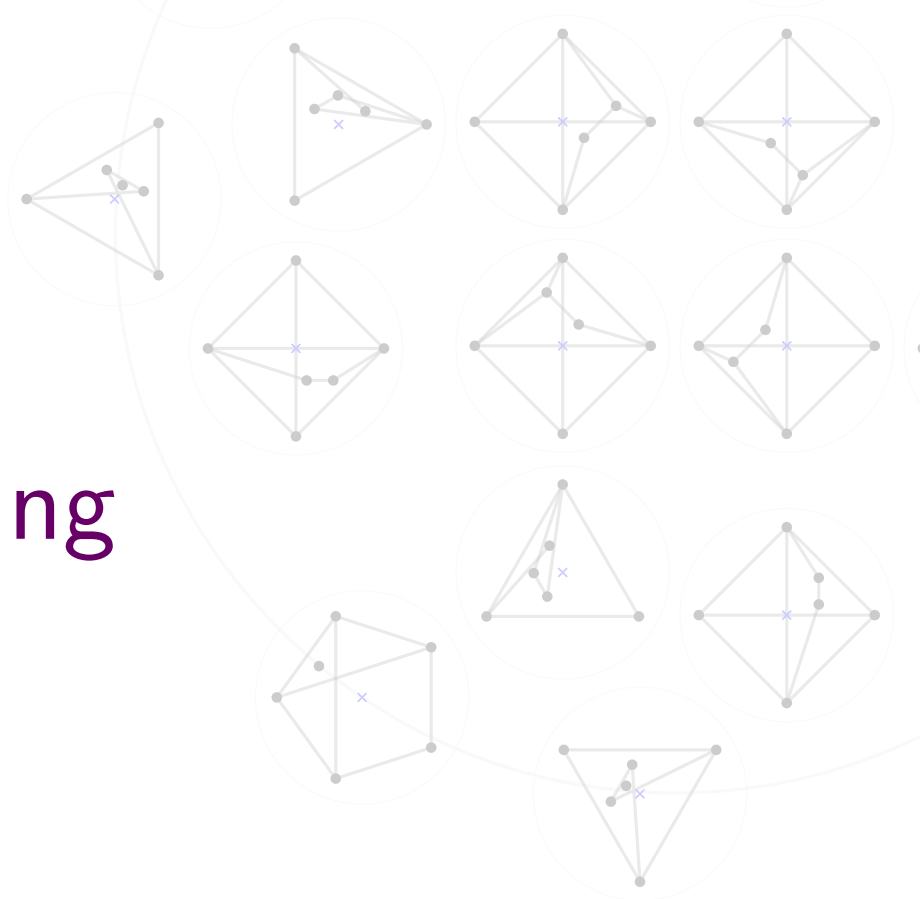


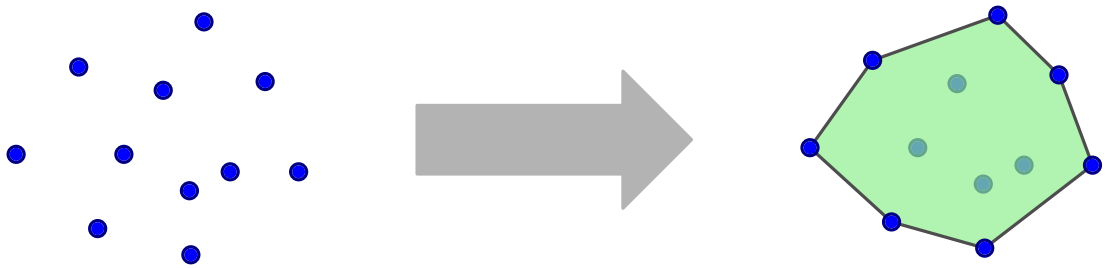
Counting and simulating planar order types

Xavier Goaoc
LORIA - Université de Lorraine

Based on joint works with

- Alfredo Hubard, Rémi de Joannis de Verclos, Jean-Sébastien Sereni and Jan Volec.
- Olivier Devillers, Philippe Duchon and Marc Glisse.
- Emo Welzl.

Geometric algorithms extract **combinatorial** information out of **numerical** data.



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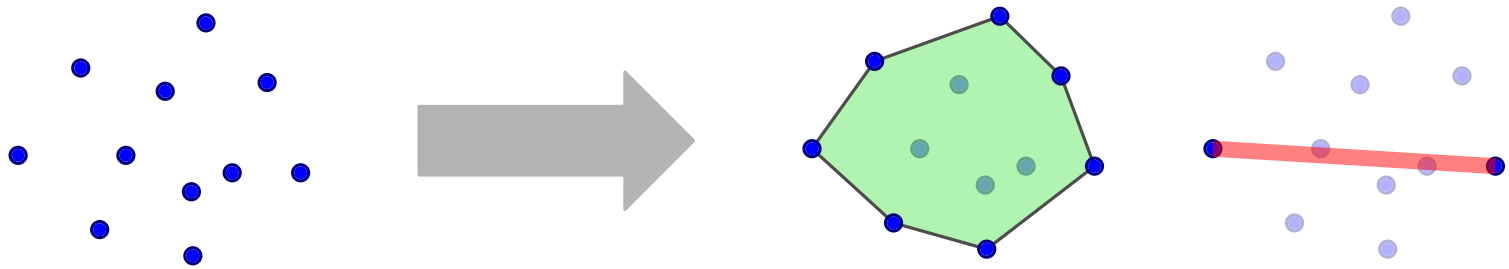

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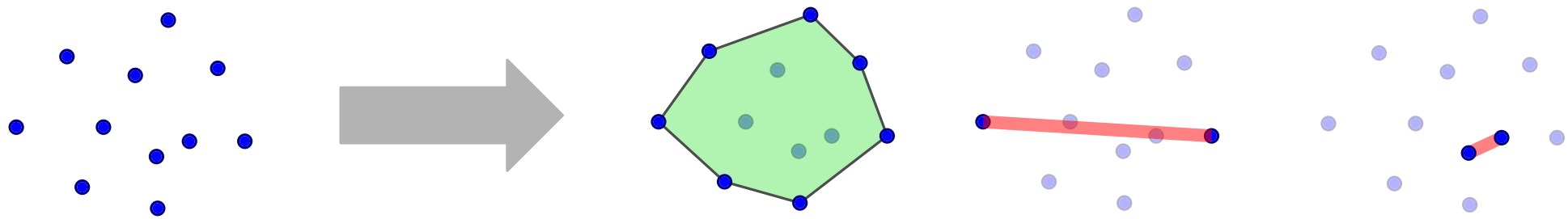

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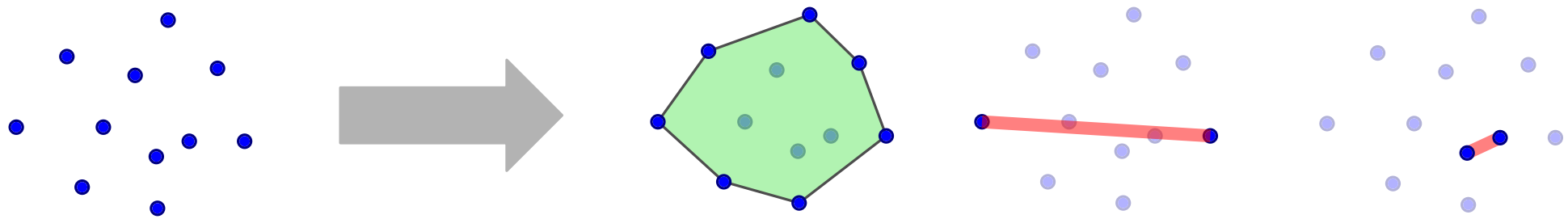

Algorithms based on the geometry of $\mathbb{R}^2$ are run on finite precision arithmetic...

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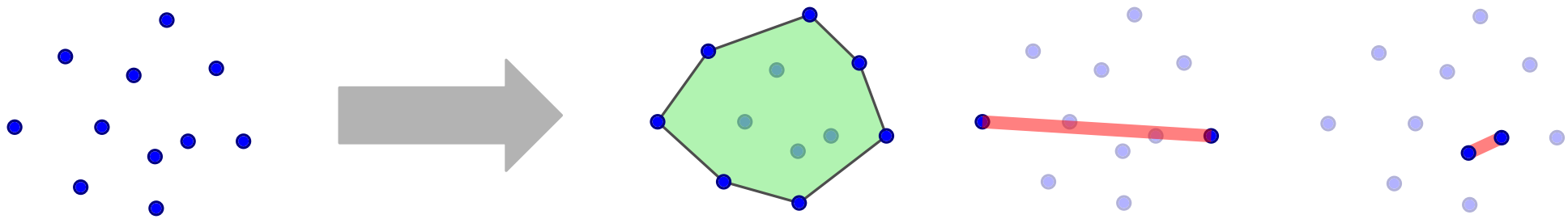

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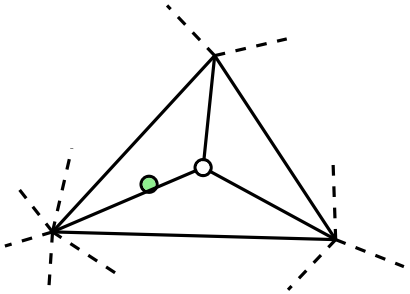

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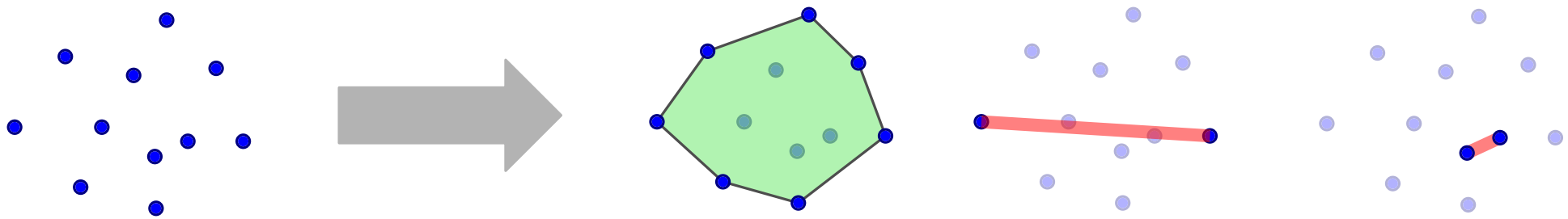

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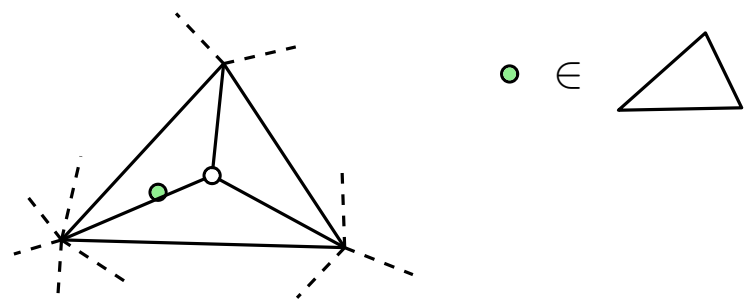

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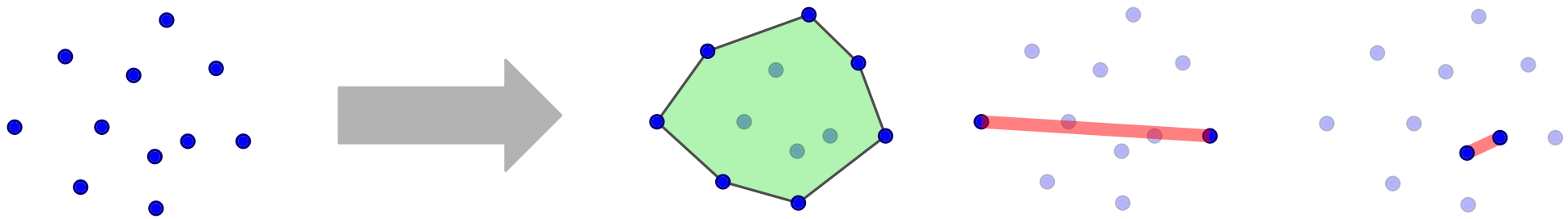

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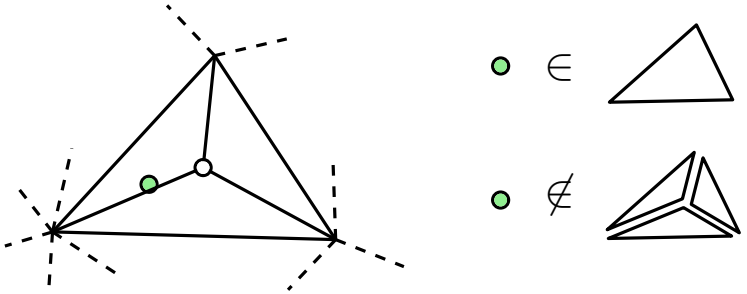

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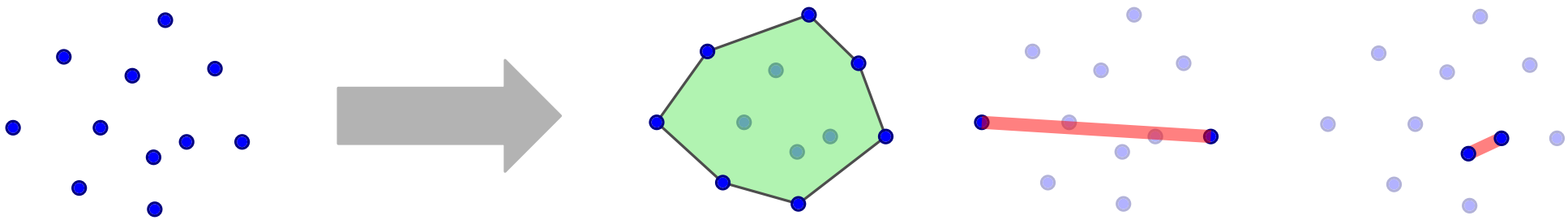

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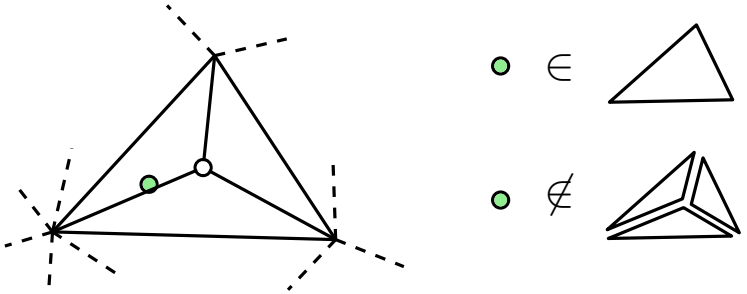

GOOD PRACTICE: algorithms make **decisions** based on **input data**, not **intermediate constructions**.

CGAL

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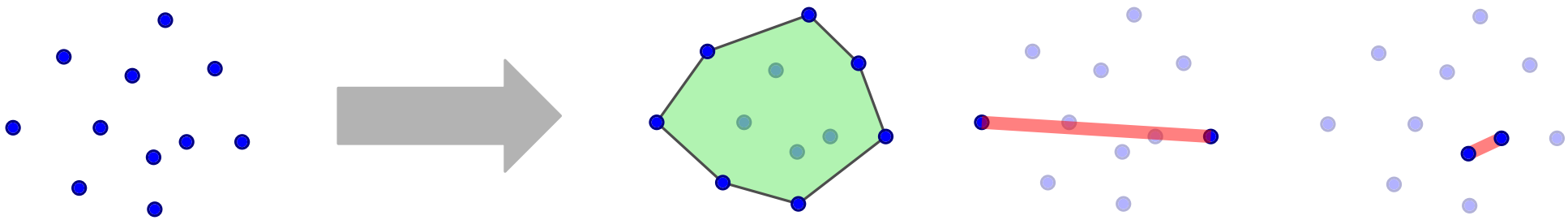

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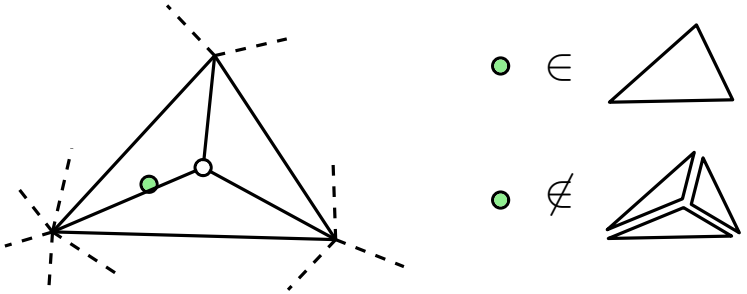

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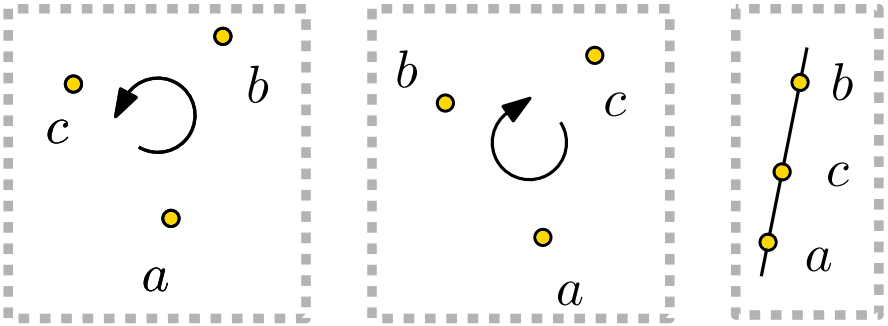
Example of decision: orientations



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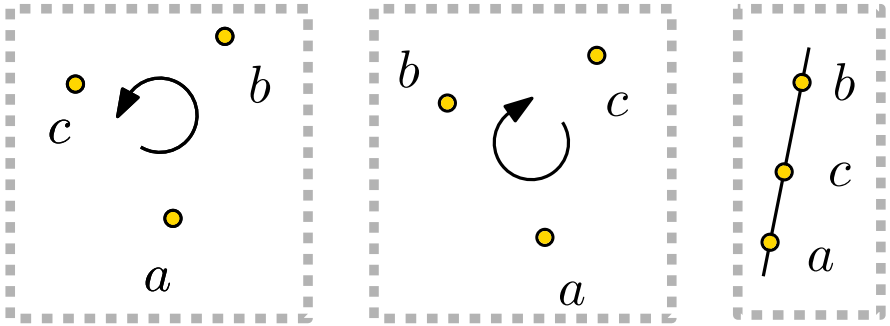

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Can be **certified**.

Signs of polynomials.
Interval arithmetic.
Exact computation.

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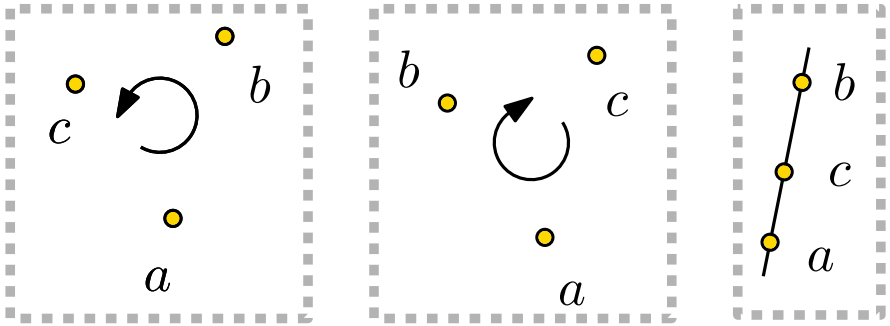

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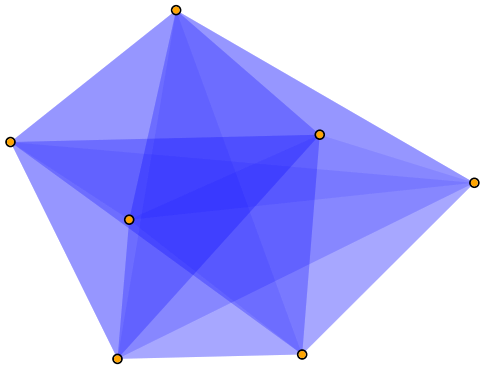
Determine convex hulls, onion peelings, segment crossings, halfspace/simplicial depth, ...



When all you know are orientations

two point sequences
$p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$
have the same chirotope

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$[p_i, p_j, p_k] = [q_i, q_j, q_k]$
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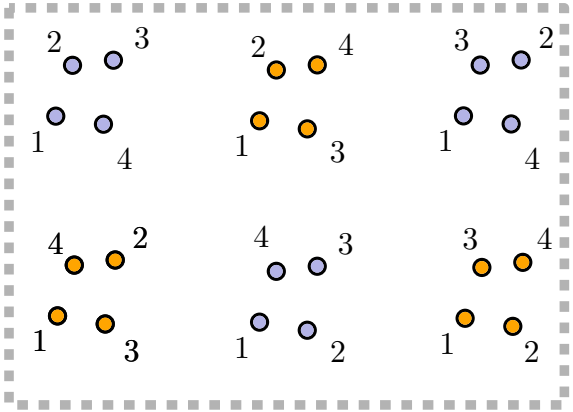

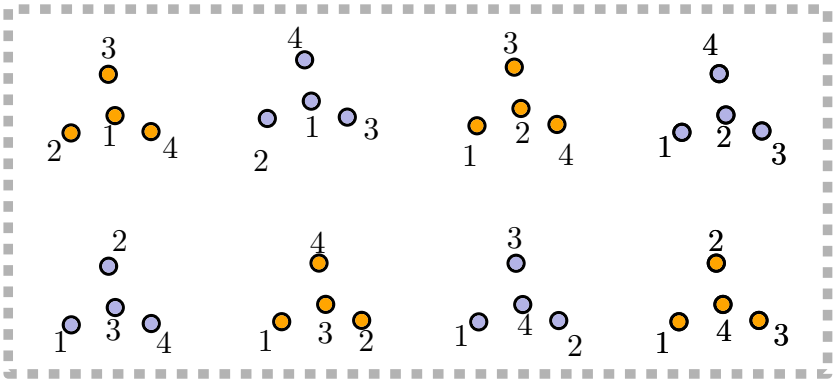

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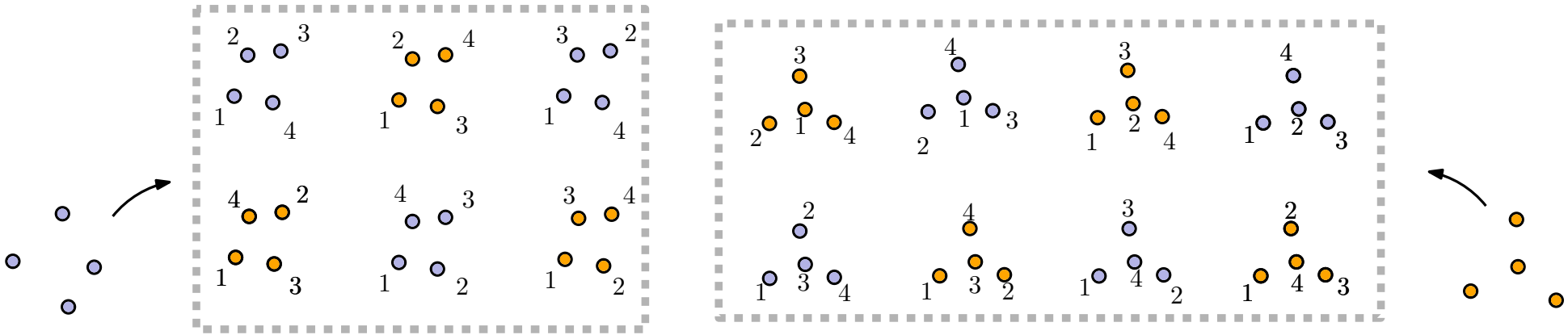

two finite subsets
P and Q of $\mathbb{R}^2$
have the same order type

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$\exists$ a bijection $f : P \to Q$ such that
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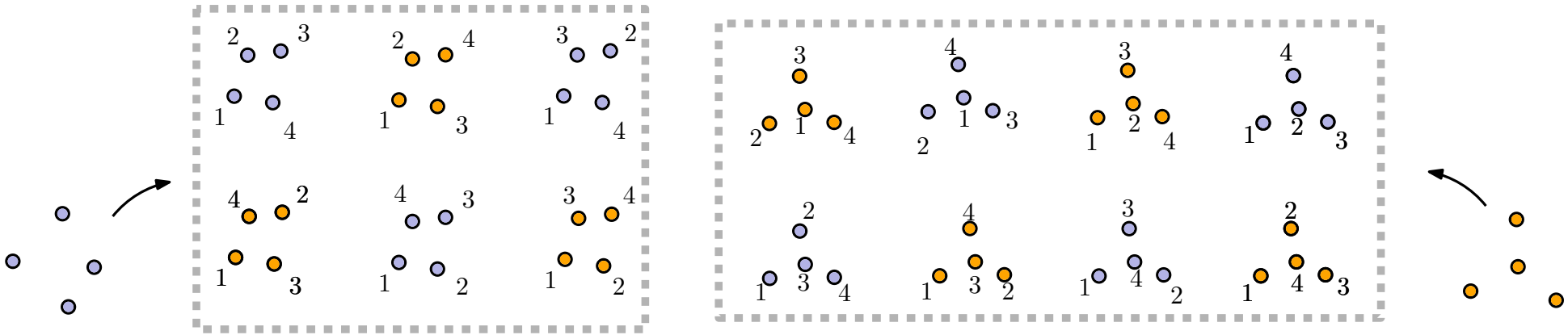

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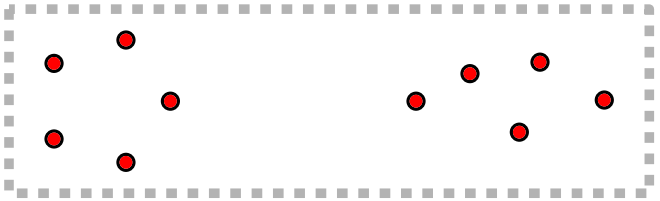
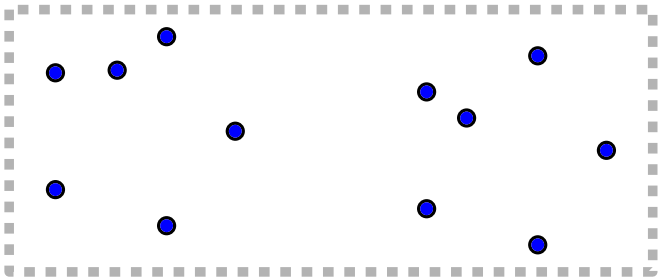
Chirotopes $\simeq$ labeled order types

Practice





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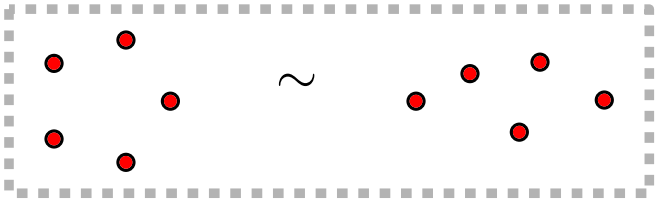

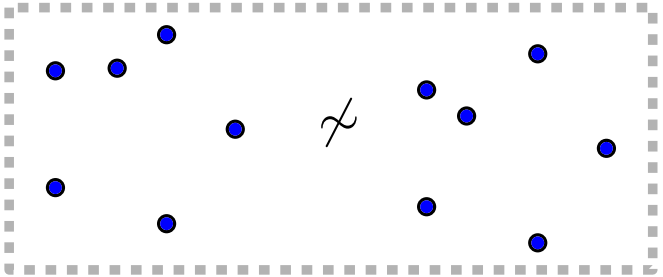

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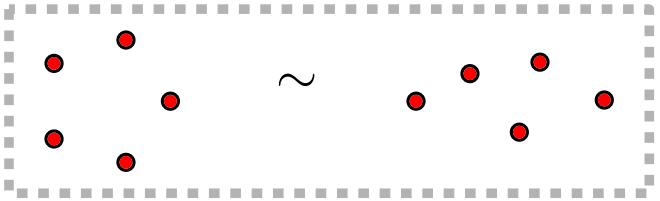

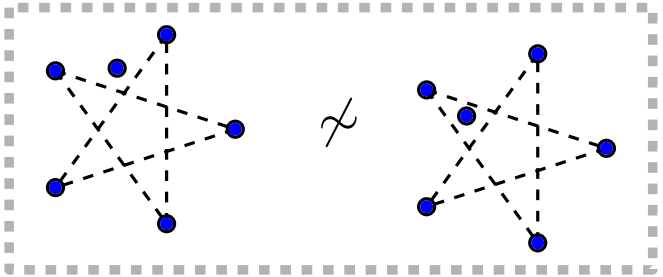

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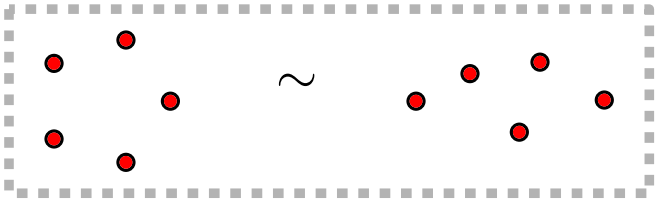

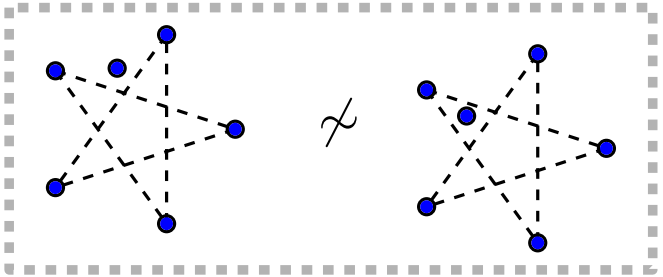

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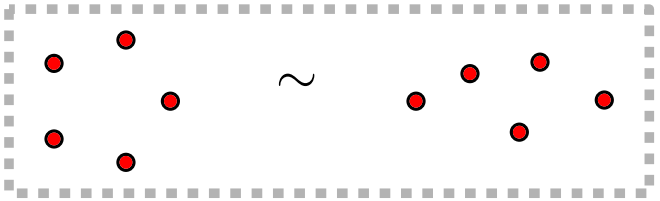
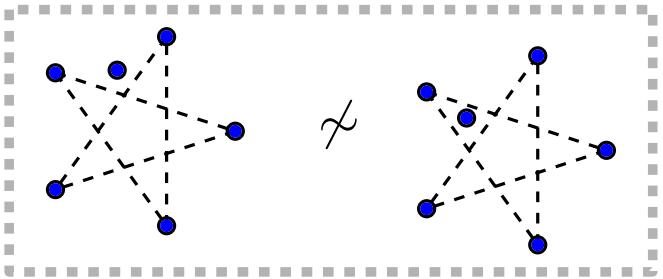
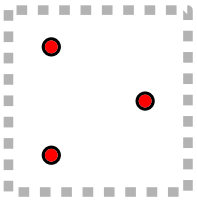
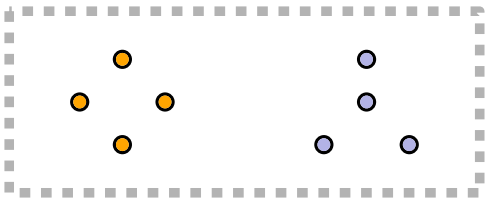
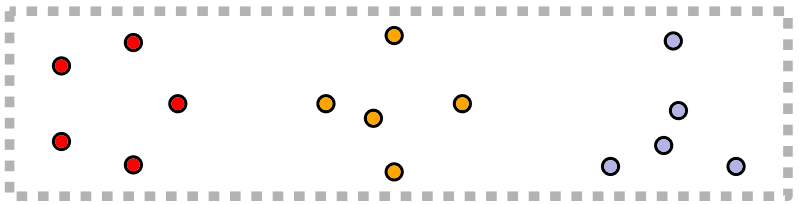




A chirotope/order type is **simple** if no three points are aligned.







Why care?

Reduce the infinitely many n-point sets to **finitely** many configurations.

Model what geometric algorithms operate on.

Questions

Count, enumerate, sample, recognize, ...

Understand their **realization spaces**.

Isotopy? Small-coordinates realizations? ...

Use to study **discrete geometry** questions.

Counting triangulations, Erdös-Szekeres conjecture, empty hexagon problem, ...

Warm-up: counting

There are $\geq n^{4n-o(n)}$ simple chirotopes of size n. [GP'86]

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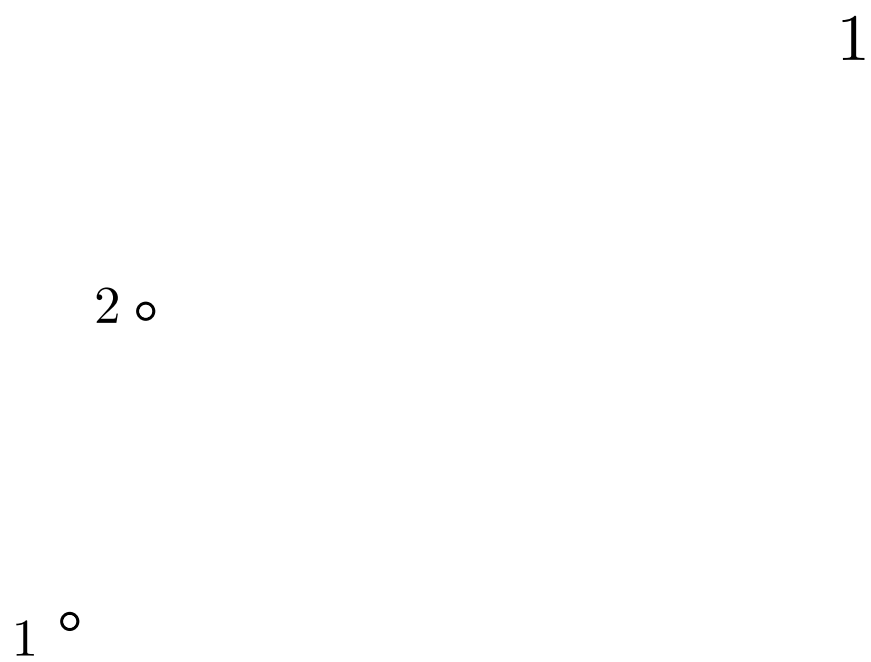

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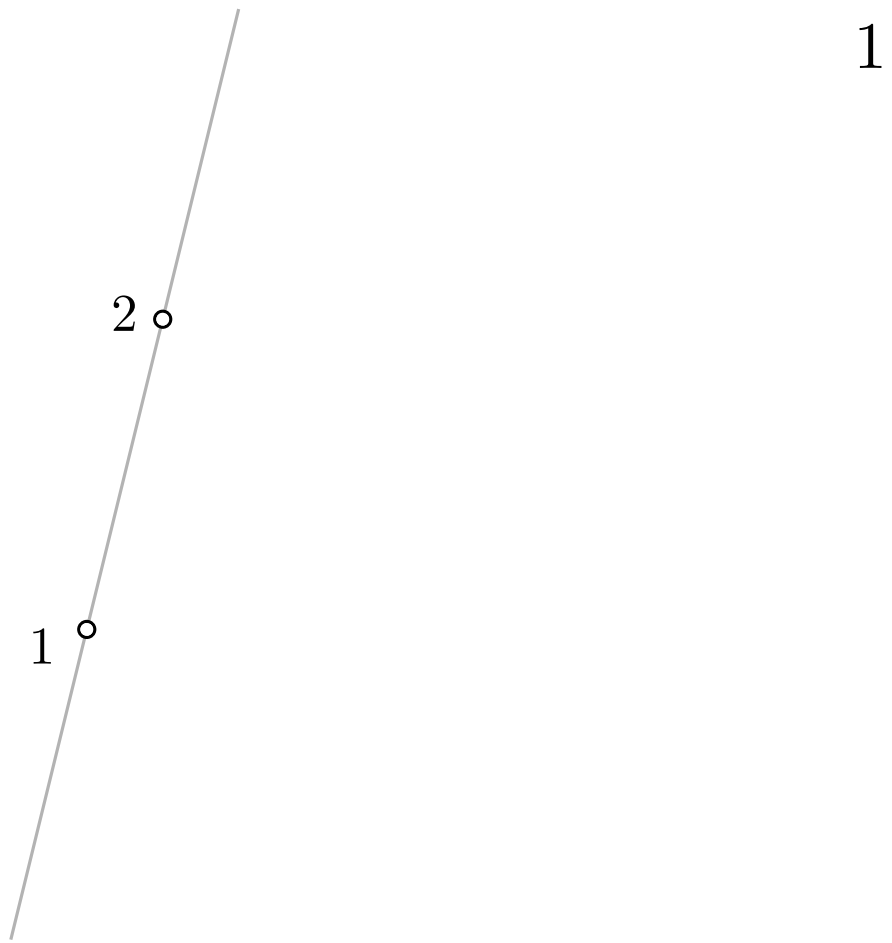

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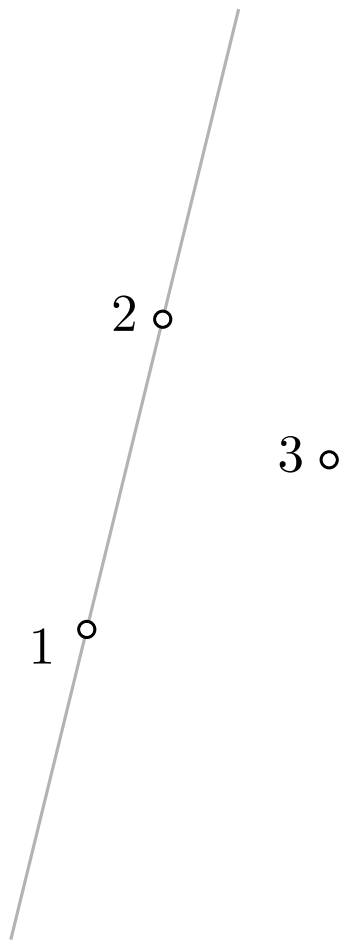

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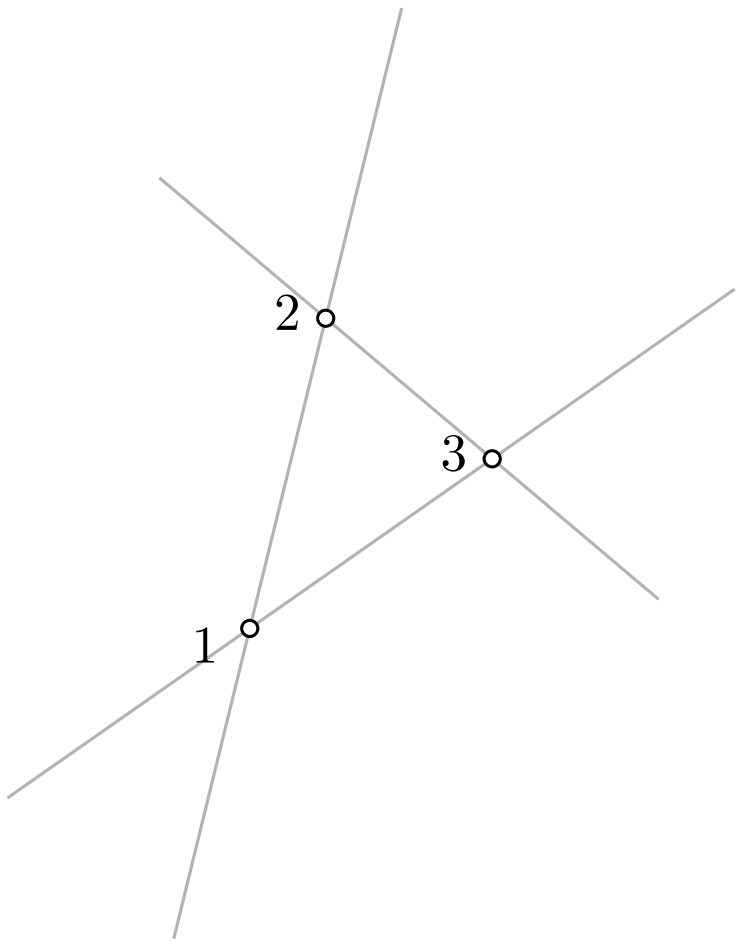

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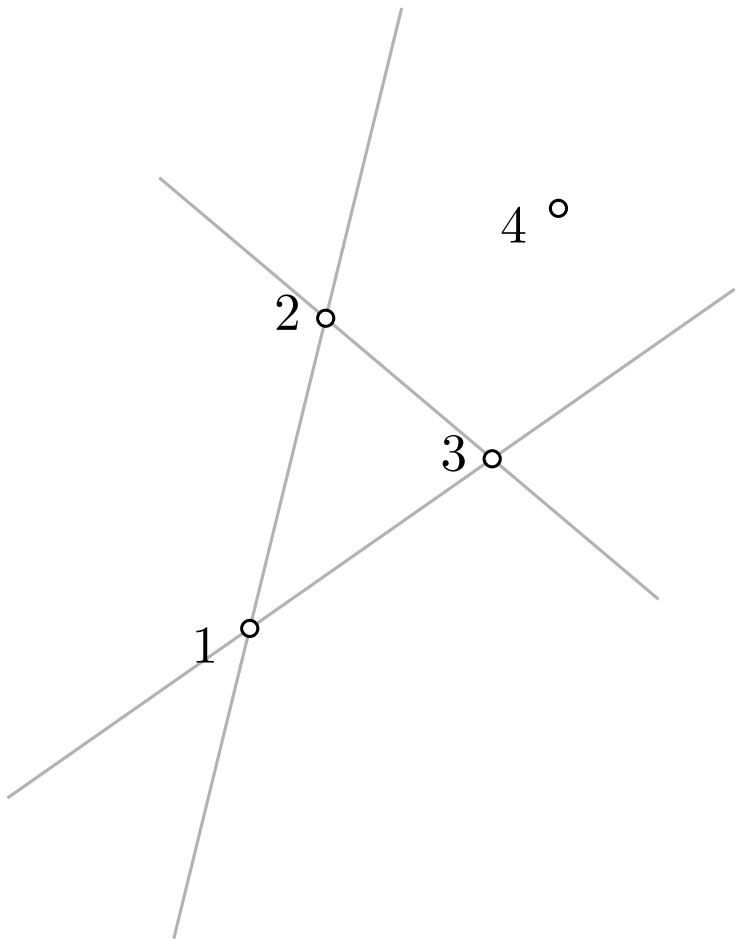

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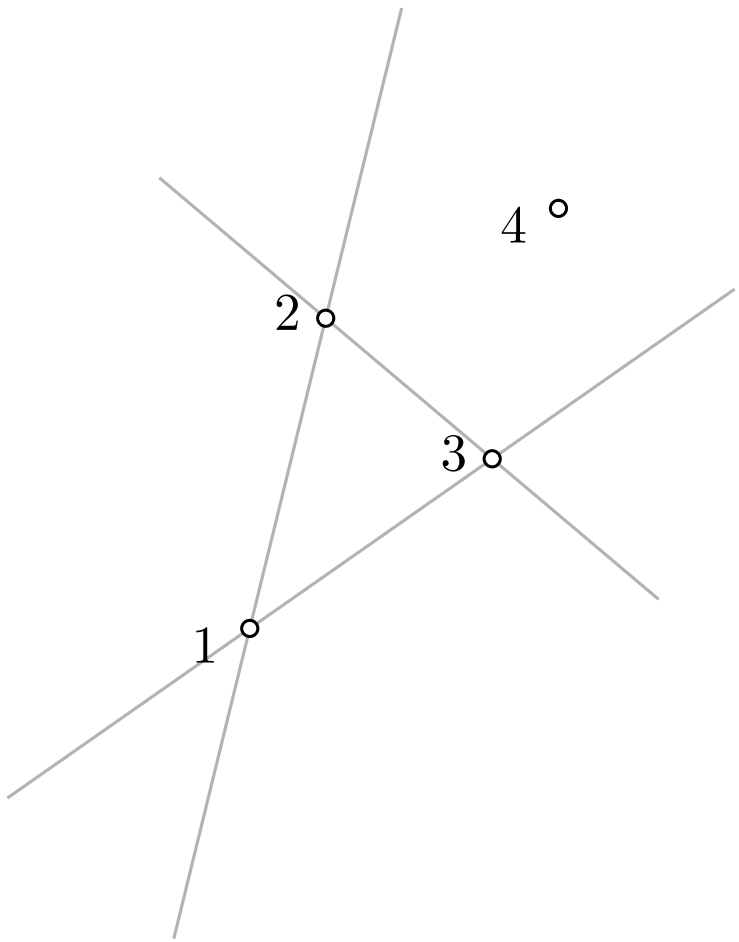

$1 *2 \ *7...$

Add the points one by one

When adding the $(n+1)$th point, pick a cell in the arrangement of the $\binom{n}{2}$ lines through 2 points.

$\#$ cells is $\Omega(n^4)$

$\#$ chirotopes grows as $n^4 \cdot (n-1)^4 \cdot \ldots \simeq (n!)^4$

Chirotopes are sign-patterns of polynomials

Work in the **space of** n-point sequences:

$$p_1, p_2, \ldots, p_n \text{ in } \mathbb{R}^2 \leftrightarrow \tilde{p} = (p_{1x}, p_{1y}, p_{2x}, \ldots, p_{ny}) \text{ in } \mathbb{R}^{2n}.$$

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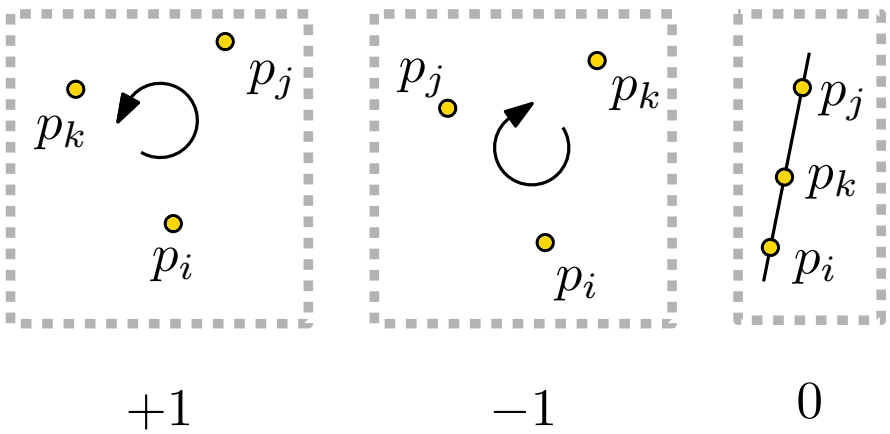

Triple's orientation are determined by polynomials in these $2n$ variables:

$$F_{i,j,k}(\tilde{p}) = \begin{vmatrix} p_{ix} & p_{jx} & p_{kx} \\ p_{iy} & p_{jy} & p_{ky} \\ 1 & 1 & 1 \end{vmatrix}$$

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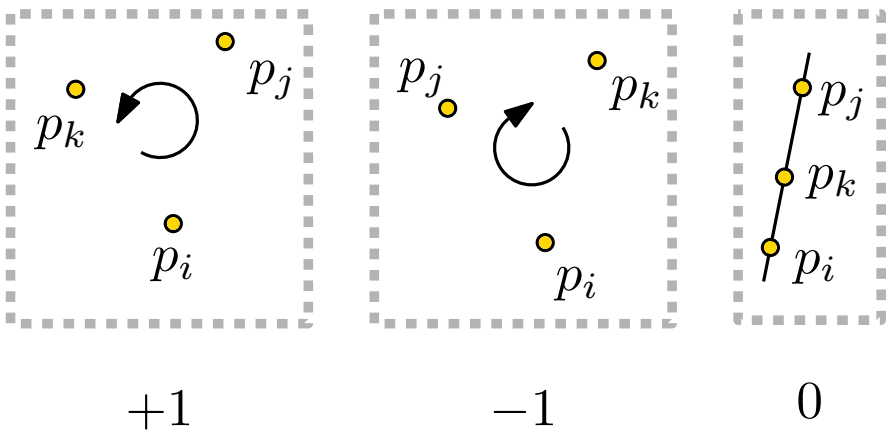

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Consider **sign sequences**:

$$\tilde{p} \in \mathbb{R}^{2n} \mapsto \sigma(\tilde{p}) = (\text{sign} F_{1,2,3}(\tilde{p}), \ldots, \text{sign} F_{n-2,n-1,n}(\tilde{p})) \in \{-1, +1\}^{\binom{n}{3}}$$

Counting

# sign sequences of $\{P_i\}_{i=1,\ldots,t}$ in $\{-1,+1\}^t$	$\leq$	# connected components of the complement of $\bigcup_{i=1}^{t}\{P_i = 0\}$.

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Theorem.[W'68] Let $P_1, \ldots, P_t$ be polynomials of degree $\leq \delta$ in v variables. If $t \geq v$, the number of connected components of $\mathbb{R}^v \setminus \left(\bigcup_{i=1}^t P_i = 0\right)$ is at most $\left(\frac{4et\delta}{v}\right)^v$.

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Theorem.[W'68] Let $P_1, \ldots, P_t$ be polynomials of degree $\leq \delta$ in v variables. If $t \geq v$, the number of connected components of $\mathbb{R}^v \setminus \left(\bigcup_{i=1}^t P_i = 0\right)$ is at most $\left(\frac{4et\delta}{v}\right)^v$.

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Some landmarks

$$n^{4n-o(n)} \leq \#\text{ simple } n\text{-point chirotopes} \leq n^{4n+o(n)}.$$

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Order types enumerated up to size 11 (mod. mirror images).

3	4	5	6	7	8	9	10	11
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That much precision is sometimes needed.

Von Staudt's constructions

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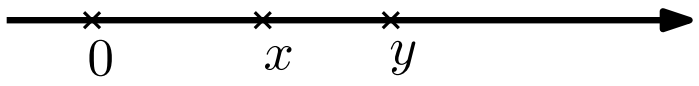
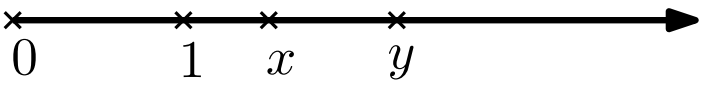
If we represent real numbers by points on a line in $\mathbb{R}^2$, $+$ and $*$ can be constructed by reporting parallel lines:





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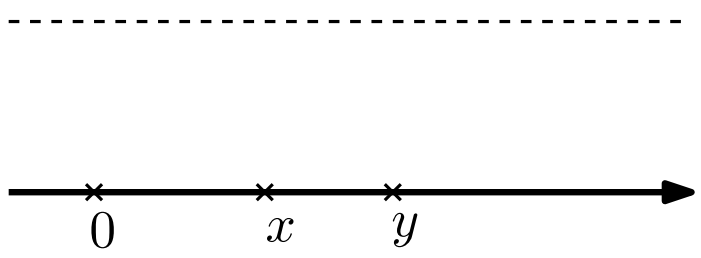

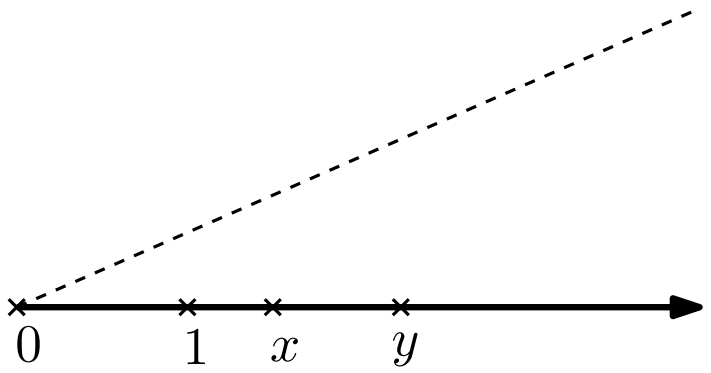

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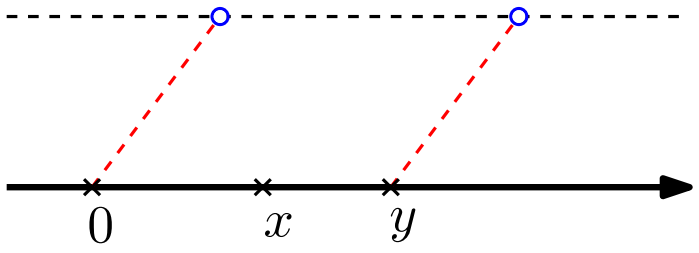

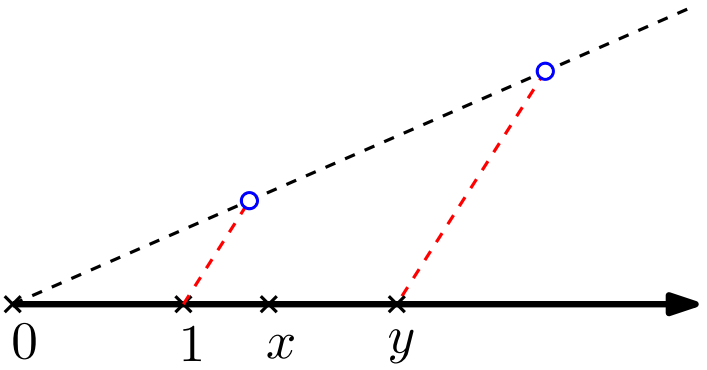

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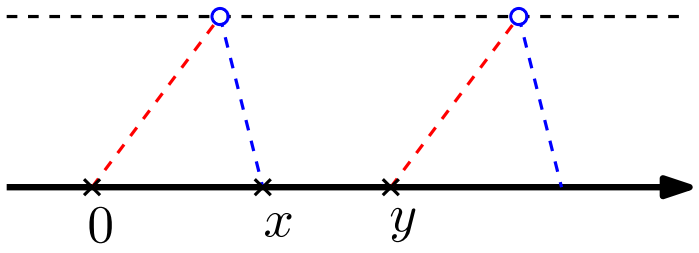

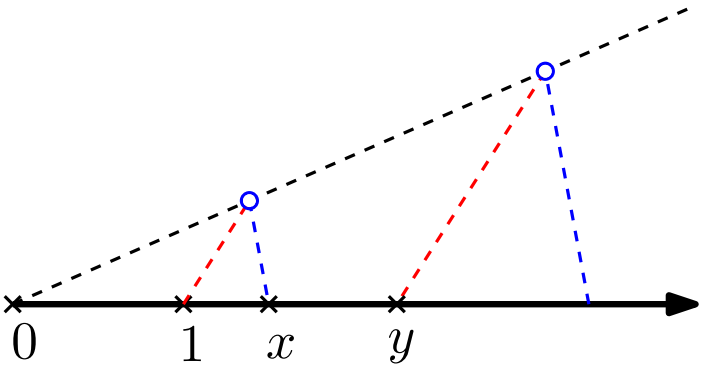

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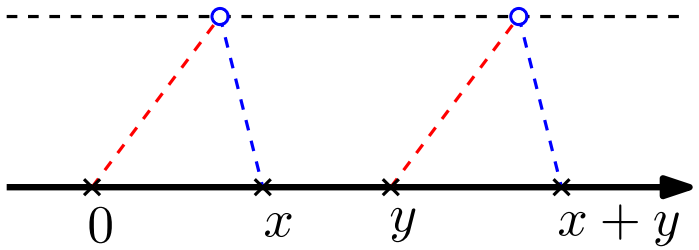

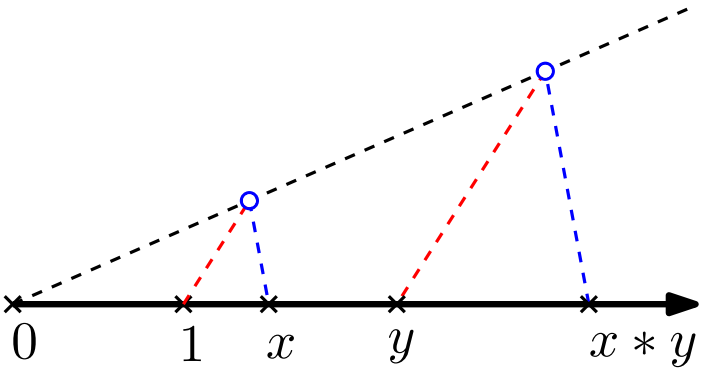

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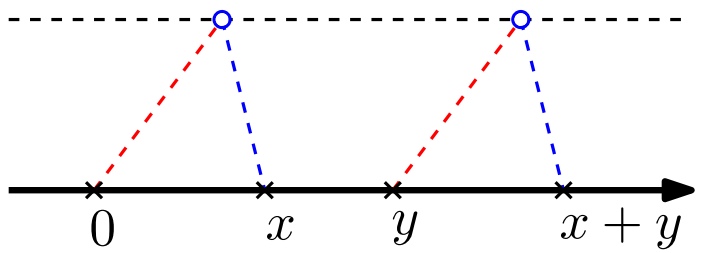

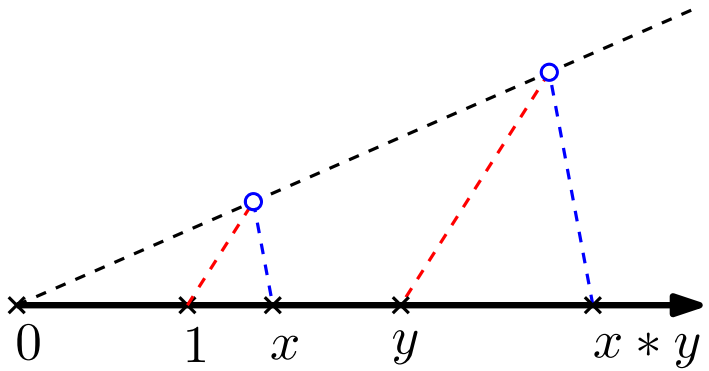

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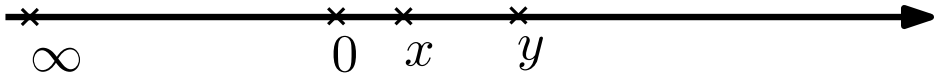

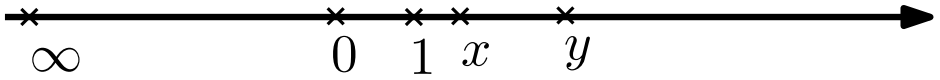

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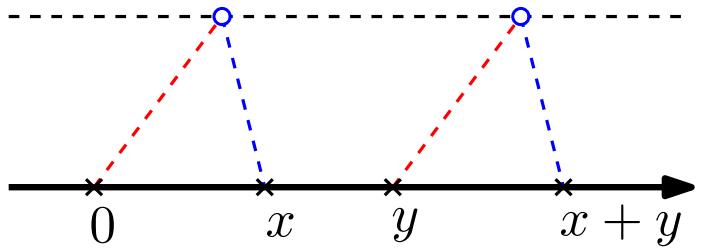

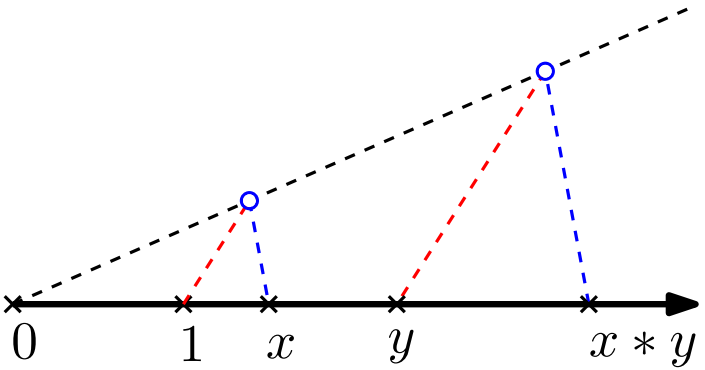

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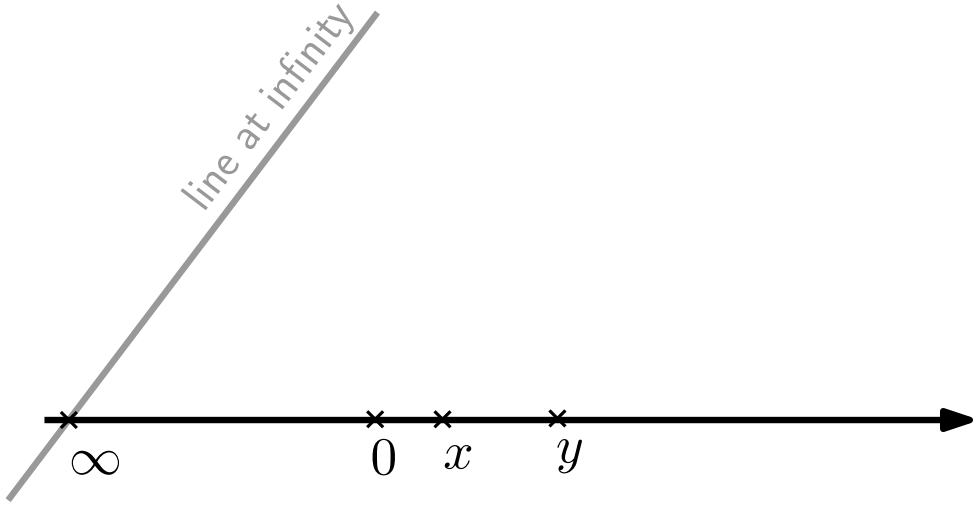

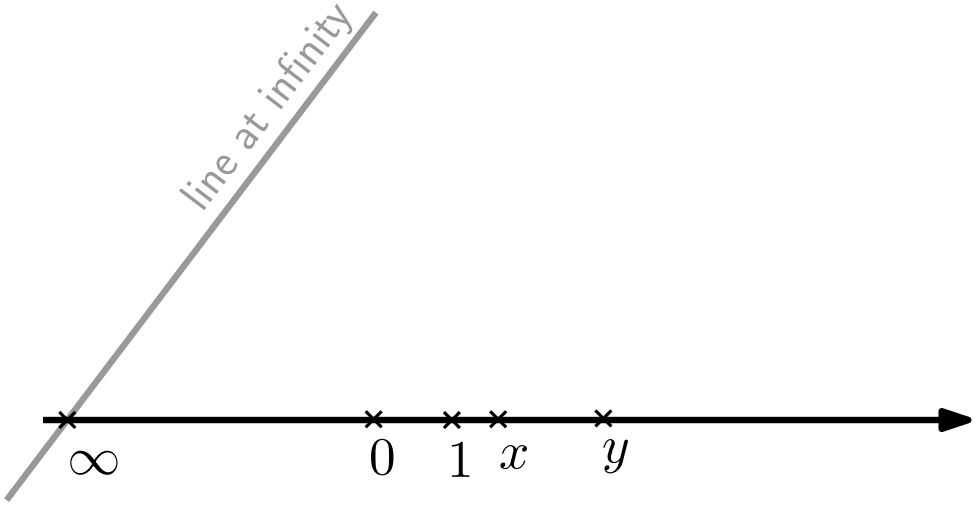

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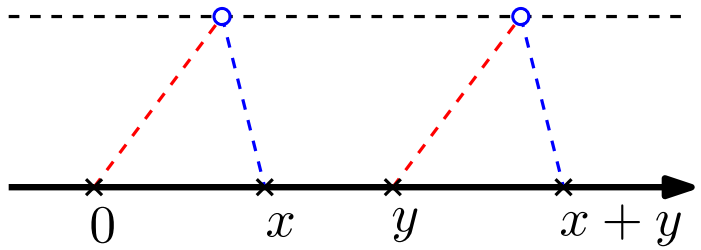

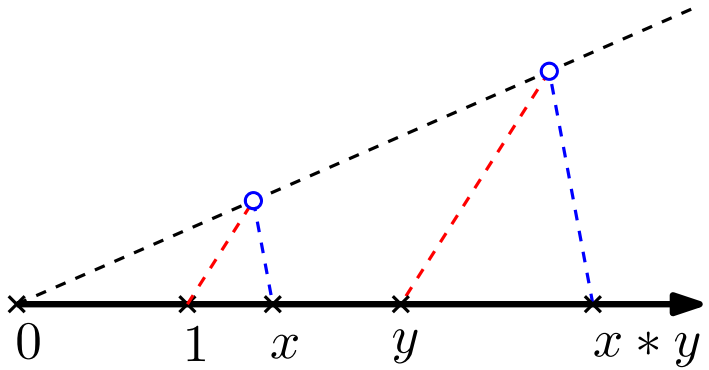

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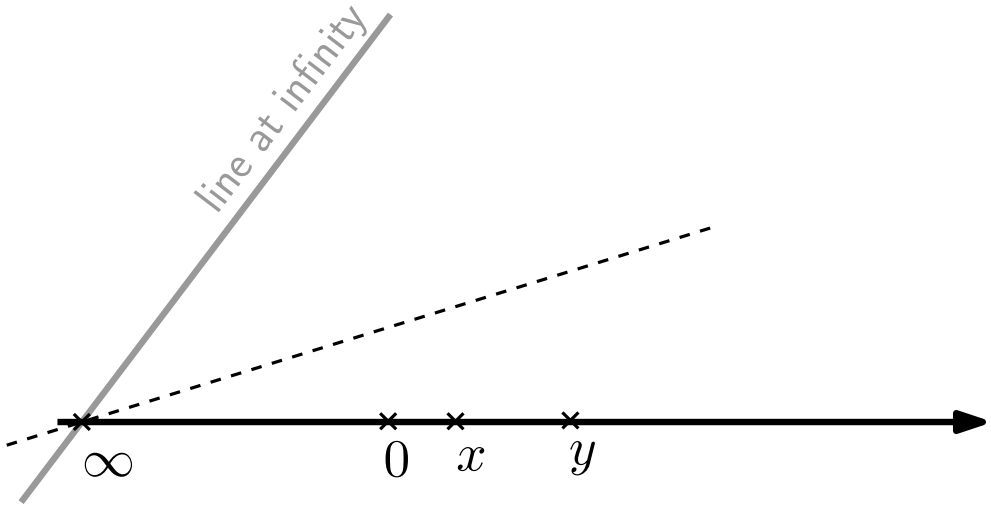

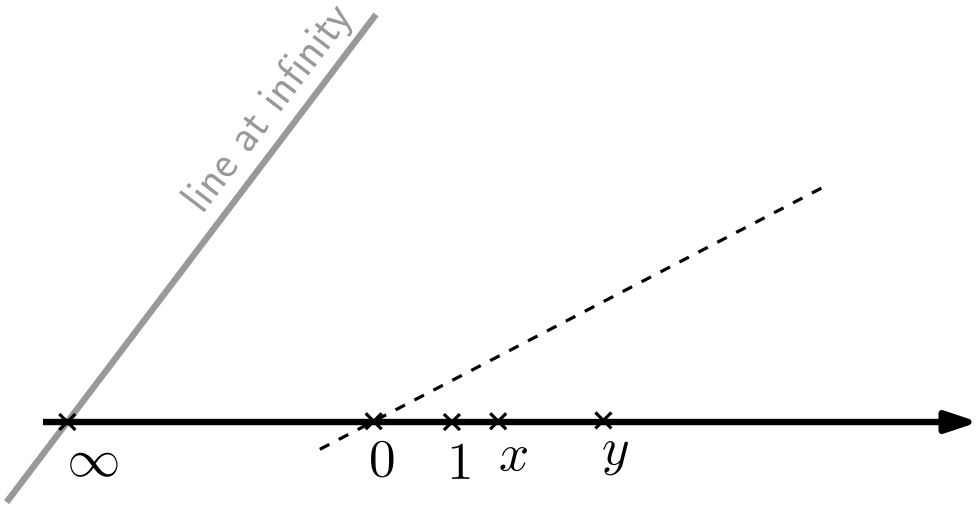

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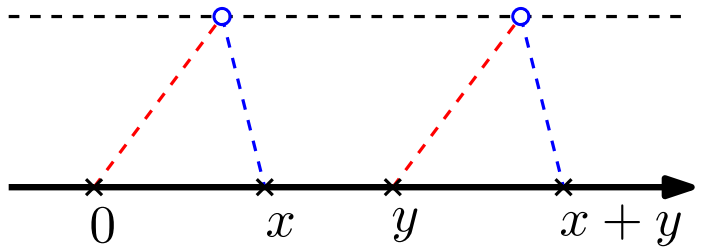

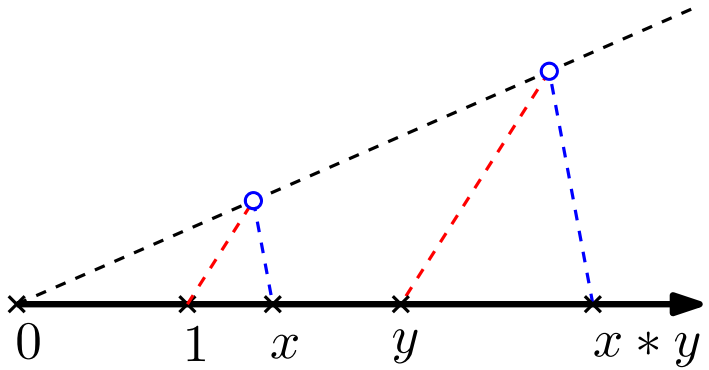

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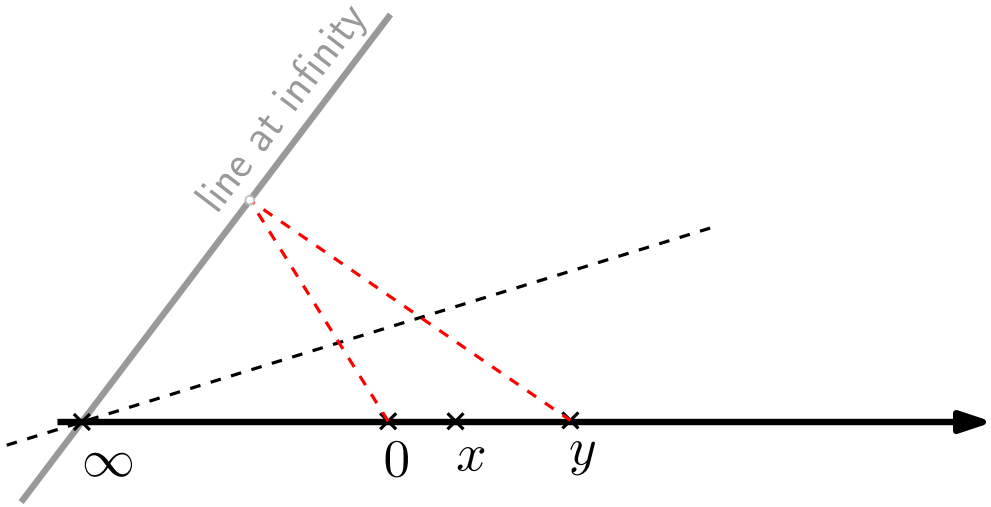

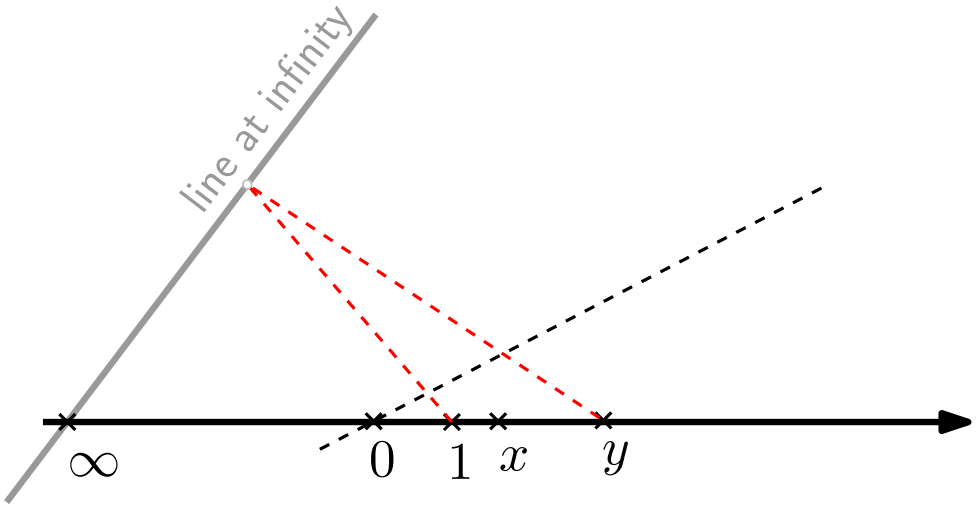

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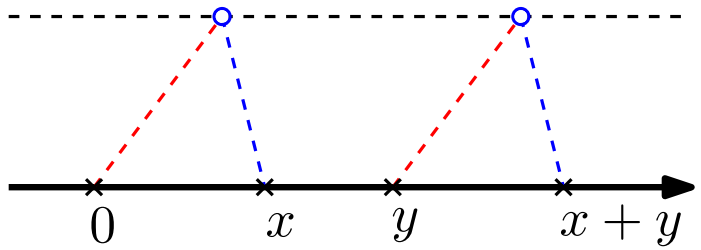

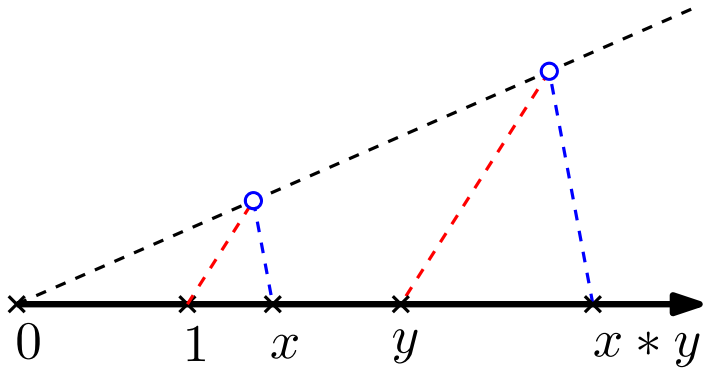

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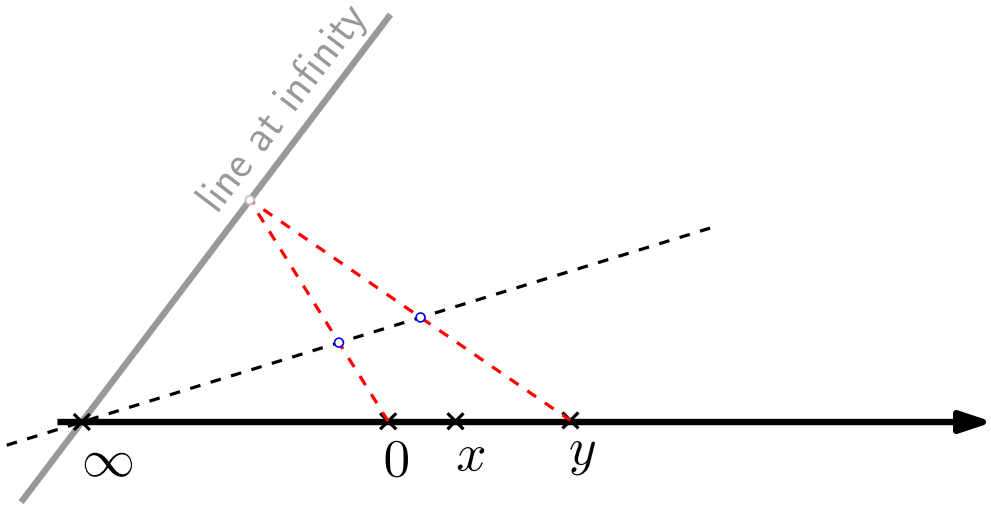

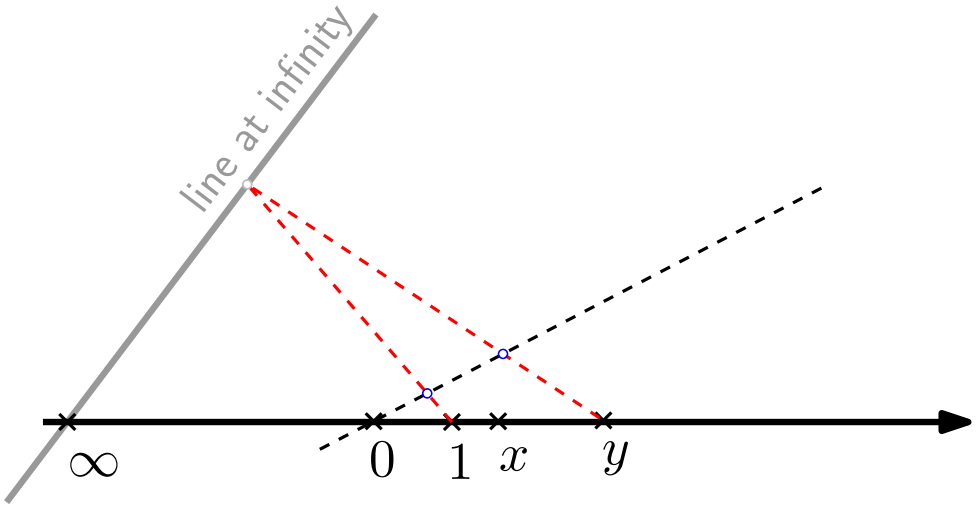

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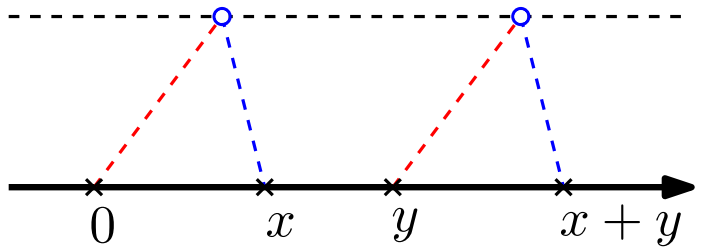

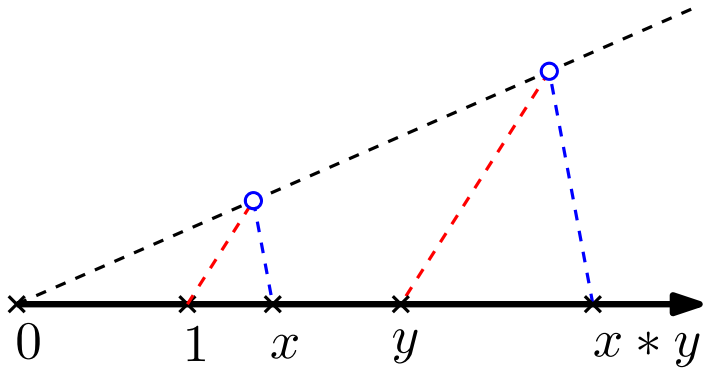

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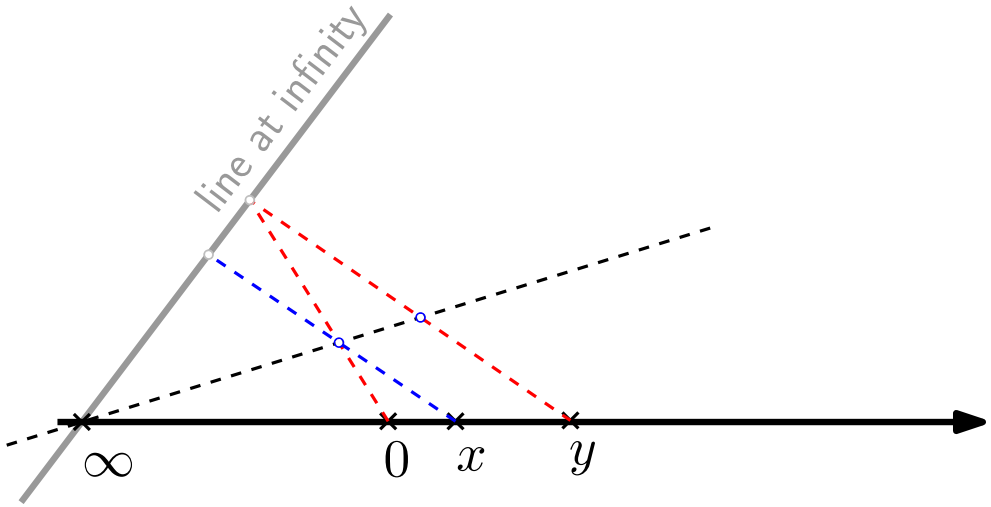

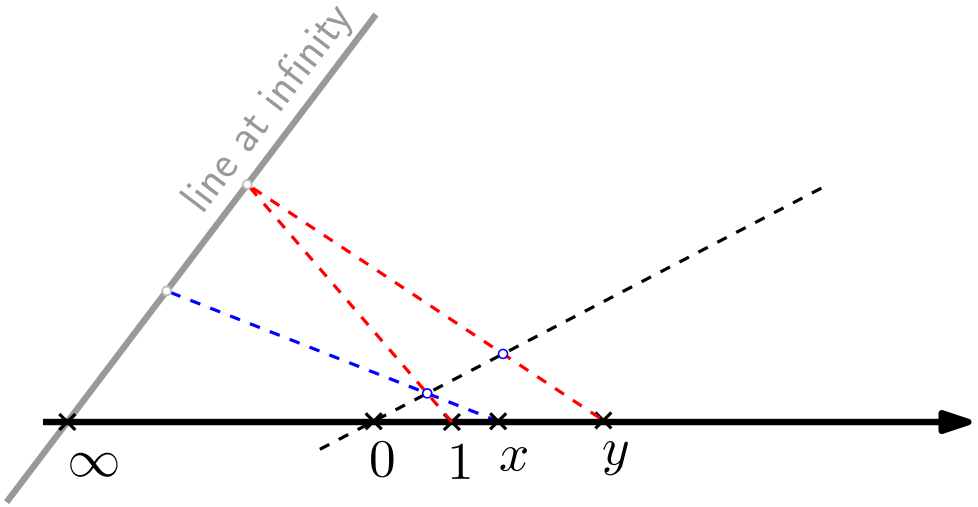

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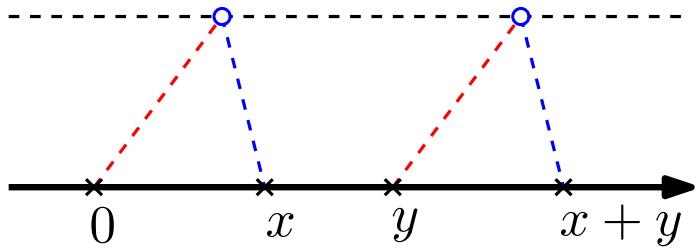

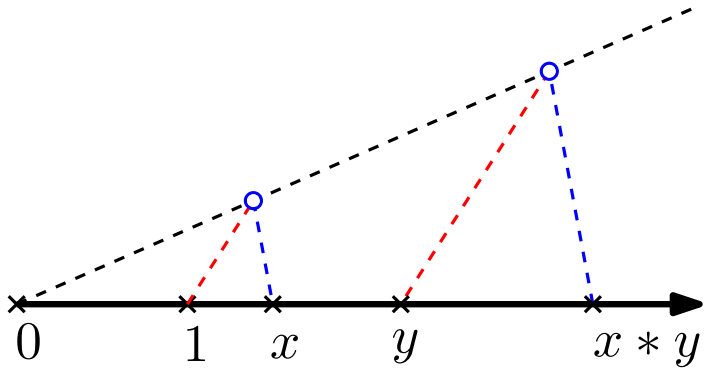

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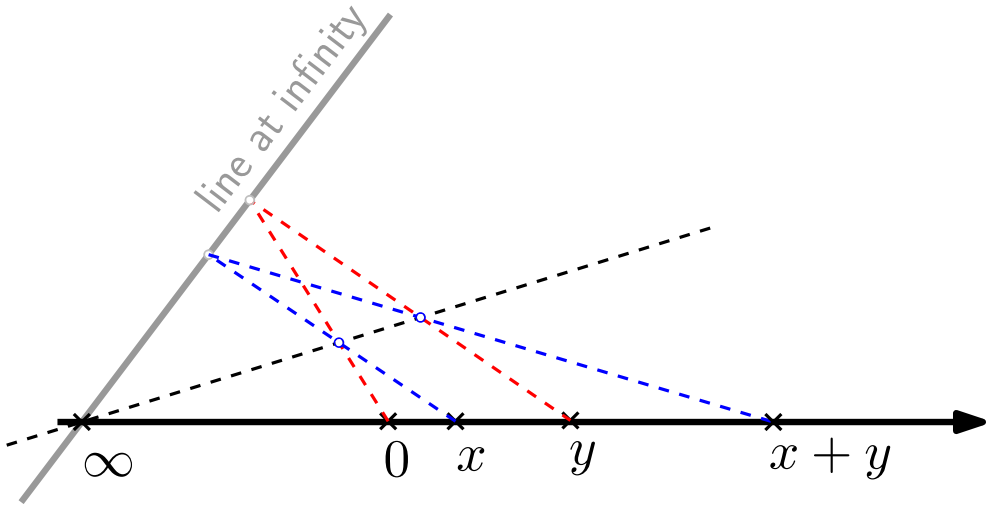

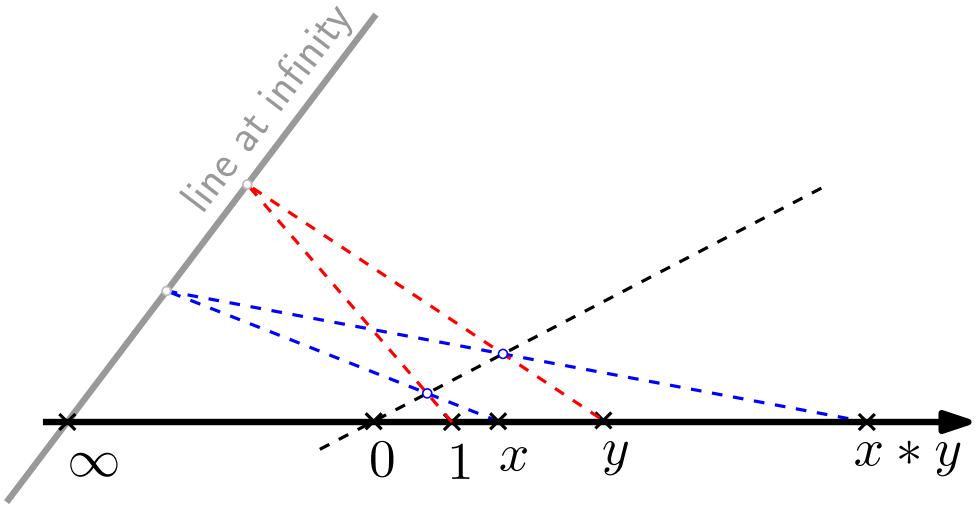

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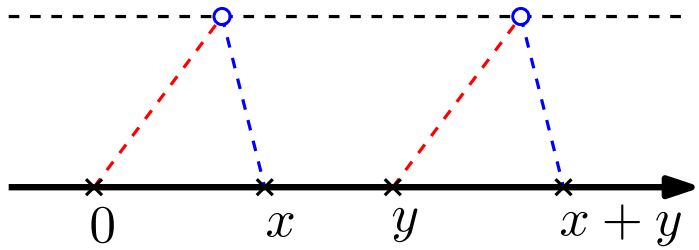

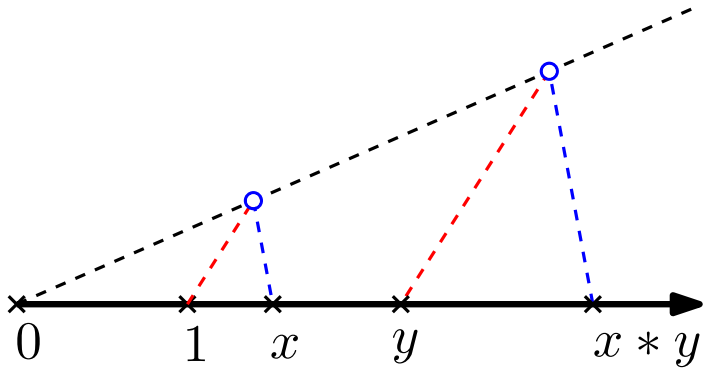

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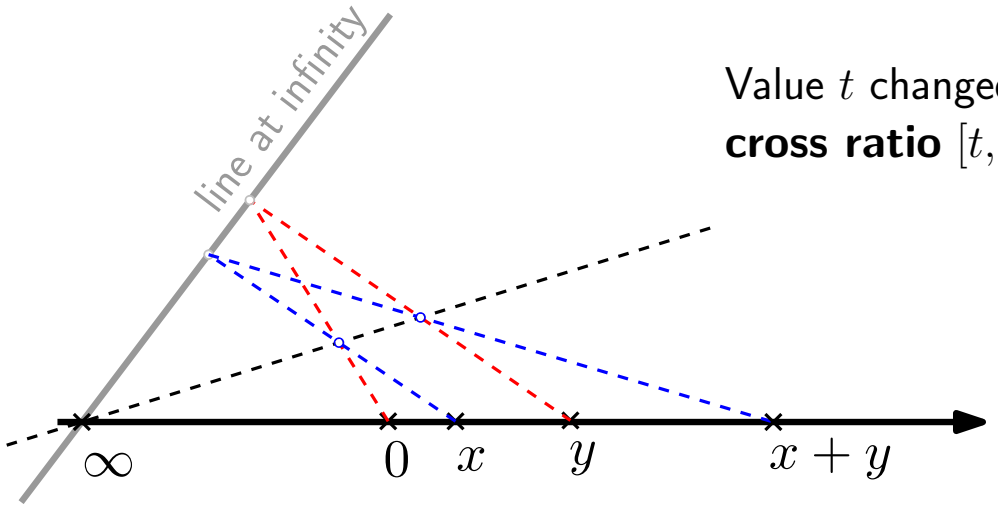
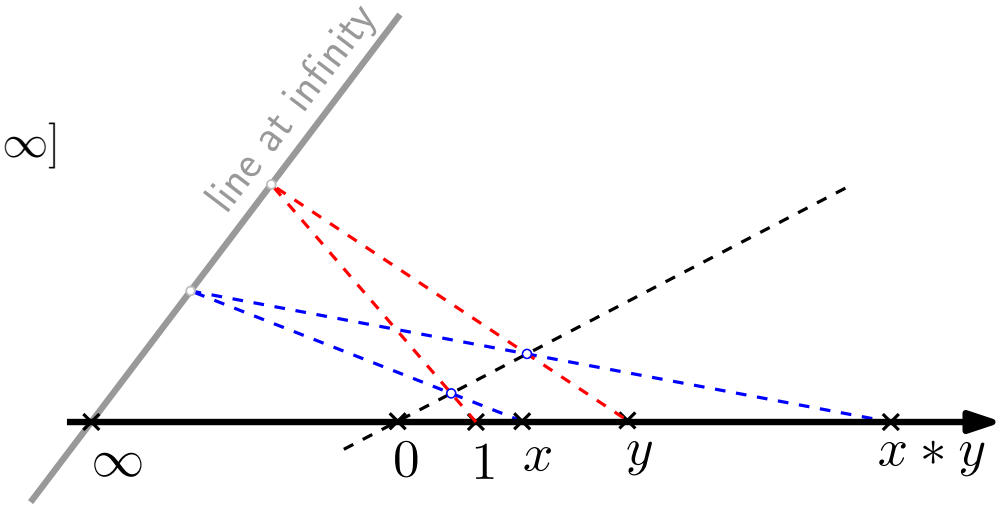
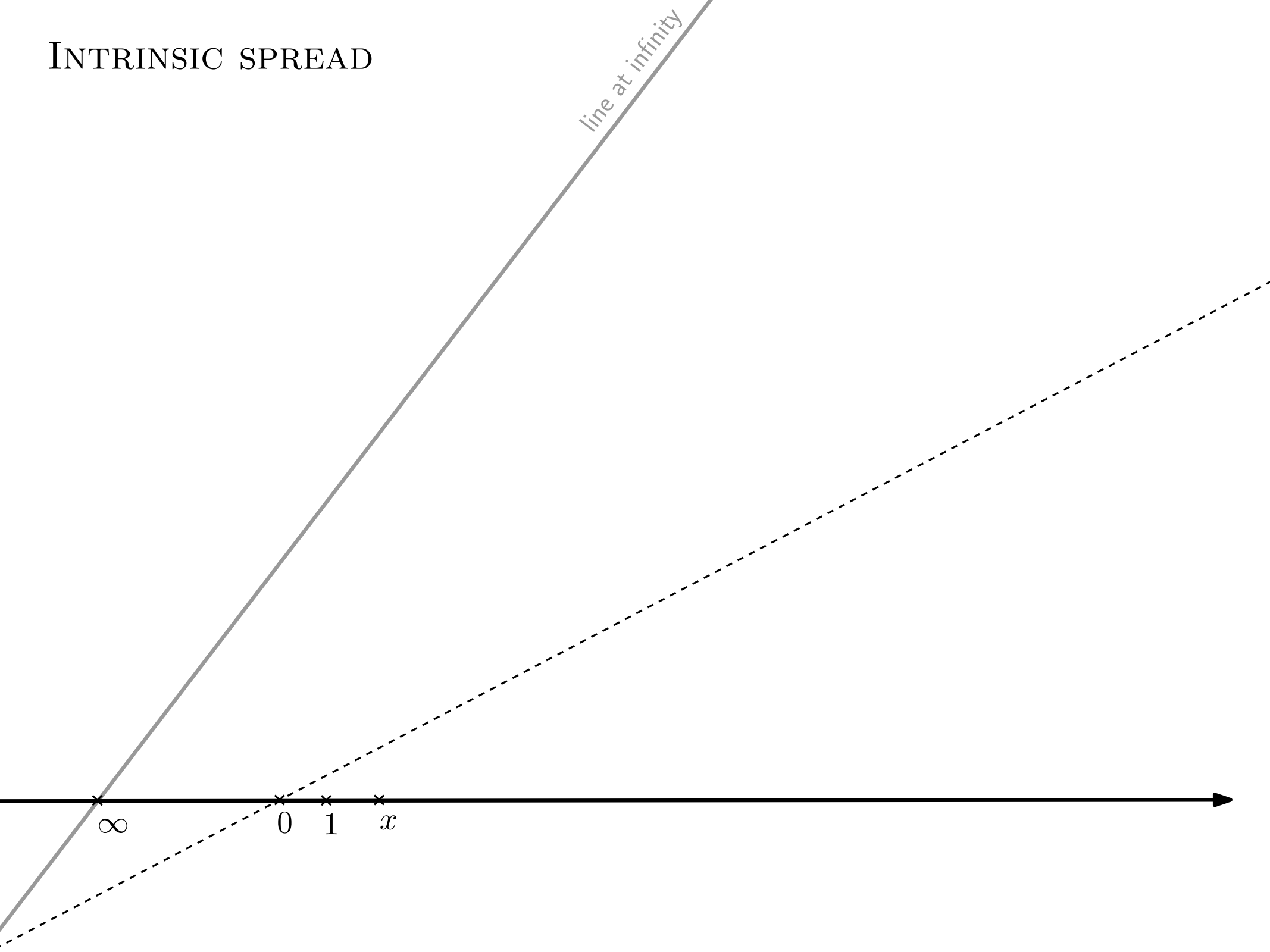


Value t changed to
cross ratio $[t, 1, 0, \infty]$



Intrinsic spread



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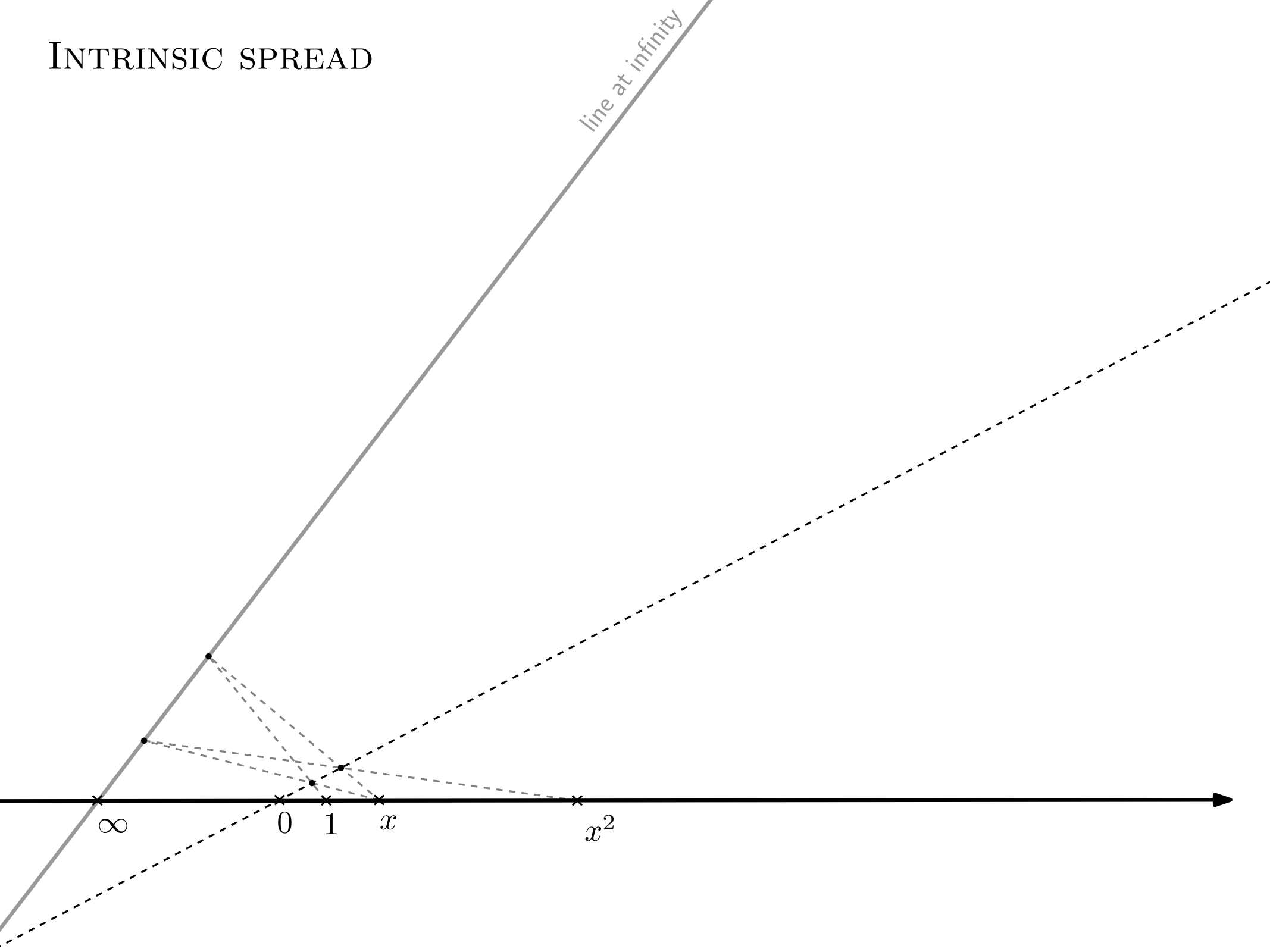

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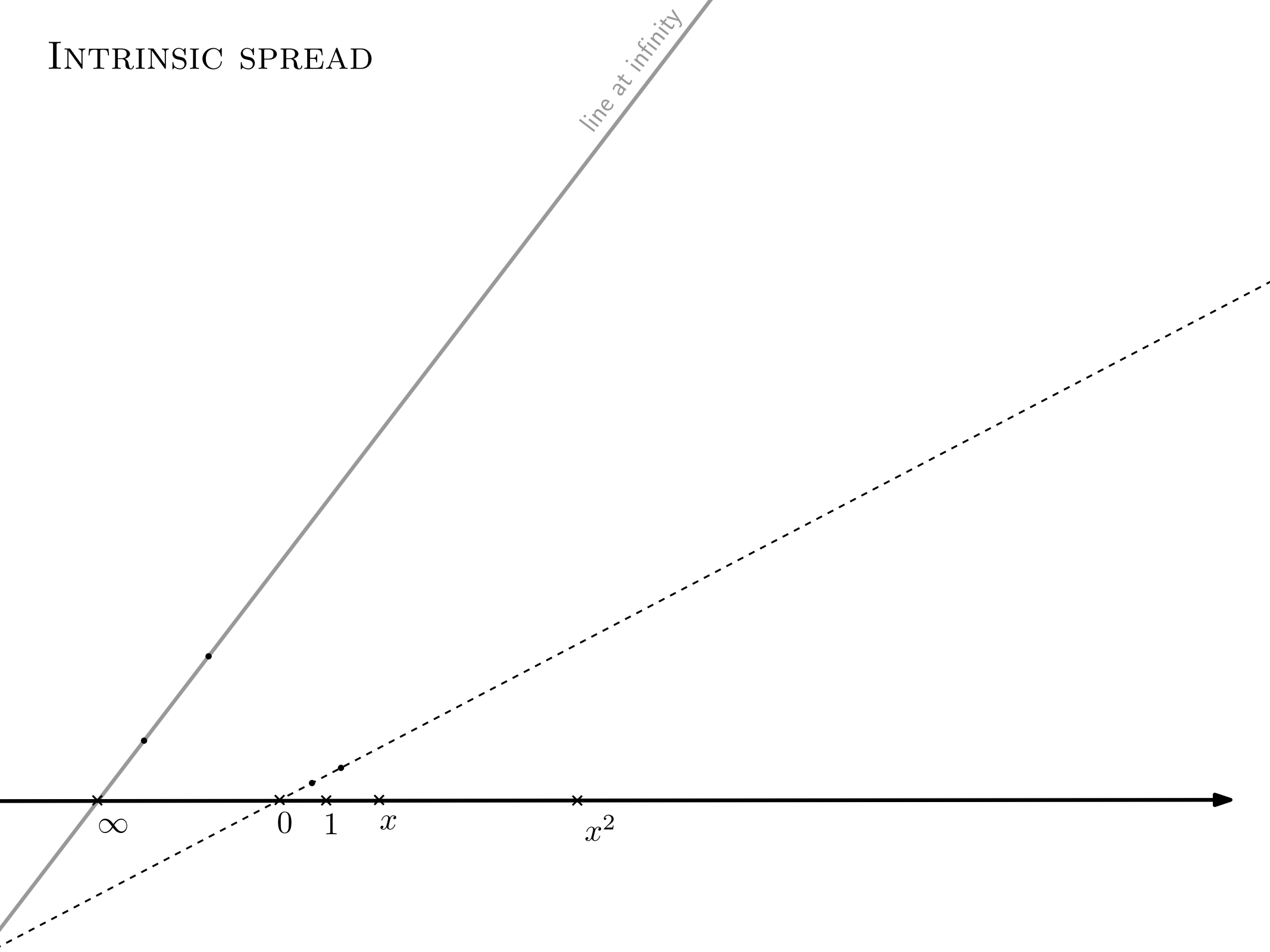

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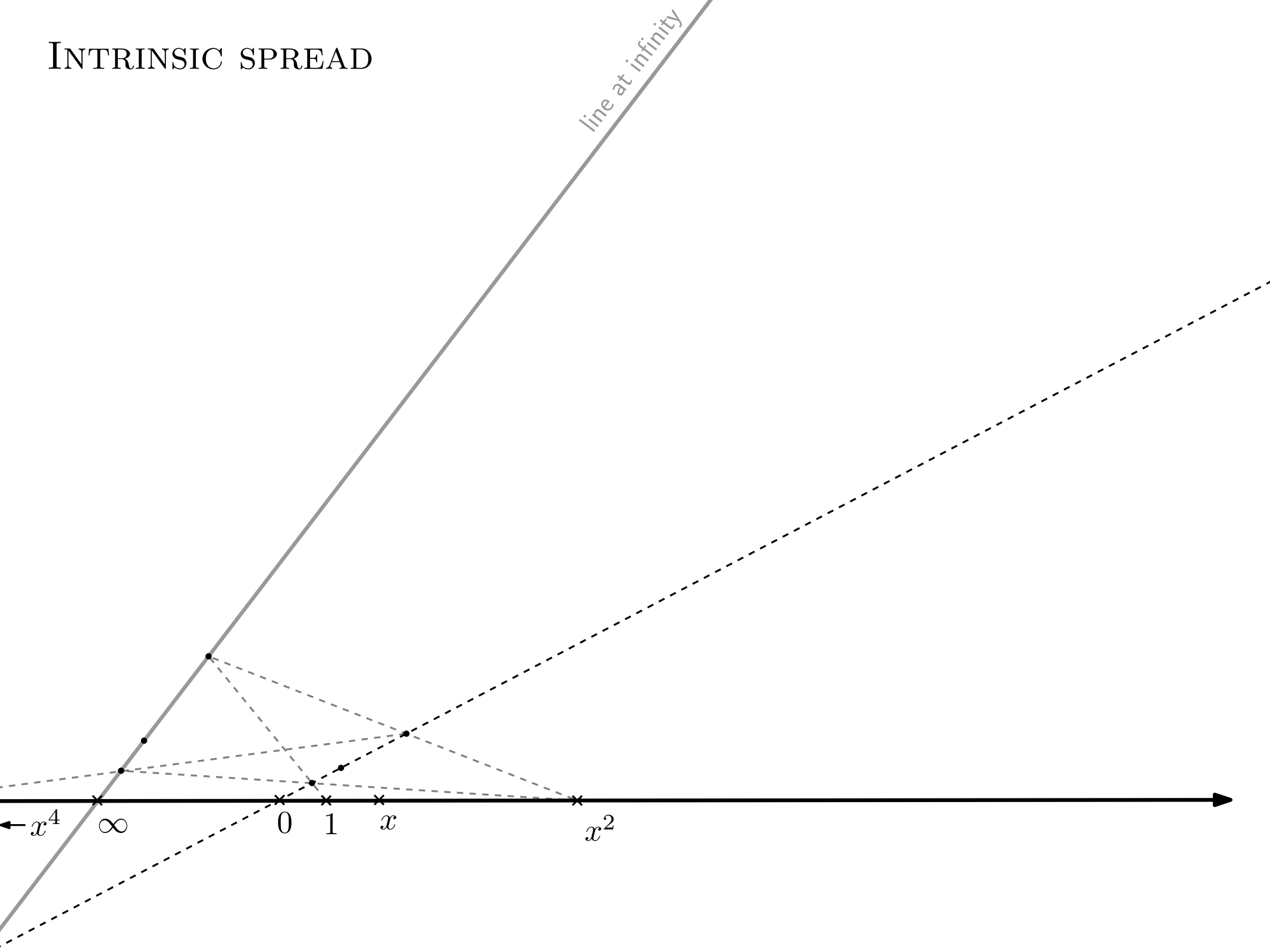

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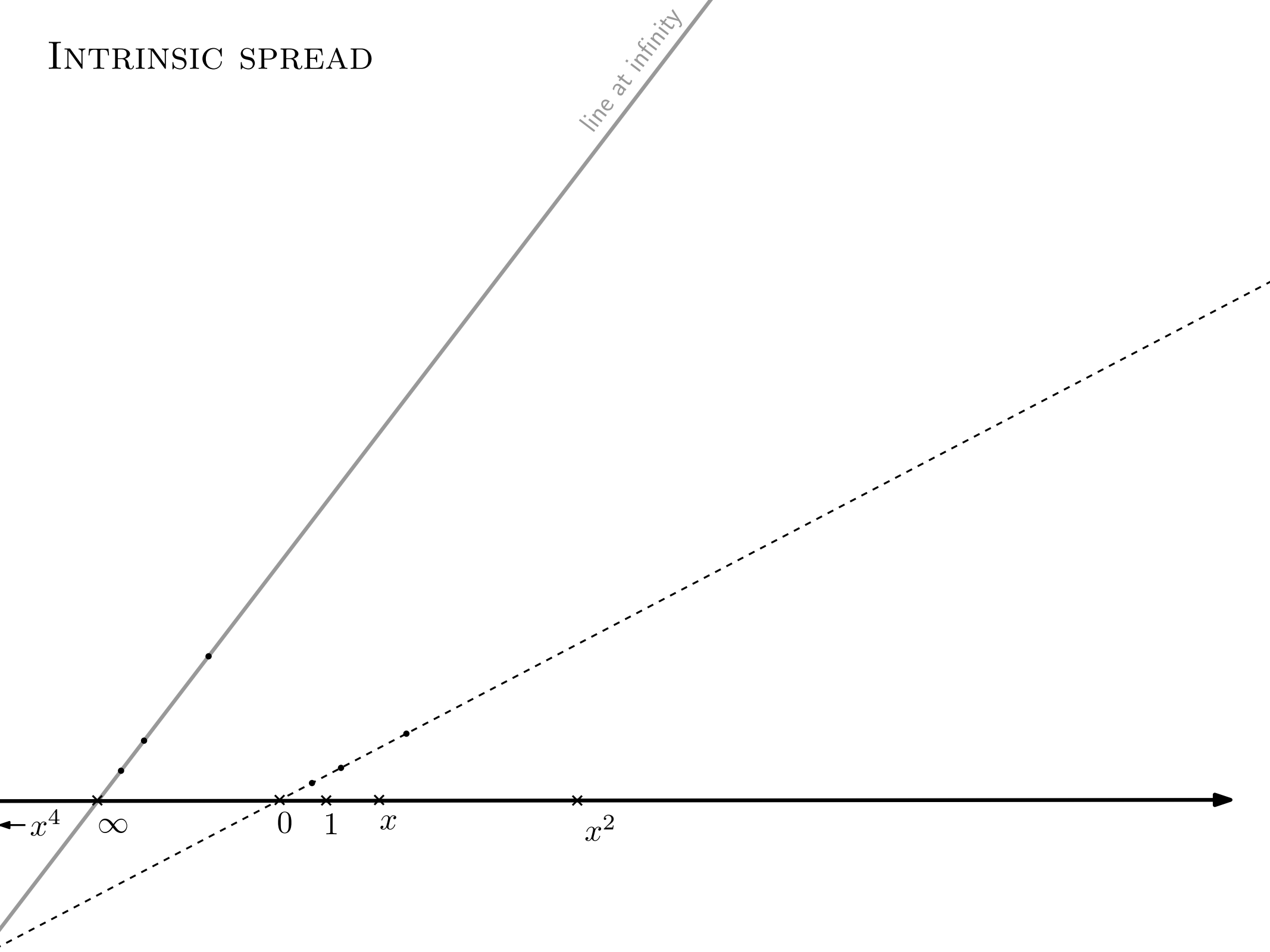
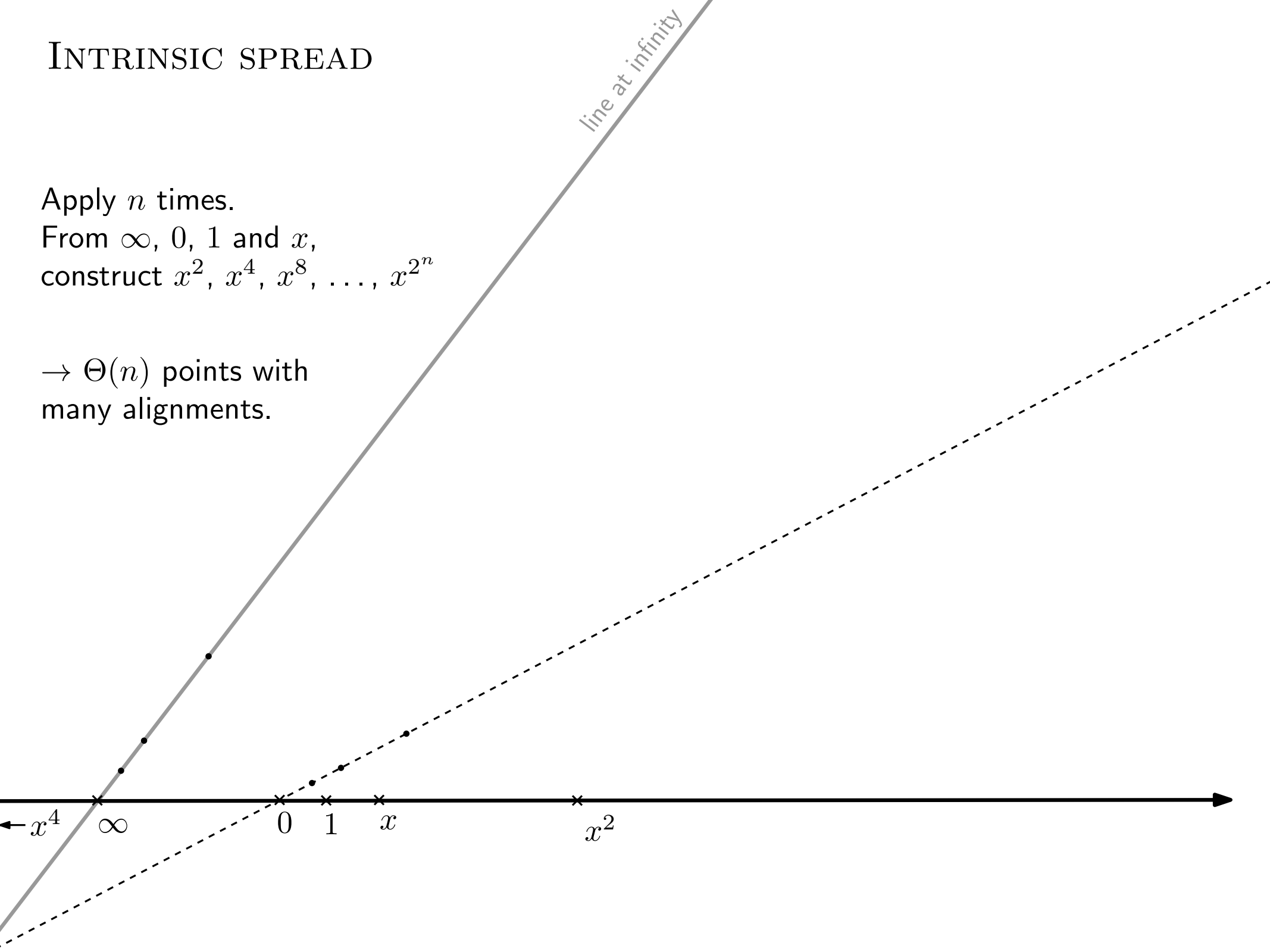


Intrinsic spread

Apply n times.
From ∞, 0, 1 and x,
construct x^2, x^4, x^8, $\ldots$, x^{2^n}

$\rightarrow \Theta(n)$ points with
many alignments.



Intrinsic spread

Apply n times.
From ∞, 0, 1 and x,
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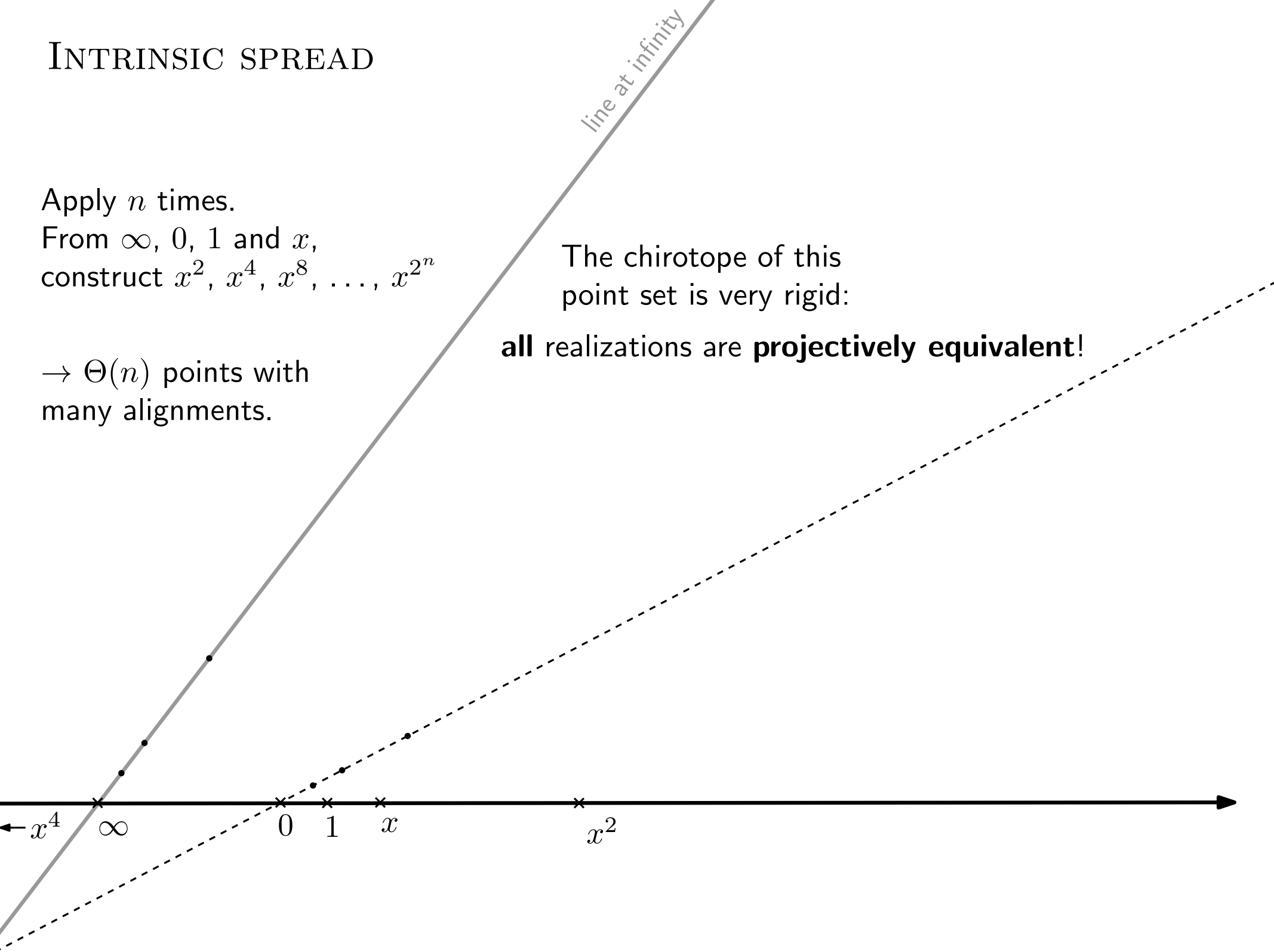
all realizations are **projectively equivalent**!



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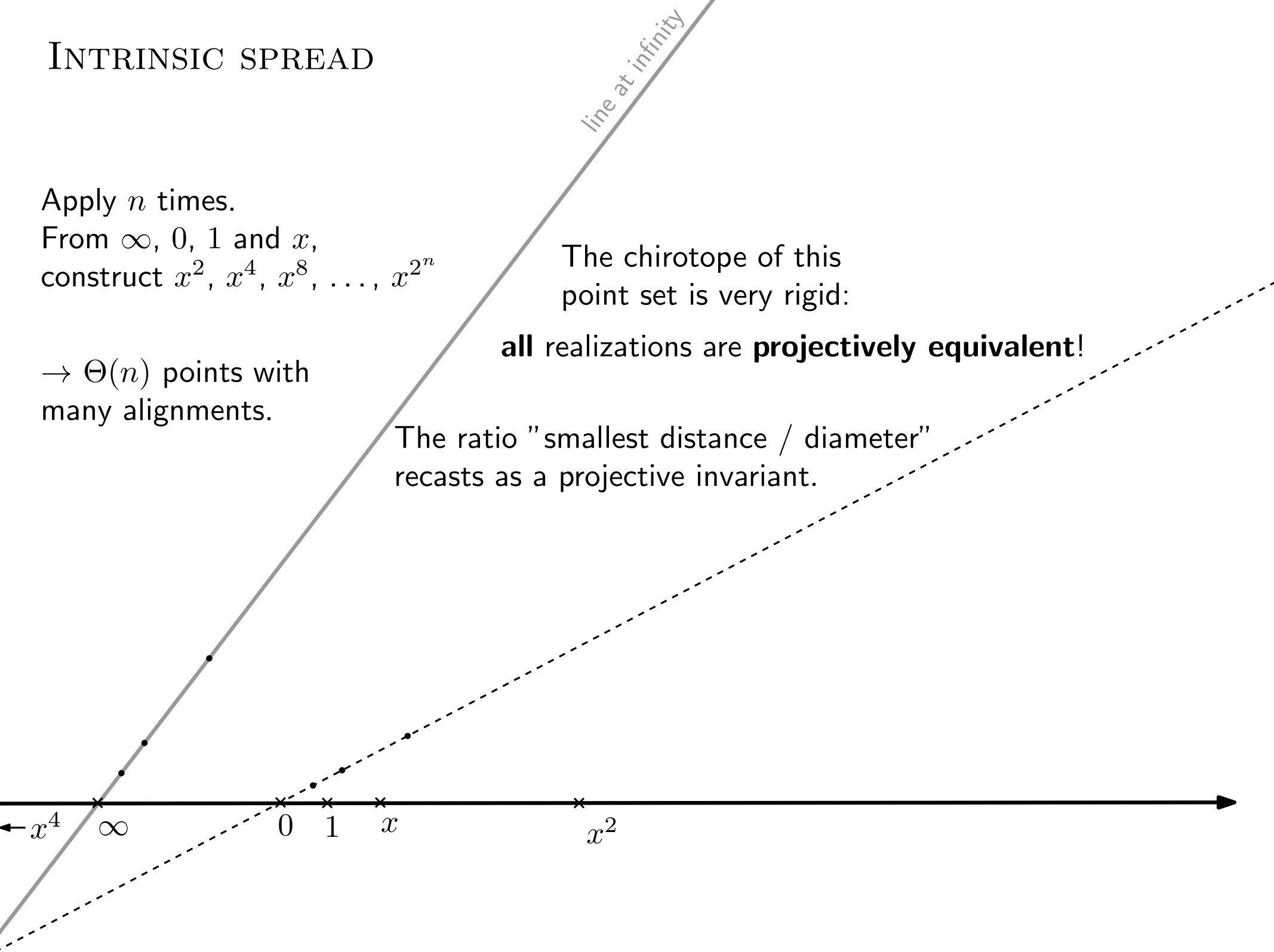

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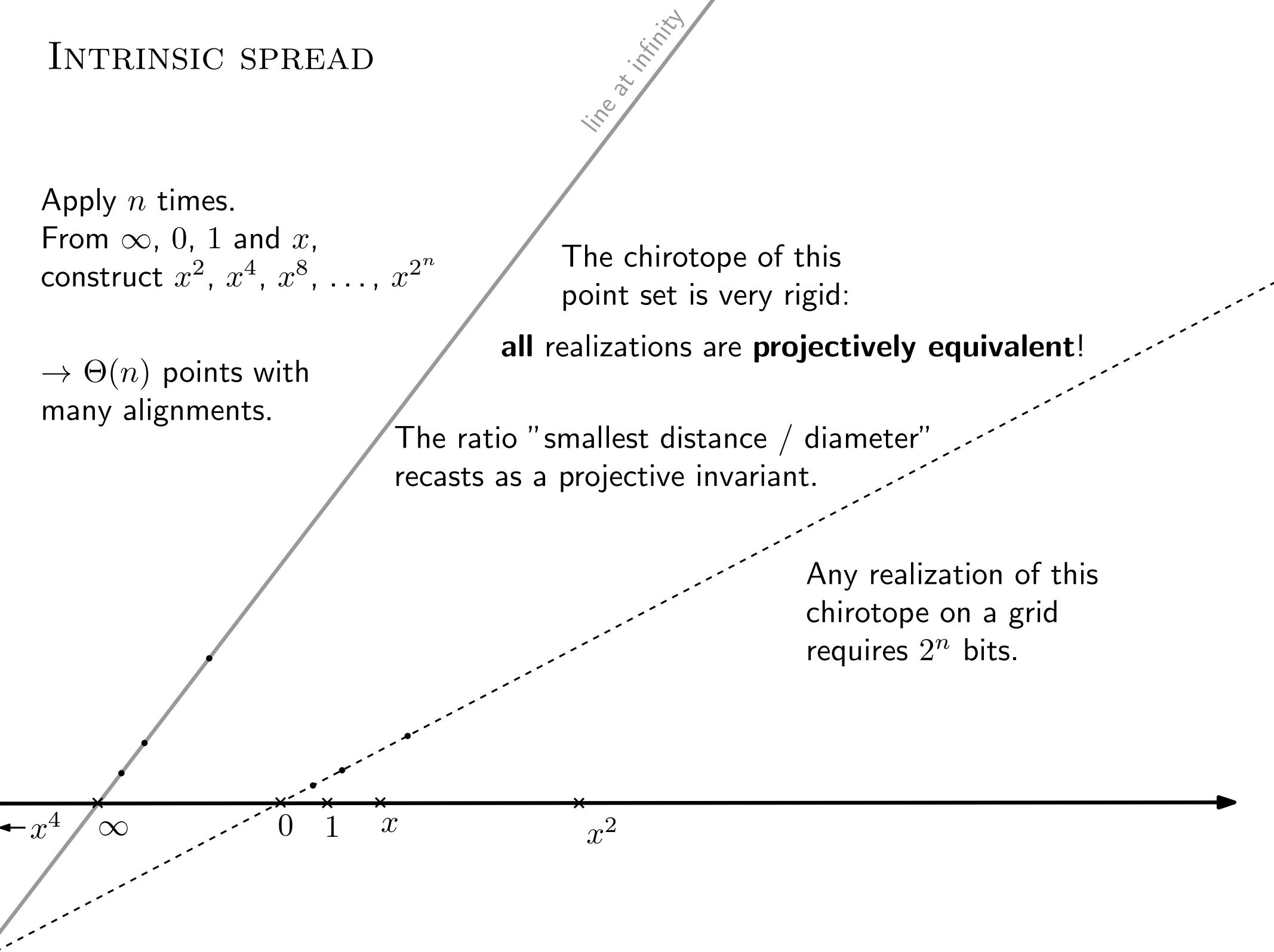

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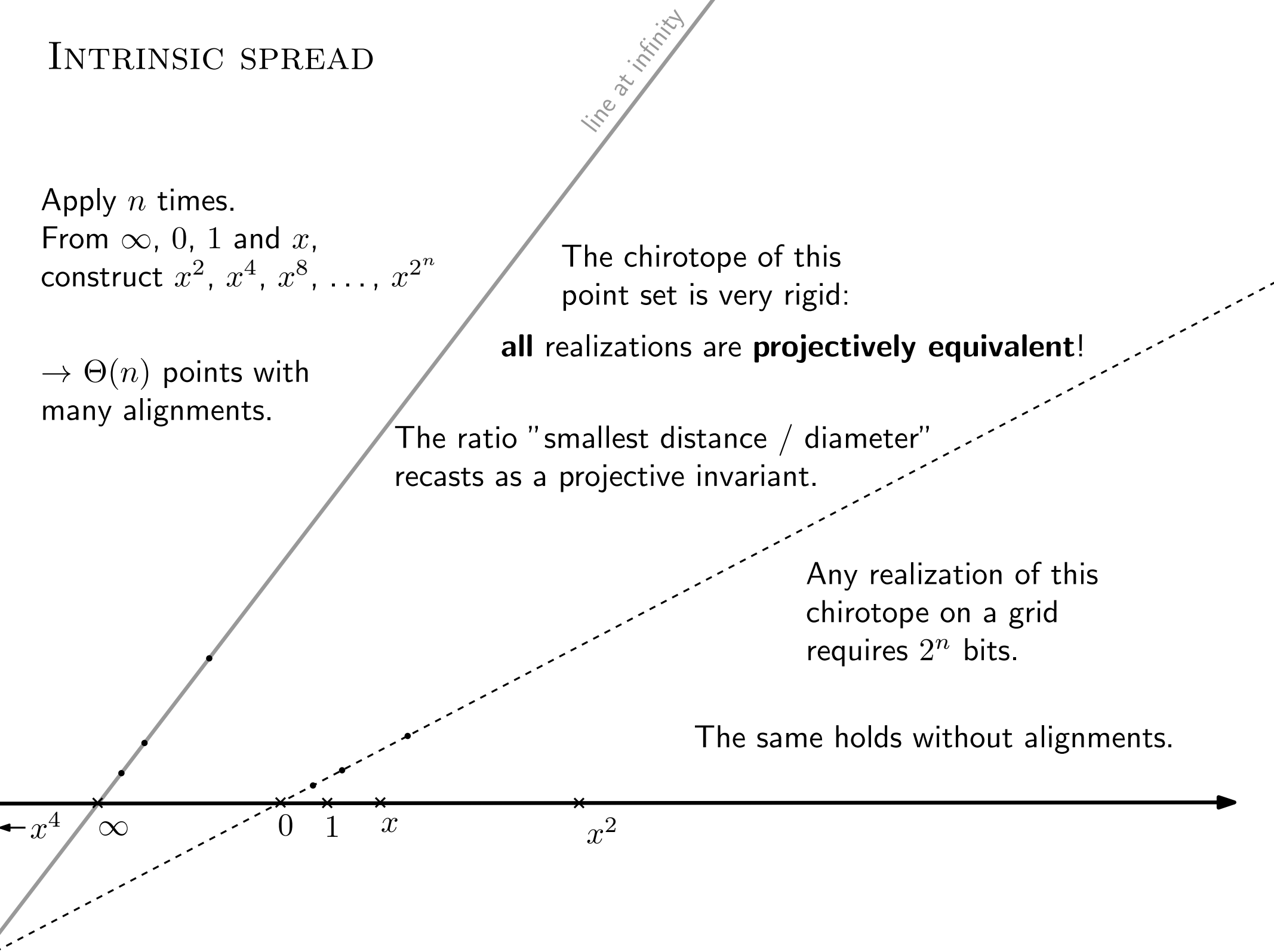

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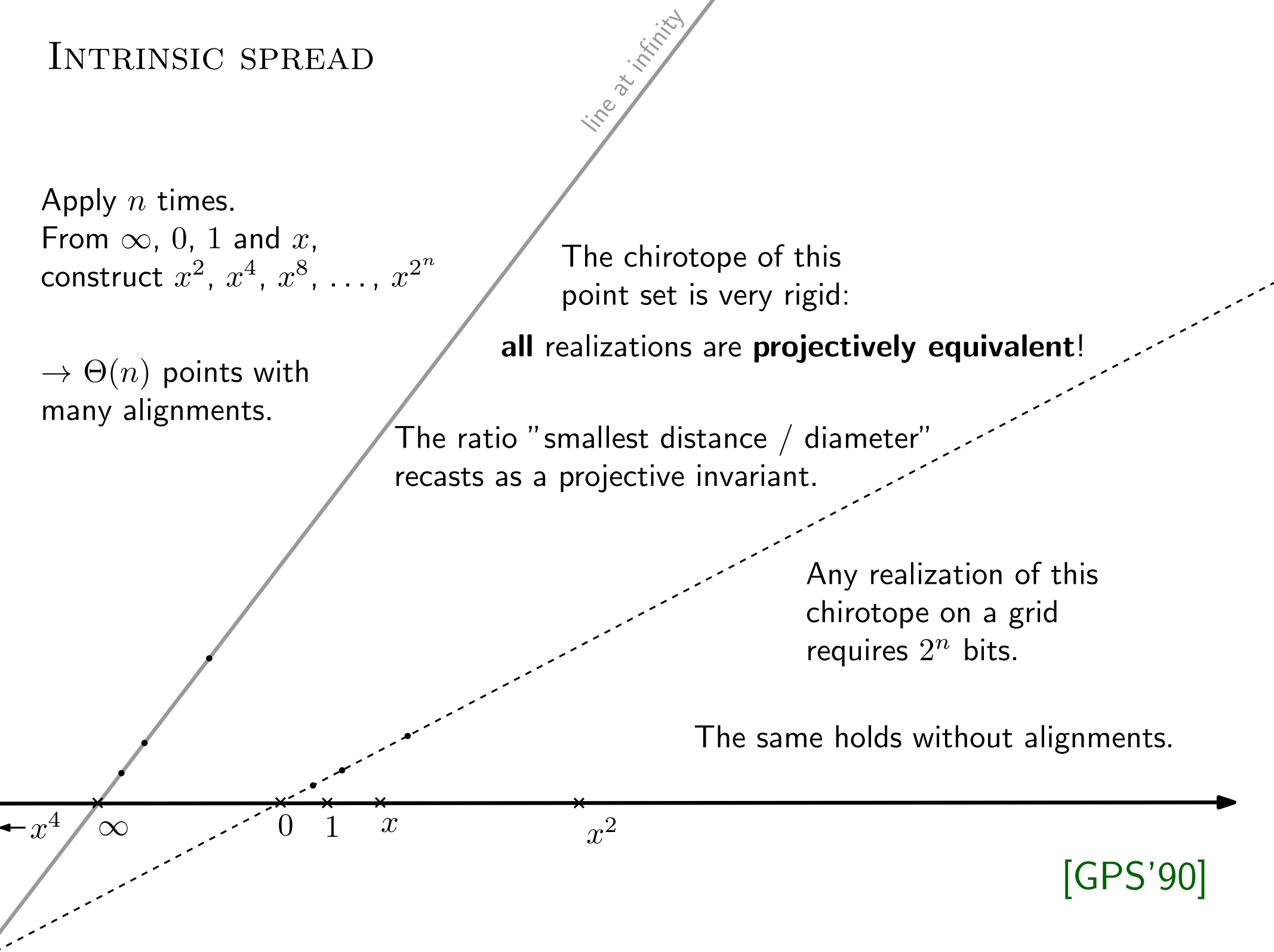
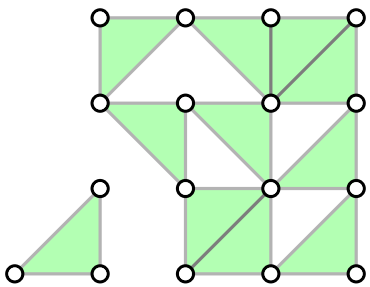


[GPS'90]

Mnëv's universality theorem

Theorem. [M'88] For every finite simplicial complex K, there exists a 2D chirotope whose space of realizations has the same homotopy type as K.

$$((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) \in (\mathbb{R}^2)^n$$
$$\leftrightarrow$$
$$(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) \in \mathbb{R}^{2n}$$



Disproved the conjecture that realization spaces were **connected** [R56].

Also holds for **simplicial polytopes** [AP17].

Random generation

The problem

$\mathcal{O}_n$ the set of n points order types.

$\{\mu_k\}_{k\geq 1}$ where μ_n is a probability on $\mathcal{O}_n$.

$\{\mu_k\}_{k\geq 1}$ **exhibits concentration** if there exists $A_n \subset \mathcal{O}_n$

s.t. $\frac{|A_n|}{|\mathcal{O}_n|} \to 0$ and $\mu_n(A_n) \to 1$.

Can we **sample** order types **efficiently** and **avoid concentration**?

Counting is only up to superexponential multiplicative error.

For combinatorial representations, membership testing is NP-hard.

Geometric representation requires exponential storage.

Order types of random point sets?

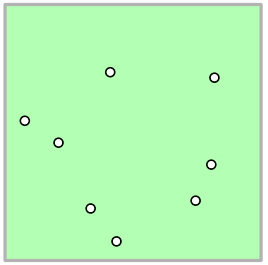
$\mathcal{O}_n$ the set of n points order types.

μ a probability over $\mathbb{R}^2$ that charges no line.

Sample n random points independently from μ.

Read off their order type or chirotope.

$\hookrightarrow$ a probability μ_n over $\mathcal{O}_n$



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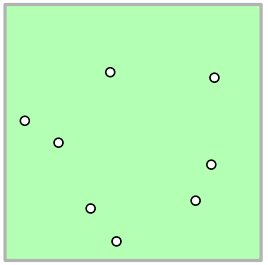
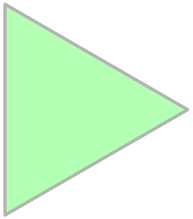
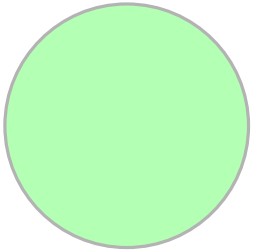


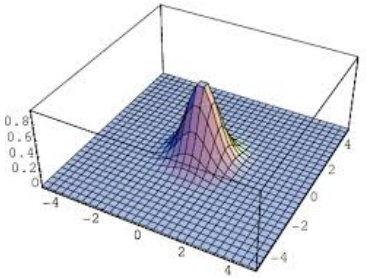
Conjecture. This family of distributions exhibits concentration.

[DDGG'18]

Same for







Evidence of concentration

Theorem. $\forall \mu$, $\exists$ order types ω_1, ω_2 of size 6 s.t. $\mu_6(\omega_1) > 1.8\mu_6(\omega_2)$. [GHJSV15]

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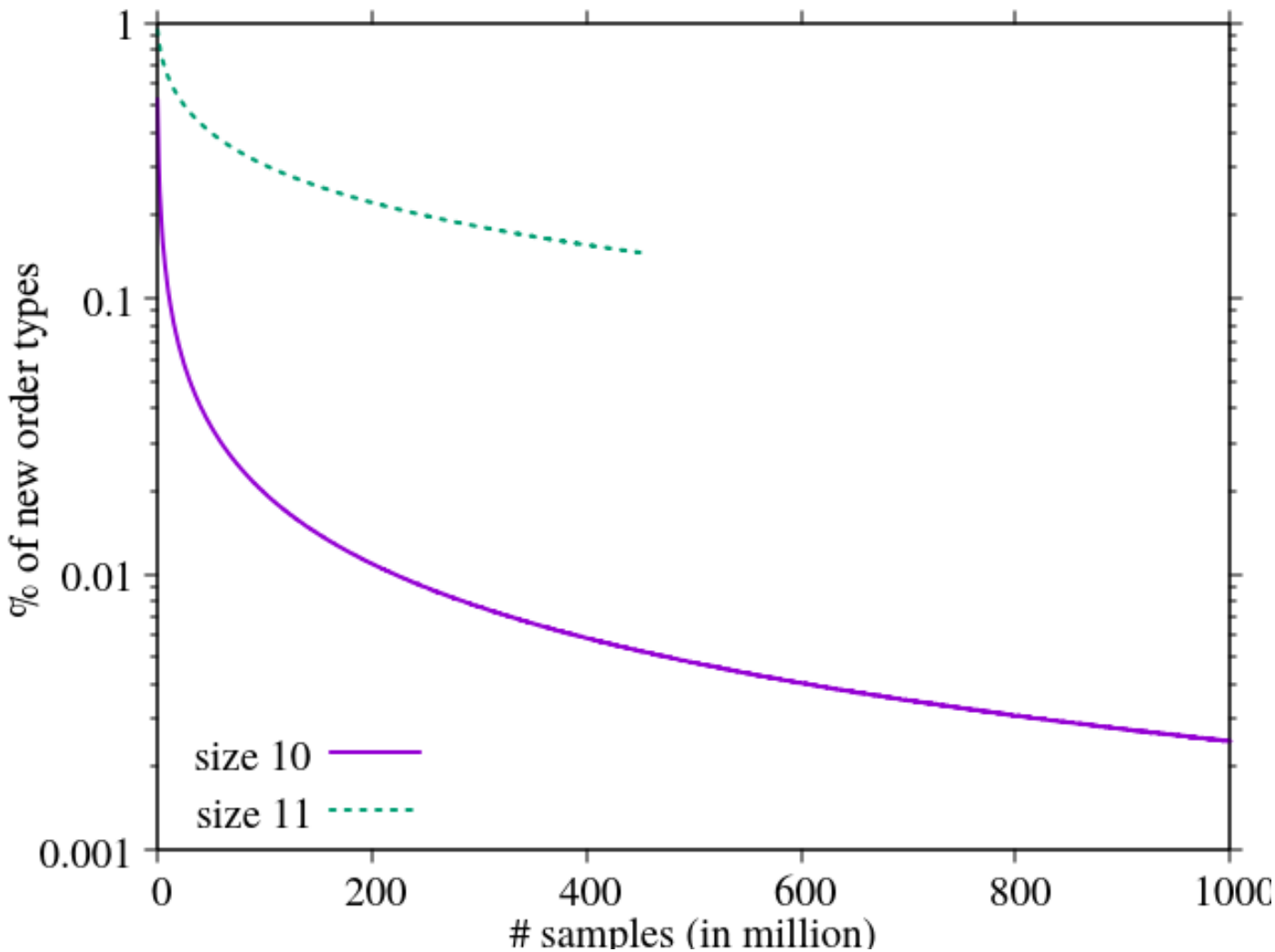
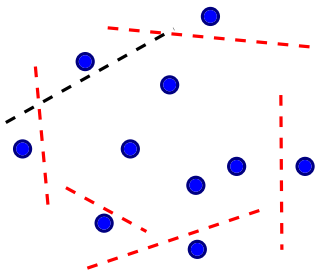


	Size 10		Size 11	
	s	d	s	d
50%	1	0.5	26	16.5
33%	1	0.5	84	39.4
25%	2	0.8	157	60.1
15%	6	1.6	432	112
10%	11	2.2	-	-
5%	31	3.6	-	-
4%	41	4	-	-
3%	60	4.7	-	-
2%	98	5.6	-	-
1%	221	7.3	-	-
0.5%	476	9.1	-	-
0.25%	991	10.8	-	-

[DDGG'18]

Proof of concentration

Extreme points



$p \in P$ is **extreme** in P

$\Leftrightarrow$

p can be **separated** from $P \setminus \{p\}$ by a line

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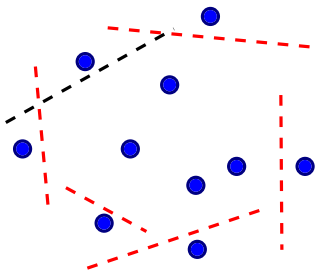

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Probabilistic geometry studied the number K_n of extreme points in n random points chosen uniformly from a compact convex set K.

$$\mathbb{E}[K_n] \sim \begin{cases} \log n & \text{if } K \text{ is a polygon} \\ n^{1/3} & \text{if } K \text{ is smooth} \end{cases}$$

$$\mathrm{Var}[K_n] = \Theta\left(\mathbb{E}[K_n]\right) \quad \text{if } K \text{ is smooth or polygonal.}$$

Extreme points

Theorem. The average number of extreme points in a simple order type of size n in the plane is at most $4 + o(1)$.

The average number of extreme points in a simple chirotope of size n in the plane equals $4 - \frac{8}{n^2-n+2}$.

The **uniform** *distribution on* $\mathcal{O}_n$.

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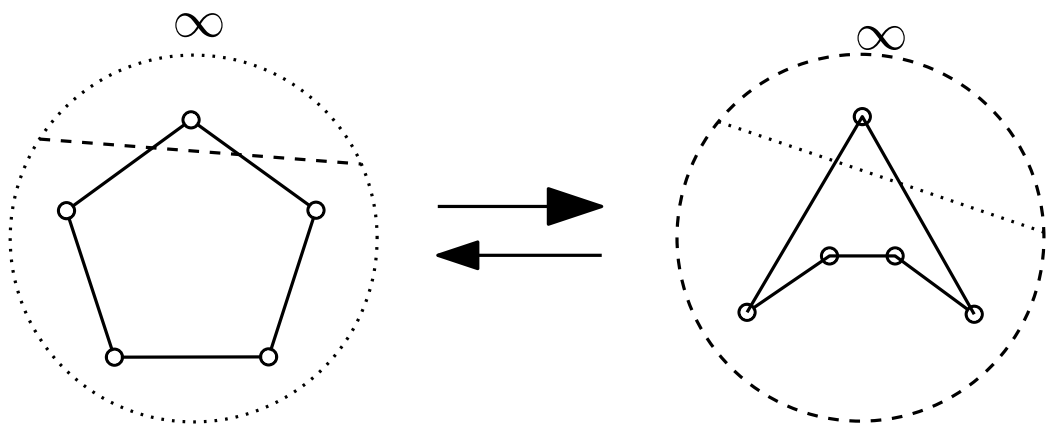
The **uniform** *distribution on* $\mathcal{O}_n$.

Corollary. Order types and chirotopes read off random samples of polygonal or smooth compact convex sets exhibit concentration.

[GW20]

Approach

Match order types!

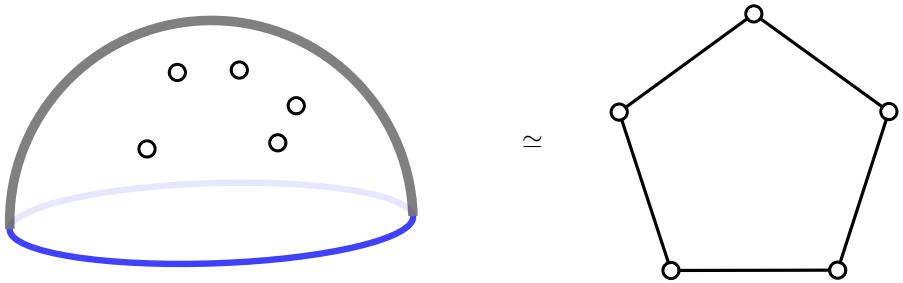


Lemma. Let A be a finite planar point set in general position and $g : \mathbb{R}^2 \to \mathbb{R}^2$ a projective transform that sends no point of A to infinity. If A and $g(A)$ have different order types, then there are at most 4 extreme vertices of A whose images are also extreme in $g(A)$.

Go projective

A subset of $\mathbb{S}^2$ is **affine** if it is contained in an **open** hemisphere.

Order types are the same as in $\mathbb{R}^2$.



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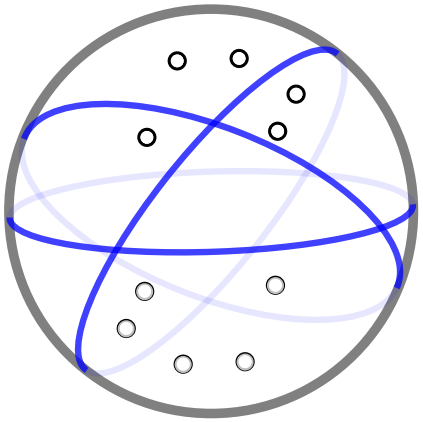
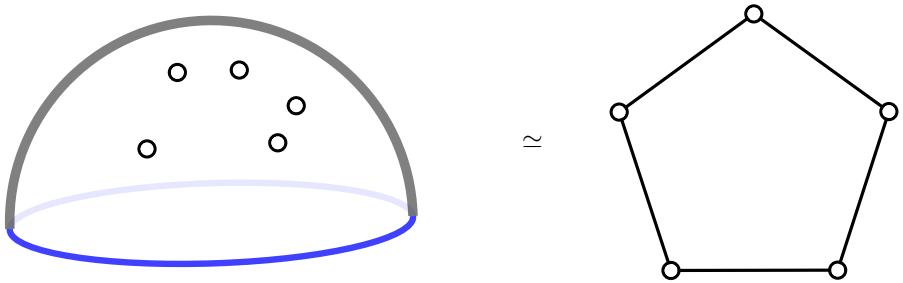
Complete an affine set A into a **projective** set $A \cup -A$.

Study together the affine sets with the same projective completion.





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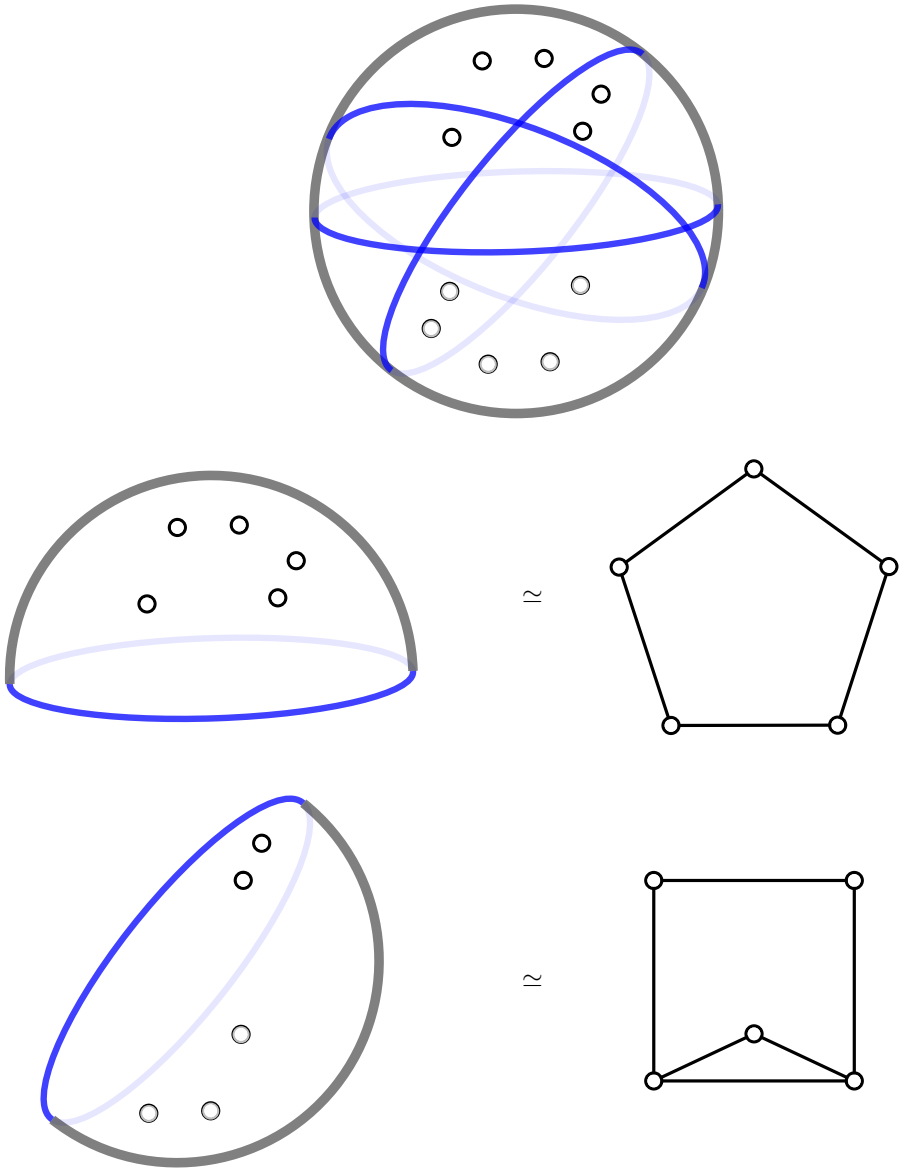

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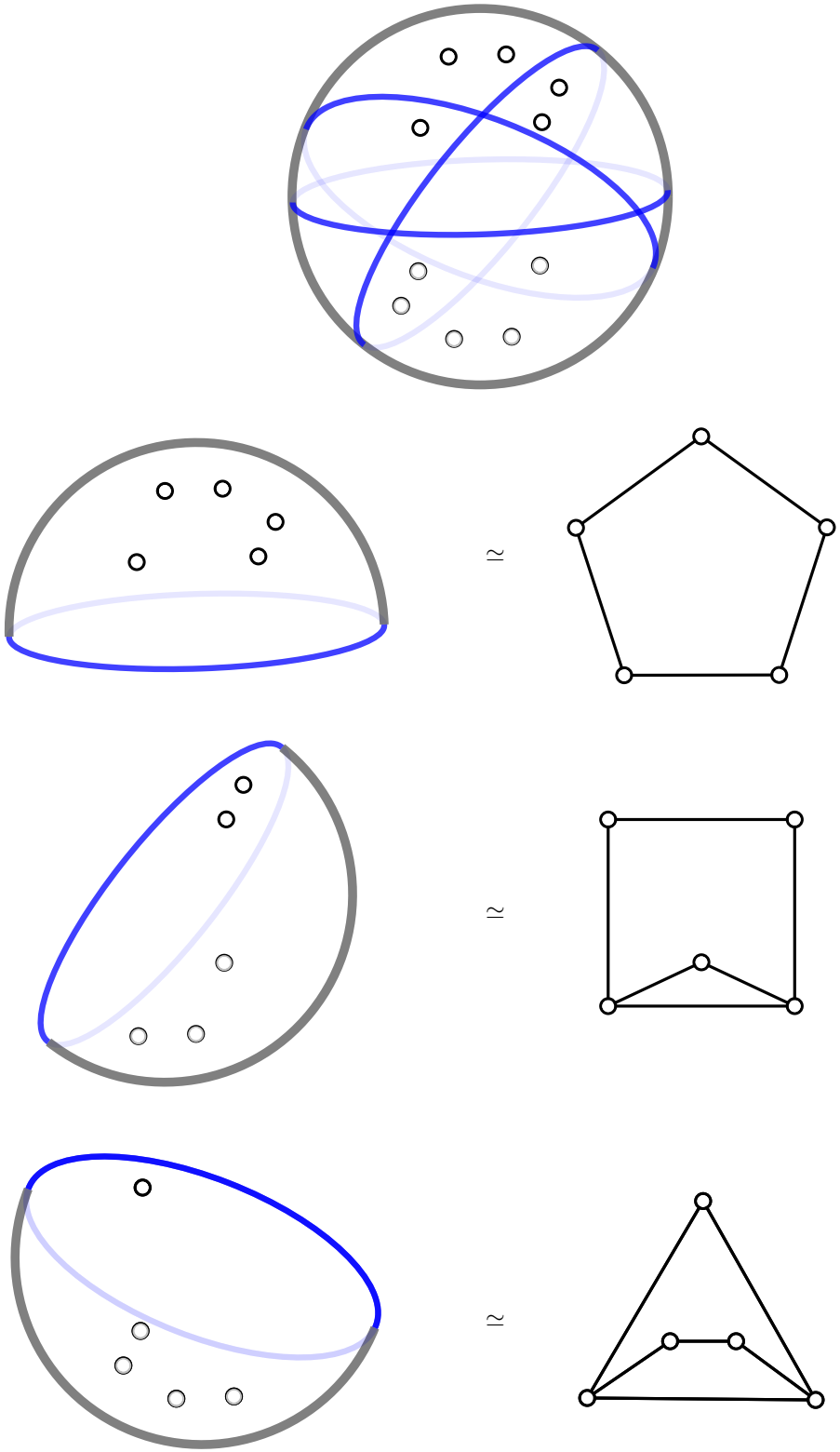

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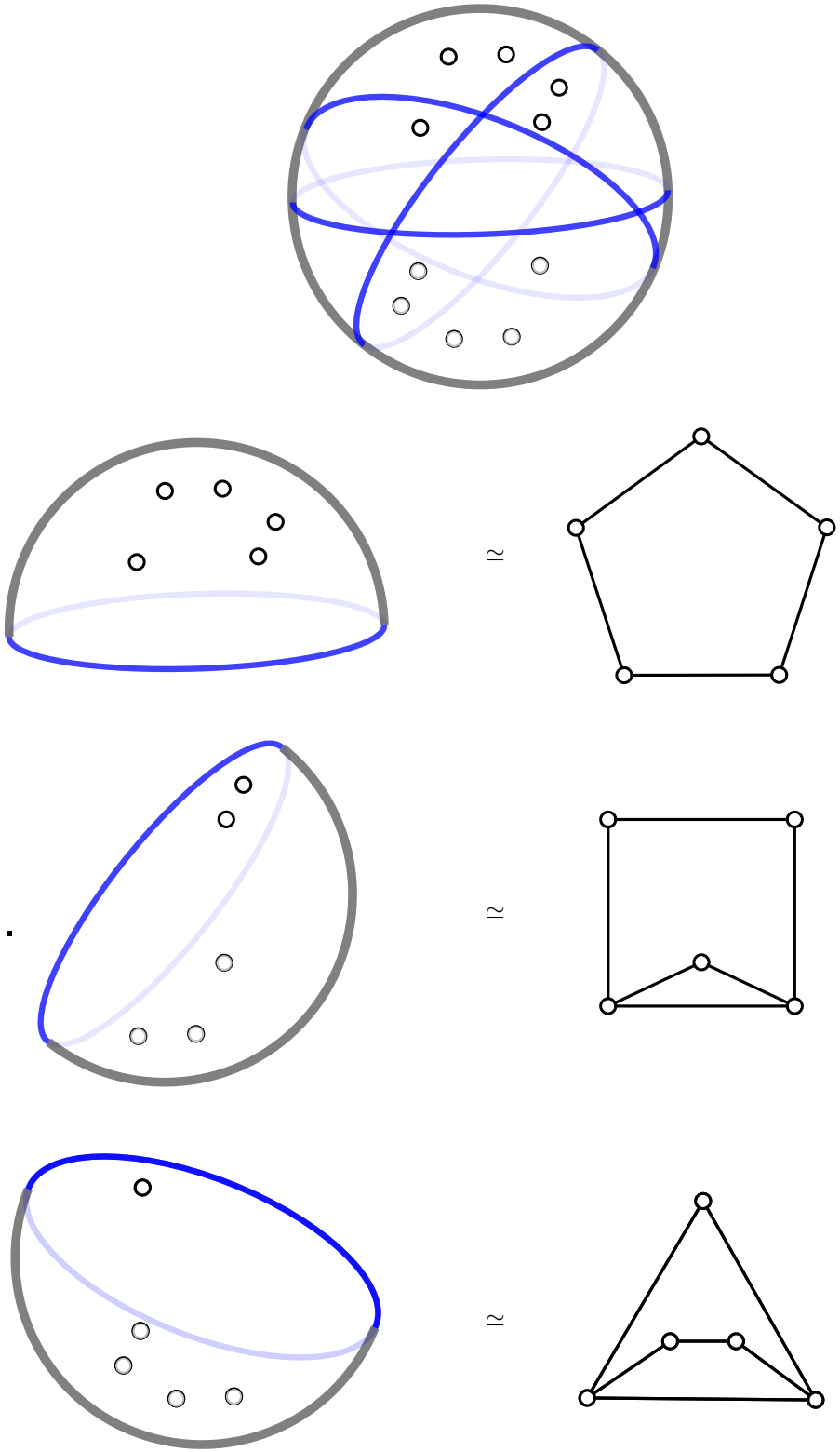

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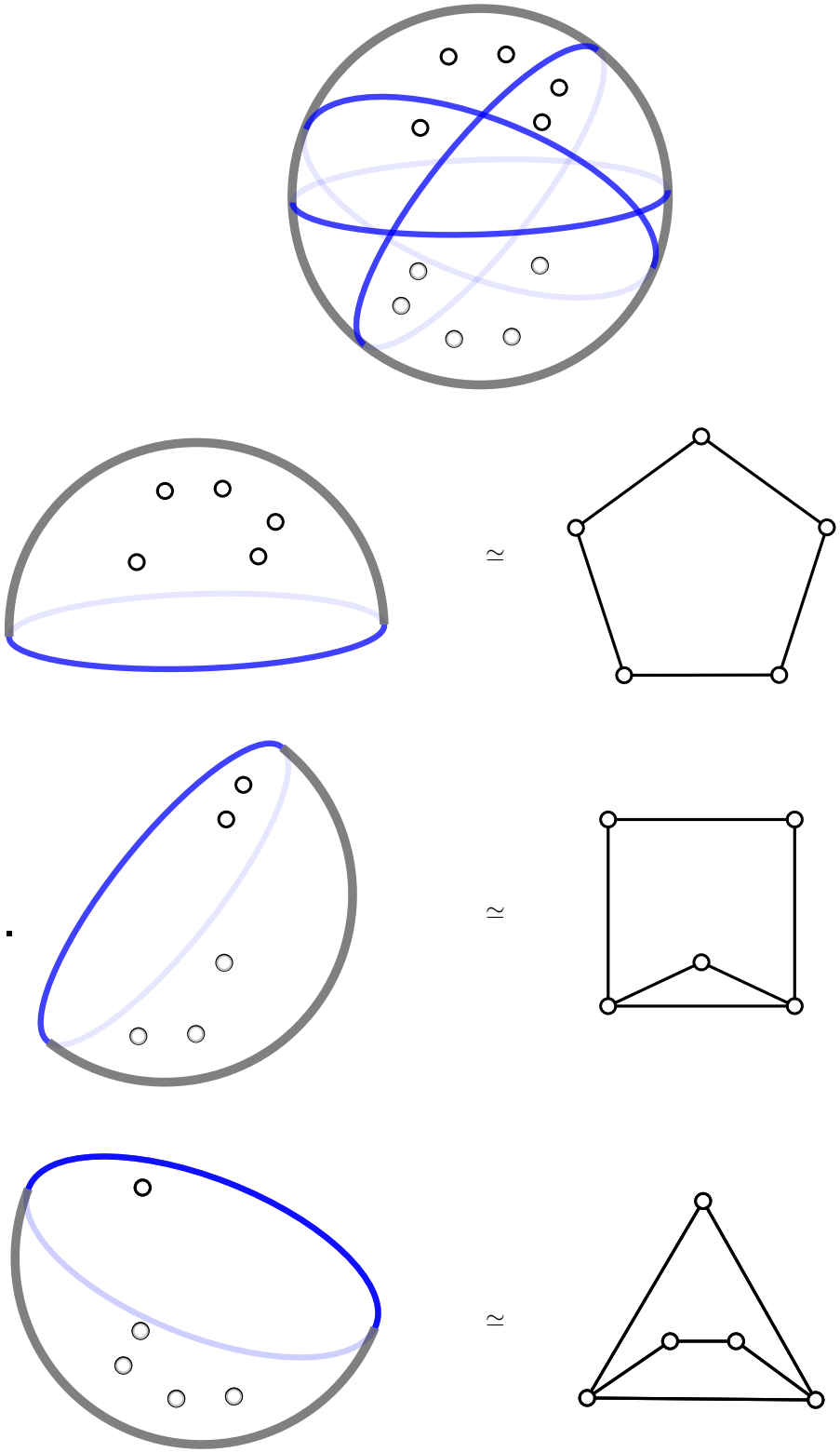

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Chirotopes:

duality + **miracle** + averaging.

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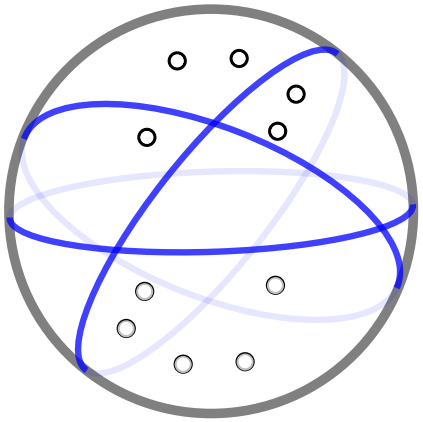

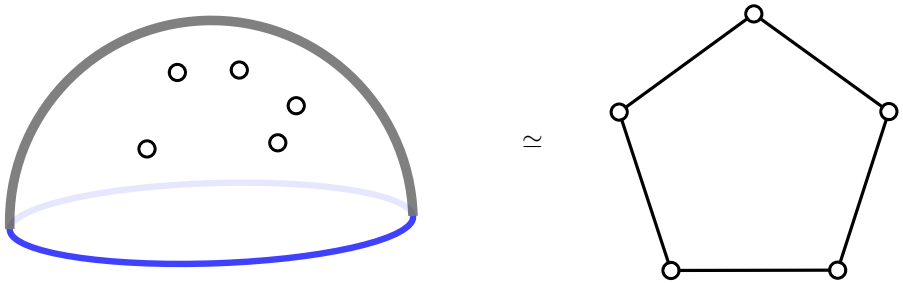

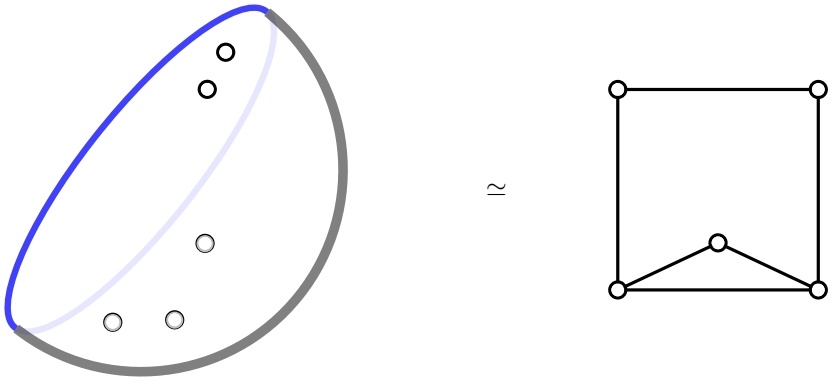

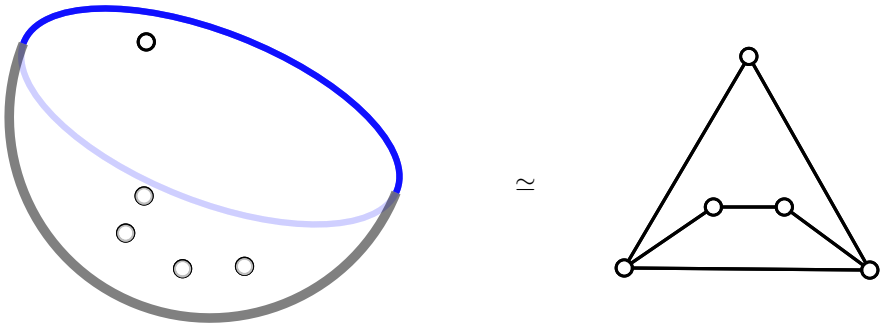

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Order types:

Klein's proof + ...

To conclude...

A combinatorial structure with a geometric twist
and algorithmic meaning.

A wonderful playground for all kinds of algebra.

Milnor-Thom, Von Staudt, semi-algebraic graphs, flag algebras, finite subgroups of $SO(3)$, ...

We do not know how to count.

We do not know how to sample efficiently.

And now we know that we don't know.

Thank you

for

your attention!