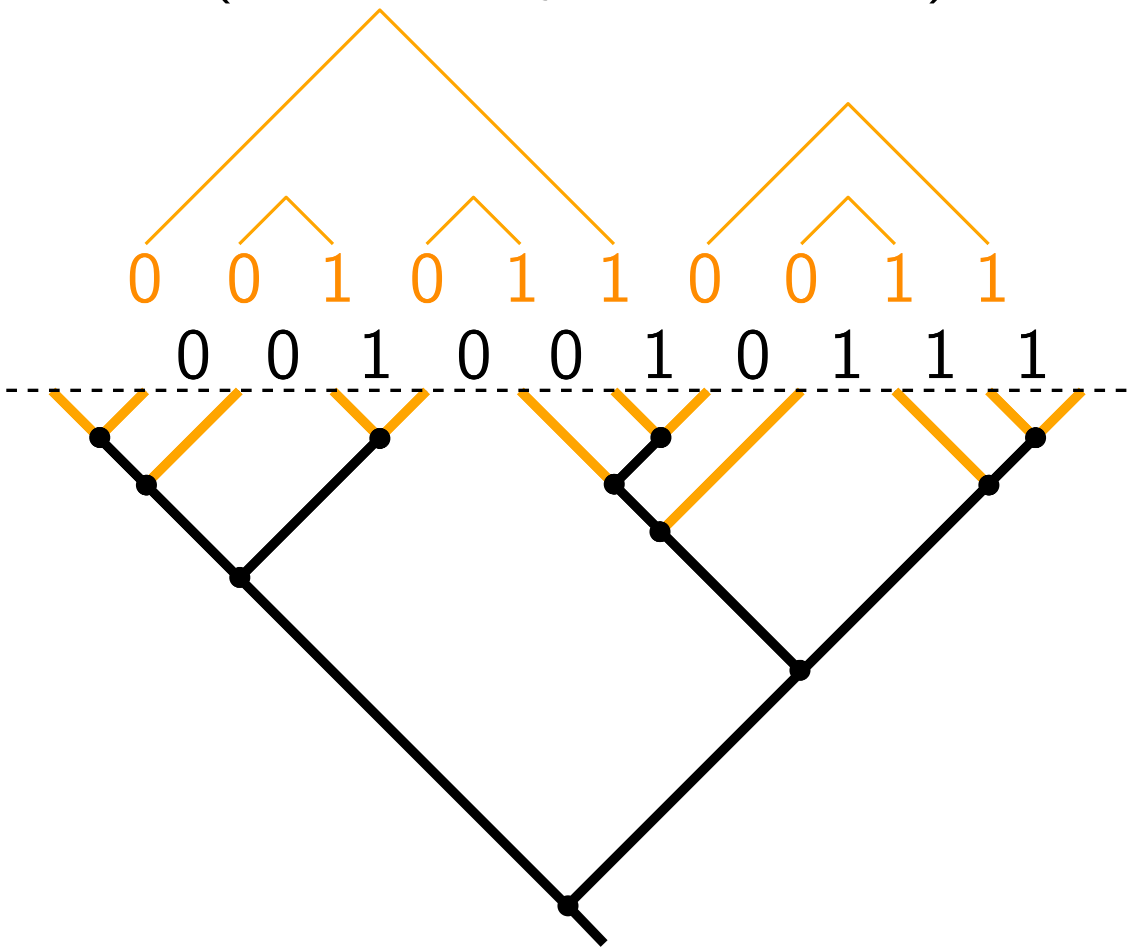

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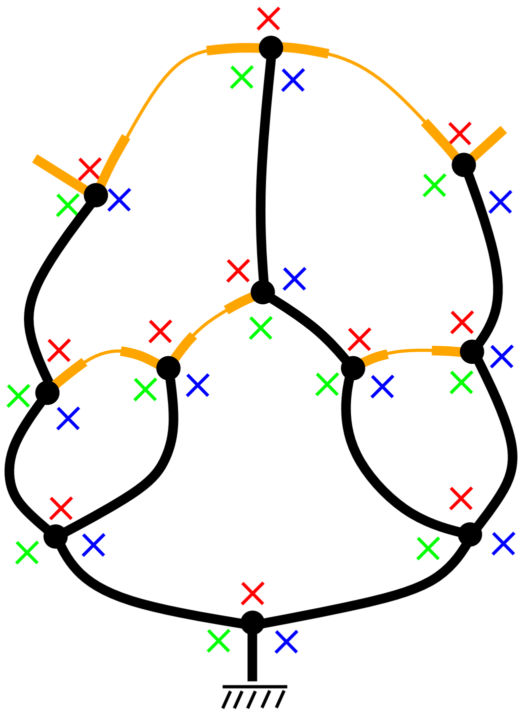

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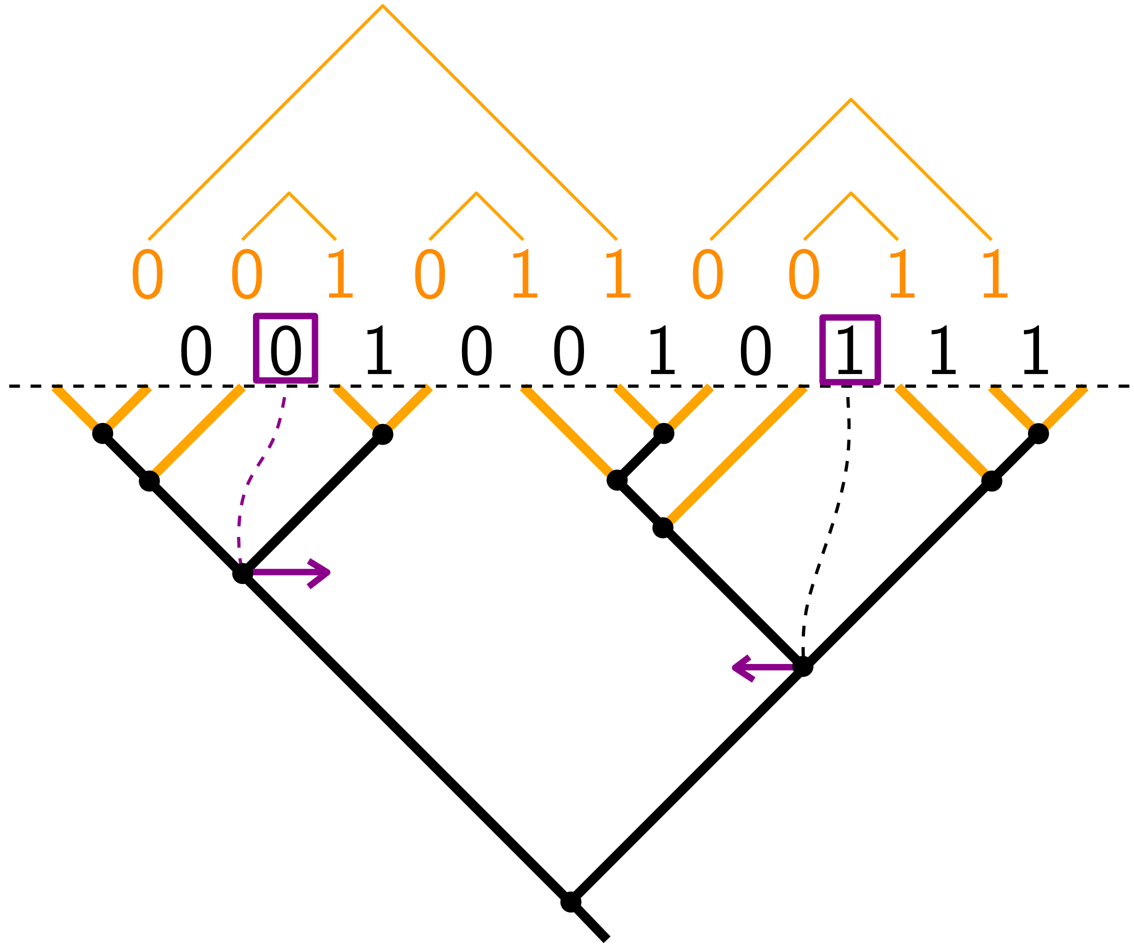

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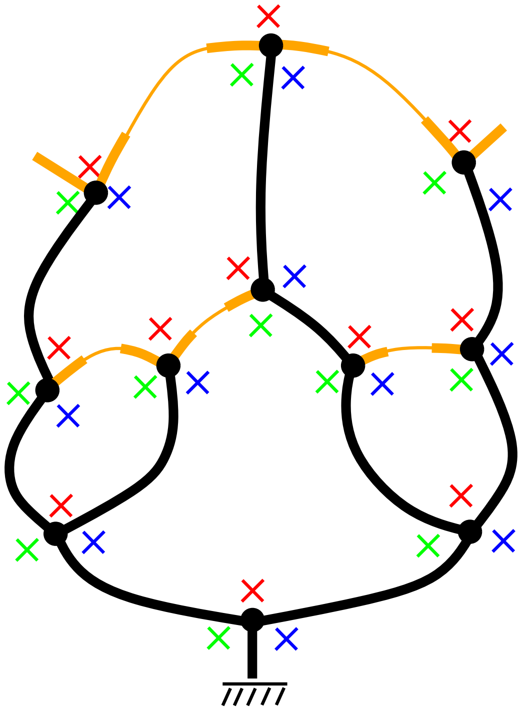

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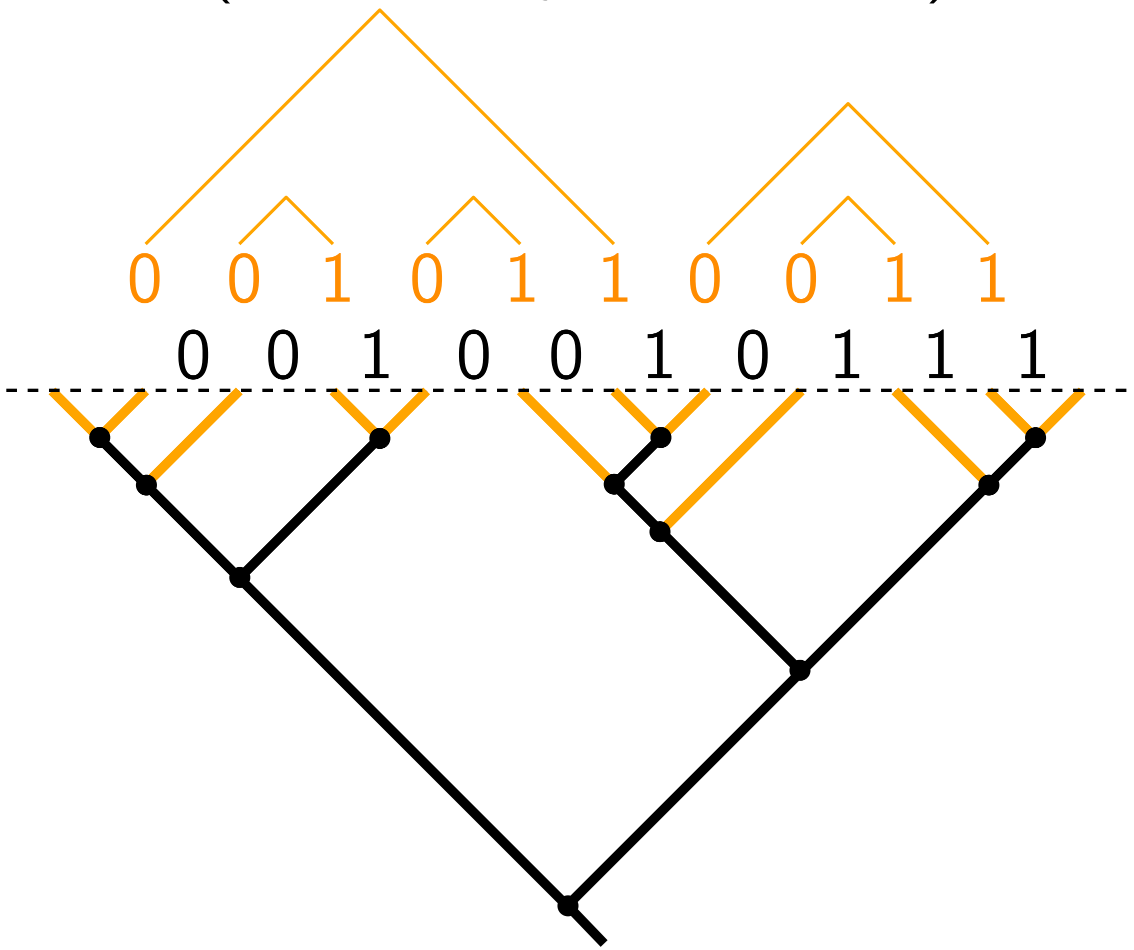

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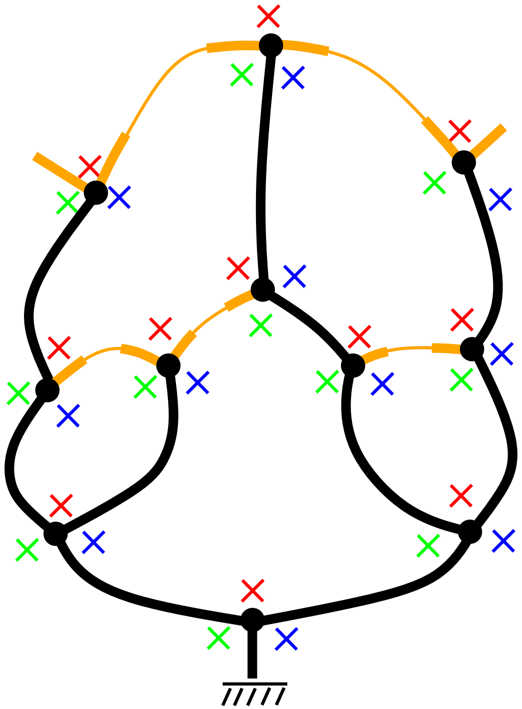

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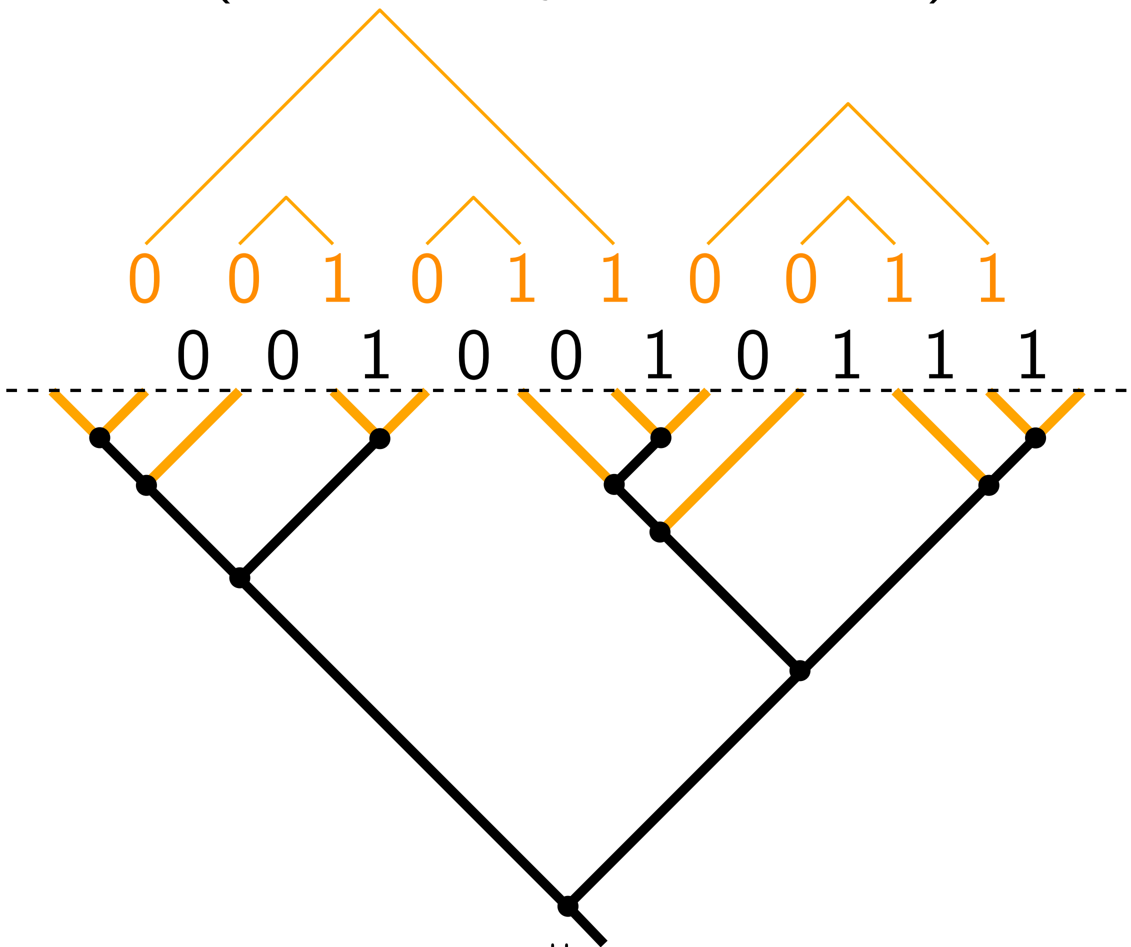
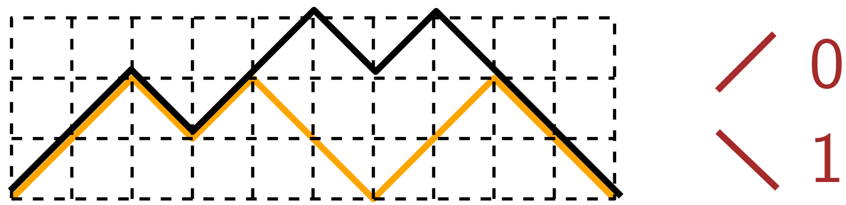


$\Downarrow$

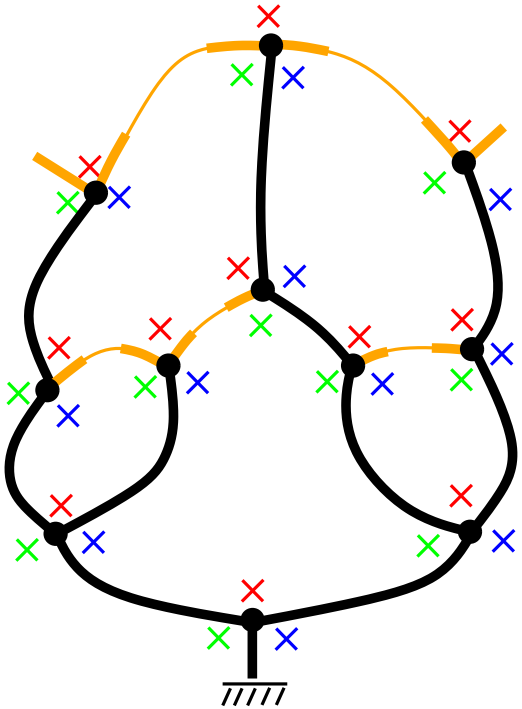


non-crossing pair of Dyck paths

Bijection for Schnyder woods

[Bonichon'02]

binary tree such that there is a parenthesis **matching of the leaves**



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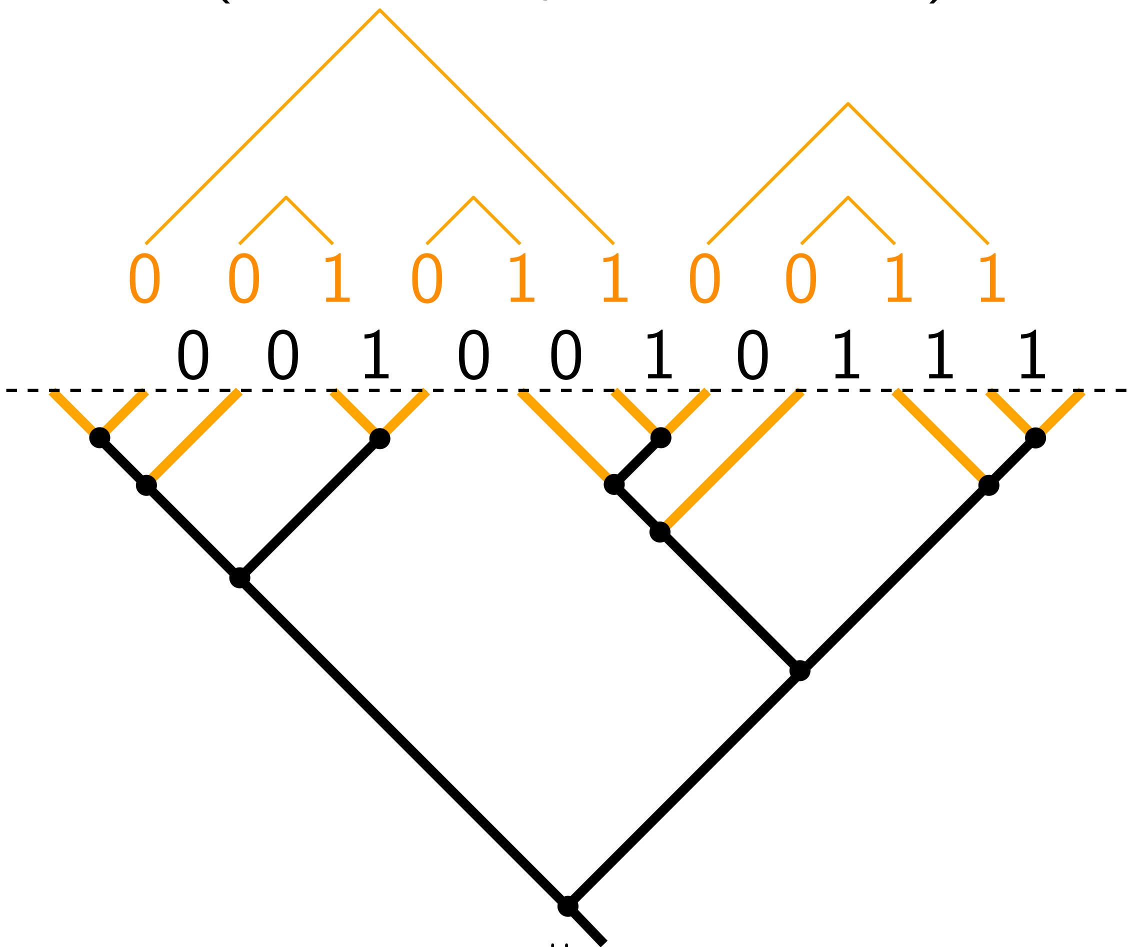
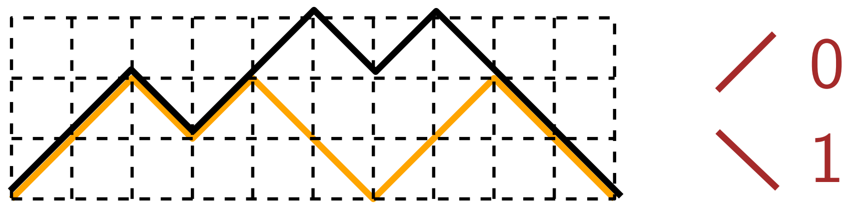




non-crossing pair of Dyck paths

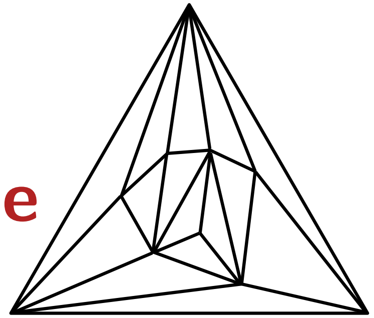
Total number s_n of **Schnyder woods** over triangulations with $n+3$ **vertices** is

$$\boxed{\begin{aligned} s_n &= \mathrm{Cat}_n \mathrm{Cat}_{n+2} - \mathrm{Cat}_{n+1} \mathrm{Cat}_{n+1} \\ &= \frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!} \end{aligned}}$$

Lattice property for Schnyder woods

[Ossona de Mendez'94], [Brehm'03]

Theorem: Let T be a simple triangulation. Then the set of Schnyder structures of T is a **distributive lattice**



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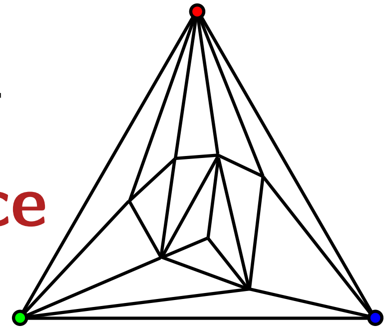

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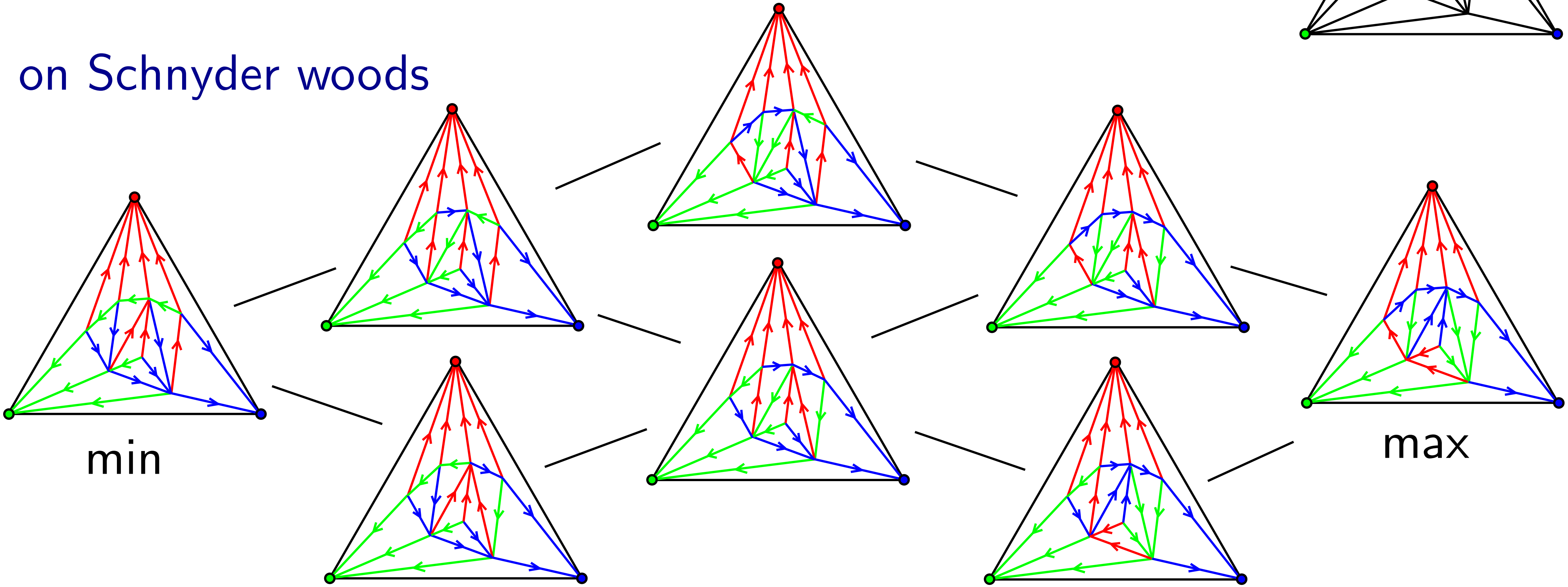
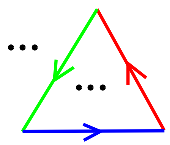
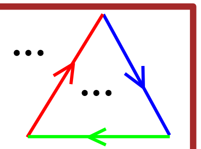
on Schnyder woods



Flip:  to 

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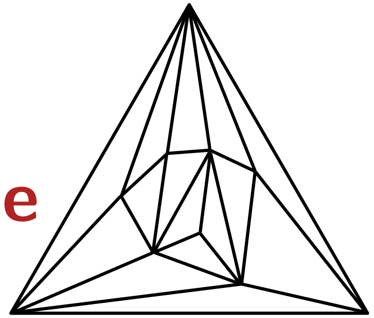
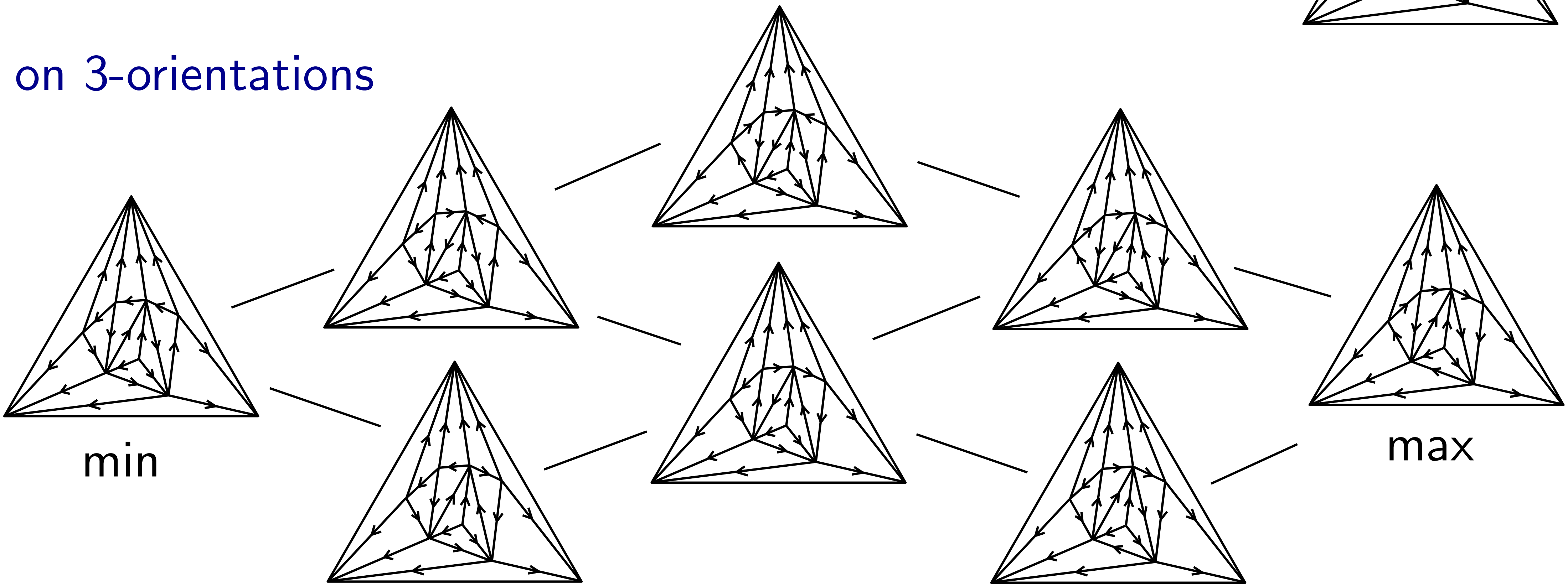
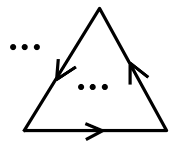
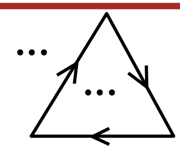


on 3-orientations



Flip:  to 

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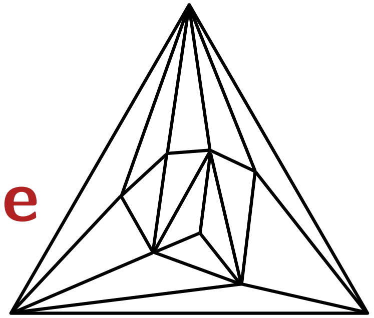

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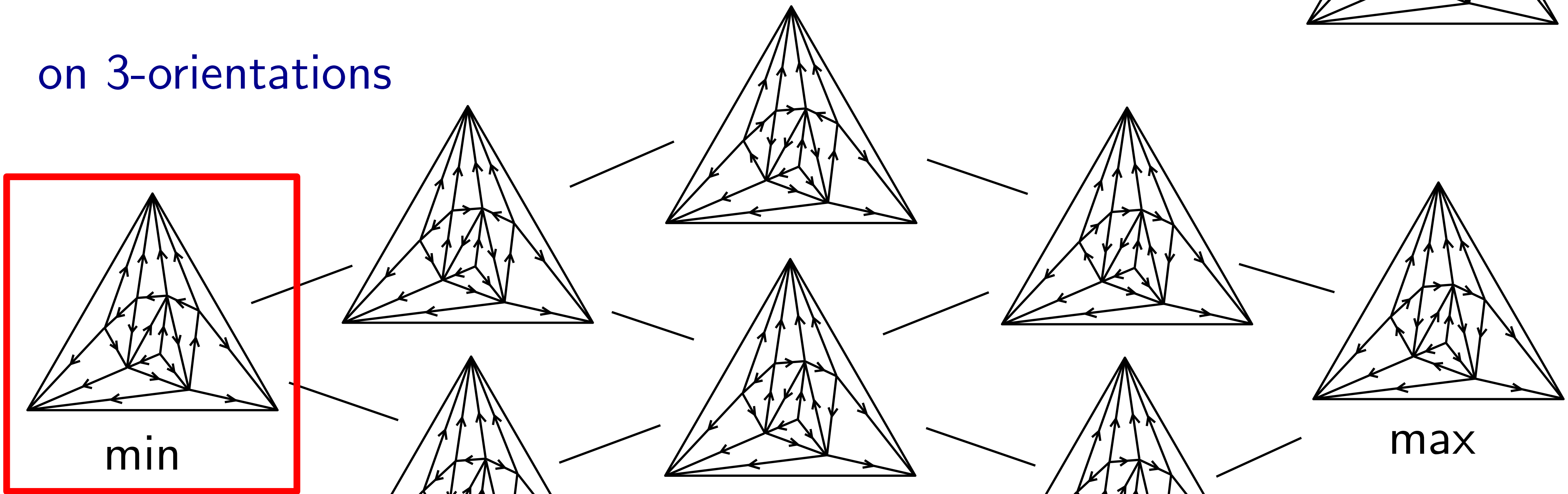

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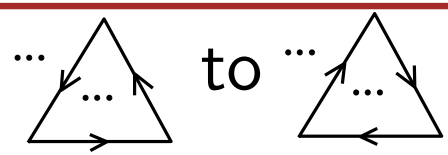
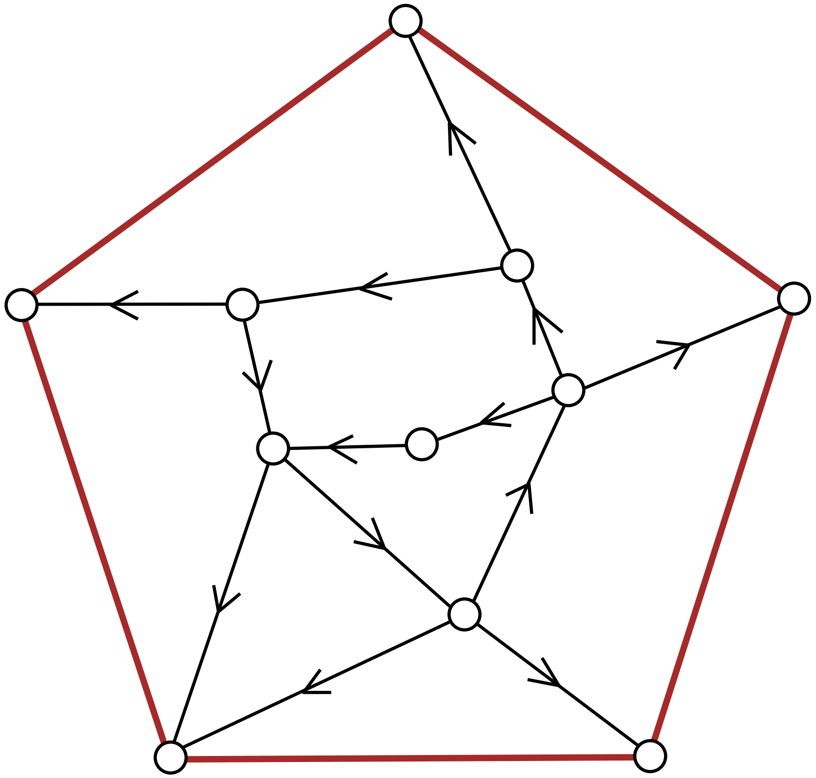


The min is the **unique 3-orientation** of T with **no clockwise circuit**

Orientations and mobiles

Let $\mathcal{O}$ be the set of **orientations** on planar maps such that:

- there is **no clockwise circuit**
- Each inner vertex can **access** the outer (unoriented simple) cycle
- the outer cycle is a **sink**



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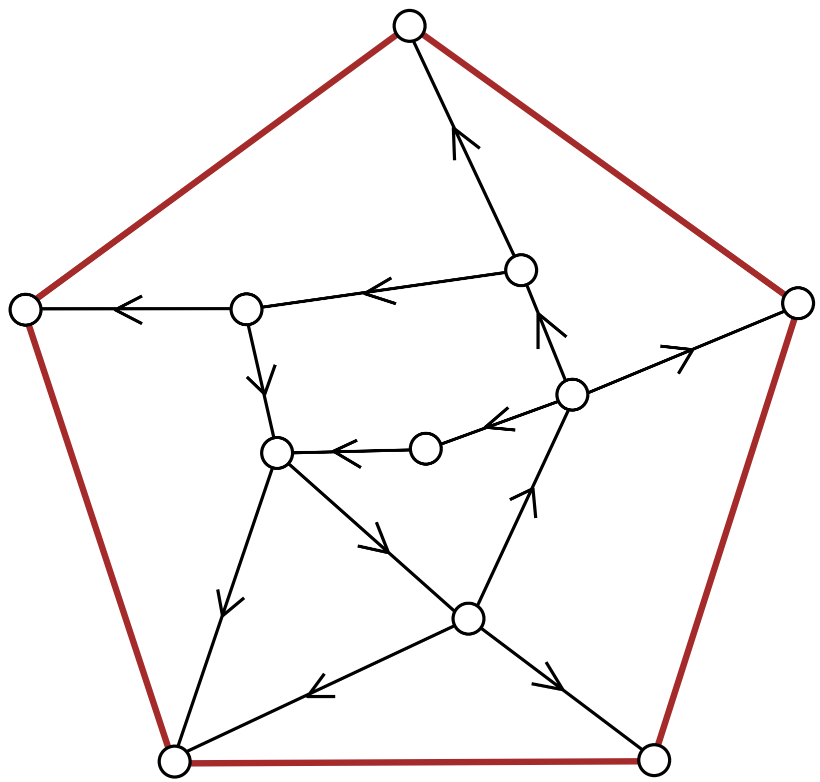
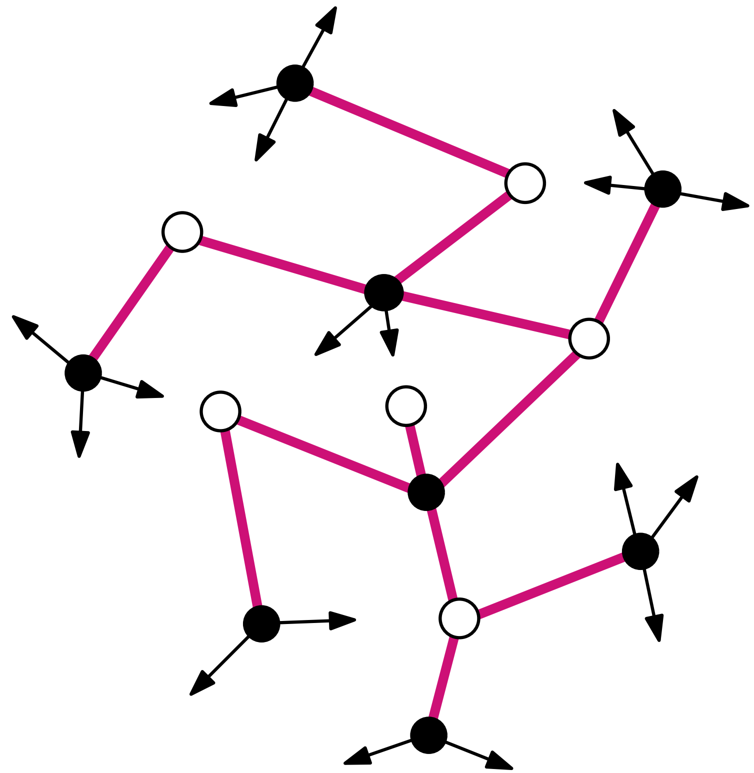
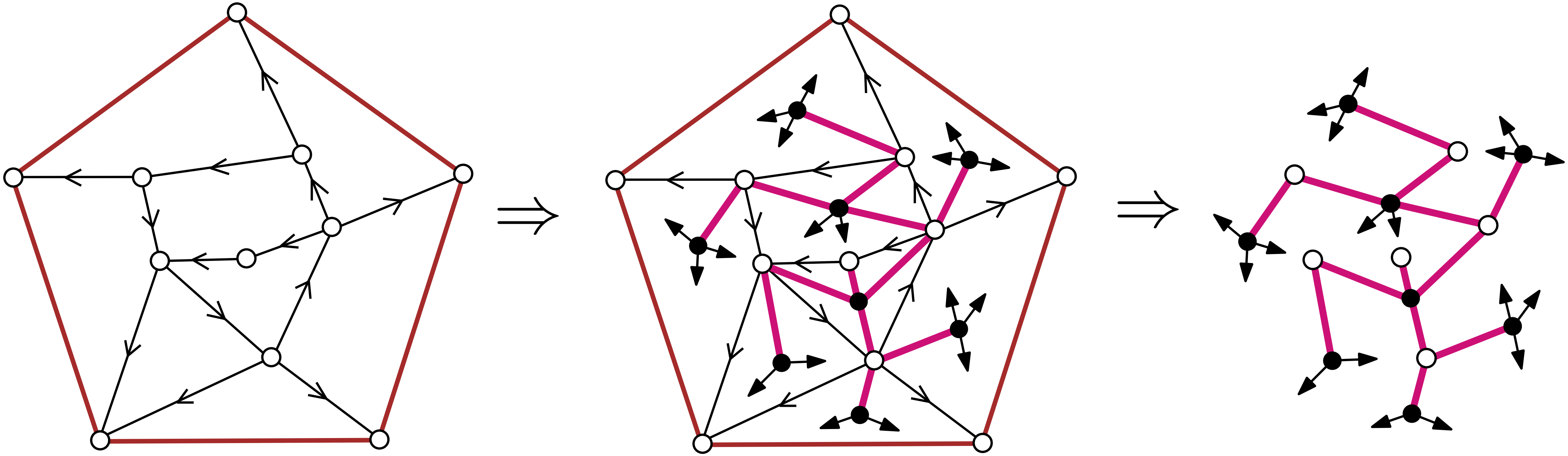
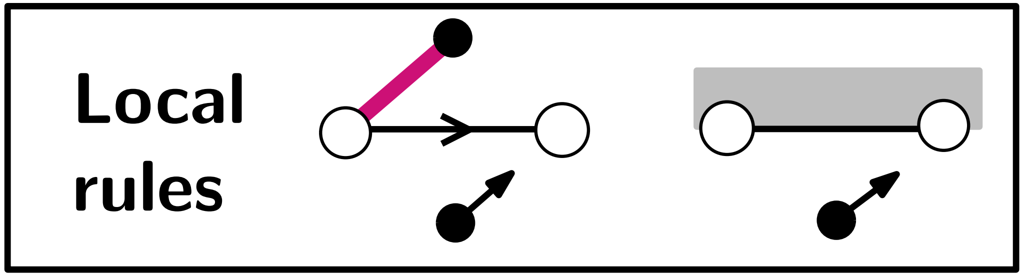


Let $\mathcal{M}$ be the set of **mobiles**, i.e., bipartite plane trees with **arrows** (called buds) at **black vertices**



“Master bijection” [Bernardi,F’10]





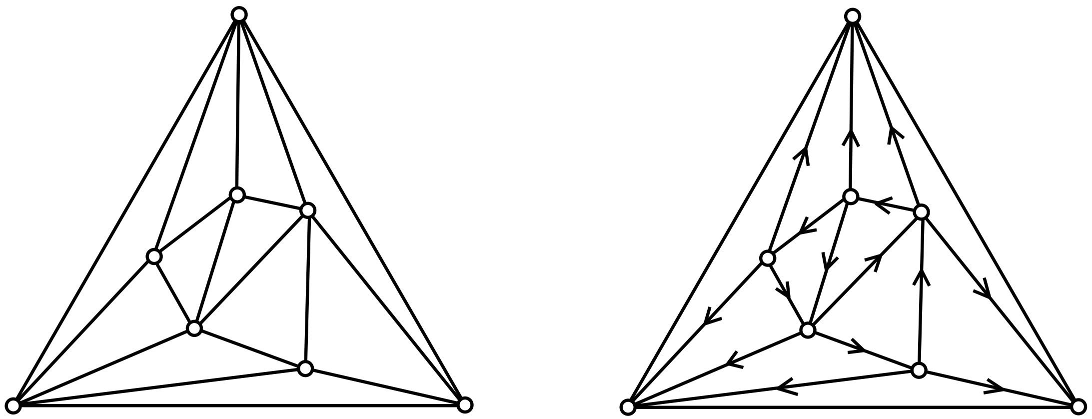
Theorem: The above construction Φ is a **bijection** between $\mathcal{O}$ and $\mathcal{M}$.
Moreover,

degrees of inner faces $\longleftrightarrow$ degrees of black vertices

outdegrees of inner vertices $\longleftrightarrow$ degrees of white vertices

Specialization to simple triangulations

- From the lattice property (**taking the min**) we have
  family of simple triangulations $\leftrightarrow$ **subfamily $\mathcal{F}$ of $\mathcal{O}$** where:
  - faces have degree 3
  - inner vertices have outdegree 3



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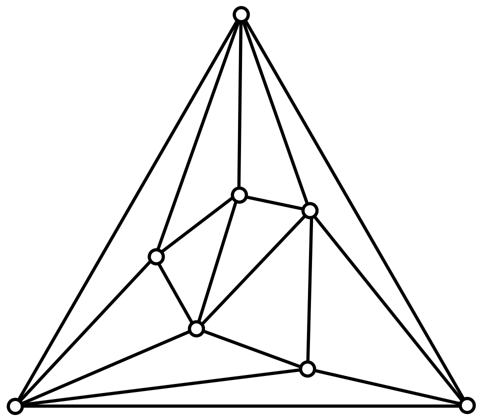
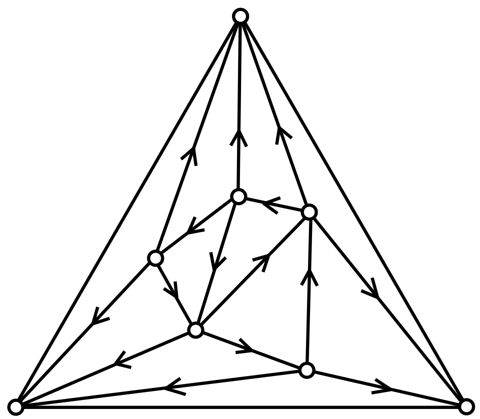
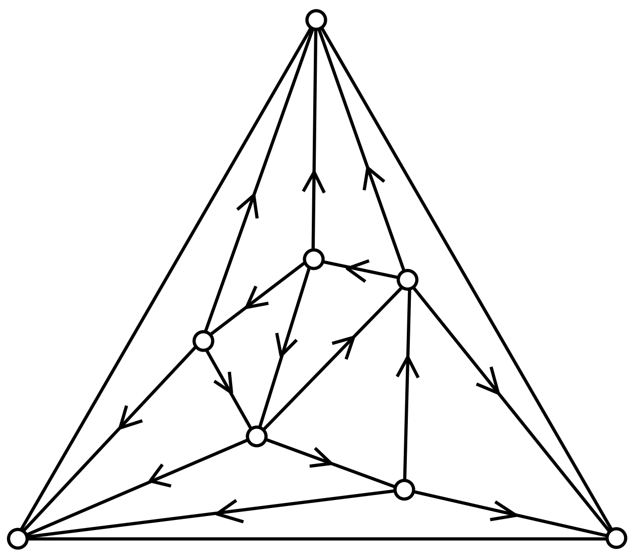
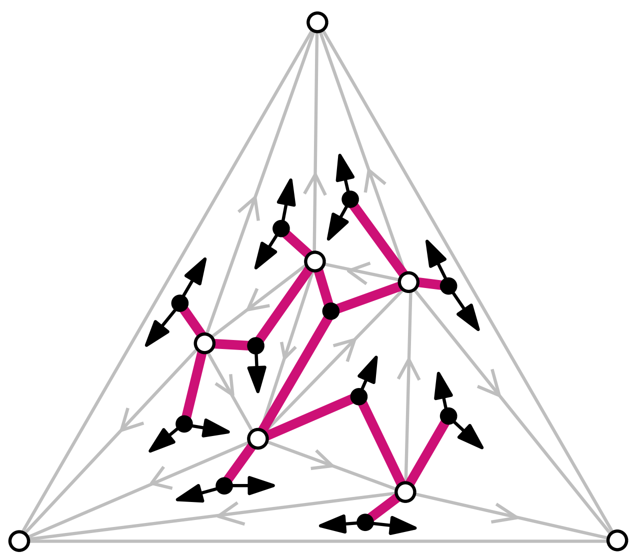
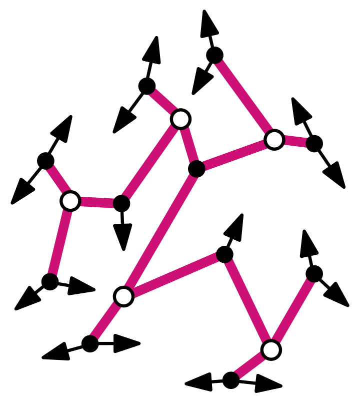



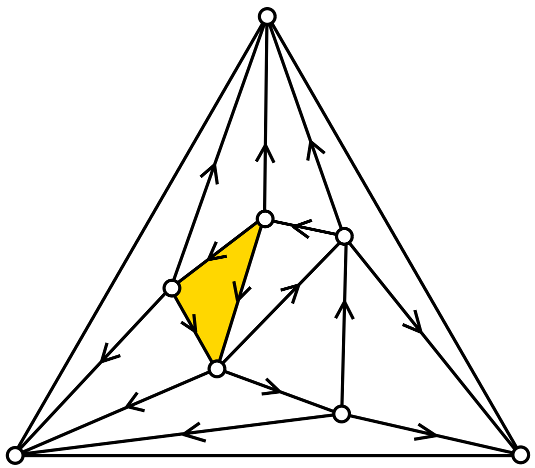
- From the **master bijection specialized to $\mathcal{F}$**, we have
  $\mathcal{F}$ $\leftrightarrow$ **subfamily of mobiles** where all vertices have **degree 3**

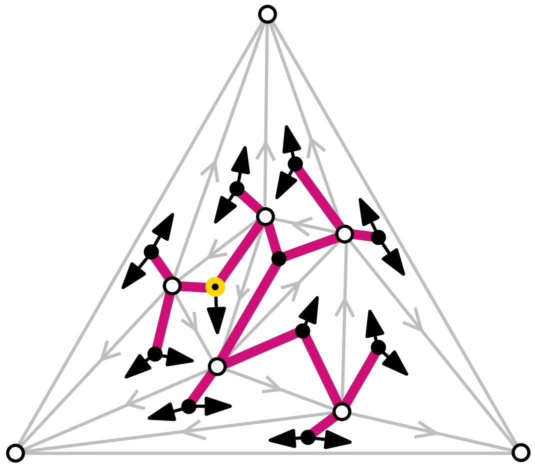




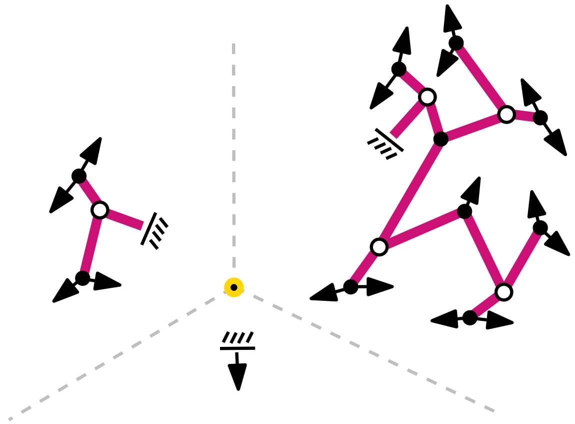
[F, Poulalhon, Schaeffer'08], other bijection in [Poulalhon, Schaeffer'03]

Counting formula

The **bijection** when there is a **marked inner face**:







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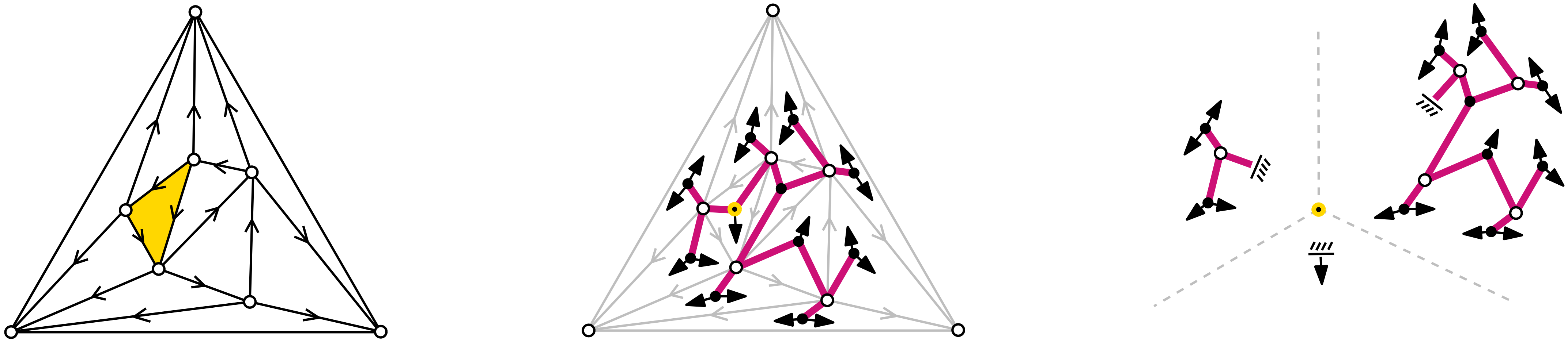
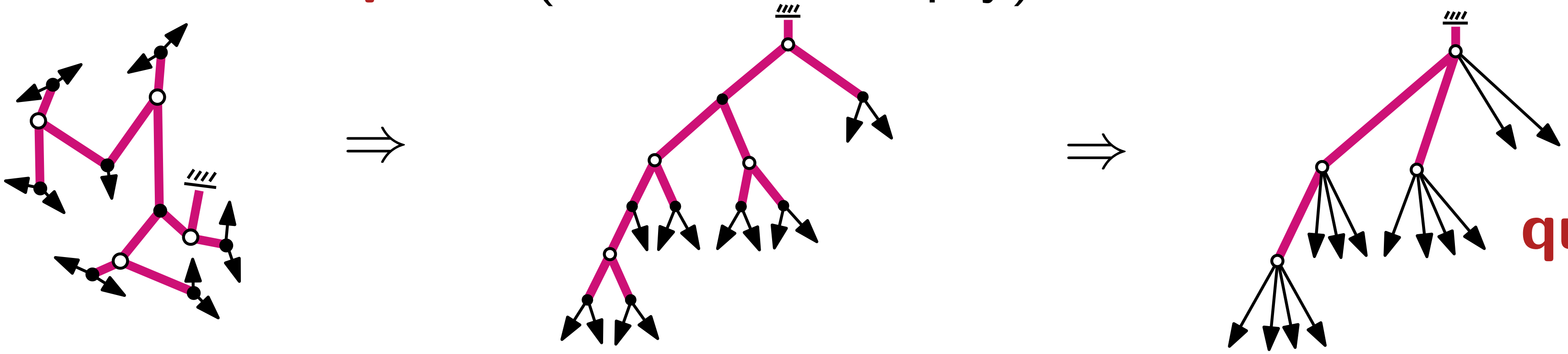


Each of the **3 parts** (when non empty) is **of the form**



quaternary tree

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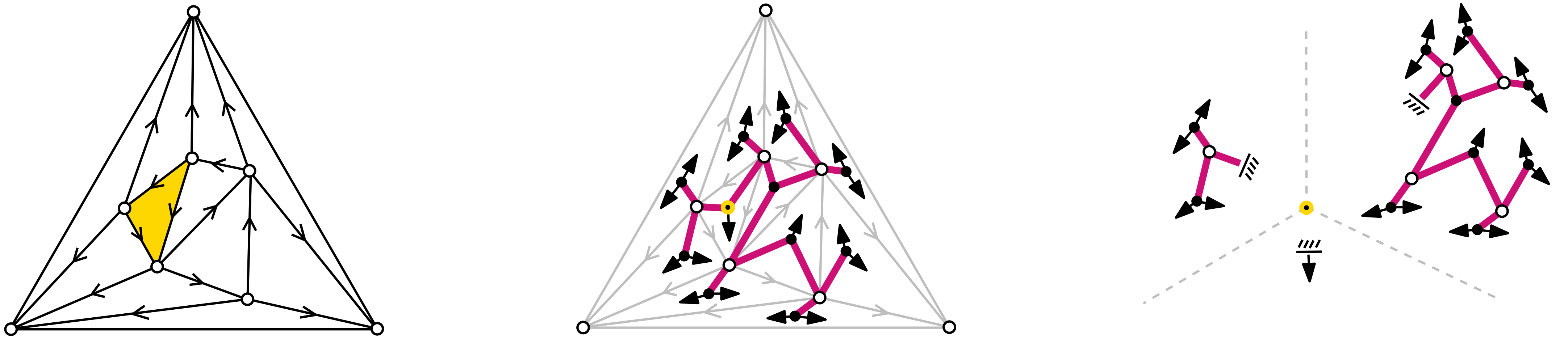

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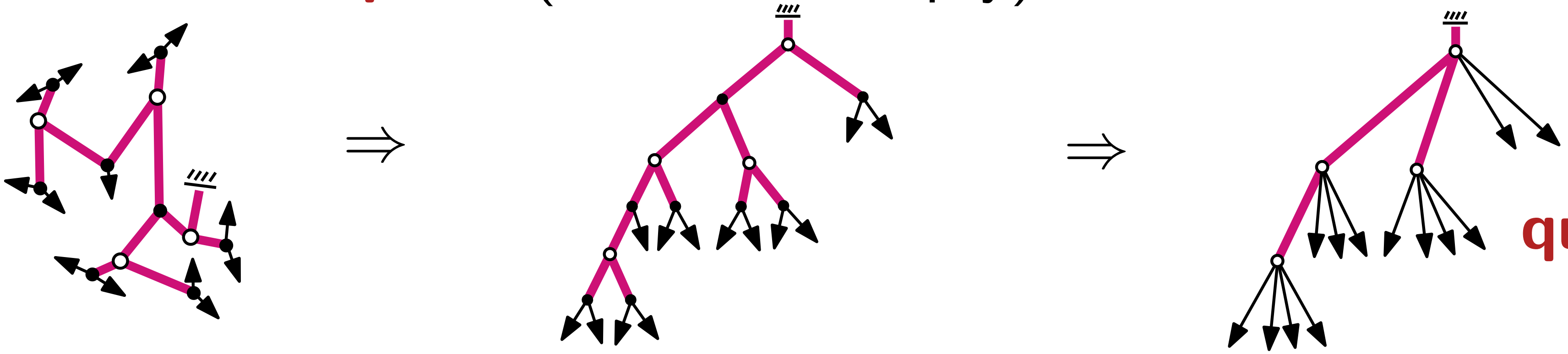

quaternary tree

Let $t_n = \#\{\text{(rooted) triang. with } n+3 \text{ vertices}\}$, $F(x) = \sum_n t_n x^{2n+1}$

Then $\boxed{F'(x) = (1+u)^3}$ where $u = u(x)$ is specified by $\underbrace{u = x^2(1+u)^4}_{\text{quat. trees}}$

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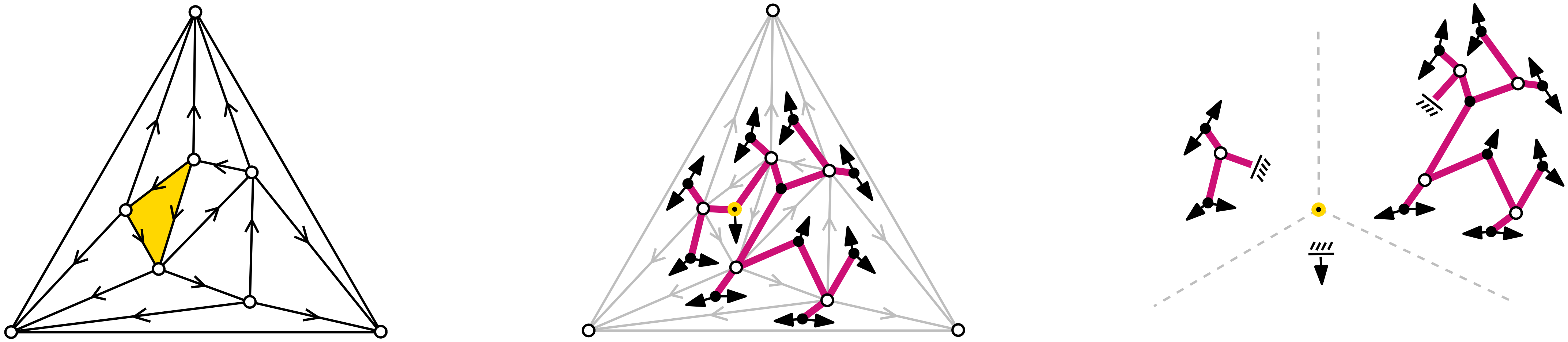

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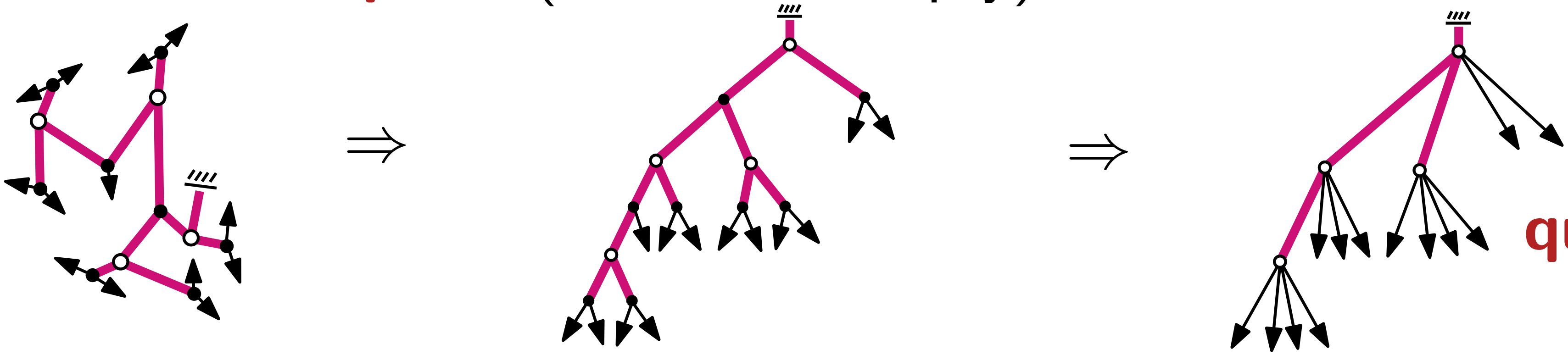

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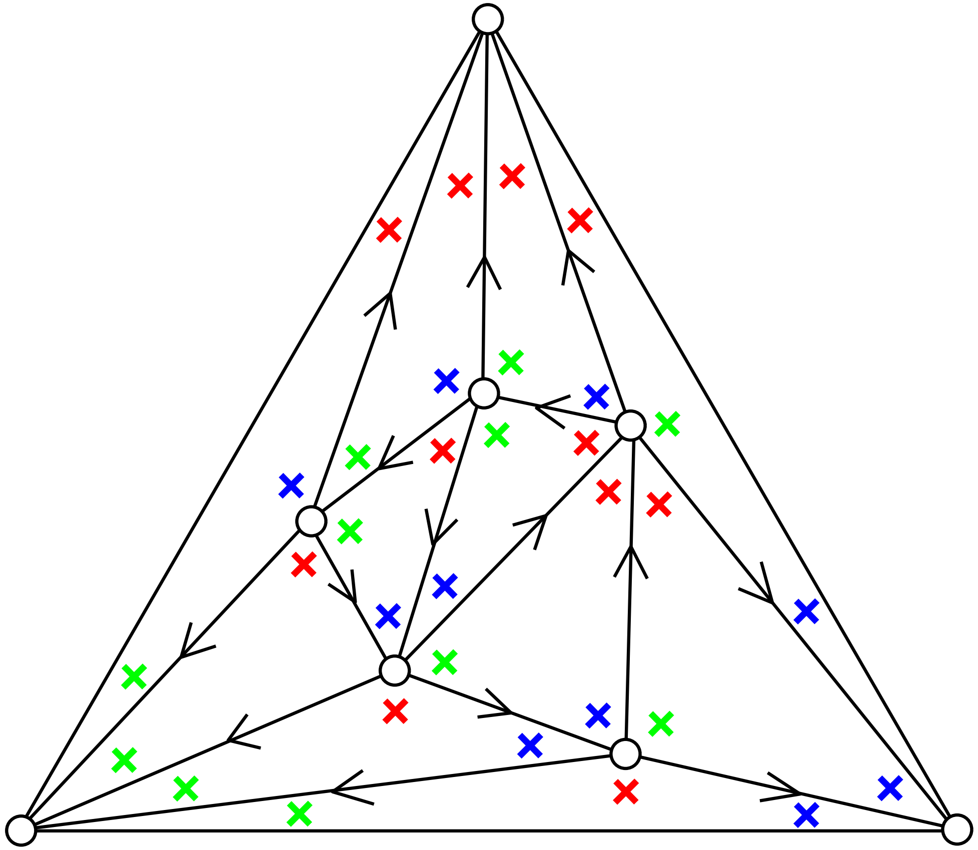
$\Rightarrow$ (Lagrange) $\boxed{t_n = \dfrac{2(4n+1)!}{(n+1)!(3n+2)!}}$ [Tutte'62]

Colored formulation of the bijection

Take the **Schnyder labelling** corresponding to the **minimal 3-orientation**



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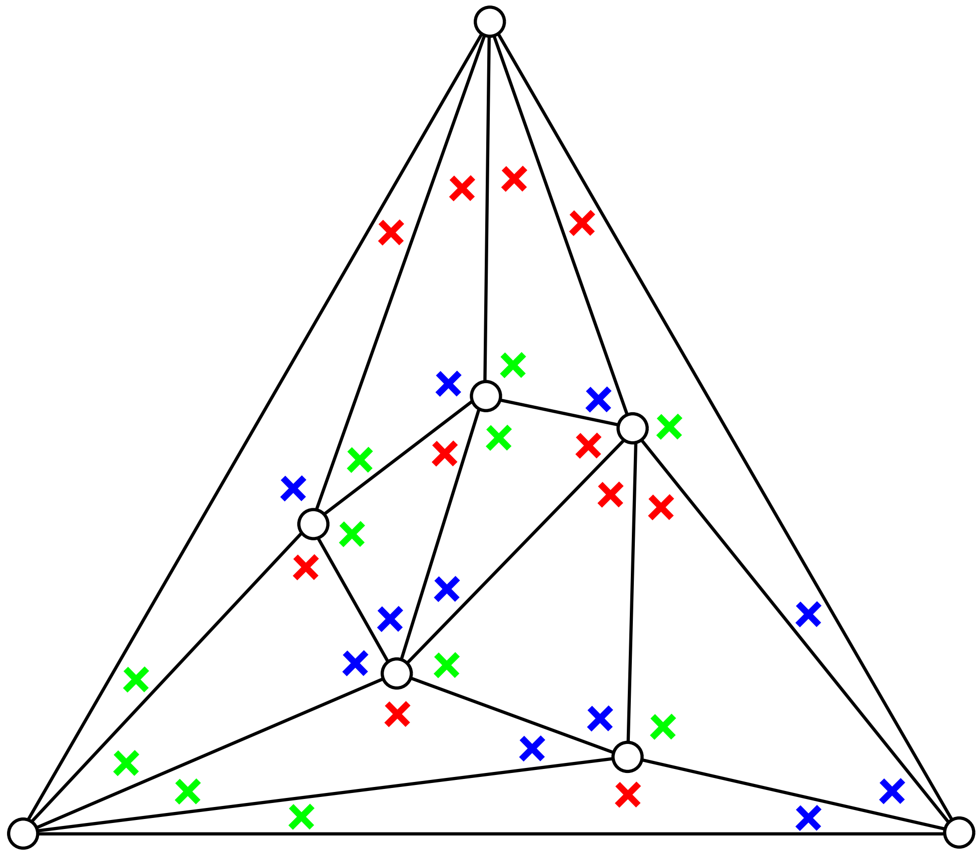

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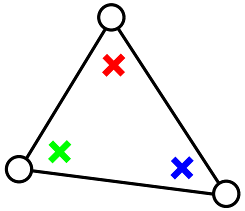
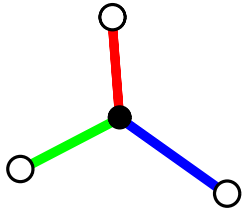
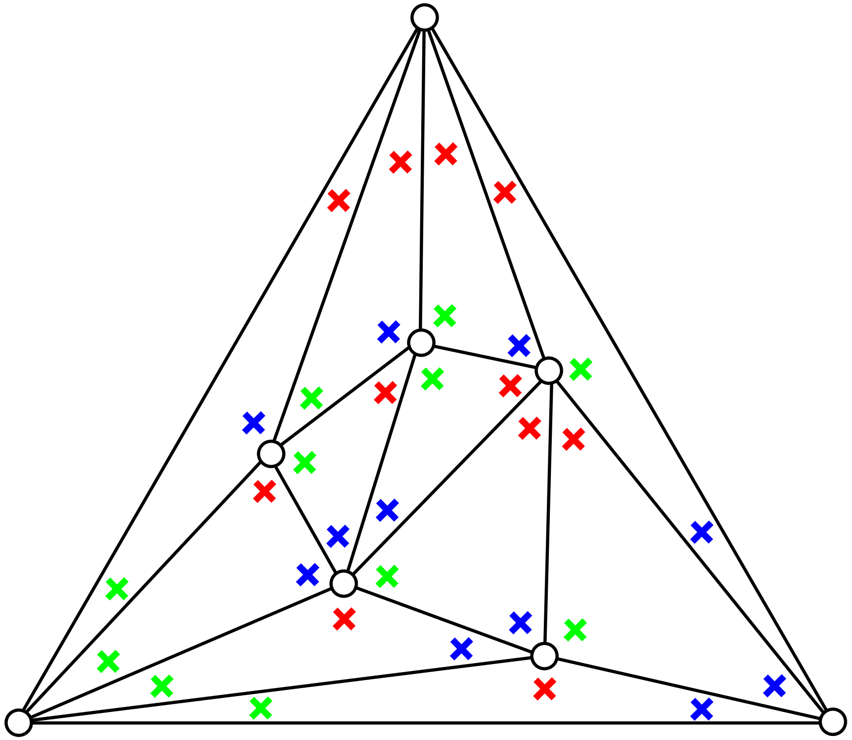
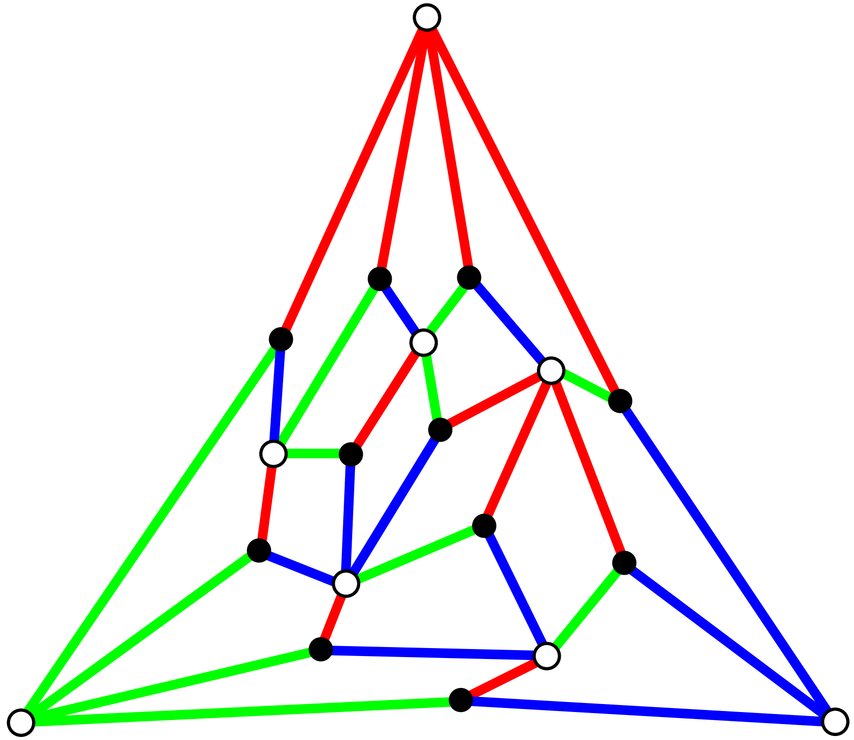
Replace each  by 



$\Rightarrow$



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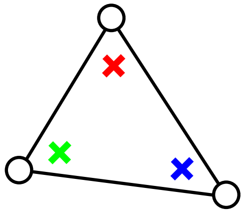
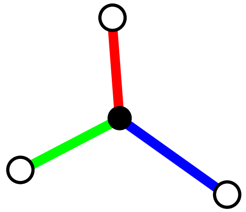
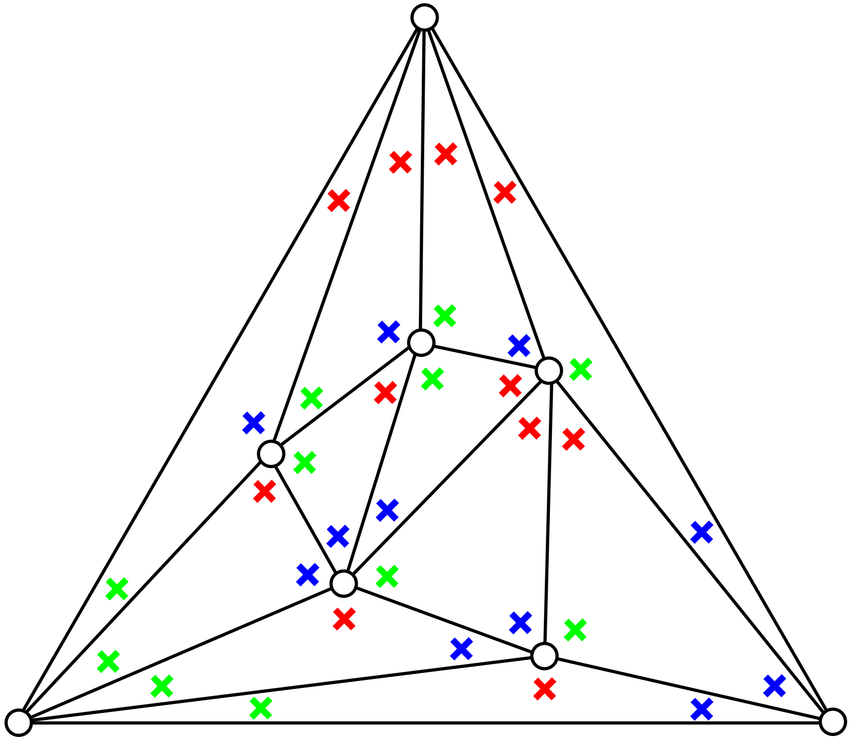
Replace each  by 



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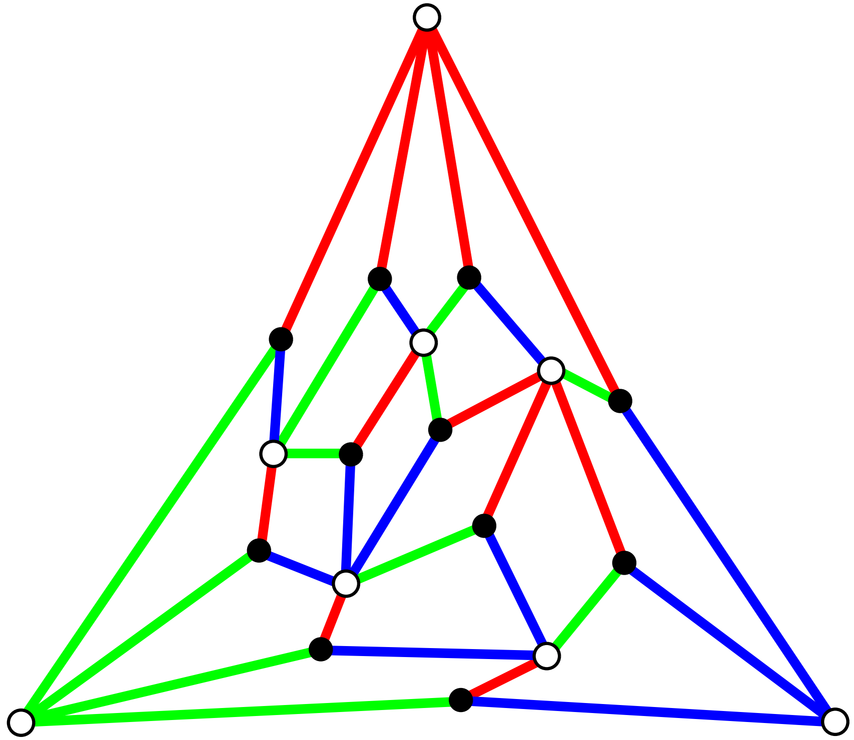
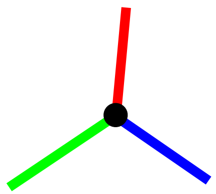
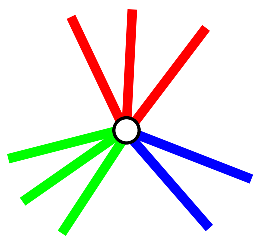
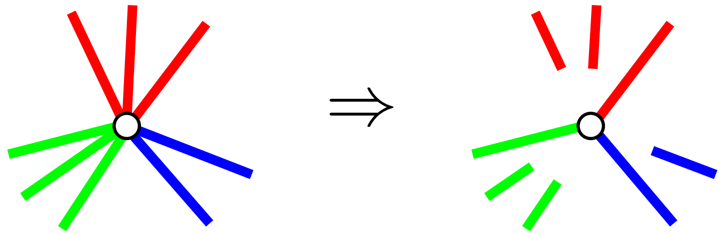
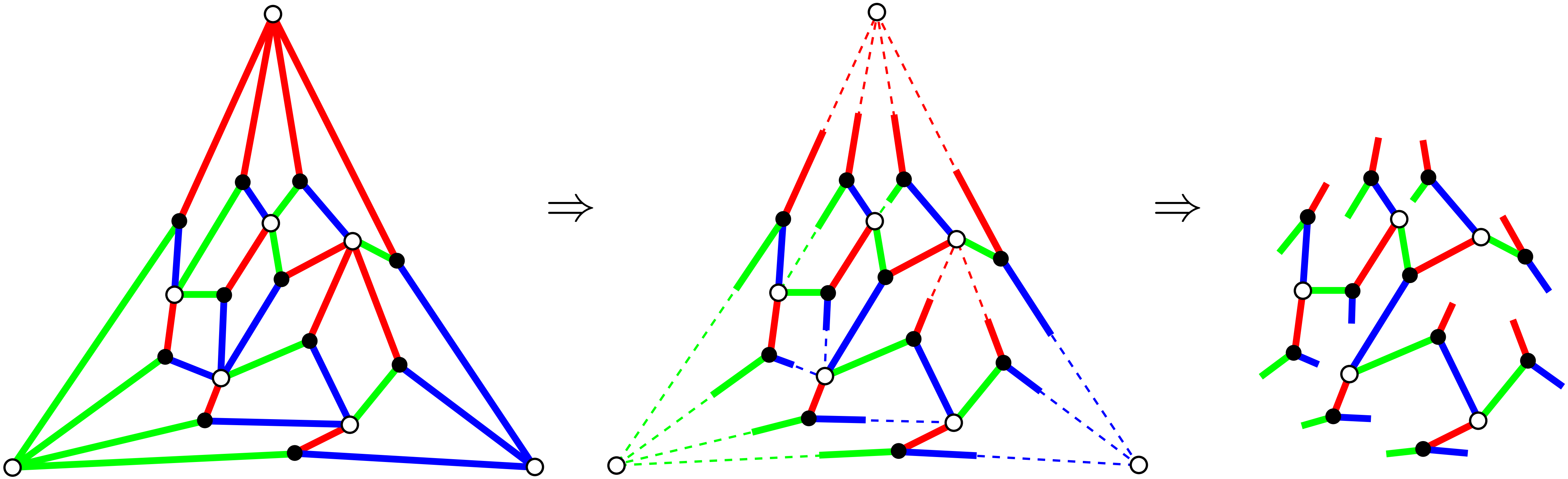


Local rules: 



Colored formulation of the bijection

- **Apply**  to each **inner white vertex**
- **Erase** the 3 **outer vertices** and their incident half-edges



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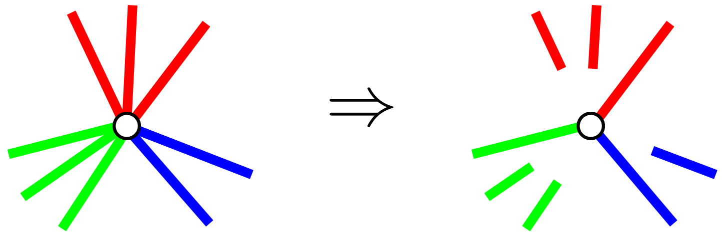
- **Apply**  to each **inner white vertex**

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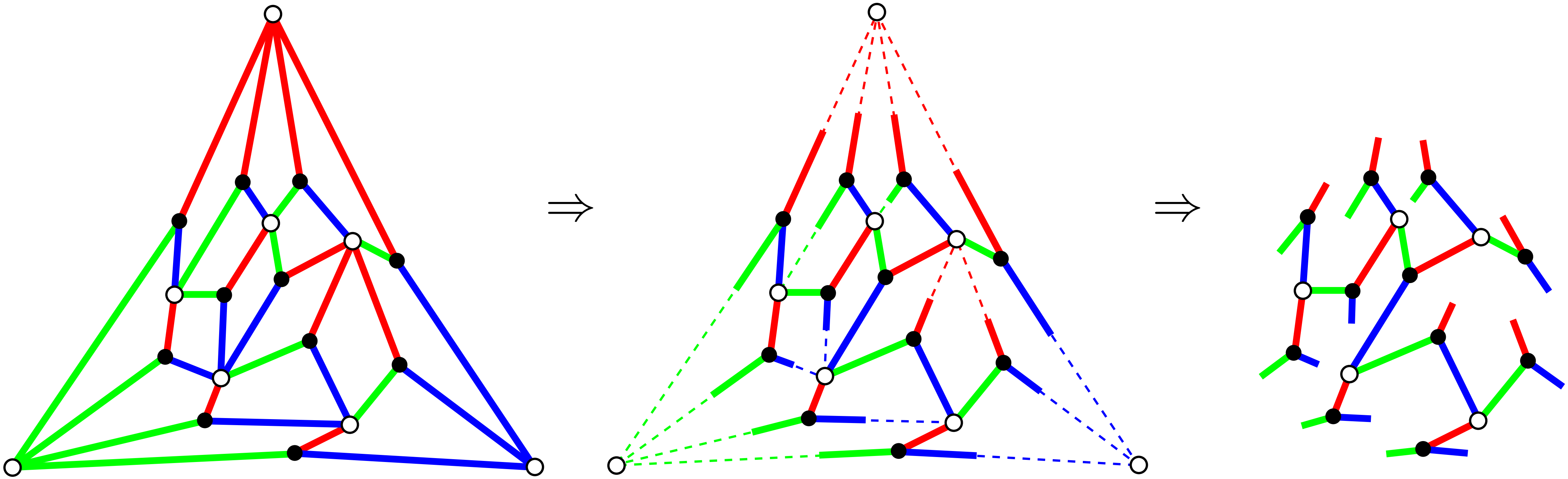
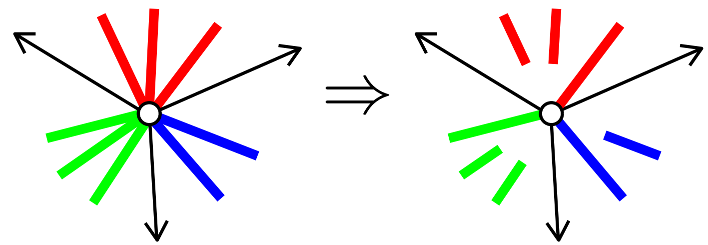
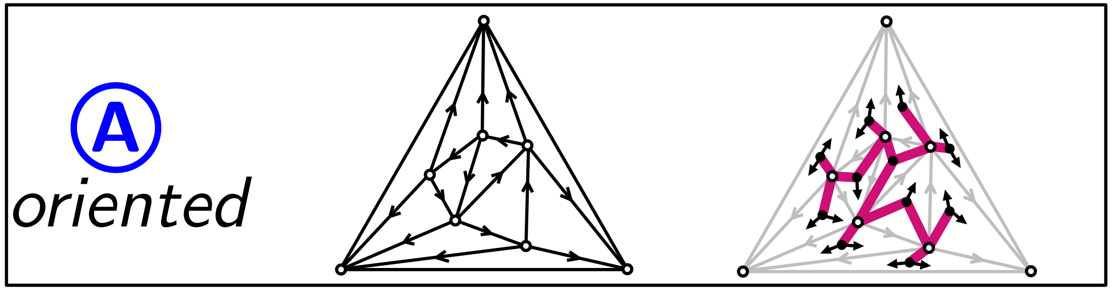
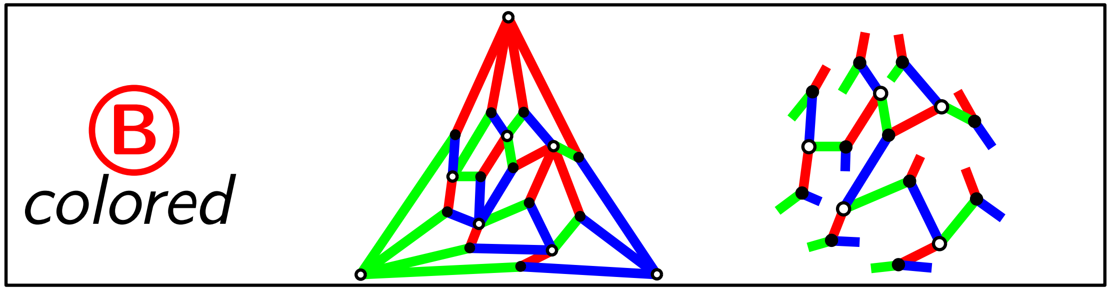


Same bijection as before, because 

Summary and extensions

- We have **two formulations of a bijection** for (simple) triangulations





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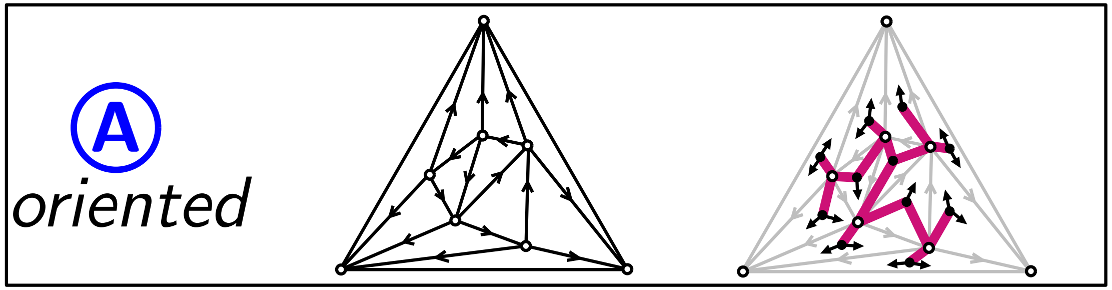

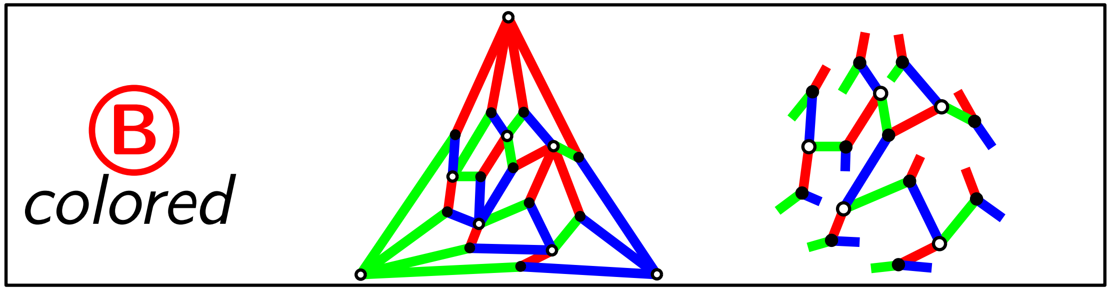

Let $t_n = \#\{\text{(rooted) triang. with } n+3 \text{ vertices}\}$, $F(x) = \sum_n t_n x^{2n+1}$

- Yields the **counting formulas** (one for GF, one for coefficients):

① $F'(x) = (1+u)^3$ where $u = x^2(1+u)^4$

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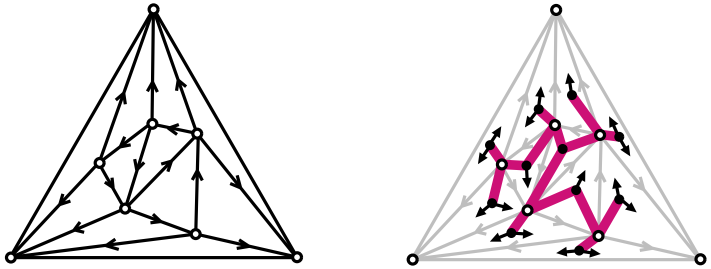
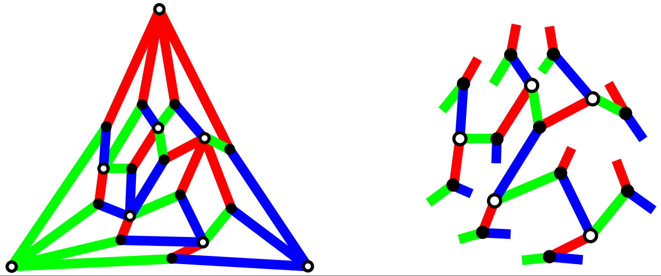


B *colored*



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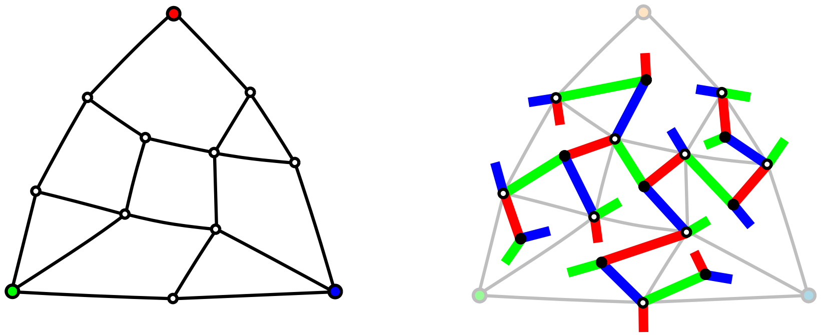
- We now give **two extensions**:

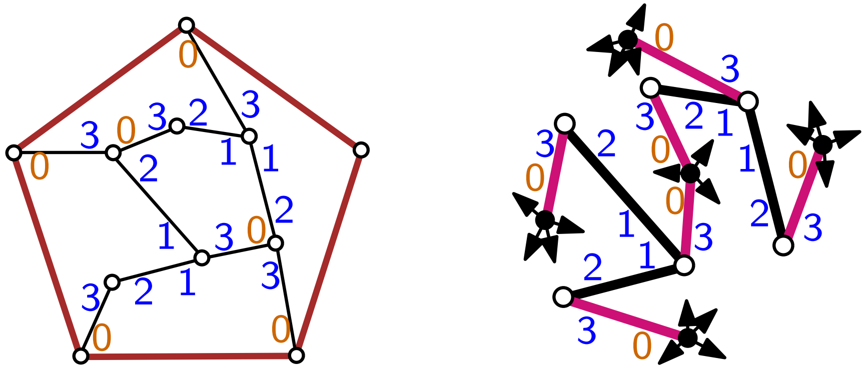
3-connected maps



Bijection extends B

Counting: (bivariate) extends 2

d-angulations of girth d



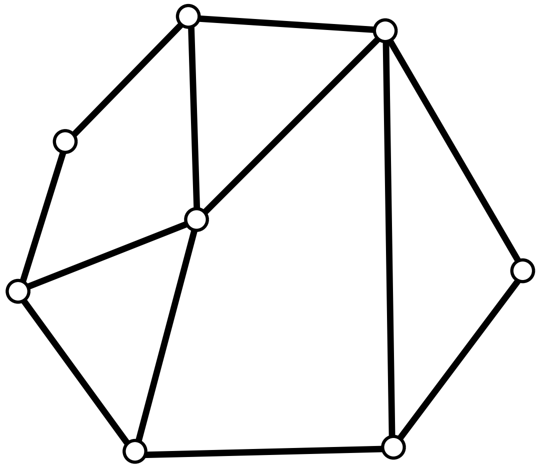
Bijection extends A

Counting: (bivariate) extends 1

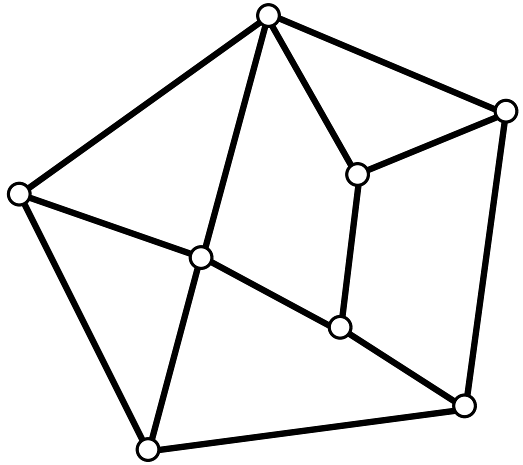
Extension to 3-connected maps

3-connectivity

3-connected graph = needs **delete** at least 3 vertices to disconnect it



not 3-connected



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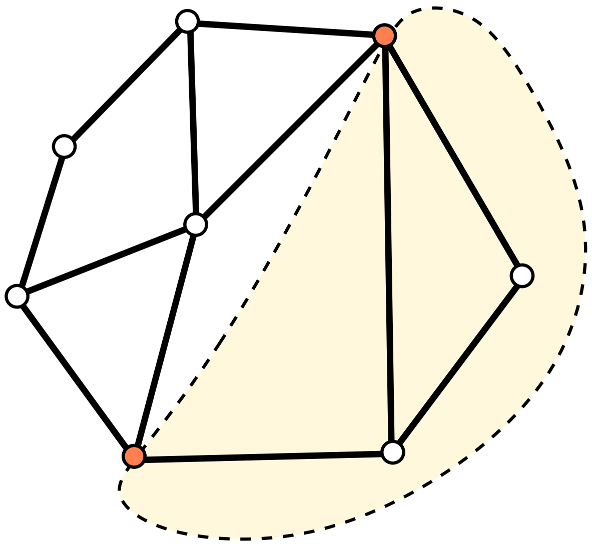
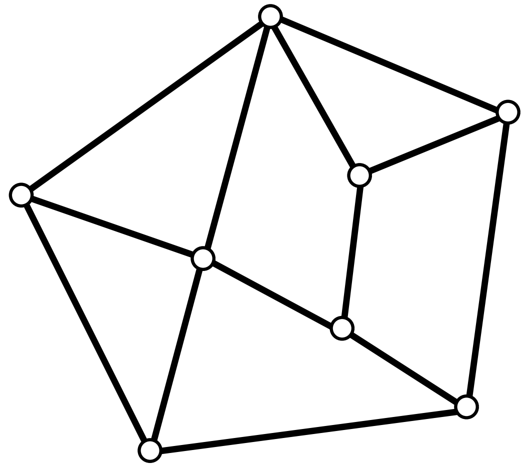


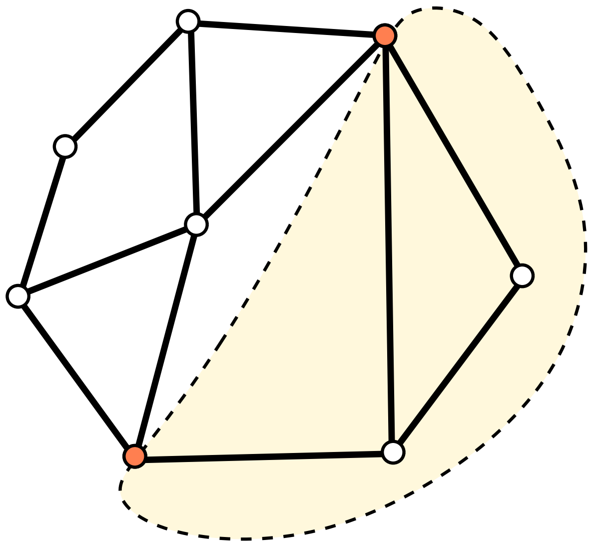
not 3-connected



3-connected

3-connectivity

3-connected graph = needs **delete** at least 3 vertices to disconnect it



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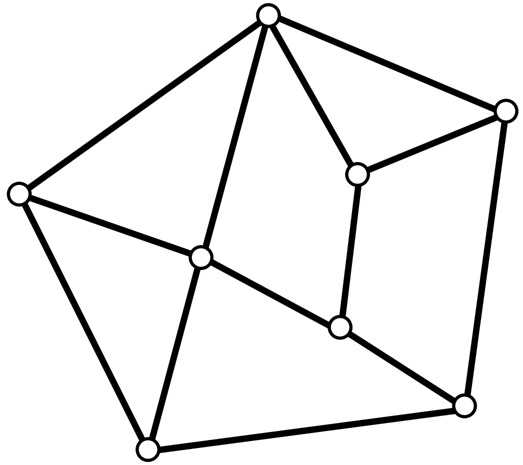
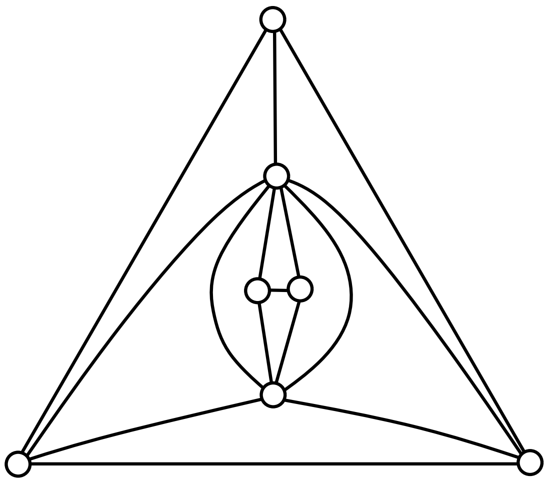
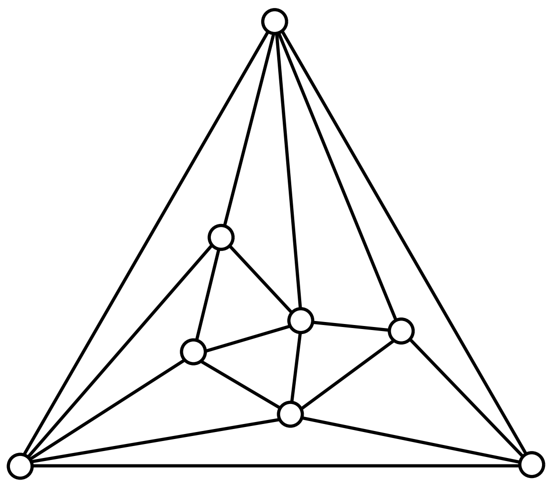


3-connected

Rk: a triangulation is **3-connected** iff it is **simple**





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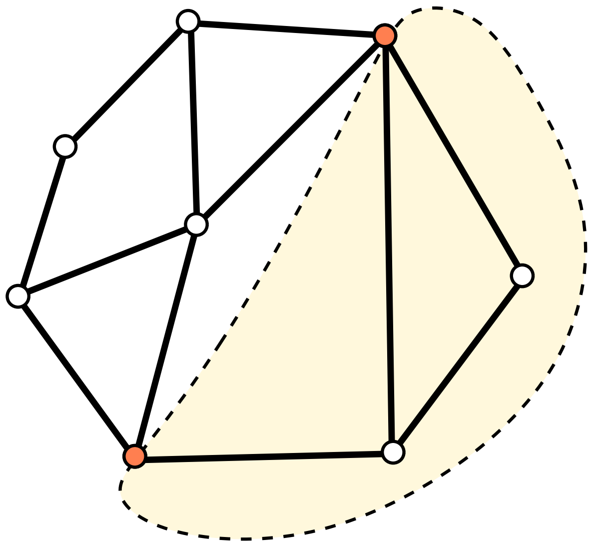

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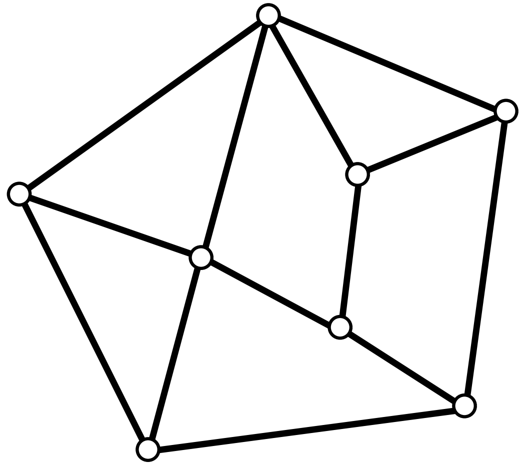

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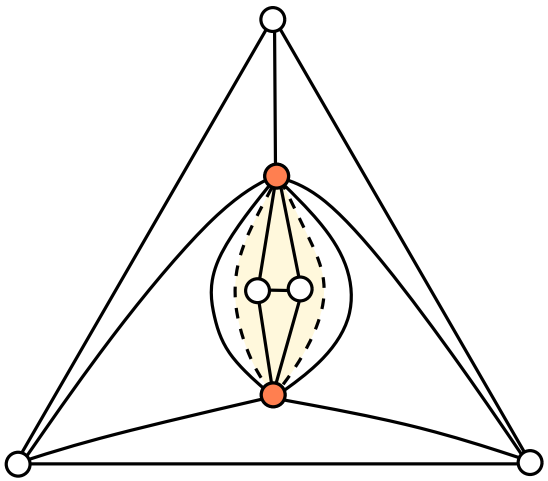
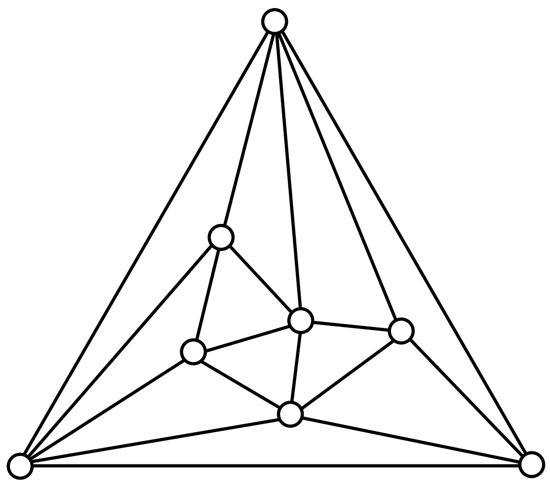
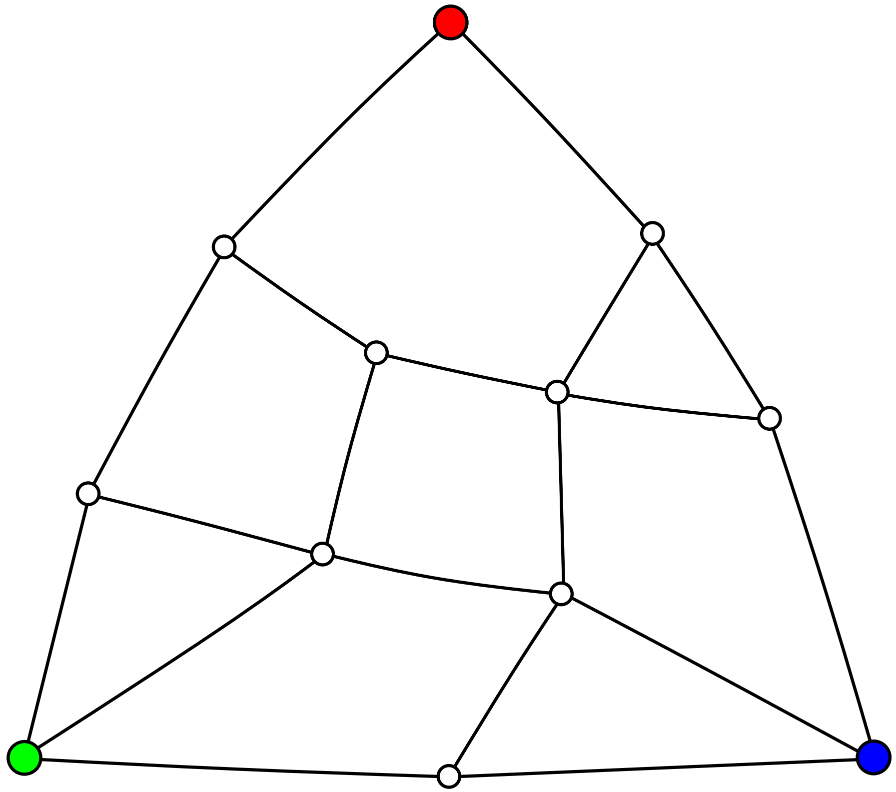




Quasi 3-connected maps

A planar map G with 3 marked outer vertices $\{R, B, G\}$ is called **quasi 3-connected** if G + triangle formed by $\{R, B, G\}$ is 3-connected



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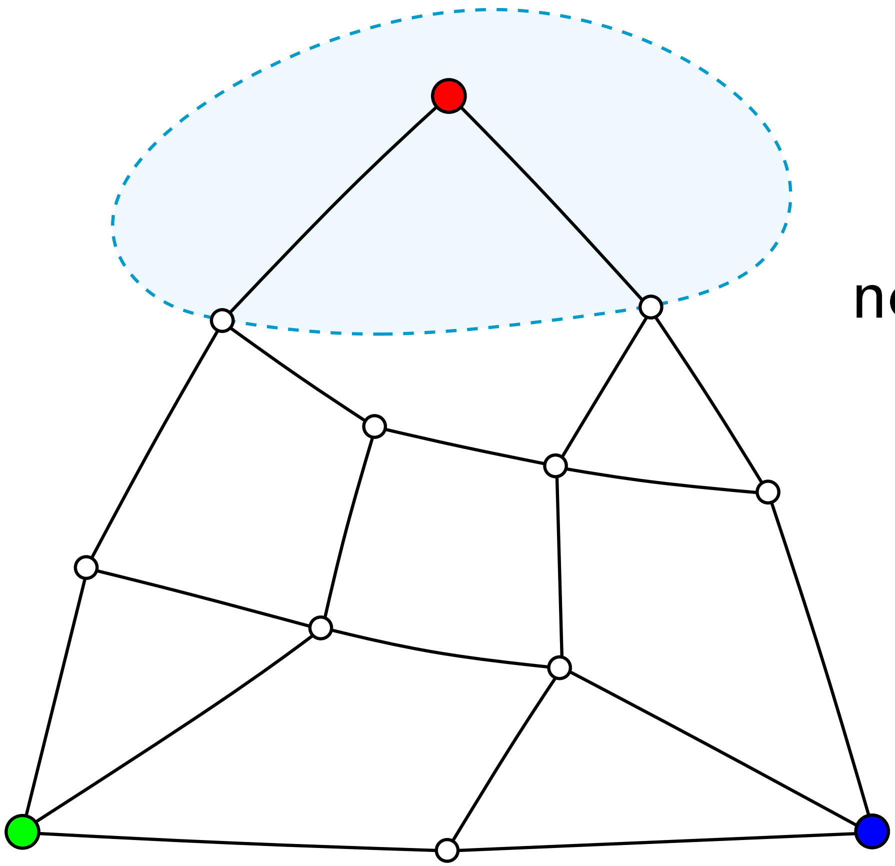

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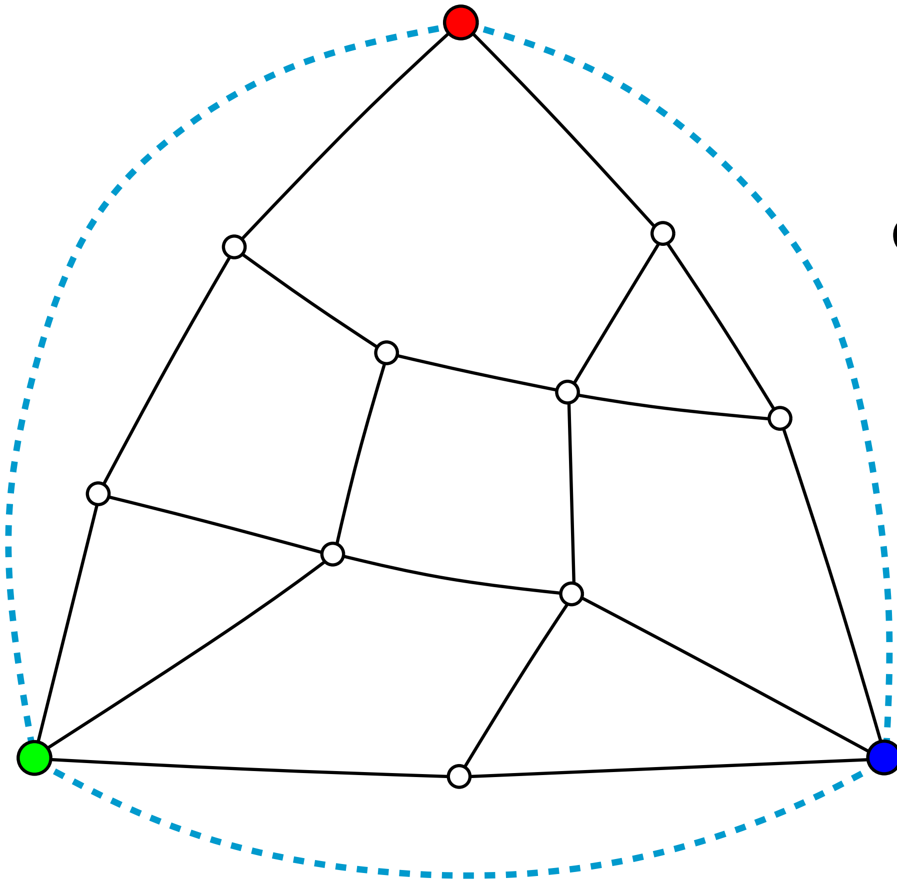

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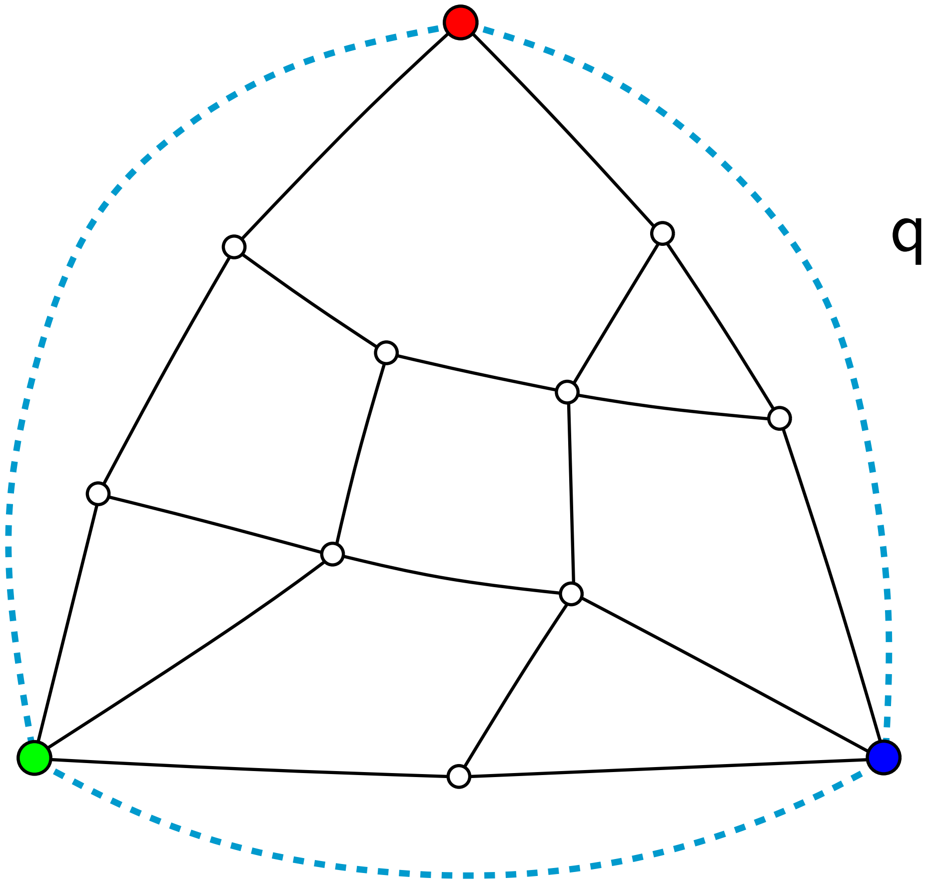

quasi 3-connected

Let $\mathcal{Q}_{i,j}$ = set quasi 3-conn. maps with $i+3$ vertices and j inner faces

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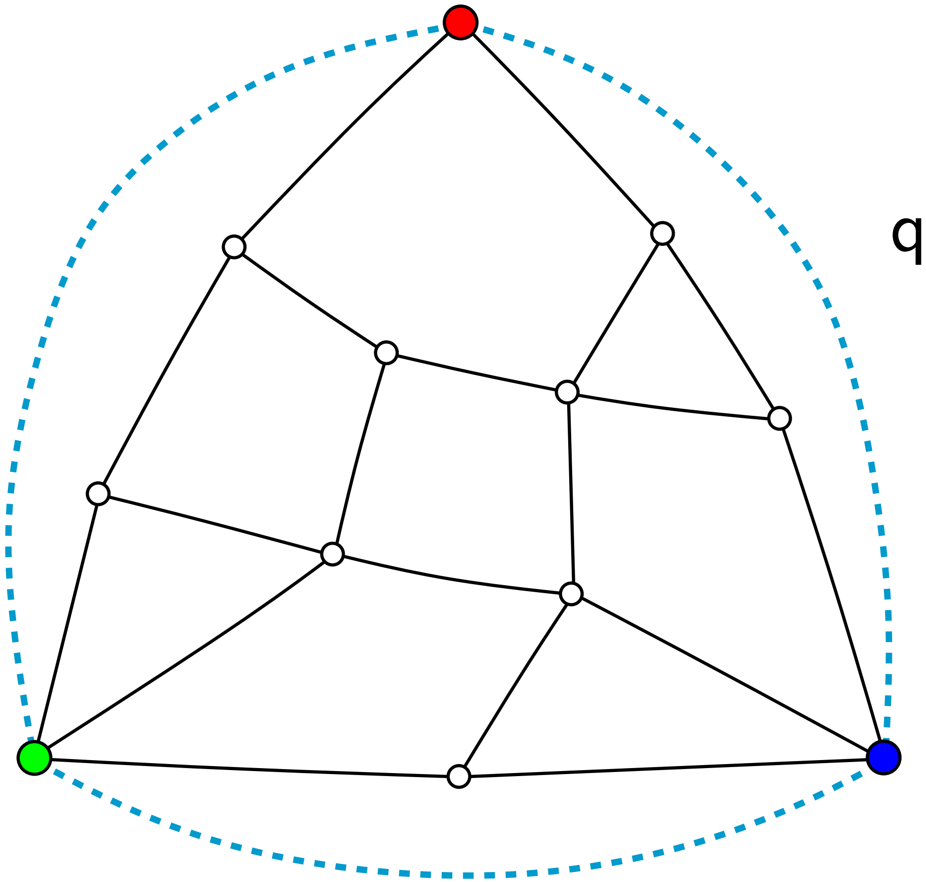


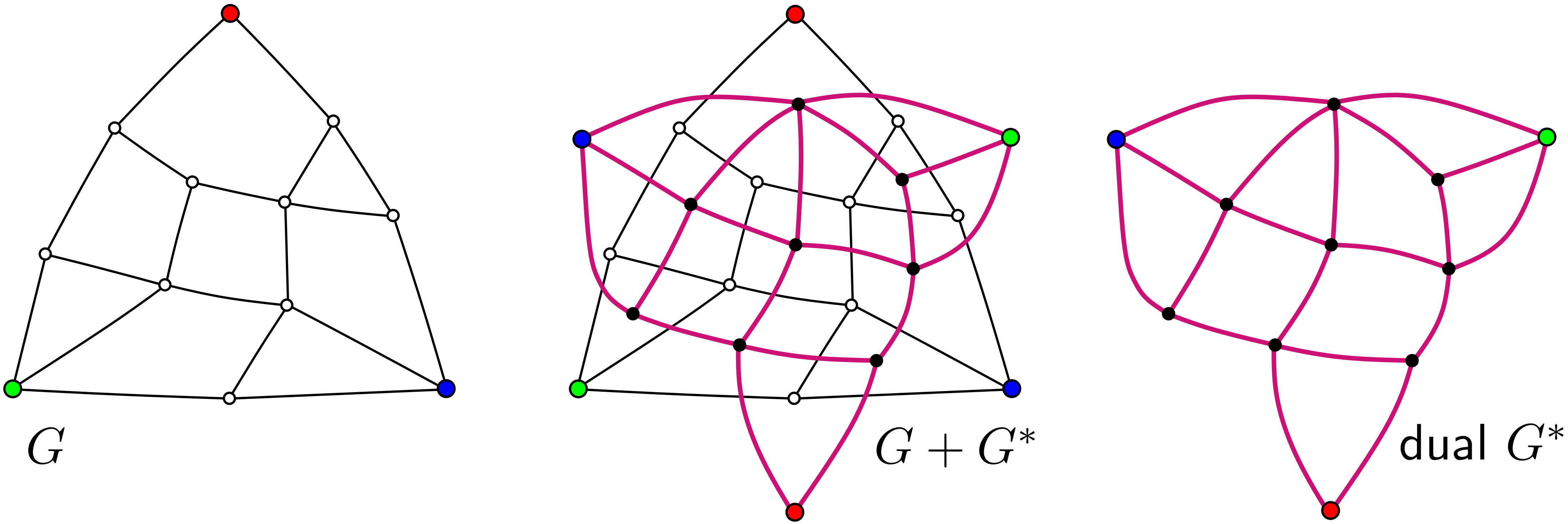
quasi 3-connected

Let $\mathcal{Q}_{i,j}$ = set quasi 3-conn. maps with $i+3$ vertices and j inner faces

Rk: Extremal case $j = 2i+1$ gives **triangulations** with $i+3$ vertices

Duality for quasi 3-connected maps

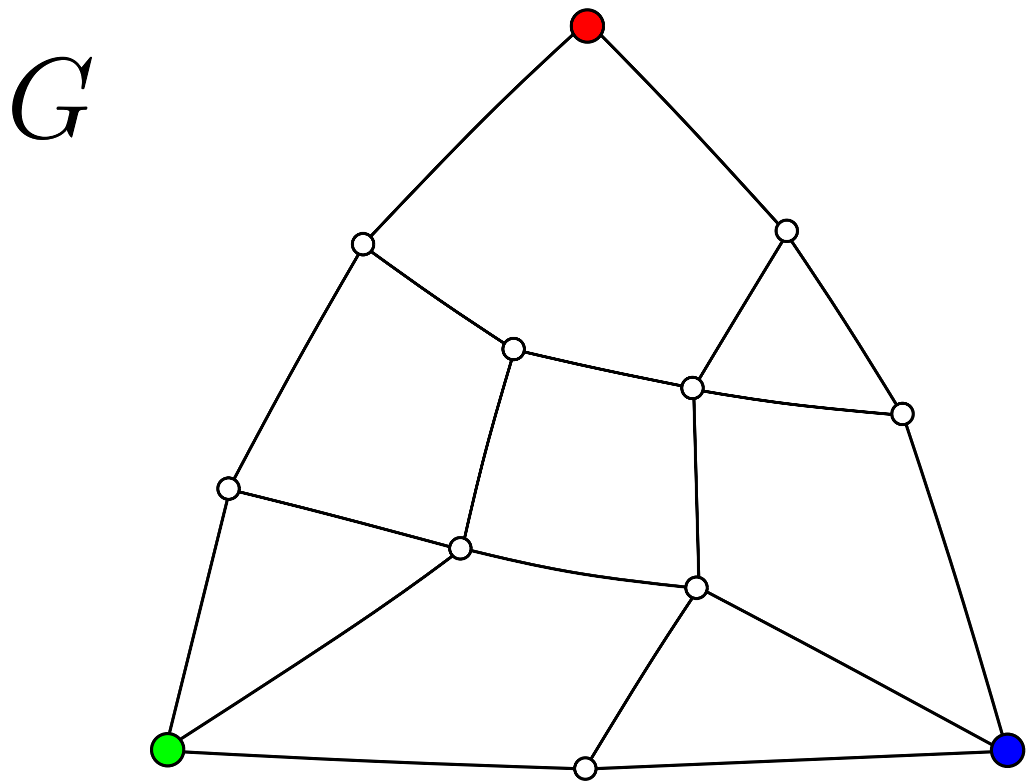
The family of quasi 3-connected maps is **stable by duality**



$$\boxed{\mathcal{Q}^*_{i,j} = \mathcal{Q}_{j,i}}$$

Duality seen with the corner-map

Corner-map: obtained by **replacing each face by a star** (3 outer faces)



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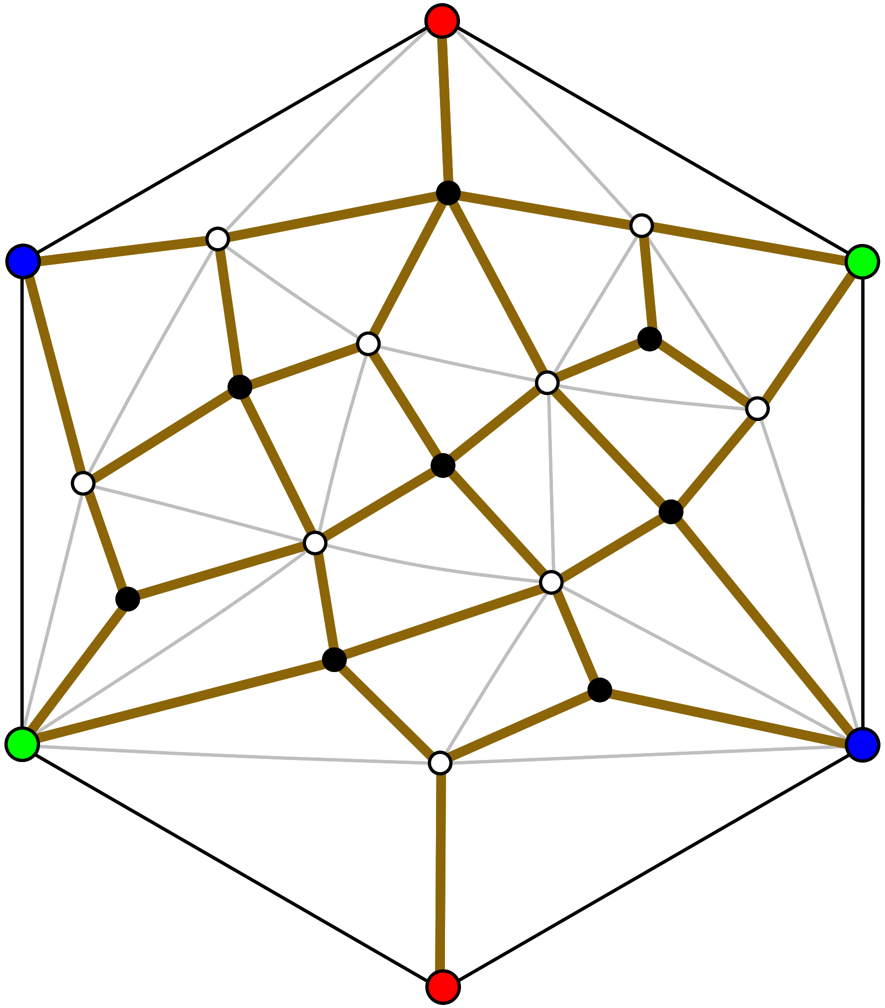

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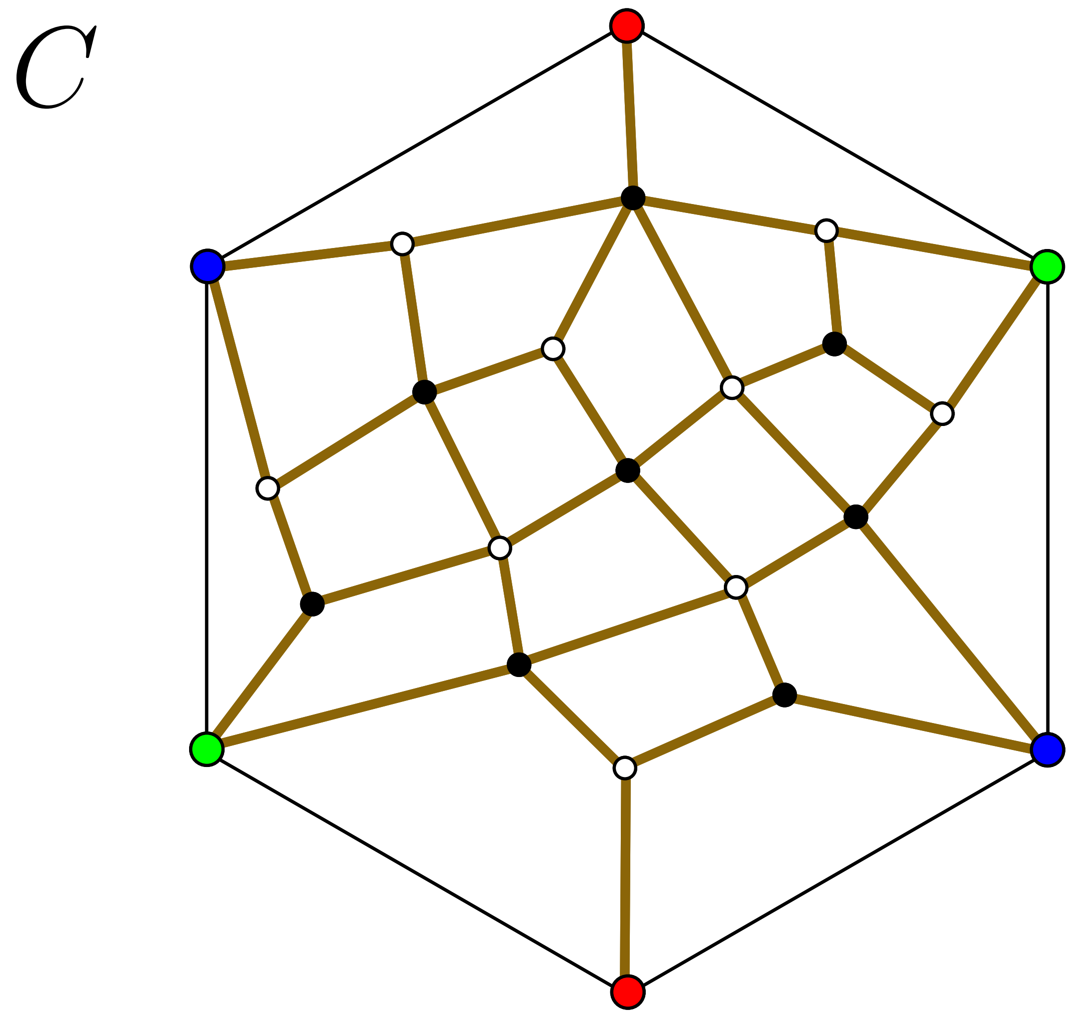
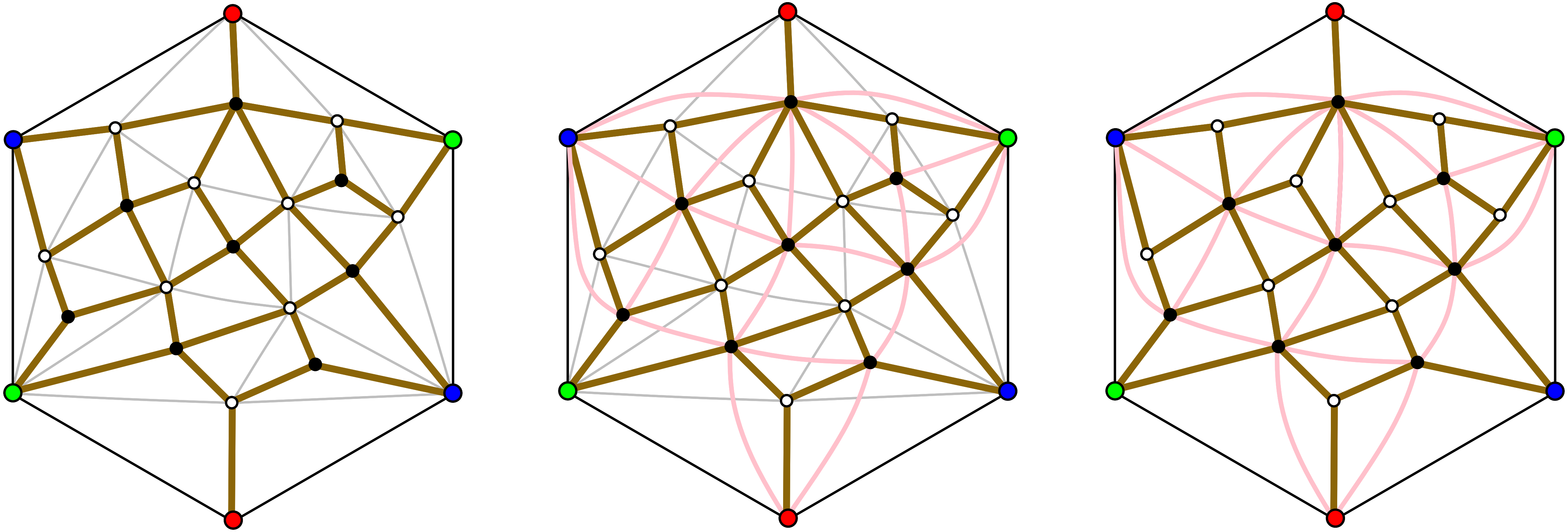


C is a dissection of an hexagon by **quadrangular faces**

Rk: quasi 3-**connectivity** of $G \Leftrightarrow$ each 4-**cycle** of C delimits a **face**

Duality seen with the corner-map

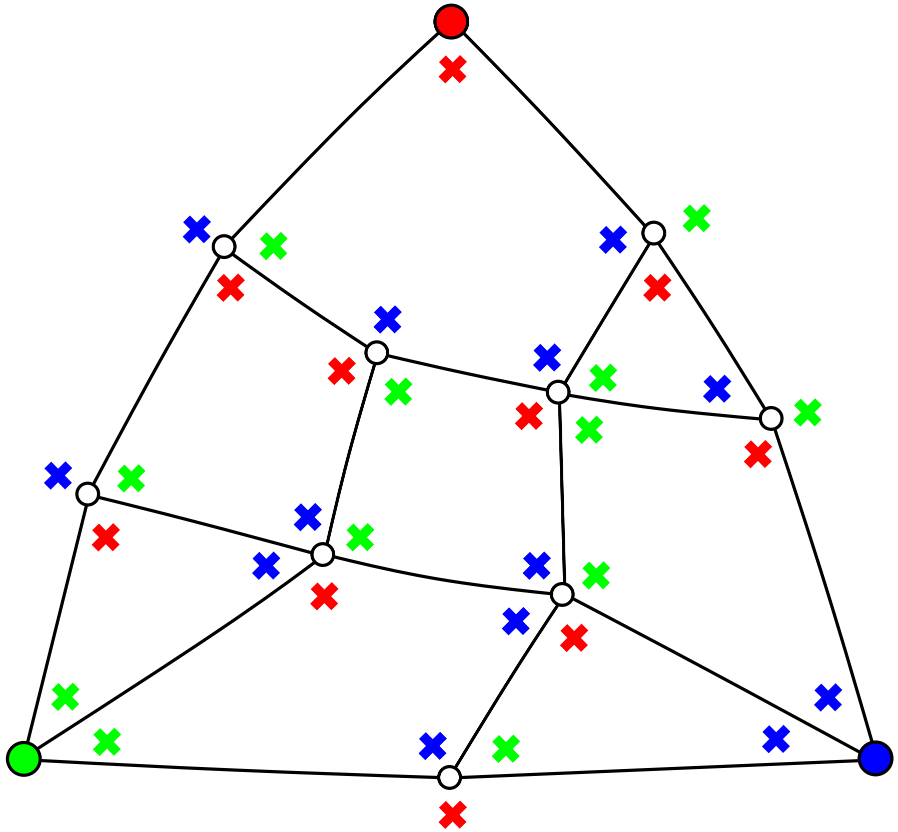
Corner-map: obtained by **replacing each face by a star** (3 outer faces)



G and G^* have the **same corner-map**

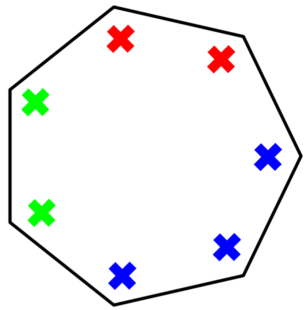
3-connected Schnyder labellings

Let G be a quasi 3-connected map. [Miller'02], [Felsner'04]

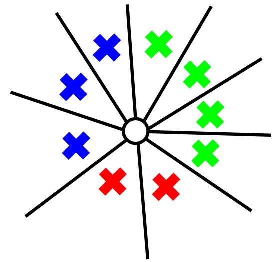


G can be endowed with a **labelling** of the **corners** by $\{\times, \times, \times\}$ such that

inner faces



inner vertices



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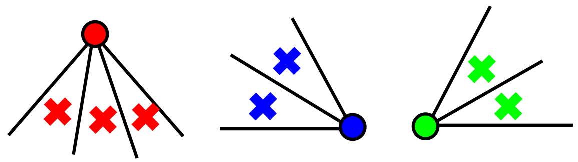
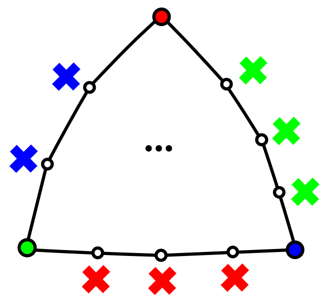


outer face(s)



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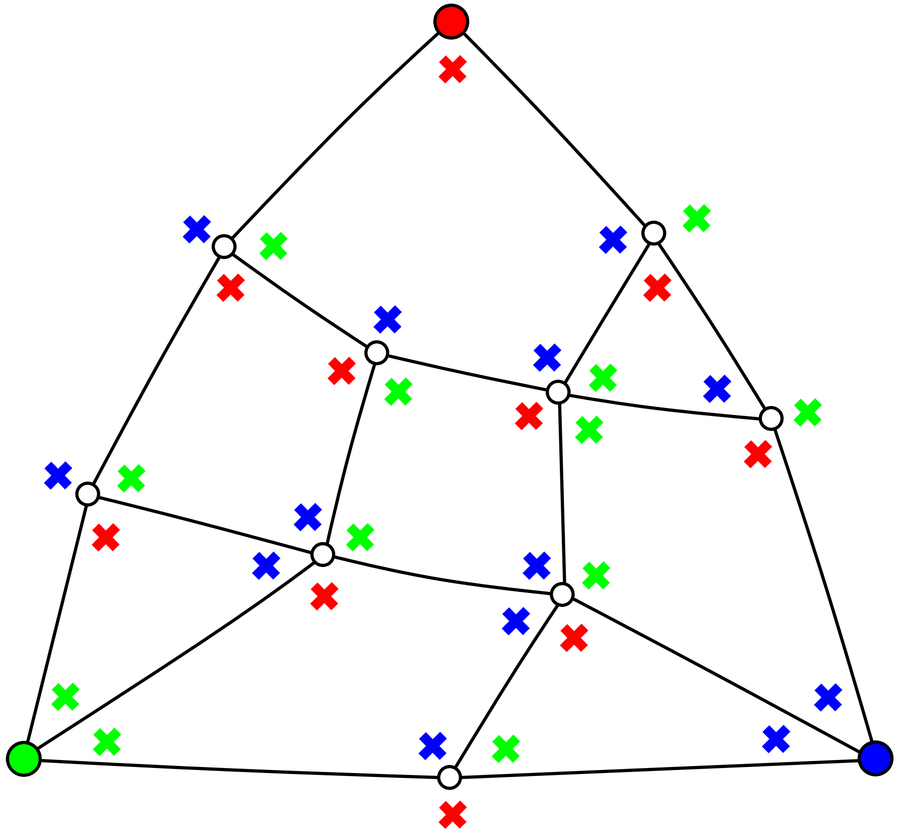

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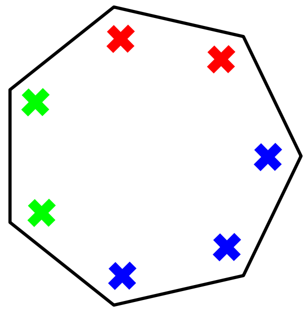

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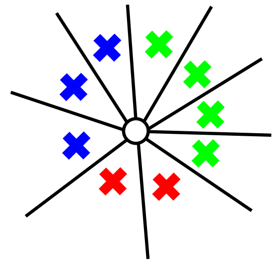
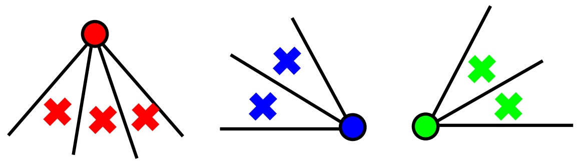


outer vertices



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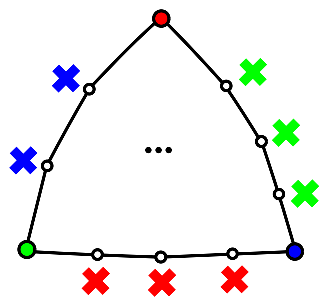
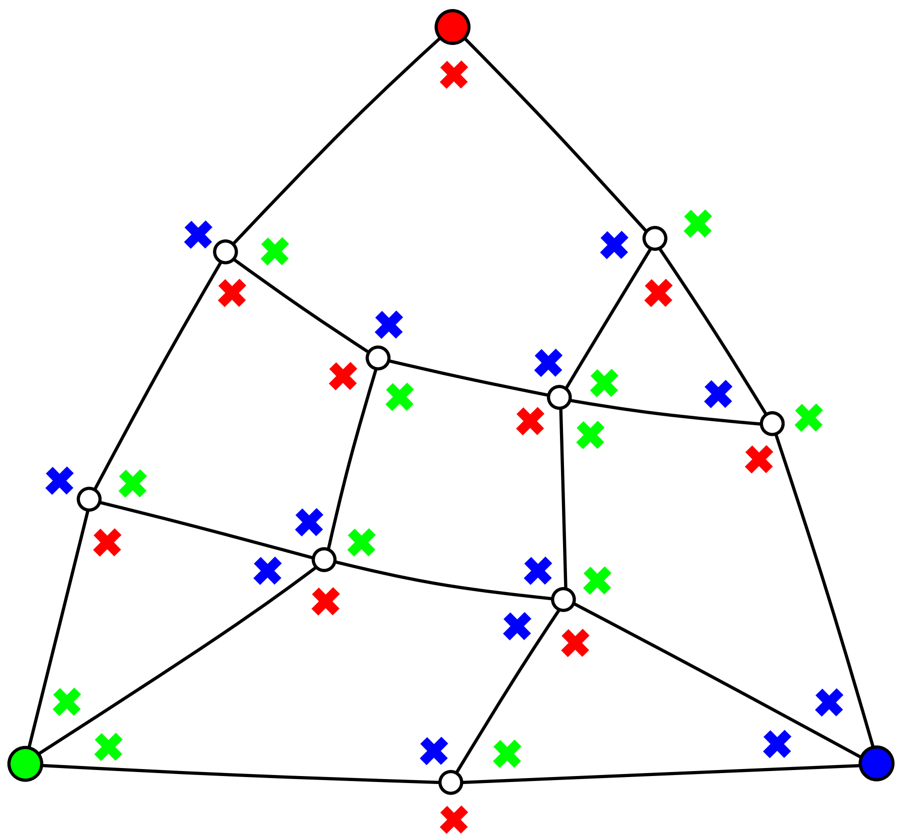


Rk: also incarnations as **Schnyder woods**, **3-orientations** (ommited)

Duality for 3-connected Schnyder labellings



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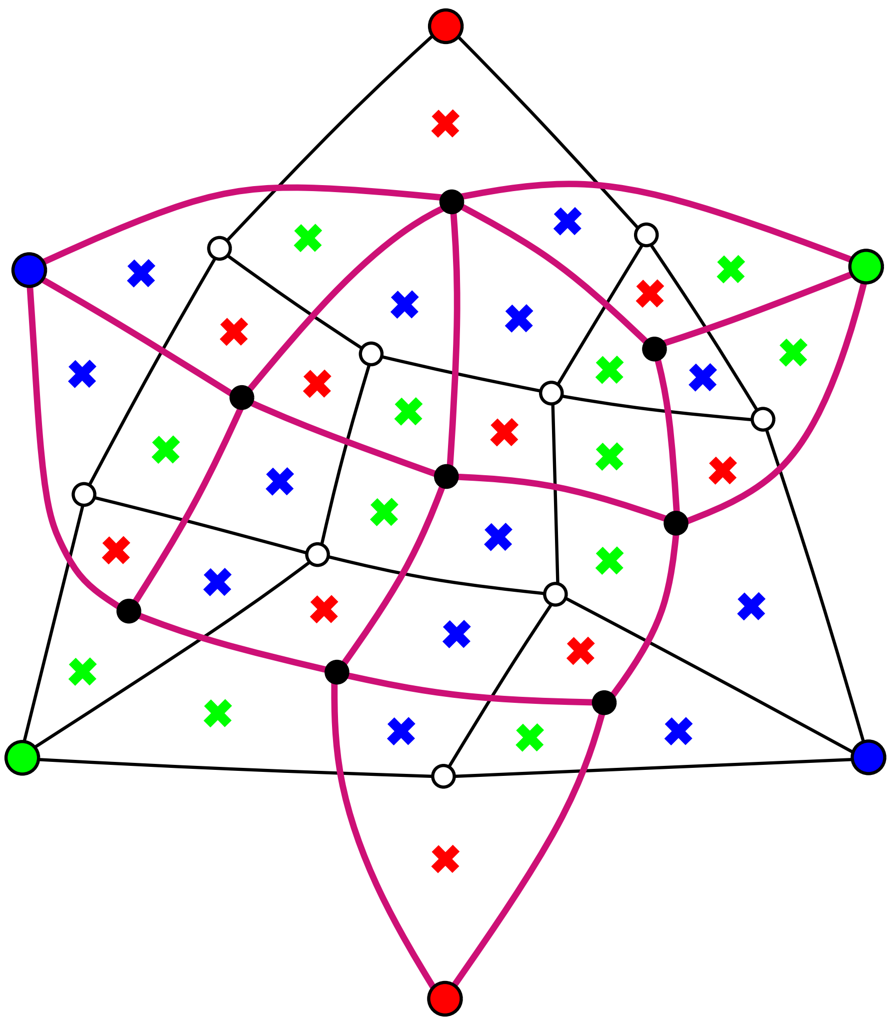

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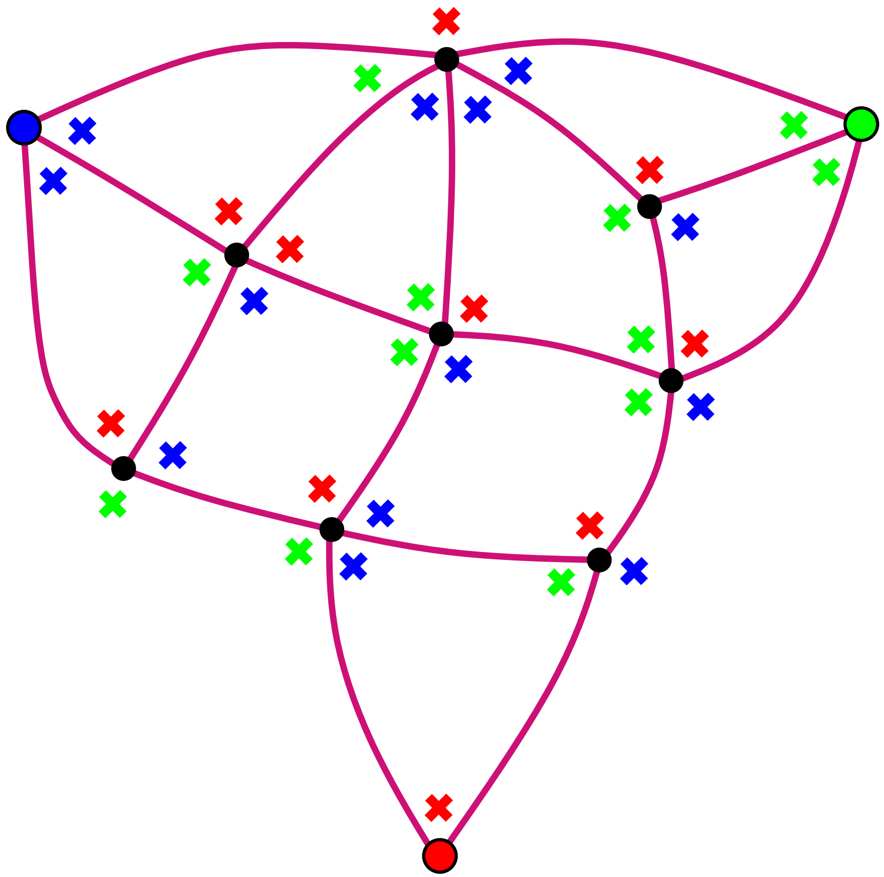

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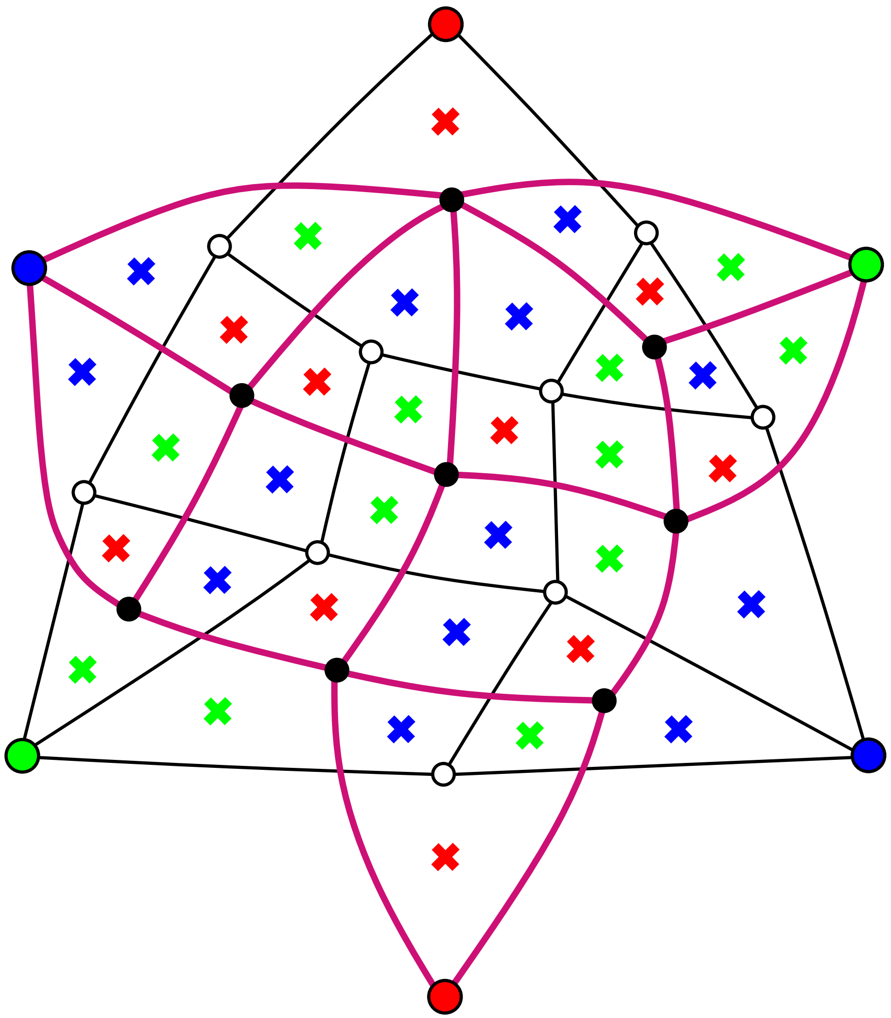
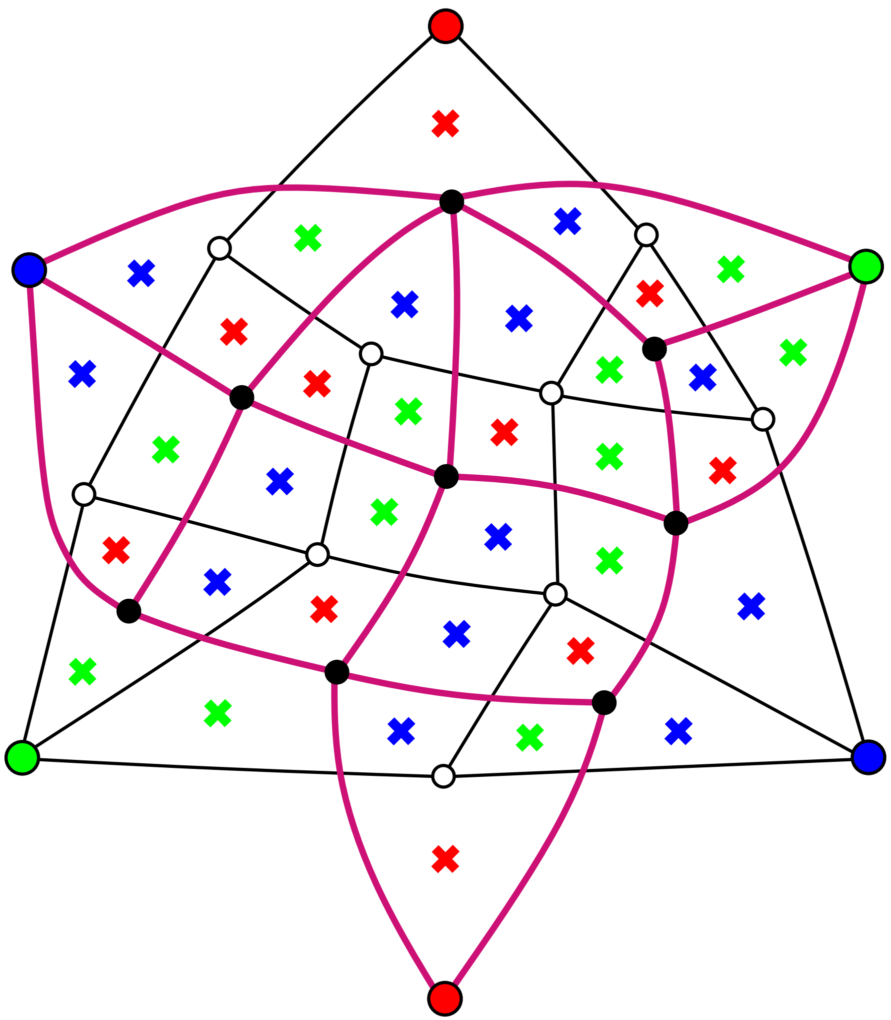
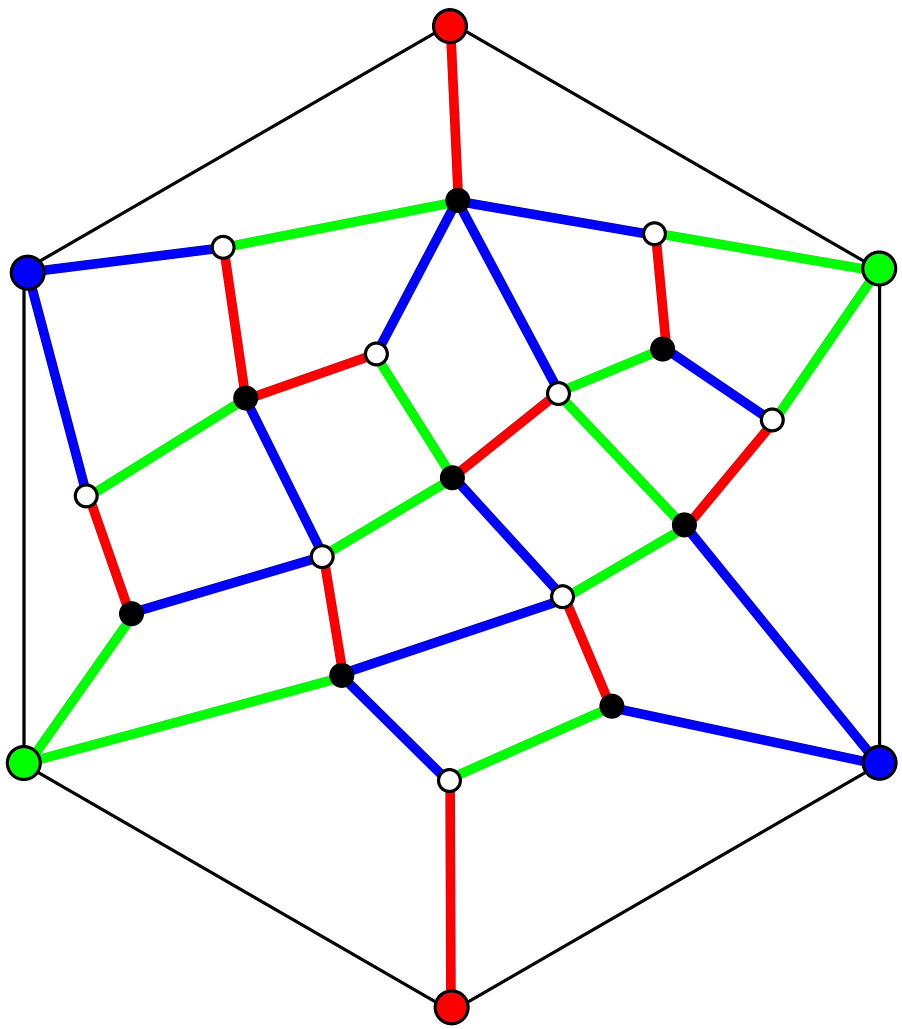


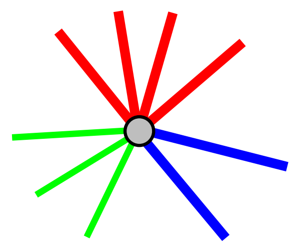
Duality for 3-connected Schnyder labellings

duality is well seen on **corner map** C





Local rule:

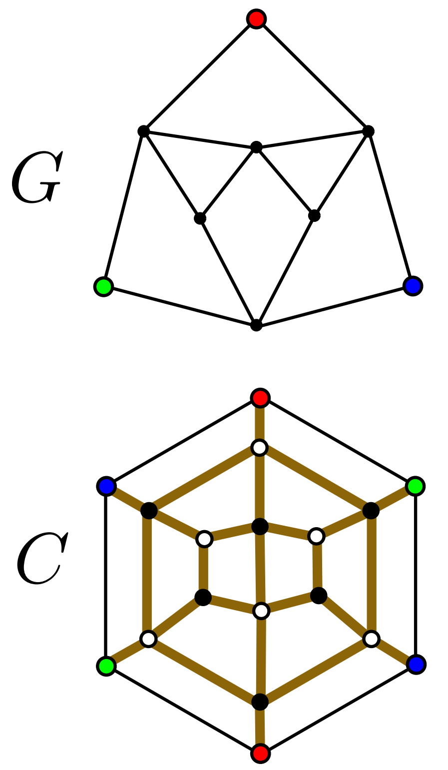


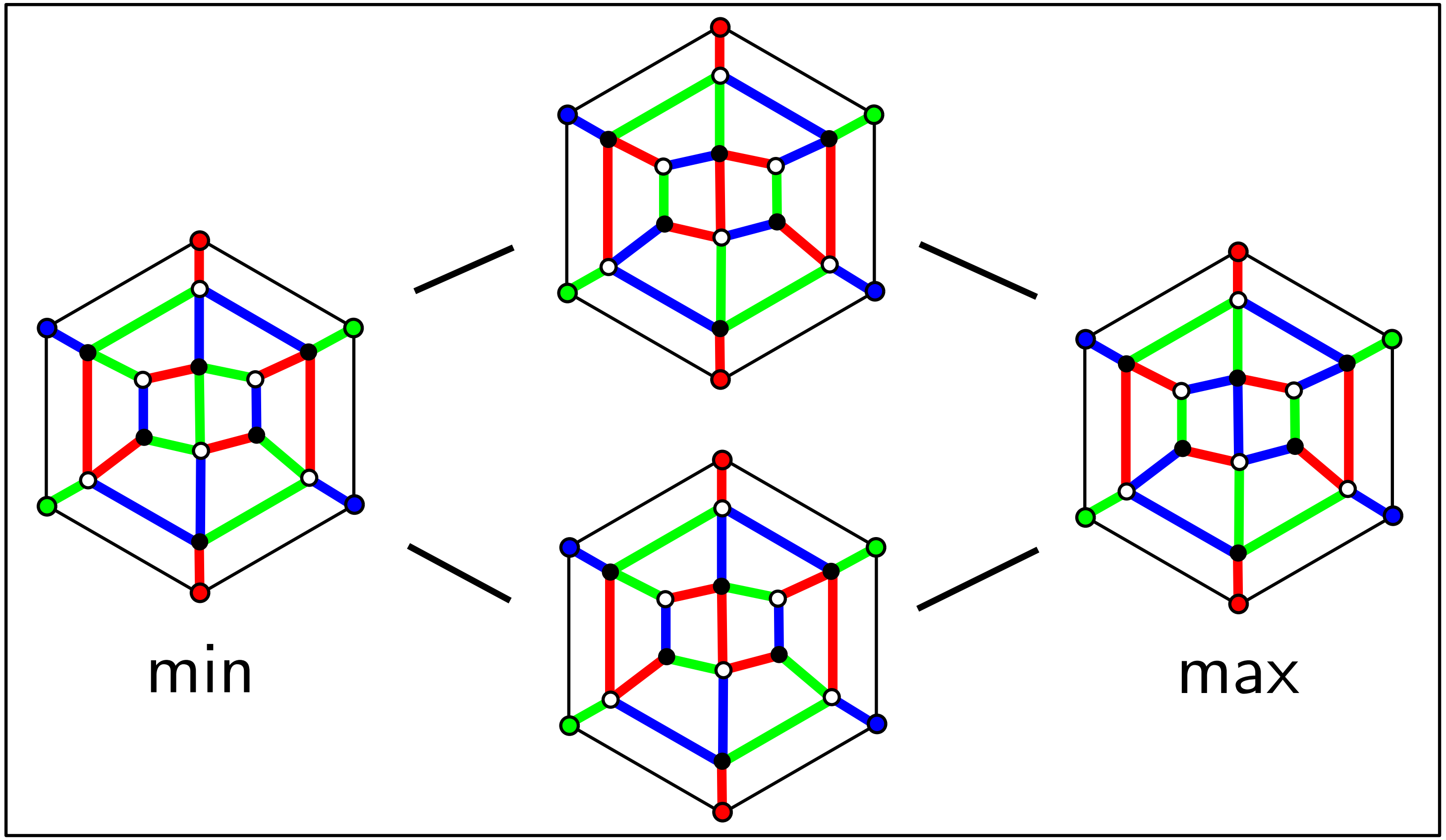
same rule at black & white vertices

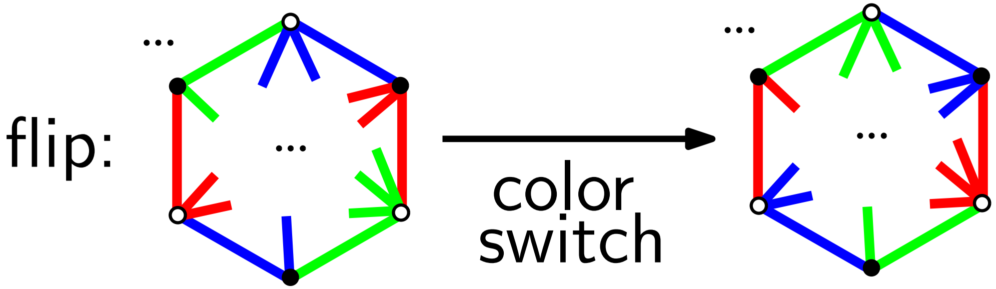
Lattice property in the 3-connected case

[Felsner'04] **formulated** on the associated **corner map** C

Theorem: Let G be a quasi 3-connected map. Then the set of Schnyder labellings of G is a **distributive lattice**







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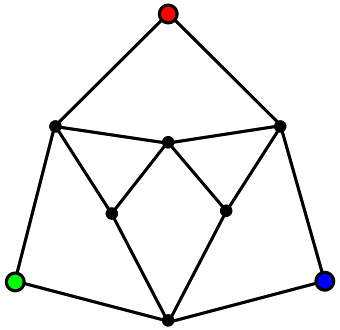
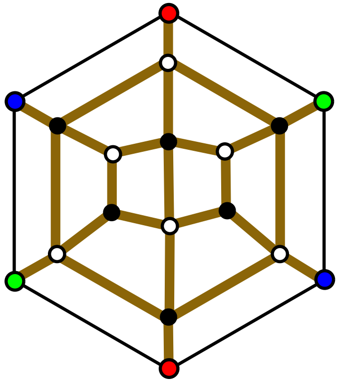
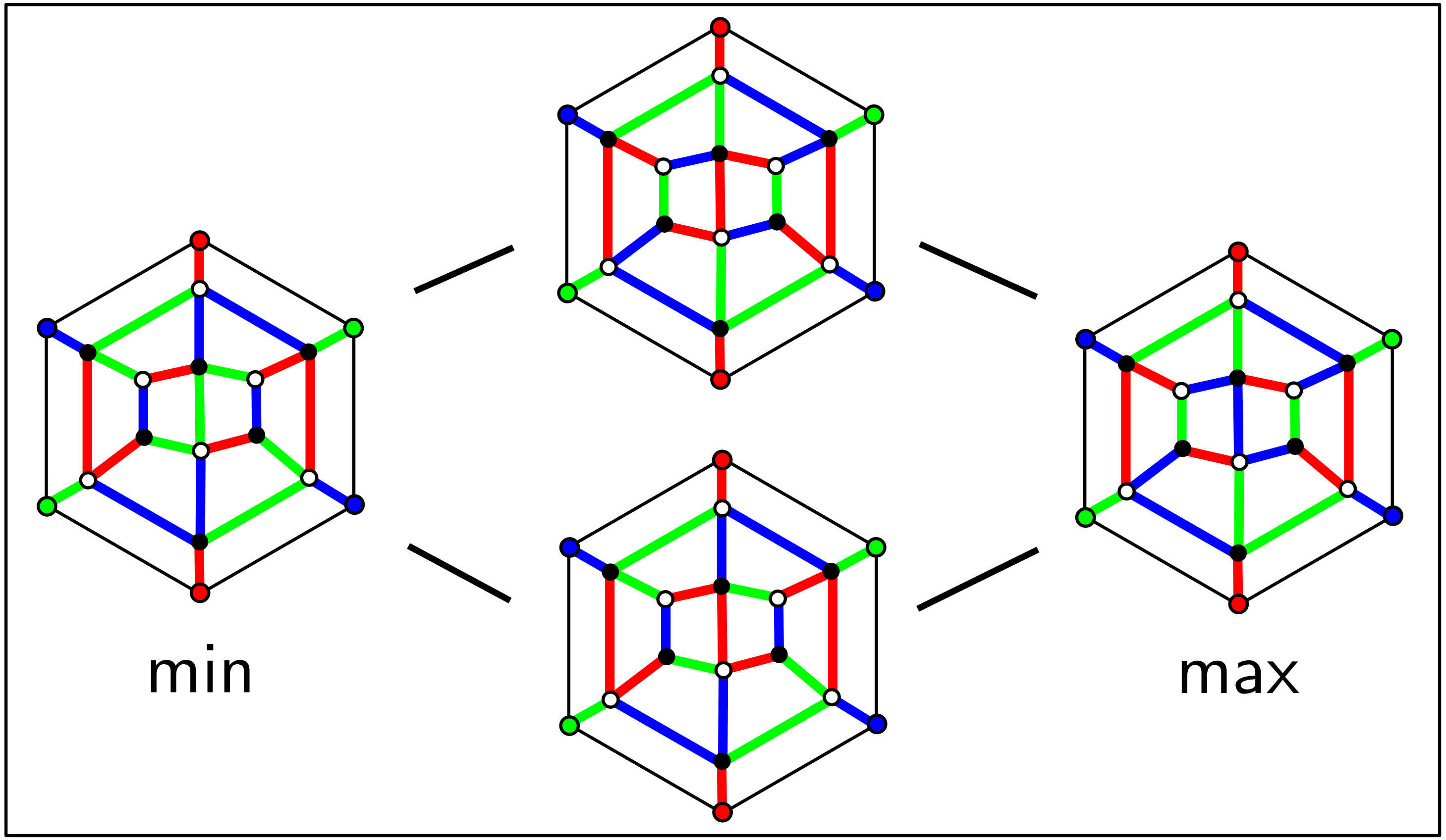
G

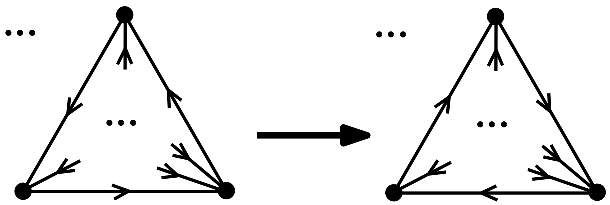


C

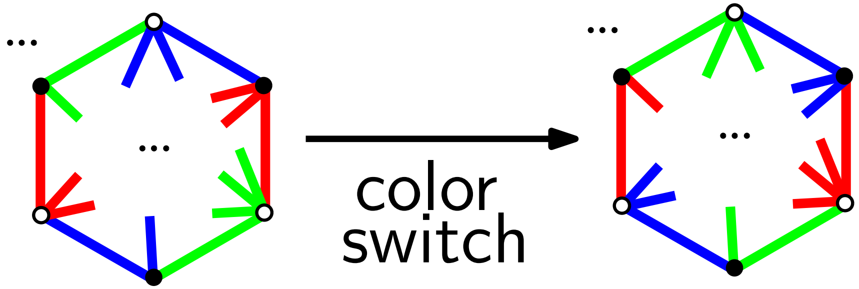




Rk: **extends** flip for triangulations



flip:



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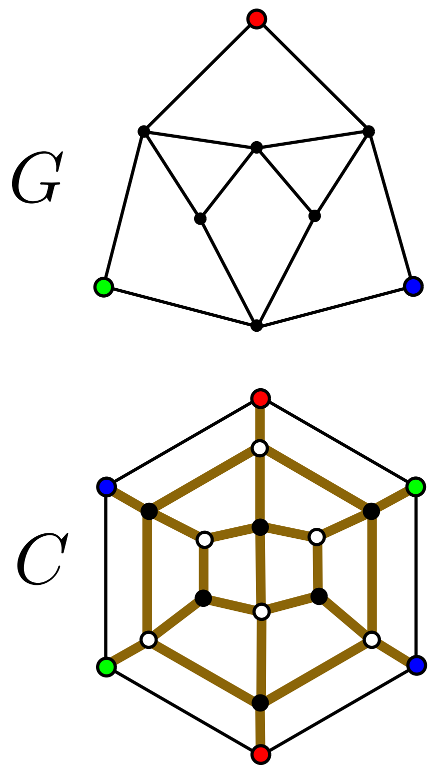

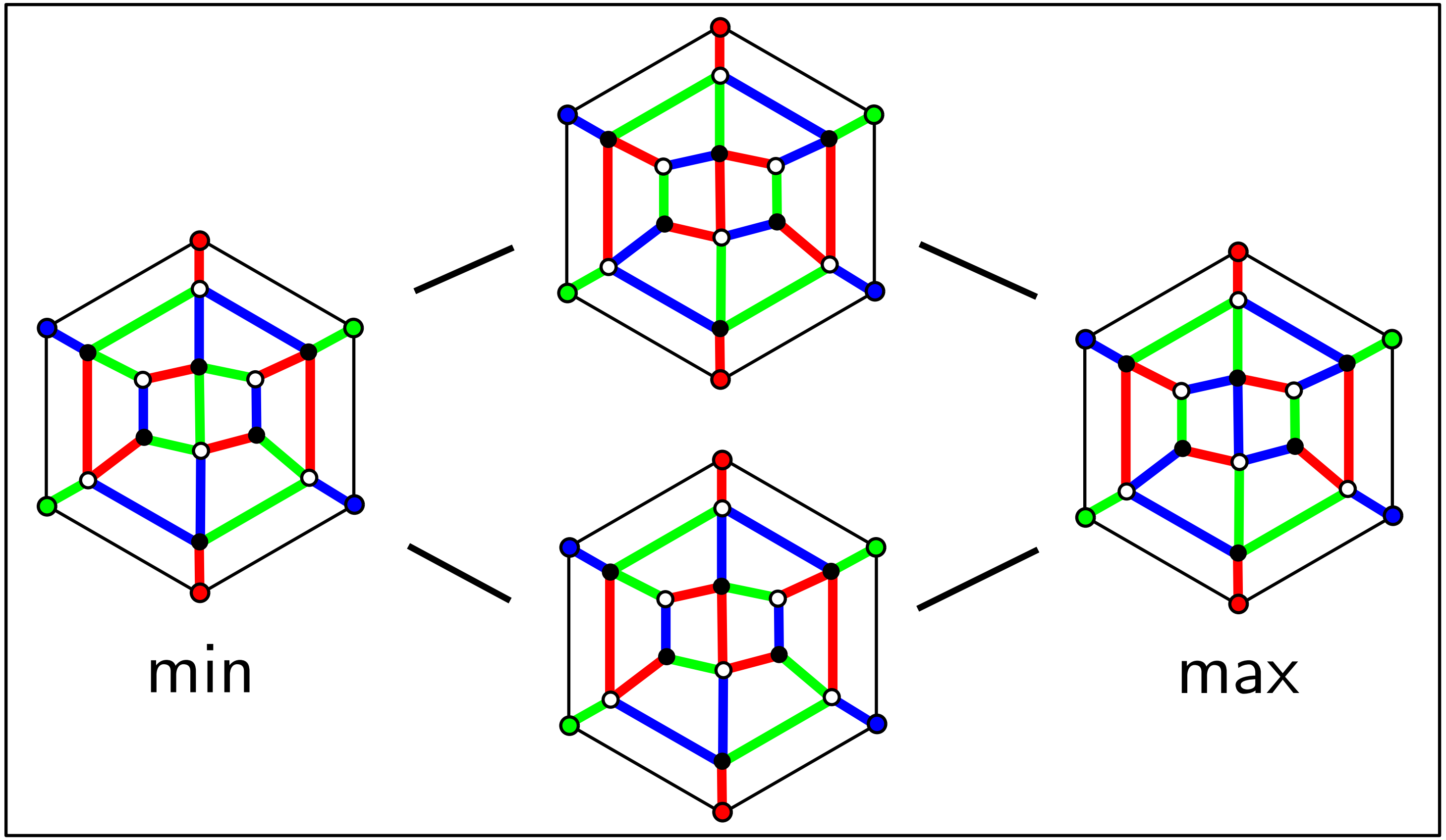

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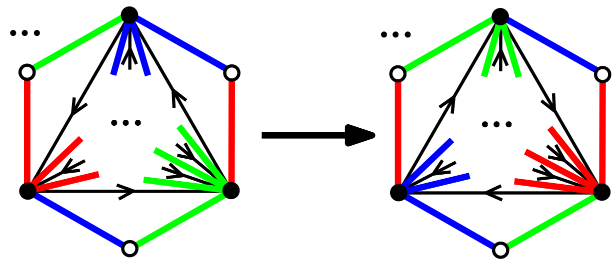
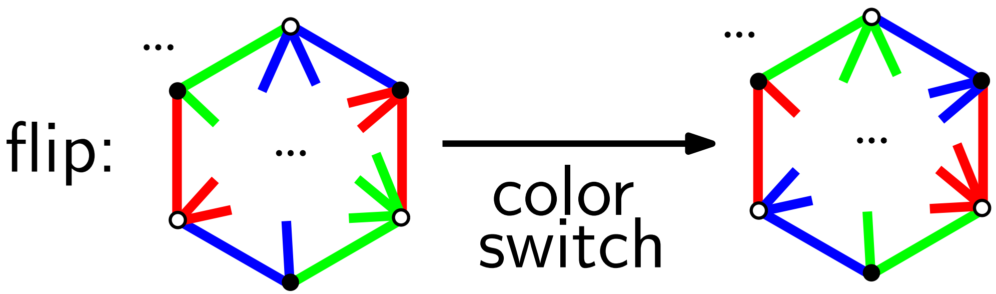
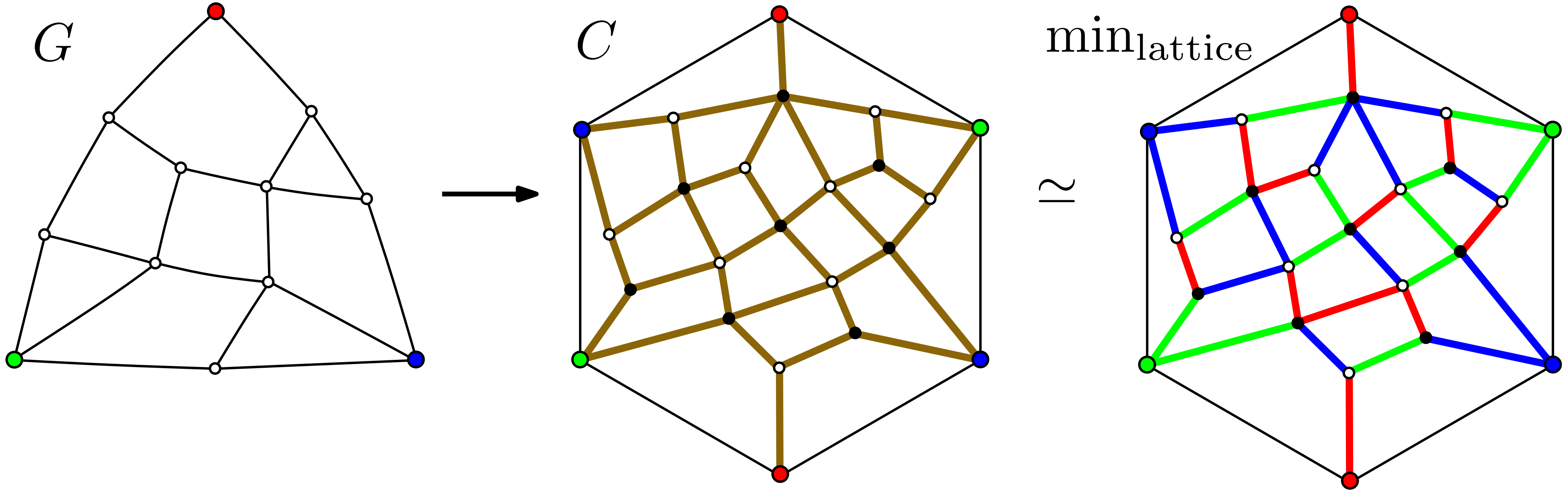




Bijection for quasi 3-connected maps

[F, Poulalhon, Schaeffer'08]



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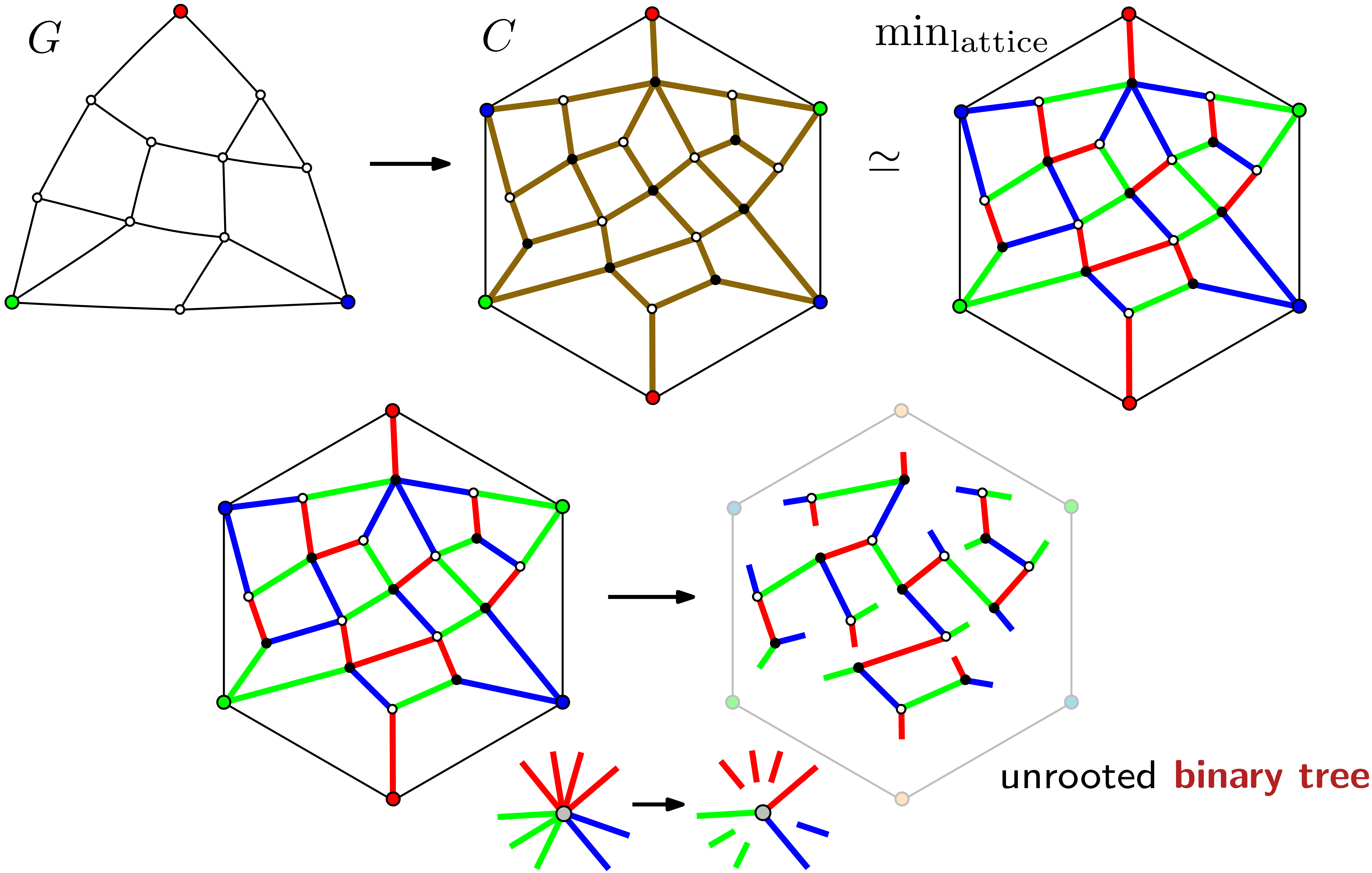

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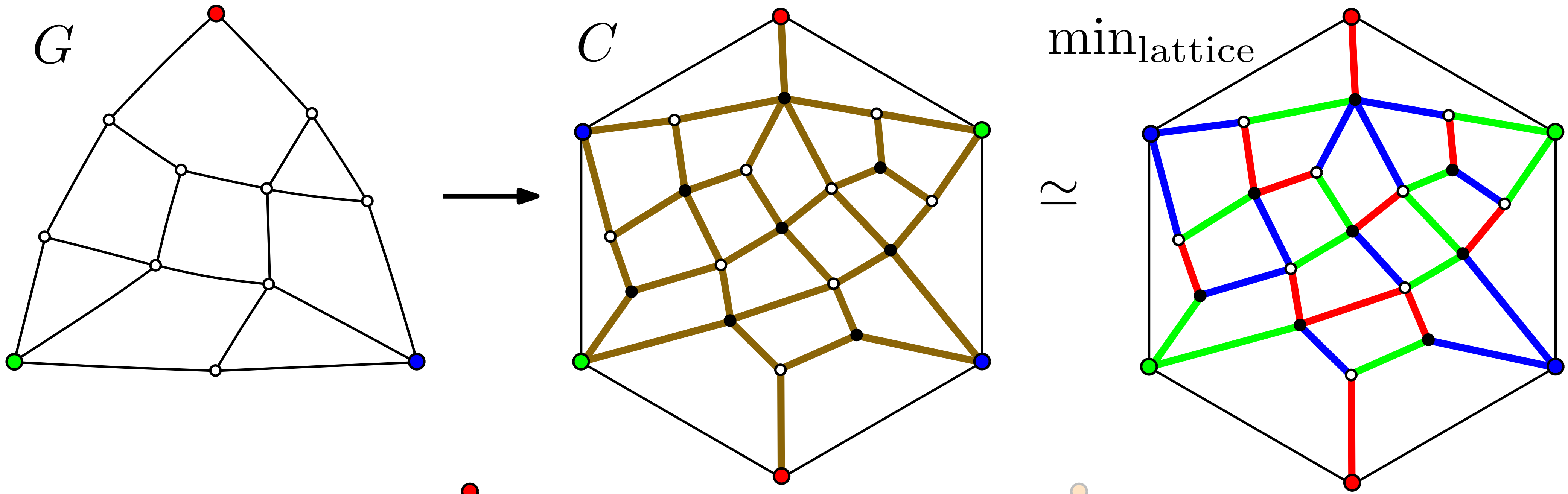

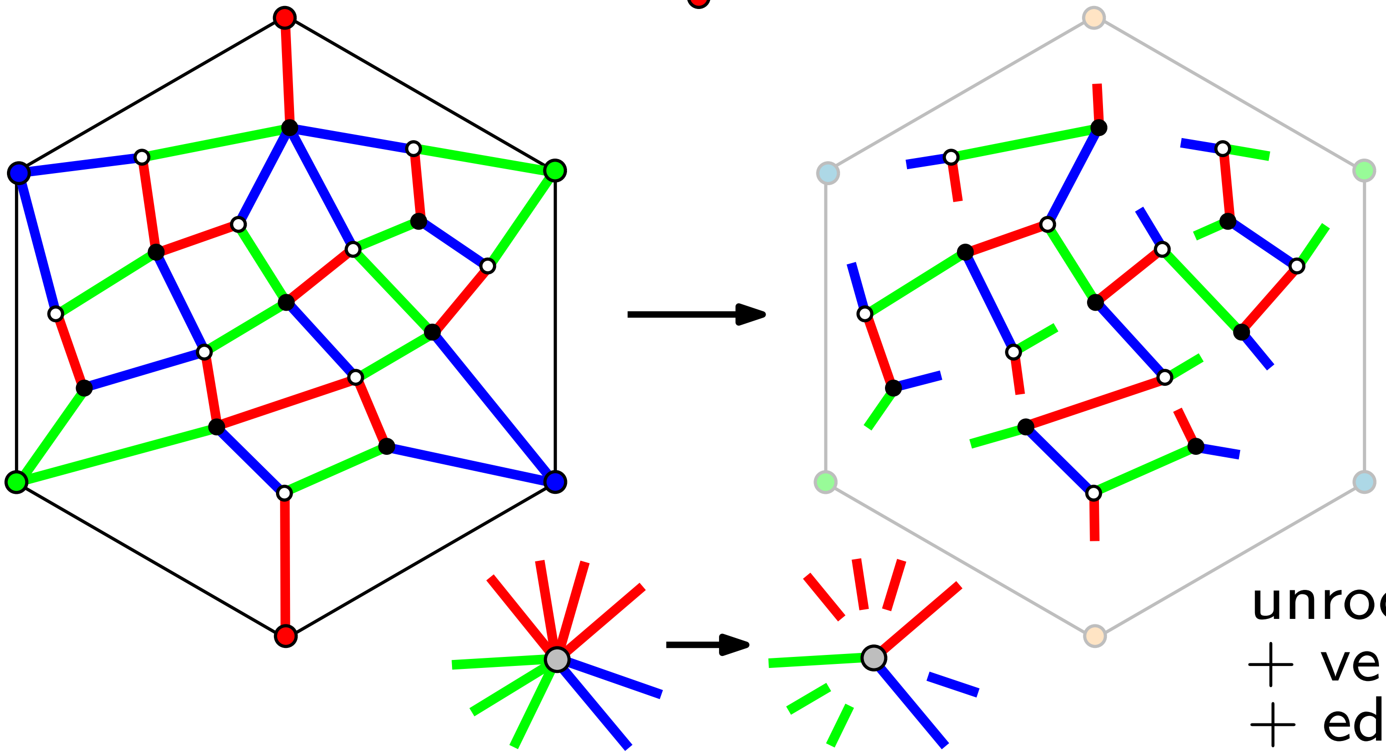

unrooted **binary tree**
+ vertex bicoloration
+ edge tricoloration

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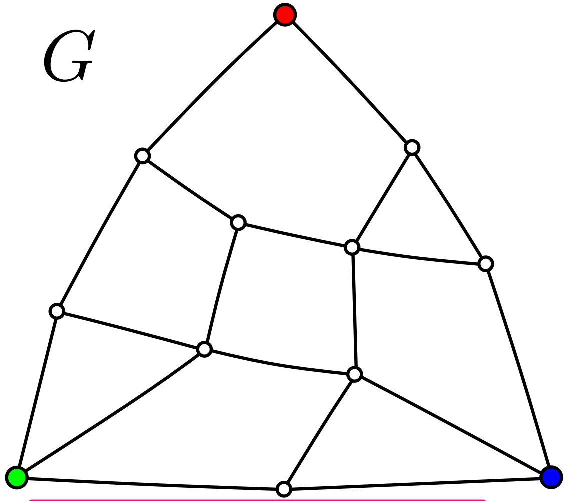
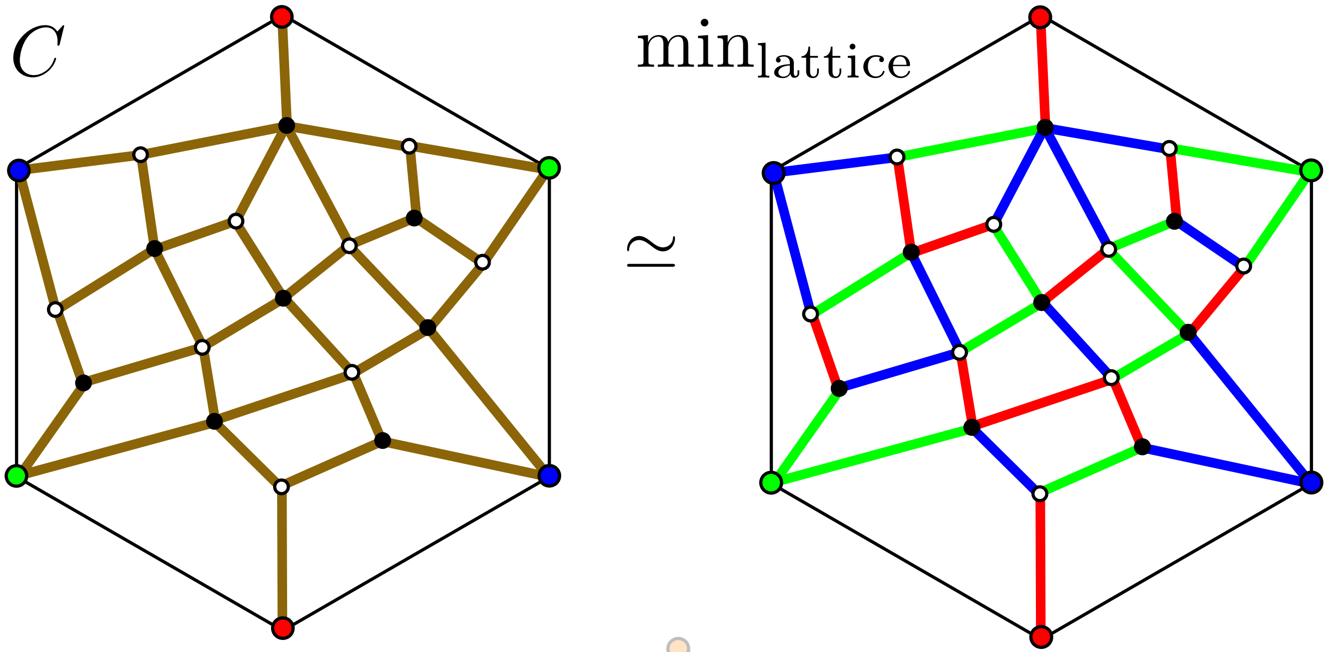
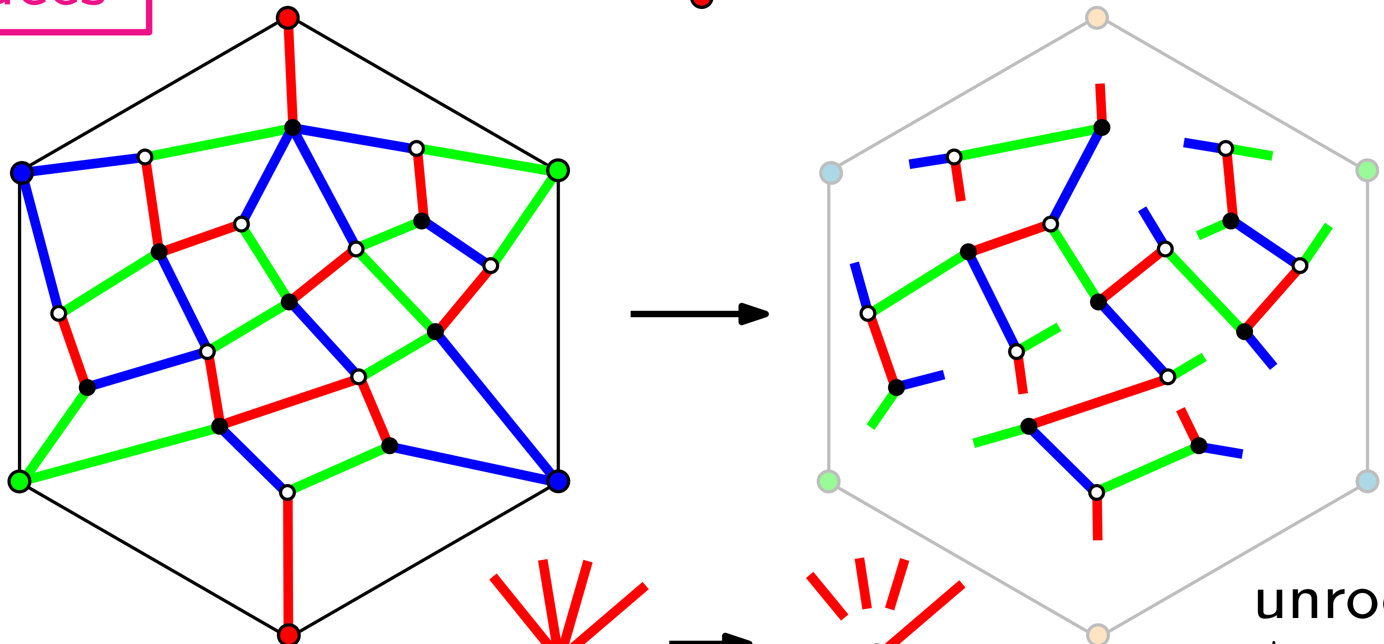
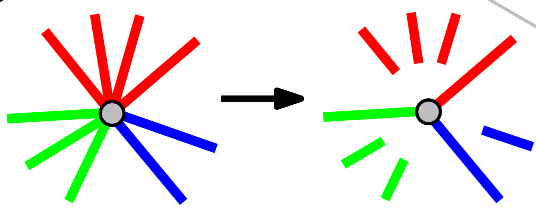


$i+3$ vertices
j inner faces



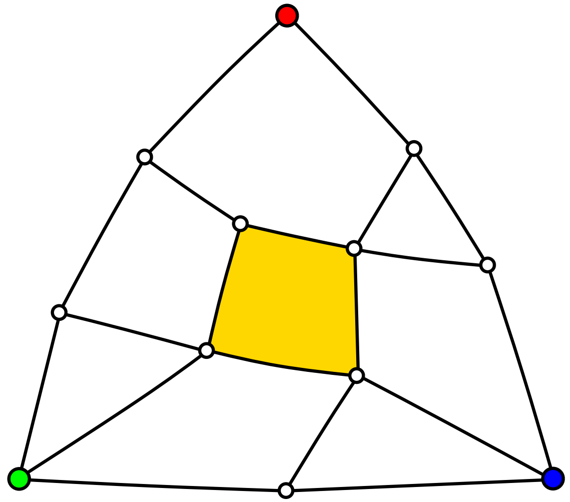


i black vert.
j white vert.

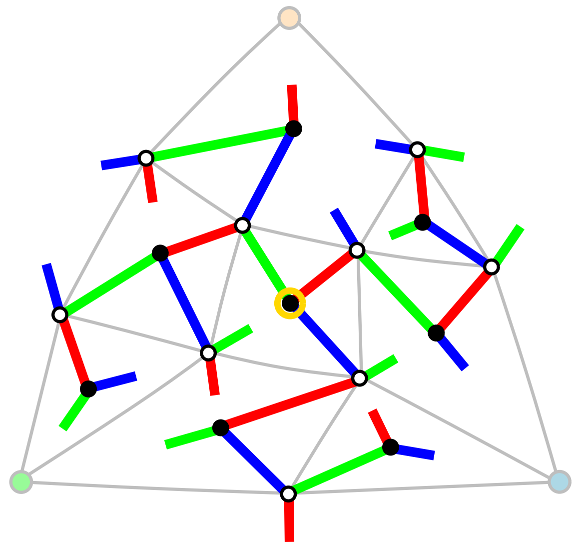


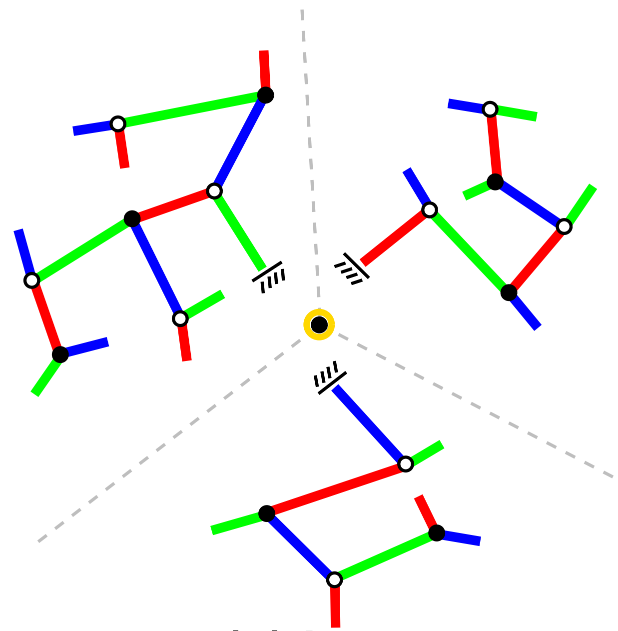
unrooted **binary tree**
$+$ vertex bicoloration
$+$ edge tricoloration

Counting formula



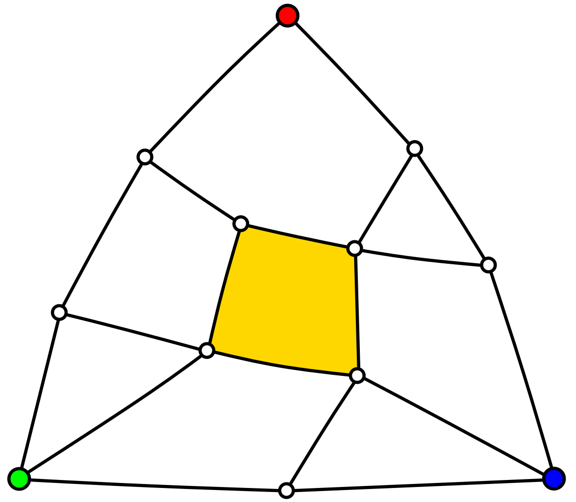
G with a marked inner face





3 rooted binary trees

Counting formula



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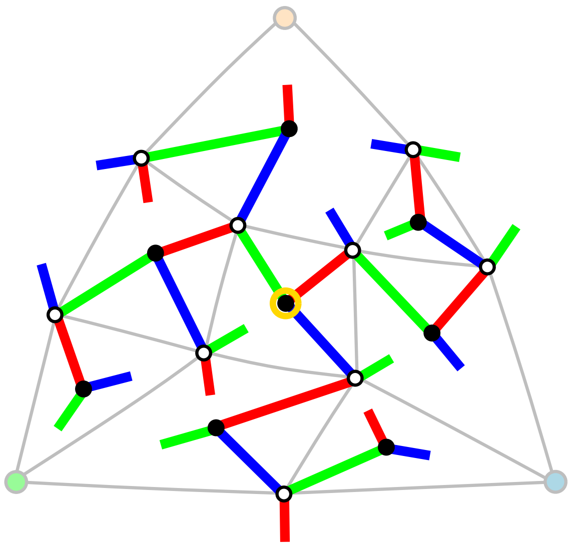
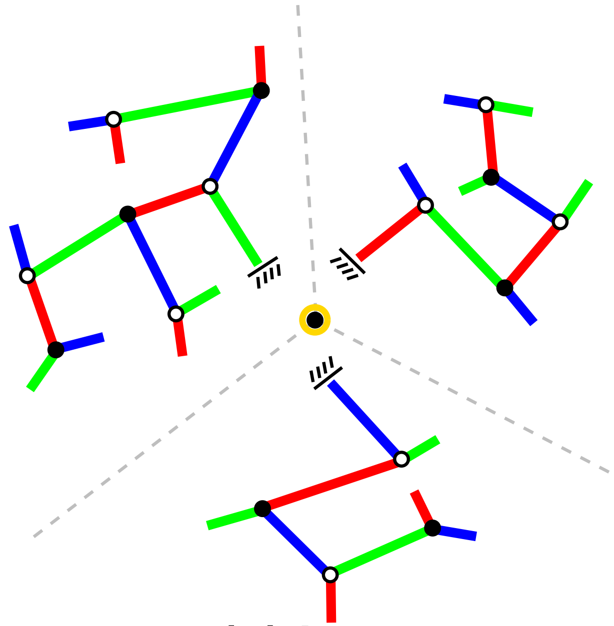




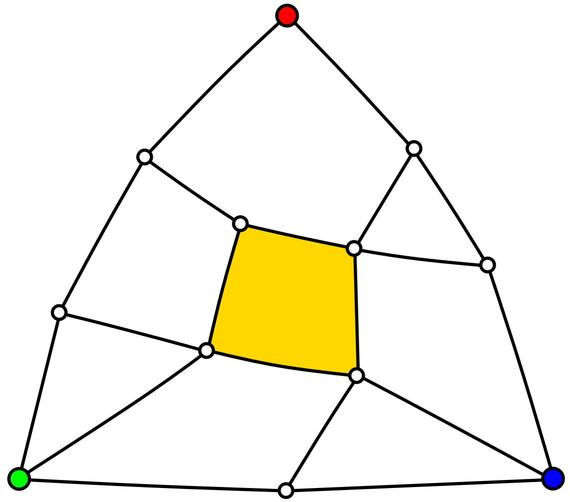
3 rooted binary trees

Let $q_{i,j} = \#\{$quasi 3-conn. maps with $i+3$ vertices and j inner faces$\}$

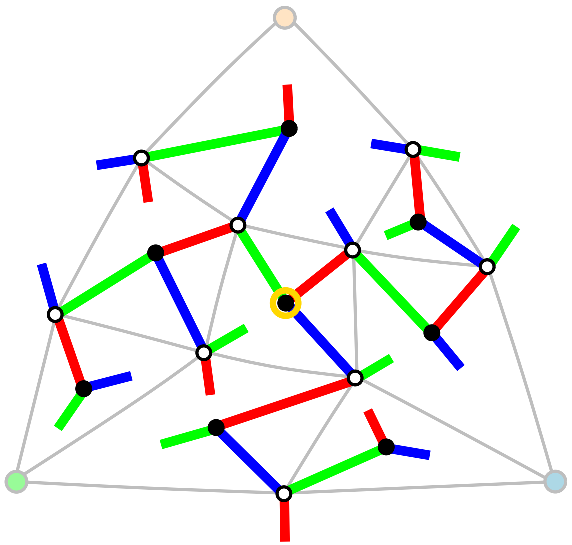
Let $F(x_\circ, x_\bullet) = \sum_{i,j} q_{i,j} x_\circ^i x_\bullet^j$

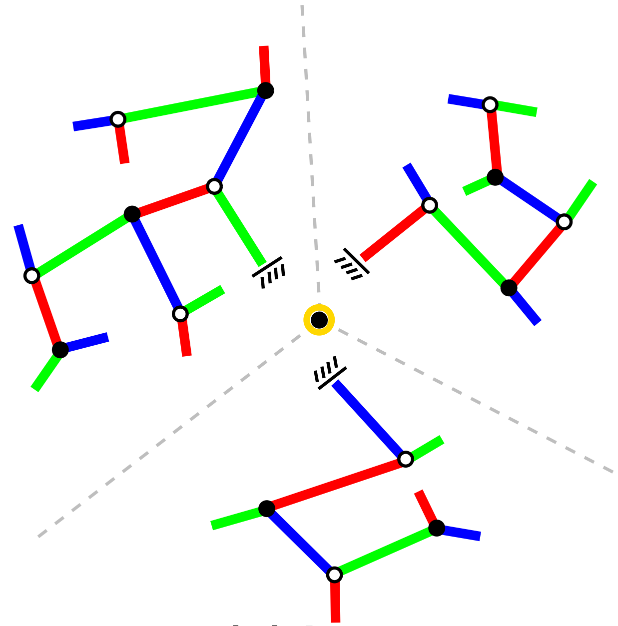
$$\boxed{\frac{\partial}{\partial x_\bullet} F(x_\circ, x_\bullet) = (1+U)^3, \text{ where } \begin{cases} U &= x_\circ \cdot (1+V)^2, \\ V &= x_\bullet \cdot (1+U)^2 \end{cases}}$$

Counting formula



G with a marked inner face





3 rooted binary trees

Let $q_{i,j} = \#\{$quasi 3-conn. maps with $i+3$ vertices and j inner faces$\}$

Let $F(x_\circ, x_\bullet) = \sum_{i,j} q_{i,j} x_\circ^i x_\bullet^j$

$$\boxed{\frac{\partial}{\partial x_\bullet} F(x_\circ, x_\bullet) = (1+U)^3, \text{ where } \begin{cases} U &= x_\circ \cdot (1+V)^2, \\ V &= x_\bullet \cdot (1+U)^2 \end{cases}}$$

$\Longrightarrow$
Lagrange

$$\boxed{q_{i,j} = \frac{3}{(2i+1)(2j+1)} \binom{2i+1}{j} \binom{2j+1}{i}}$$

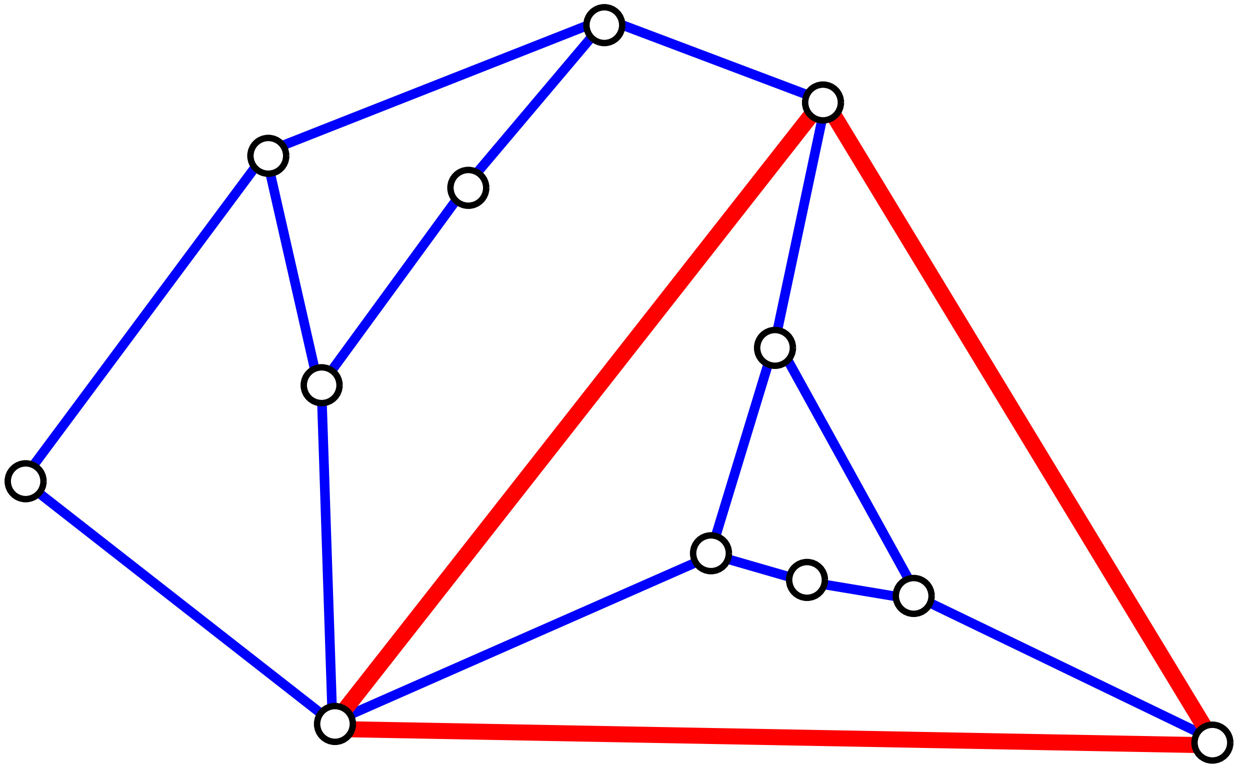
[Mullin& Schellenberg'68]

recover triangulations counting formula in the (extremal) case $j = 2i+1$

Extension to d-angulations of girth d

The girth parameter

The **girth** of a graph is the length of a shortest cycle within the graph



Girth $= 3$

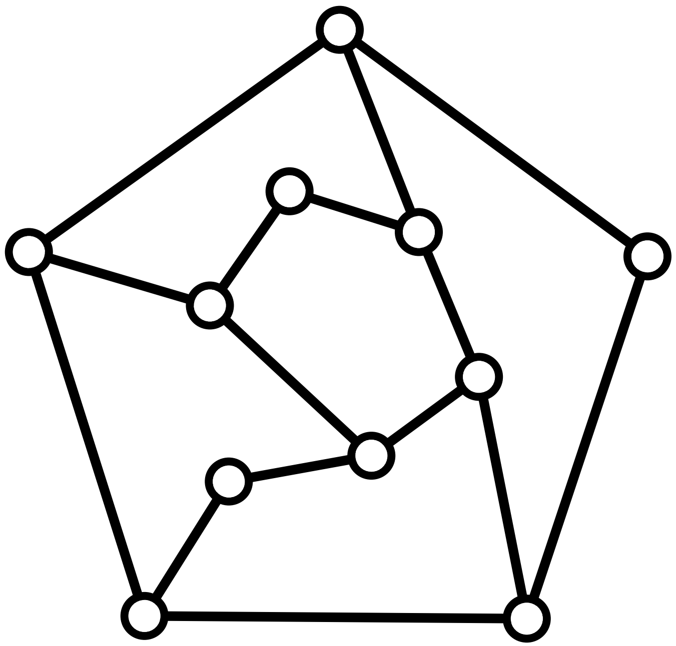
Rk: Simple $\Leftrightarrow$ girth ≥ 3

If **girth**$= d$ then all faces have **degree at least** d

(in particular a triangulation is simple iff it has girth 3)

d-angulations of girth d

For $d \geq 3$ we consider **d-angulations** (all faces have degree d) of **girth d**



a pentagulation of girth 5

d-angulations of girth d

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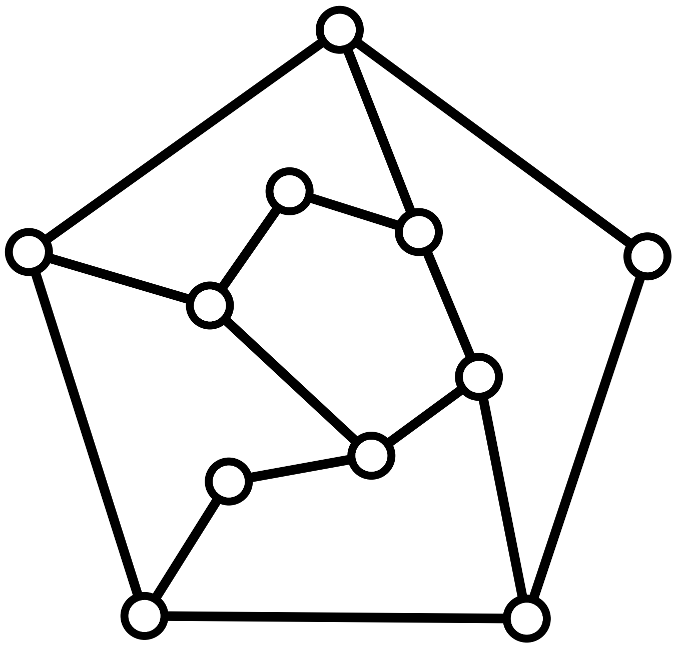


a pentagulation of girth 5

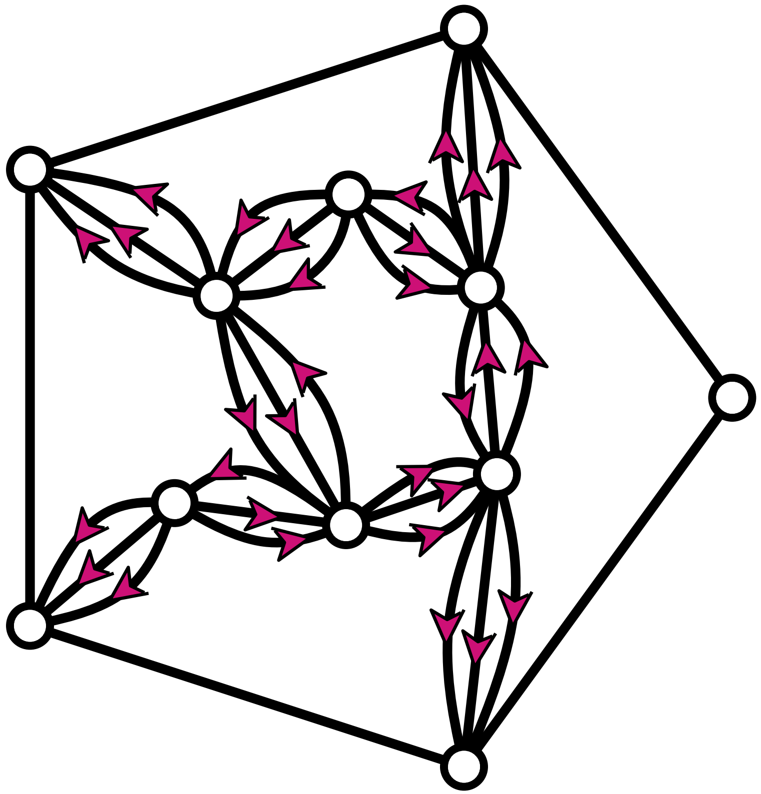
Rk: By the Euler relation, $\boxed{\dfrac{\#(\text{inner edges})}{\#(\text{inner vertices})} = \dfrac{d}{d-2}}$

$d/(d-2)$-orientations for d-angulations of girth d

[Bernardi-F'10]: Let G be a d-angulation of girth d. Then $(d-2)G$ admits an orientation where each inner vertex has **outdegree** d

Such an orientation is called a $d/(d-2)$-**orientation**

$d = 5$



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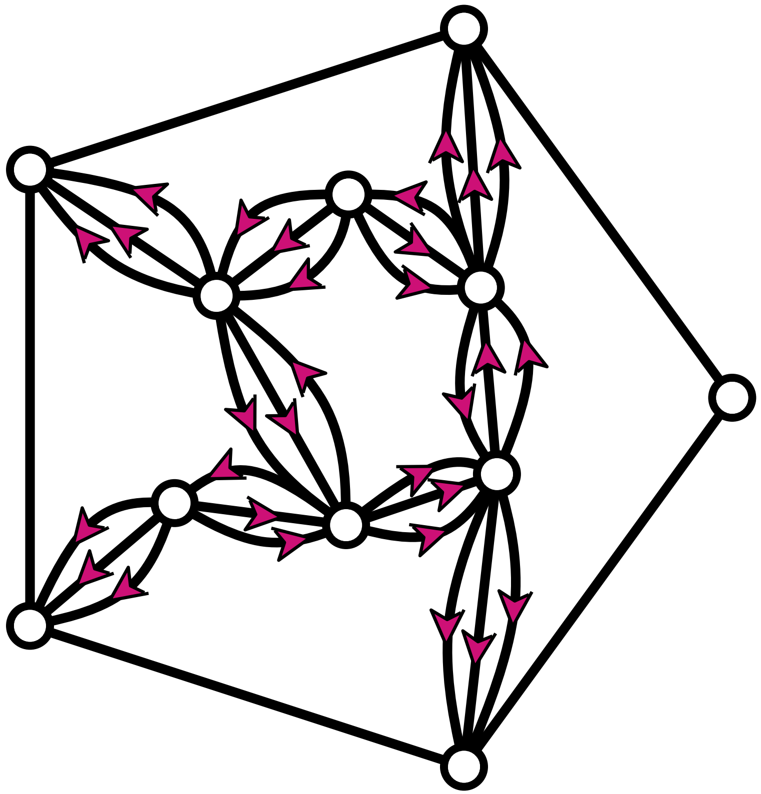
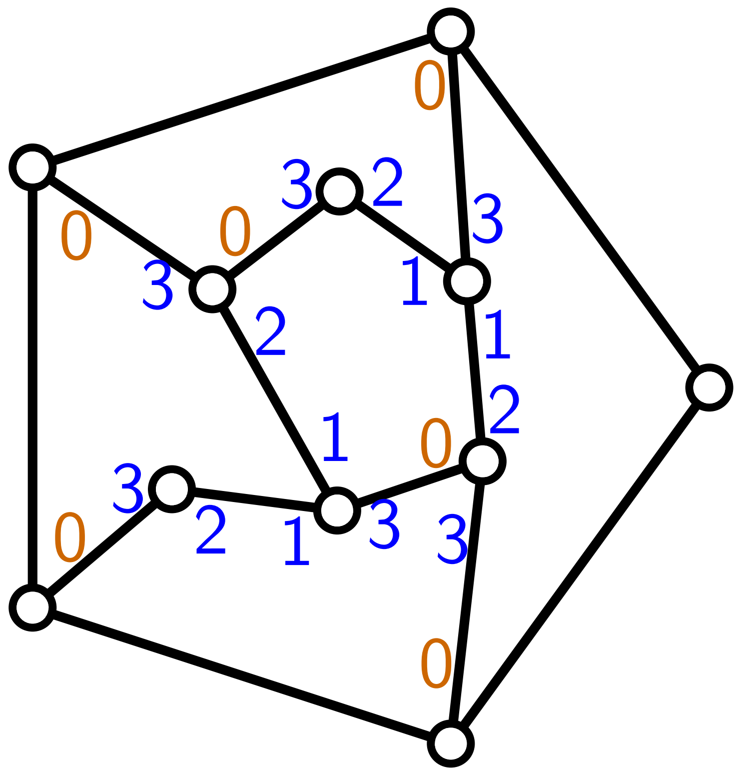
$\Leftrightarrow$ **assignment** of (outgoing) **flows** to half-edges

$d = 5$





total flow at inner edge $= d-2$
total flow at inner vertex $= d$

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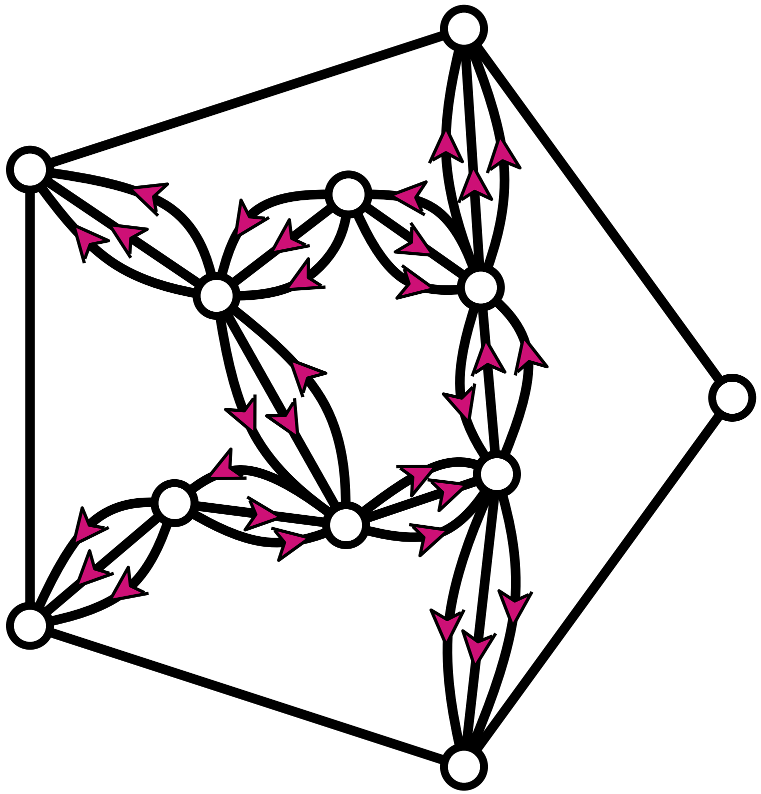
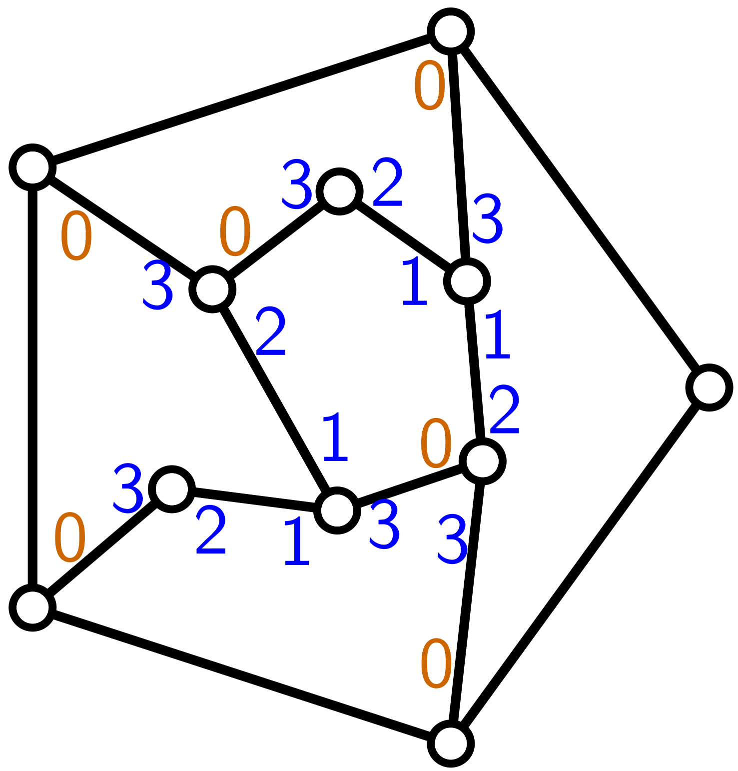
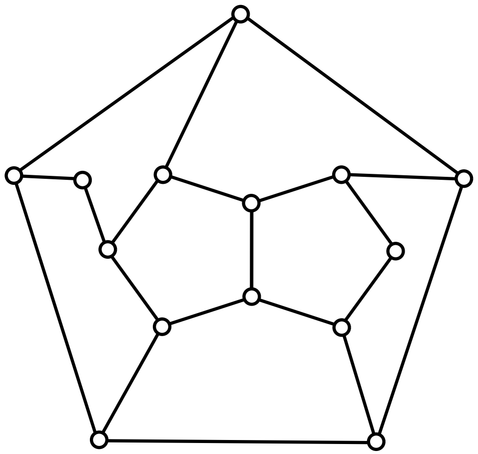




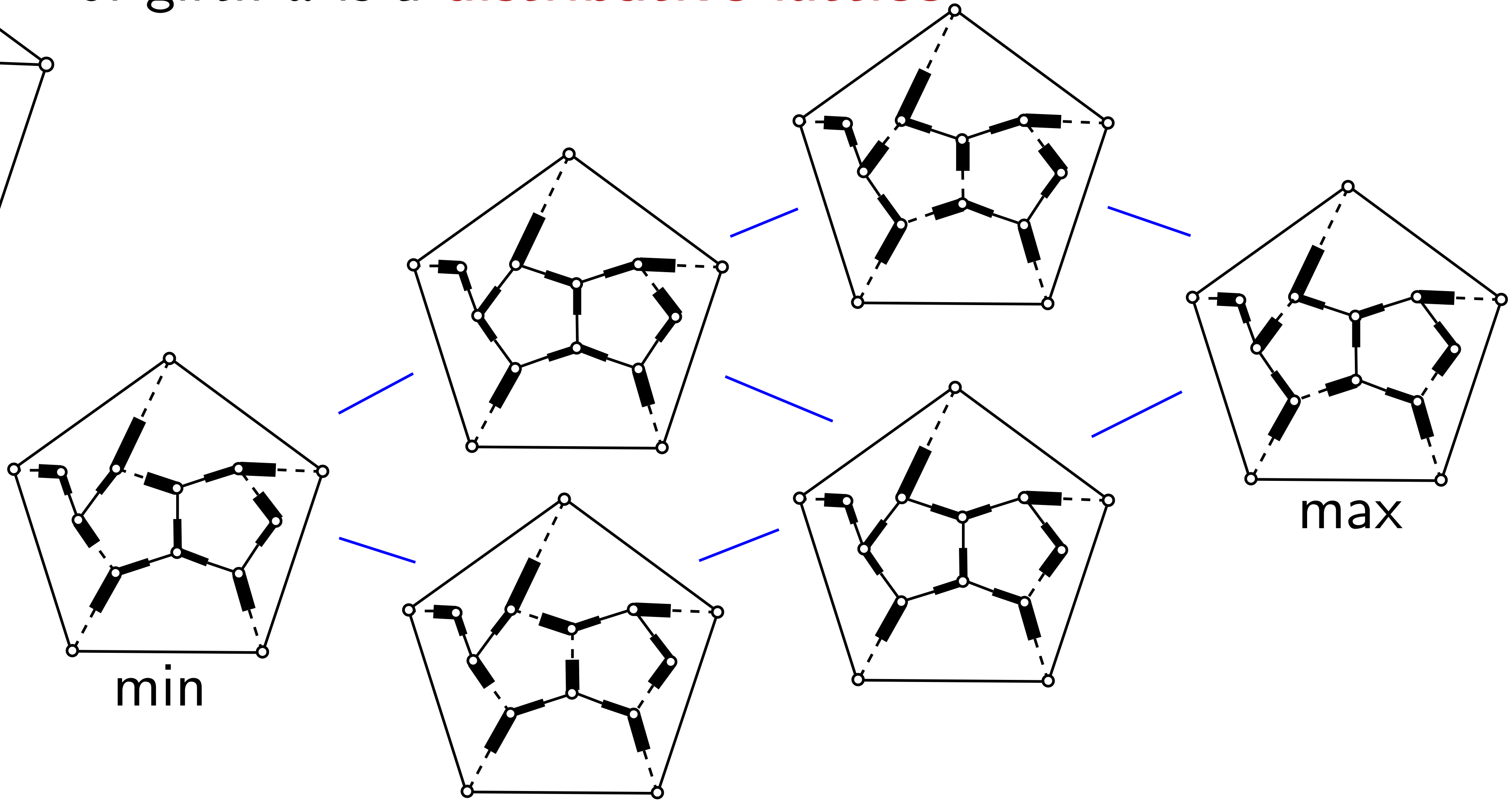
Rk: also formulations as Schnyder labellings/woods

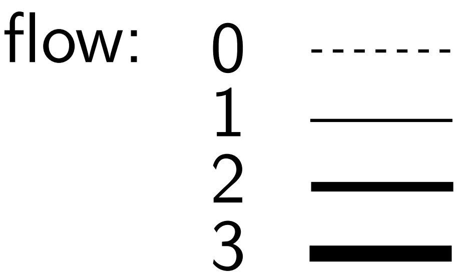
total flow at inner edge $= d-2$
total flow at inner vertex $= d$

Lattice property for $d/(d-2)$-orientations

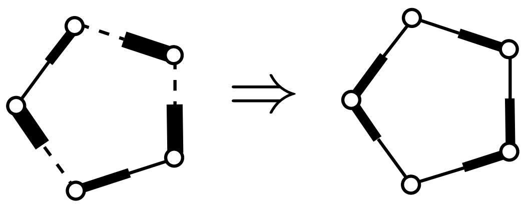


The set of $d/(d-2)$-orientations of a fixed d-angulation of girth d is a **distributive lattice**



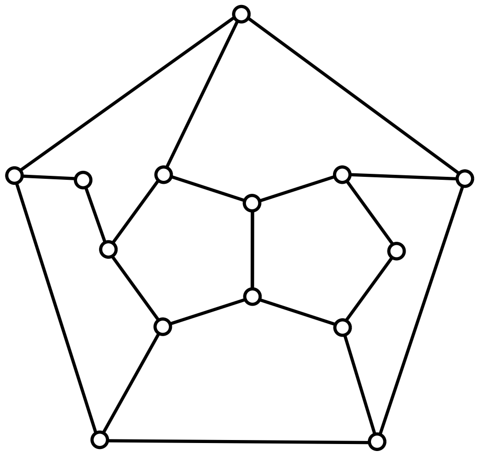


flip:

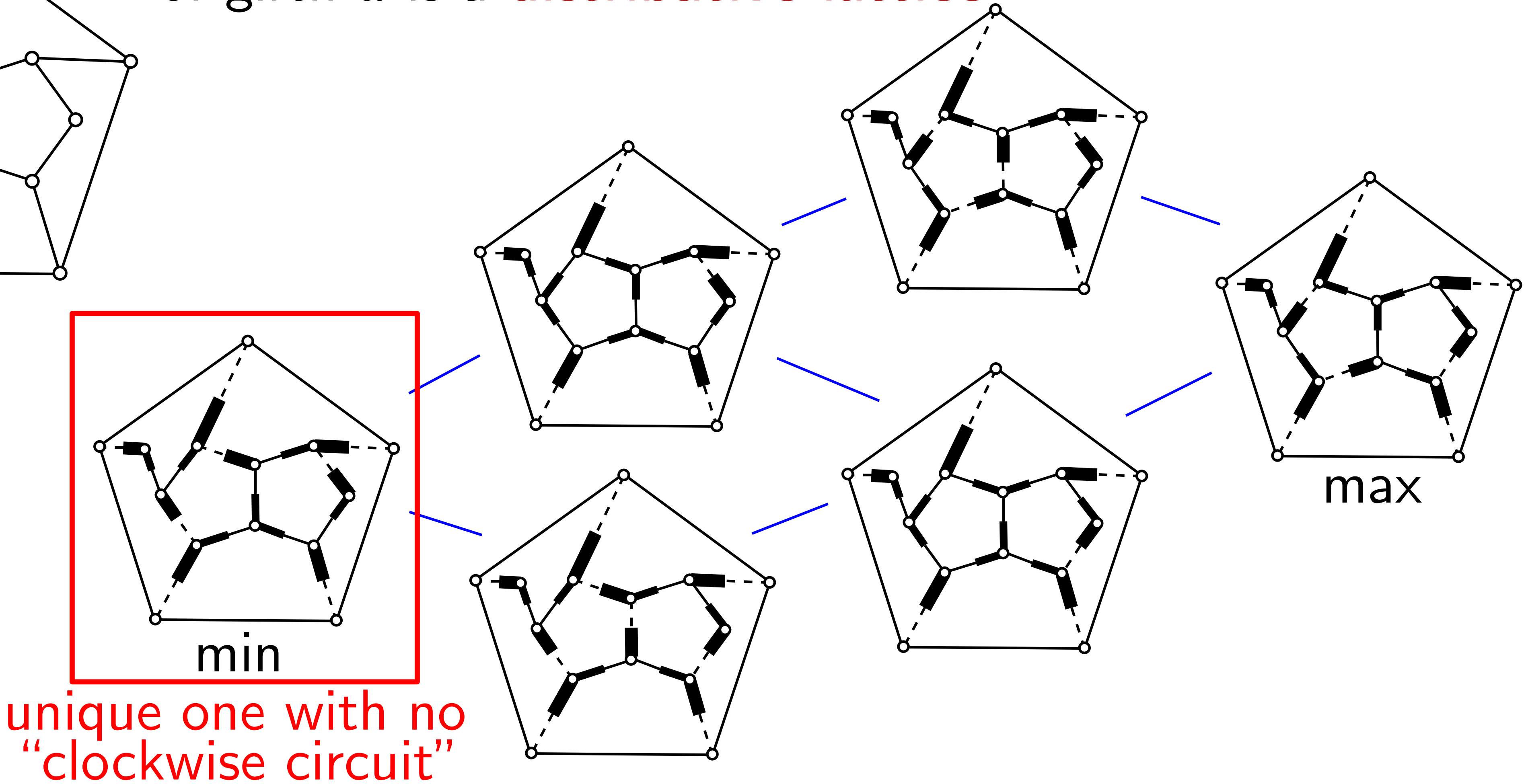


increase flow by 1 clockwise

Lattice property for $d/(d-2)$-orientations

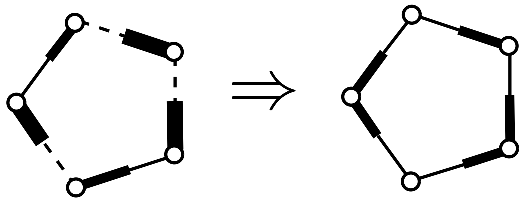


The set of $d/(d-2)$-orientations of a fixed d-angulation of girth d is a **distributive lattice**



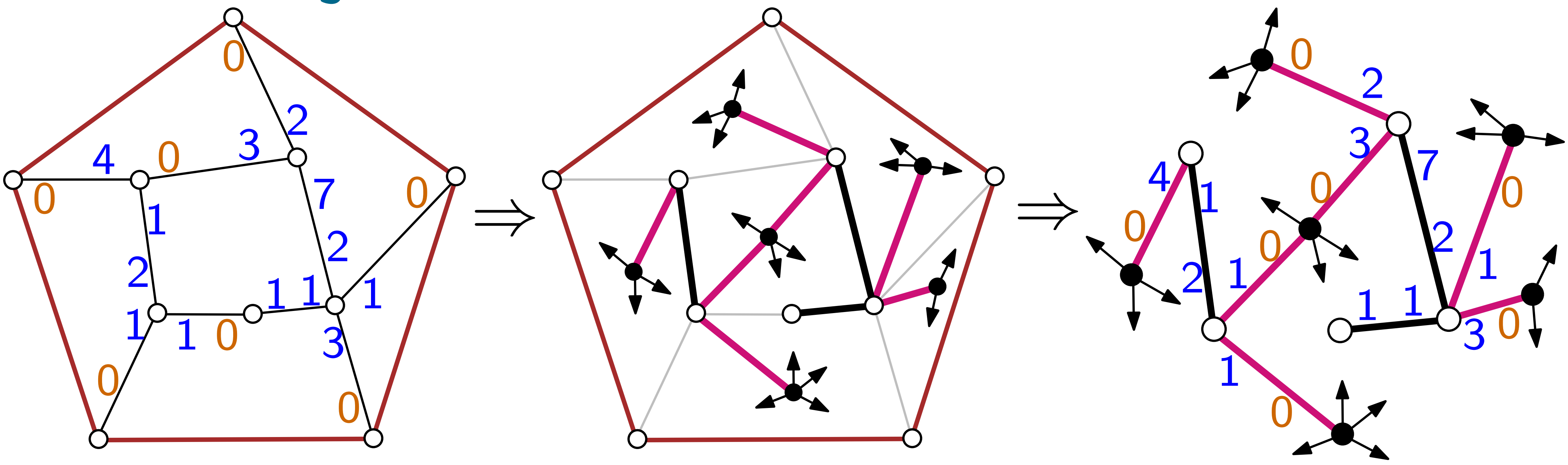
flow: 0 1 2 3

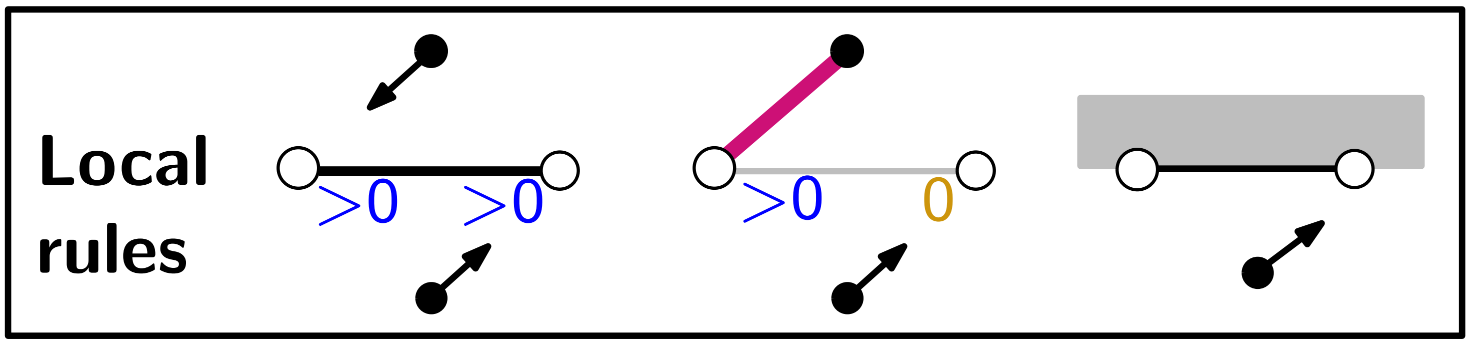
flip:



increase flow by 1 clockwise

Master bijection in the flow-formulation

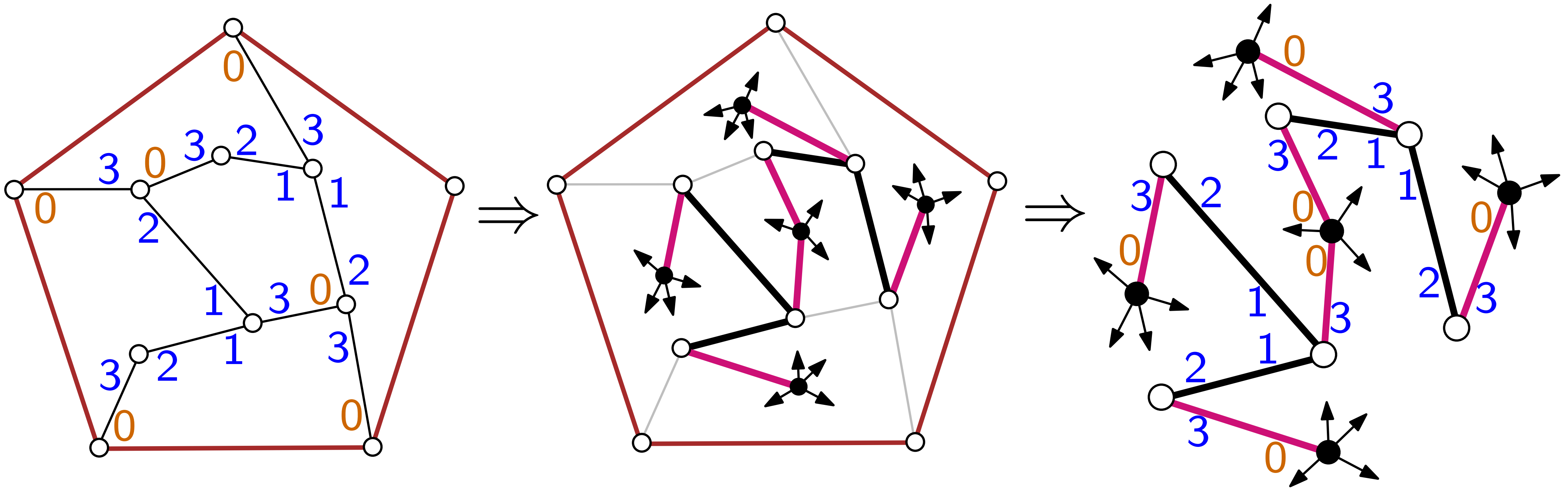




degrees of inner faces ⟷ degrees of black vertices

total flows at inner vertices ⟷ total weights at white vertices

Specialization to d-angulations of girth d



Bijection **d-angulations of girth d** $\leftrightarrow$ **weighted mobiles** such that

- each black vertex has degree d
- each white vertex has total weight d
- each edge has total weight $d-2$ (weight>0 at $\circ$, weight$=0$ at $\bullet$)

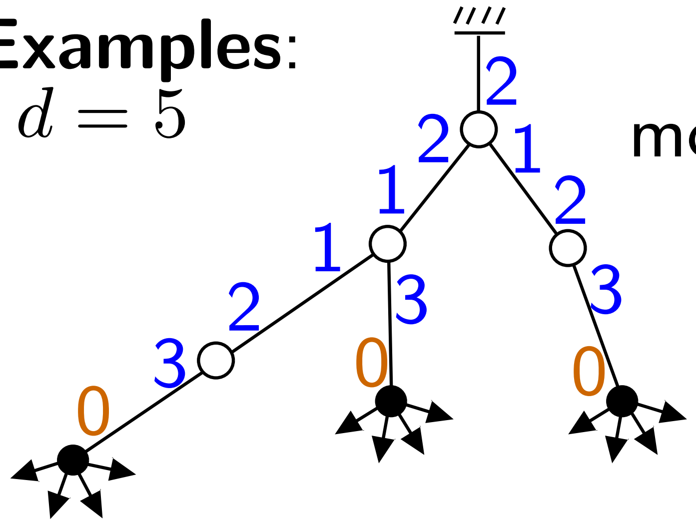
[Albenque, Poulalhon'11]: other bijection (with blossoming tree)

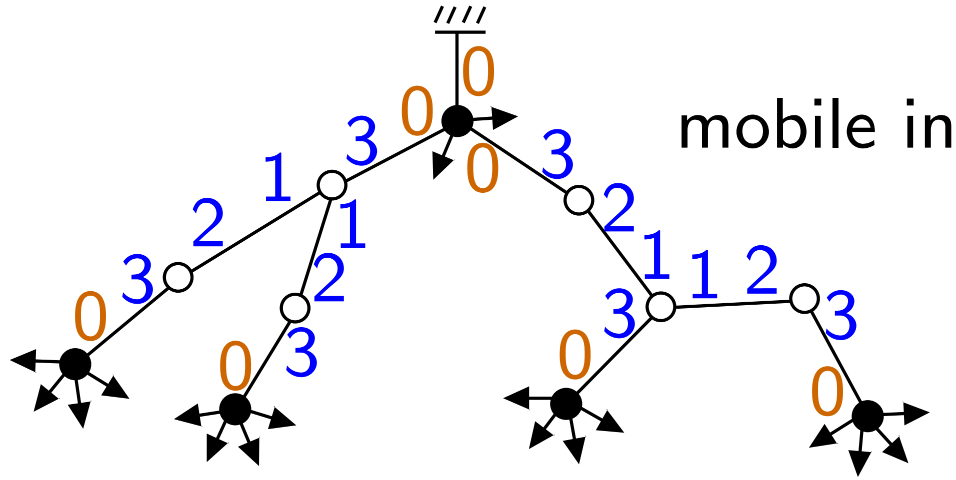
Generating function expression

For $i \in [0..d]$, $\mathcal{L}_i :=$ family of such mobiles with a **root-leg of weight i**

Let $L_i(x)$ be the GF of $\mathcal{L}_i$ where x marks black nodes

Examples:
$d = 5$





For $d \geq 3$, $F_d(x) :=$ GF of (rooted) d-angulations of girth d by inner faces

- **Bijection** when an **inner face** is **marked**

$$\Rightarrow \boxed{F'(x) = (1 + L_{d-2})^d}$$

- **Root-decomposition** of mobiles in $\mathcal{L}_i \Rightarrow (L_0, L_1, \ldots, L_d)$ are given by

$$\boxed{\begin{cases} L_0 & = & x \cdot (1 + L_{d-2})^{d-1}, \\ L_d & = & 1, \\ L_i & = & \displaystyle\sum_{j>0} L_{d-2-j} L_{i+j} \quad \text{for } i = 1..d-1 \end{cases}}$$