

Higher genus meanders

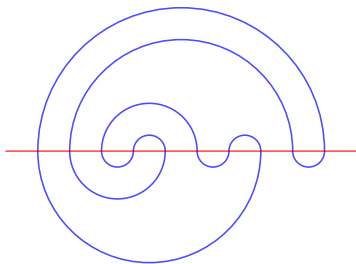
work in progress with V. Delecroix, A. Zorich, P. Zograf

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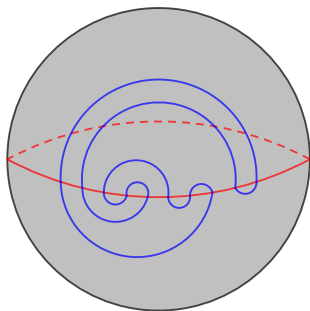
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Meanders



topological configuration of a line and a simple closed curve in the plane, intersecting transversally



topological configuration of a pair of transverse simple closed curves on the 2-sphere (+ marking)

Conjectural asymptotics

Conjecture (Di Francesco-Golinelli-Guitter, '97)

The number of meanders with $2n$ crossings is

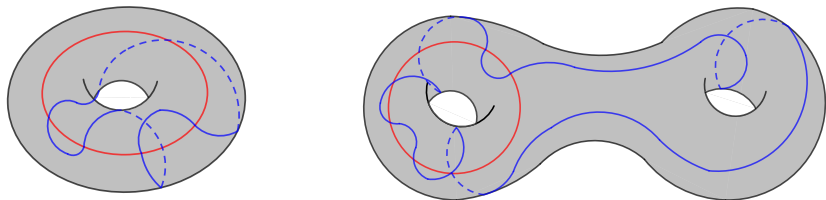
$$M_n \sim CR^n n^{-\alpha}$$

with $C > 0$, $R \simeq 12.2628$, $\alpha = \frac{29+\sqrt{145}}{12}$.

Jensen Guttman '00: extensive numerics non infirming the conjecture

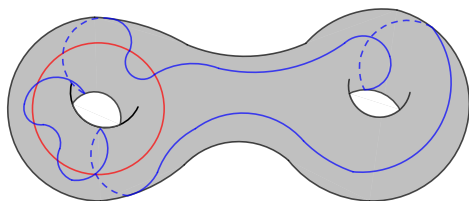
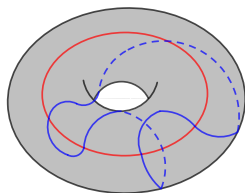
Albert-Paterson '05: $11.380 < R < 12.901$

Higher genus meanders



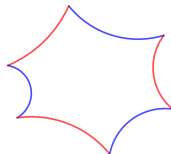
Topological configuration of a pair of transverse simple closed curves on a higher genus surface, such that the corresponding faces are topological disks (no marking).

Higher genus meanders

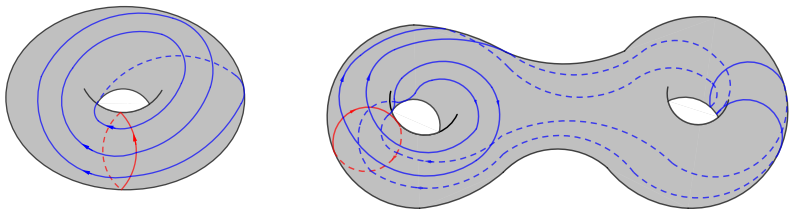


Topological configuration of a pair of transverse simple closed curves on a higher genus surface, such that the corresponding faces are topological disks (no marking).

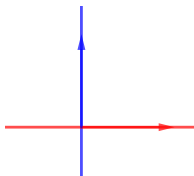
Faces have degree $2k$.



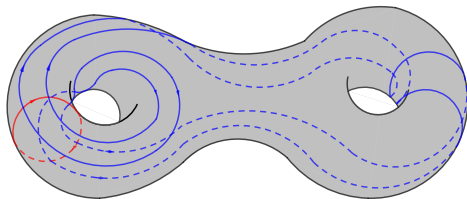
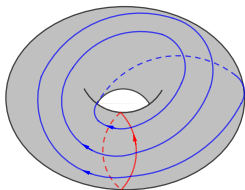
Oriented meander



Higher genus meander with a coherent orientation of the curves.

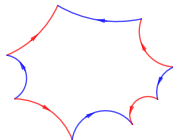
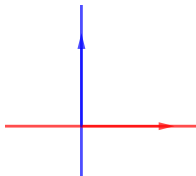


Oriented meander

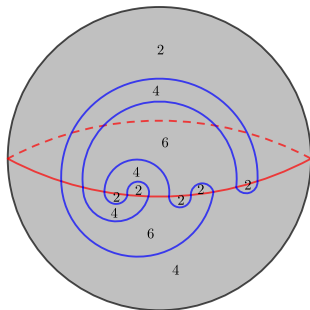
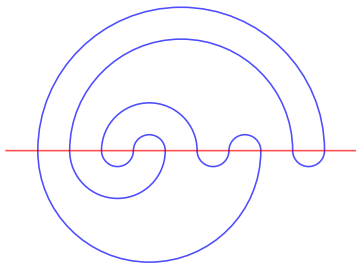


Higher genus meander with a coherent orientation of the curves.

Faces have degree $4k$.



Results: counting for a fixed number of bigons



Results: counting for a fixed number of bigons

$M_{n,p}$: Plane meanders with $2n$ crossings and p bigons

Theorem (DGZZ, '17)

$$\sum_{n \leq N} M_{n,p} = A_p N^{2p-5} + o(N^{2p-5})$$

with

$$A_p = \frac{2}{p!(p-4)!(2p-5)} \left(\frac{2}{\pi^2}\right)^{p-3} \binom{2p-4}{p-2}^2$$

and

$$A_p \sim \frac{\pi^2 e^2}{128} p \left(\frac{8e}{\pi} \cdot \frac{1}{p}\right)^{2p} \text{ as } p \rightarrow \infty.$$

Results: counting meanders with a fixed number of bigons

$M_{n,p}^g$: Genus g meanders with $2n$ crossings and p bigons

Theorem (DGZZ, '20)

$$\sum_{n \leq N} M_{n,p}^g = C_{g,p} N^{6g-6+2p} + o(N^{6g-6+2p})$$

where $C_{g,p}$ is a positive rational multiple of $\pi^{-6g+6-2n}$.

Moreover, $C_{g,p}$ satisfies the two asymptotics

$$\text{for any fixed } g \geq 0, \quad C_{g,p} \sim b_g \cdot \frac{1}{p^{\frac{5}{2}g-1}} \cdot \left(\frac{8e}{\pi} \cdot \frac{1}{p} \right)^{2p} \quad \text{as } p \rightarrow \infty$$

$$\text{for any fixed } p \geq 0, \quad C_{g,p} \sim B_p \cdot \frac{1}{g^{\frac{5}{2}p-3}} \left(\frac{2e}{3} \cdot \frac{1}{g} \right)^{4g} \quad \text{as } g \rightarrow \infty,$$

Results: counting oriented meanders

M_n^{g+} : Oriented genus g meanders with n crossings

Theorem (DGZZ, '20)

$$\sum_{n \leq N} M_n^{g+} = C_g^+ N^{4g-3} + o(N^{4g-3}),$$

where C_g^+ is a positive rational multiple of π^{-2g} .

Moreover, C_g^+ satisfies the following asymptotics

$$C_g^+ \sim \frac{1}{4\sqrt{\pi}} \cdot \frac{1}{\sqrt{g}} \cdot \left(\frac{e}{4} \cdot \frac{1}{g}\right)^{2g} \text{ as } g \rightarrow \infty.$$

Higher genus arc systems

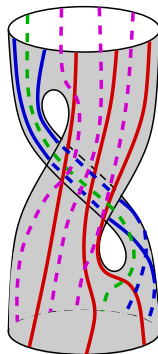


Finite collection of pairwise nonintersecting segments on a genus g surface with two boundary components s.t.

- all the endpoints of all segments are located on the boundary
- each segment lands to the boundary transversally
- the numbers of endpoints of the segments on the two boundary components is the same
- the segments cut the surface into a collection of topological disks.

Higher genus oriented arc systems

Arc systems + orientation : all intersections of arcs with one boundary component are positive and all intersections with the other component are negative (considering the natural orientation of the boundary components).



Proportion of meanders

Fix N , g , p and consider all possible couples (for all $n \leq N$):

(arc system of genus g with p bigons and n endpoints; identification)

$$P_{g,p}(N) = \frac{\text{number of such couples giving rise to meanders}}{\text{total number of different couples}}.$$

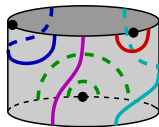
Theorem

$$\lim_{N \rightarrow \infty} P_{g,p}(N) = p_{g,p},$$

where $p_{g,p}$ is a positive rational multiple of $\pi^{-6g+6-2p}$.

For any fixed g , $p_{g,p} \sim \tilde{b}_g \cdot p^{\frac{g-1}{2}} \cdot \left(\frac{8}{\pi^2}\right)^p$ as $p \rightarrow \infty$

For any fixed p , $p_{g,p} \sim \frac{1}{4} \sqrt{\frac{2\pi}{3}} \cdot \frac{1}{\sqrt{g}}$ as $g \rightarrow \infty$.



Proportion of meanders: oriented case

Fix N and g , and consider all possible couples (for all $n \leq N$):

(oriented arc system of genus g with n endpoints; identification)

$$P_g^+(N) = \frac{\text{number of such couples giving rise to oriented meanders}}{\text{total number of different couples}}.$$

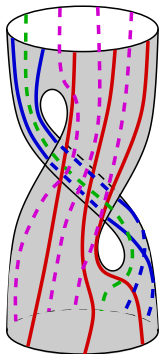
Theorem

$$\lim_{N \rightarrow \infty} P_g^+(N) = p_g^+,$$

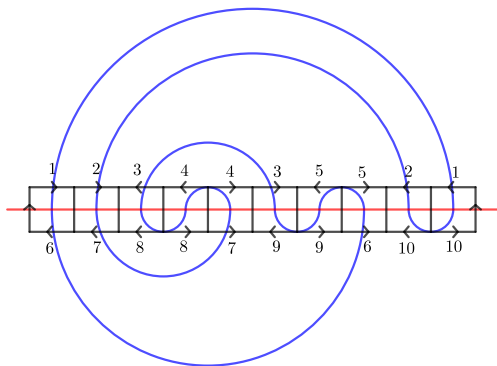
where p_g^+ is a positive rational multiple of π^{-2g} .

As $g \rightarrow \infty$:

$$p_g^+ \sim \frac{1}{4g}.$$

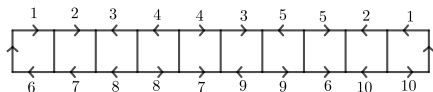


Outline of the proof: meanders as square-tiled surfaces



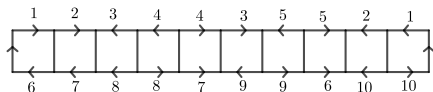
Take the dual graph: all faces have valency 4. Put a flat metric on the faces so that they are euclidean squares.

Outline of the proof: meanders as square-tiled surfaces



A **square-tiled surface** is a surface glued from squares (**vertical** \leftrightarrow **vertical**, **horizontal** \leftrightarrow **horizontal**). They form a special family of flat surfaces with conical singularities (half-translation surfaces).

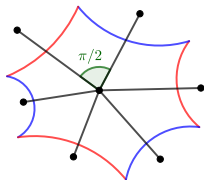
Outline of the proof: meanders as square-tiled surfaces



A **square-tiled surface** is a surface glued from squares (**vertical** \leftrightarrow **vertical**, **horizontal** \leftrightarrow **horizontal**). They form a special family of flat surfaces with conical singularities (half-translation surfaces).

Meanders with $2n$ crossings correspond to marked square-tiled surfaces of genus g with $2n$ squares and exactly one horizontal cylinder, one vertical cylinder.

$2k$ -gons correspond to singularities of angle $k\pi$.



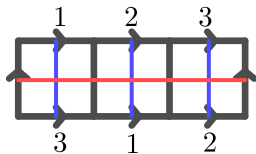
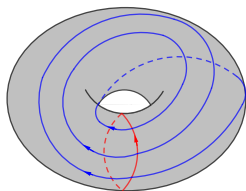
Outline of the proof: square-tiled surfaces and volumes of moduli spaces

- Such square-tiled surfaces correspond to special points (integer points) in the family $\mathcal{Q}_{g,p}$ of flat surfaces of genus g with p singularities of angle π . (moduli space for half-translation surfaces / quadratic differentials)
- $\text{Vol } \mathcal{Q}_{g,p} = c \lim_{N \rightarrow \infty} \frac{\text{Card}\{\text{SQT with } \leq N \text{ squares, genus } g \text{ and } p \text{ angles } \pi\}}{N^{6g-6+2p}}$
- Let $\text{cyl}_1(g, p)$ be the same limit for SQT with one horizontal cylinder, and $\text{cyl}_{1,1}(g, p)$ the same limit for SQT for one horizontal cylinder and one vertical cylinder (they exist and are positive).

Theorem (DGZZ '17)

$$\frac{\text{cyl}_{1,1}(g, p)}{\text{cyl}_1(g, p)} = \frac{\text{cyl}_1(g, p)}{\text{Vol } \mathcal{Q}_{g,p}} = p_{g,p}$$

Outline of the proof: oriented case



- Square-tiled surfaces corresponding to oriented meanders come with an additional orientation (the gluing of squares is **top** \leftrightarrow **bottom**, **right** \leftrightarrow **left**). They form a family of flat surfaces with conical singularities (translation surfaces).
- All previous results translate in terms of the family \mathcal{H}_g of translation surfaces of genus g (moduli space of Abelian differentials), which has dimension $4g - 3$.

Outline of the proof: computation of volumes and recent advances

$$\text{Vol}(\mathcal{Q}_{g,p}) \text{ and } \text{Vol}(\mathcal{H}_g)$$

- Eskin-Okounkov \sim '00, '05: algorithms for small dimension
- Athreya-Eskin-Zorich '12: closed formulas for $\text{Vol } \mathcal{Q}_{0,p}$
- Chen-Möller-Zagier '18 $\text{Vol}(\mathcal{H}_g)$ as $g \rightarrow \infty$ (gen. case Aggarwal '19)
- DGZZ '18, $\text{Vol}(\mathcal{Q}_{g,p})$ as a sum over stable graphs
- Chen-Möller-Sauvaget-Zagier '19: $\text{Vol}(\mathcal{H}_g)$ as Hodge integrals
- Andersen-Borot-Charbonnier-Delecroix-Giacchetto-Lewanski-Wheeler '19: topological recursion for $\text{Vol}(\mathcal{Q}_{g,p})$
- Chen-Möller-Sauvaget '19 $\text{Vol}(\mathcal{Q}_{g,p})$ as Hodge integrals, and $p \rightarrow \infty$
- Aggarwal '19 $\text{Vol}(\mathcal{Q}_{g,p})$ as $g \rightarrow \infty$ based on [DGZZ].
- Kazarian '19 , Yang-Zagier-Zhang '20: quadratic recursion for $\text{Vol}(\mathcal{Q}_{g,p})$ based on [CMS].

$$\text{cyl}_1(\mathcal{Q}_{g,p}) \text{ and } \text{cyl}_1(\mathcal{H}_g)$$

- DGZZ Explicit formula for $\text{cyl}_1(\mathcal{H}_g)$ (via characters of the symmetric group)
- DGZZ Explicit formula for $\text{cyl}_1(\mathcal{Q}_{g,p})$ via sum over stable graphs involving some intersection numbers (computed by Zograf), giving $g \rightarrow \infty$ and $p \rightarrow \infty$.

$$\text{cyl}_1(\mathcal{Q}_{g,p}) \text{ and } \text{cyl}_1(\mathcal{H}_g)$$

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$$p_{g,p} = \text{cyl}_{1,1}/\text{cyl}_1 = \text{cyl}_1/\text{Vol}$$

- We never managed to prove asymptotics of $p_{g,p}$ or p_g by direct combinatorial arguments!

Further results

- Meanders with a fixed number of 8-gons, 10-gons, etc.
 - The growth rate in N is known
 - Not the asymptotics as g or p tends to infinity (only conjectures so far!)
- Allowing the vertical curve to have several components, study the distribution of the number of components in large genus?
 - Answer only for the number of primitive components (number of cylinders VS number of bands of squares). Uncorrelation results imply that it is the same distribution as the number of primitive components of a random multicurve [DGZZ'20].