Ordre faible et cone imaginaire dans les groupes de Coxeter infinis

– Séminaire de combinatoire Philippe Flajolet –

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(en sabbatique à l'IRMA, Strasbourg)

Basé sur des travaux en collaboration avec :

- J.P Labbé (FU Berlin), V. Ripoll (Vienne) and M. Dyer (U. Notre-Dame, USA)

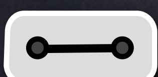
Coxeter groups

- An introductory example: the symmetric group $\mathcal{S}_n$.
  - Generators: simple transpositions $\tau_i = (i\, i+1)$;
  - Relations: $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$ (braid relat°)

    $\tau_i\tau_j = \tau_j\tau_i,\ |i-j|>1$ (commutation relat°)

So $e = (\tau_i\tau_{i+1}\tau_i)^2 = (\tau_i\tau_{i+1}\tau_i)(\tau_{i+1}\tau_i\tau_{i+1}) = (\tau_i\tau_{i+1})^3$, hence

$$\mathcal{S}_n = \langle \tau_i \mid \tau_i^2 = (\tau_i\tau_j)^2 = (\tau_i\tau_{i+1})^3 = e,\ 1 \le i < n,\ |j-i|>1 \rangle$$

-  means $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$ or $(\tau_i\tau_{i+1})^3 = e$
-  means $\tau_i\tau_j = \tau_j\tau_i$ or $(\tau_i\tau_j)^2 = e$ (they commute)

Coxeter groups

(W, S) Coxeter system of finite rank $|S| < \infty$ i.e.

- $W = \langle S \,|\, (st)^{m_{st}} = e \rangle$ group
- $m_{ss} = 1$ (s involut°); $m_{st} = m_{ts} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for $s \neq t$

A Coxeter graph Γ is given by:

- vertices S (finite)
- edges 

with $m_{st} \geq 3$ or $m_{st} = \infty$

Examples. Symmetric group $\mathcal{S}_n$ is 

- Dihedral group: $\mathcal{D}_m = \langle s, t \,|\, s^2 = t^2 = (st)^m = e \rangle$;
- Infinite dihedral group: $\mathcal{D}_\infty = \langle s, t \,|\, s^2 = t^2 = e \rangle$;
- Universal Coxeter group: $U_n = \langle a_1, \ldots, a_n \,|\, a_i^2 = e \rangle$

Coxeter groups

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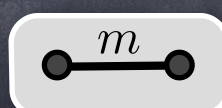
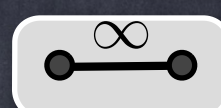
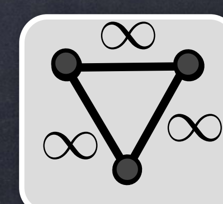
Examples. Symmetric group $\mathcal{S}_n$ is 

- Dihedral group: $\mathcal{D}_m$ is 

or 
$(m = 2)$
- Infinite dihedral group: $\mathcal{D}_\infty$ is 

- Universal Coxeter group: $U_n = \langle a_1, \ldots, a_n \,|\, a_i^2 = e \rangle$ 

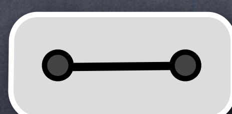
Coxeter groups

Words and Length

- any $w \in W$ is a word in the alphabet S ;
- Length function $\ell : W \to \mathbb{N}$ with $\ell(e) = 0$ and

$$\ell(w) = \min\{k \mid w = s_1 s_2 \dots s_k,\ s_i \in S\}$$

How to study words on S representing w ? Is a word $s_1 s_2 \dots s_k$ a reduced word for w (i.e. $k = \ell(w)$) ?

Examples. $\mathcal{D}_3$ is  ;

	e	s	t	st	ts	$sts = tst$
ℓ	0	1	1	2	2	3

$\ell(ststs) = 1$ since $ststs = (sts)ts = (tst)ts = t$

Proposition. Let $s \in S$ and $w \in W$, then $\ell(ws) = \ell(w) \pm 1$.

Weak order and reduced words

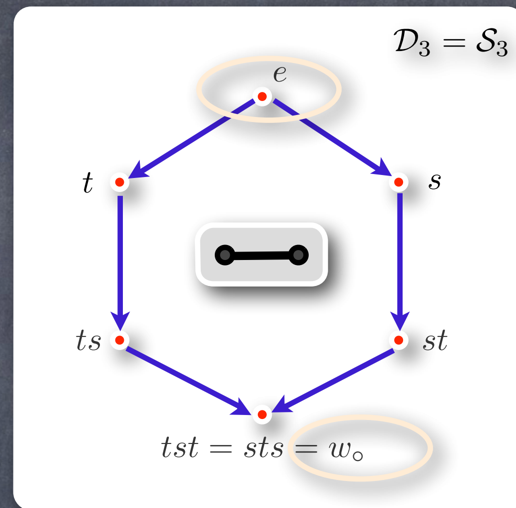
Cayley graph of $W = \langle S \rangle$ i.e.

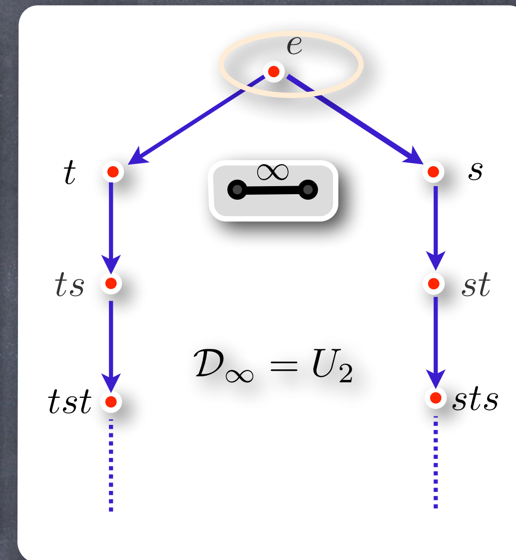
- vertices W
- edges $w \overset{s}{\text{———}} ws \quad (s \in S)$

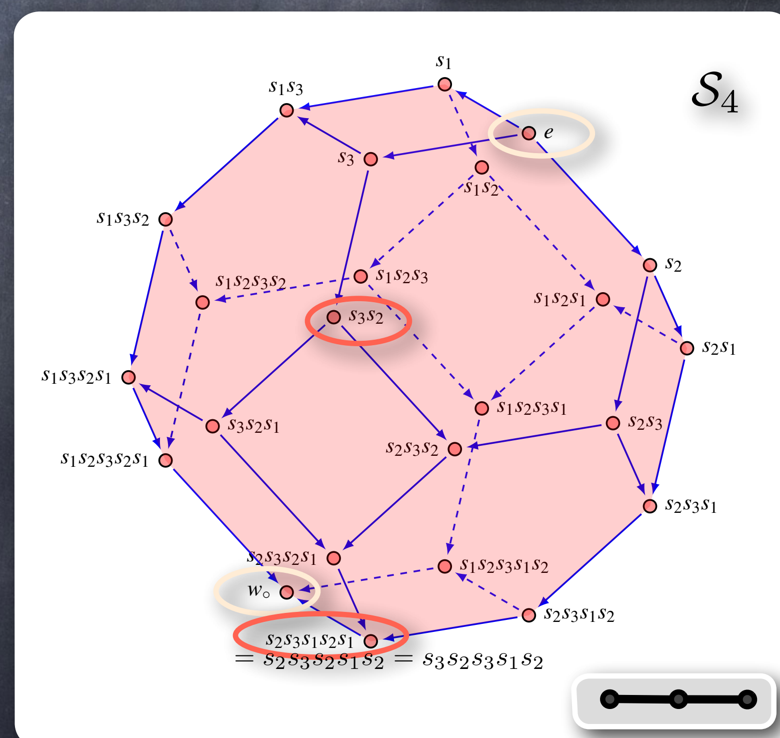
is naturally oriented by the (right) weak order:

$$w < ws \quad \text{if} \quad \ell(w) < \ell(ws)$$

write: $w \xrightarrow{s} ws$







Fact: (a) $u \leq w$ iff a reduced word of u is a prefix of a red. word of w.

(b) reduced words of w corresp. to maximal chains in the interval $[e, w]$.

Weak order and reduced words

Theorem (Björner). The weak order is a complete meet-semilattice. In particular $u \wedge v = \inf(u,v),\ \forall u,v \in W$, exists.

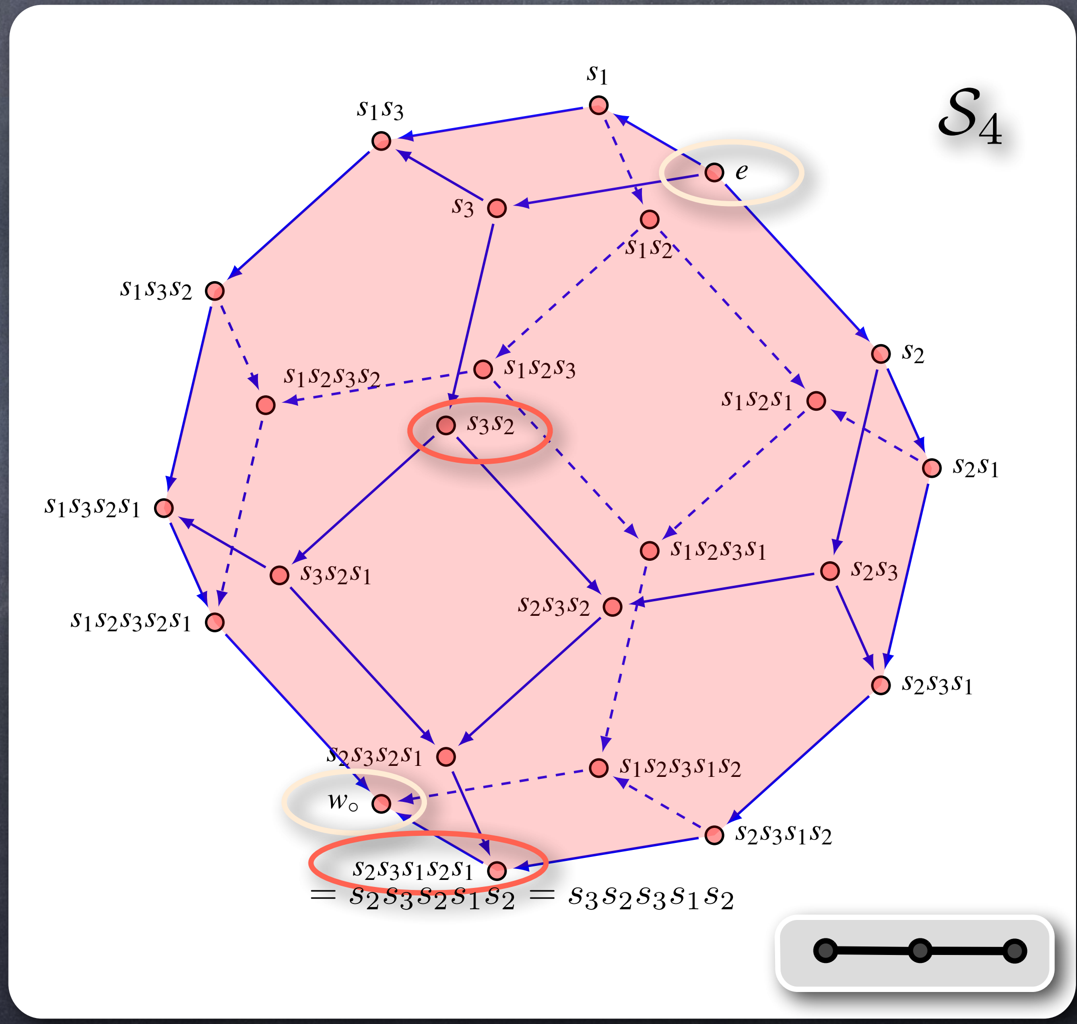
Proposition. Assume W is finite, then:

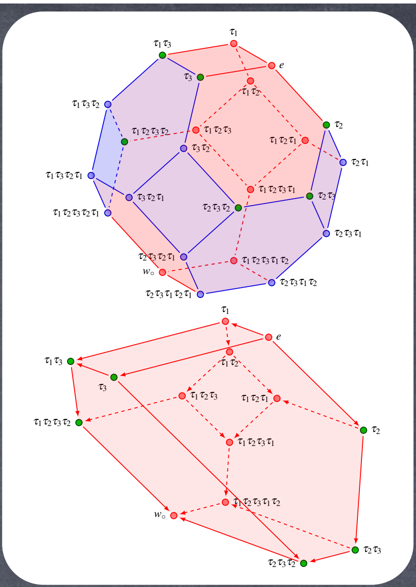
(i) there is a unique $w_\circ \in W$ such that: $u \leq w_\circ,\ \forall u \in W.$

(ii) the map $w \mapsto ww_\circ$ is a poset antiautomorphism.

(iii) the weak order is a complete lattice. In part., $u \vee v = \sup(u,v)$ exists.

(iv) $u \wedge v = (uw_\circ \vee vw_\circ)w_\circ$





Weak order: a combinatorial model

Cambrian (semi)lattices/fans in finite case (N. Reading, N. reading & D.Speyer)/Generalized associahedra in finite case (CH, C. Lange, H. Thomas): link with subword complexes (C. Ceballos, J.P. Labbé, V. Pilaud, C. Stump); recovering the corresponding Cluster algebras (S. Stella).

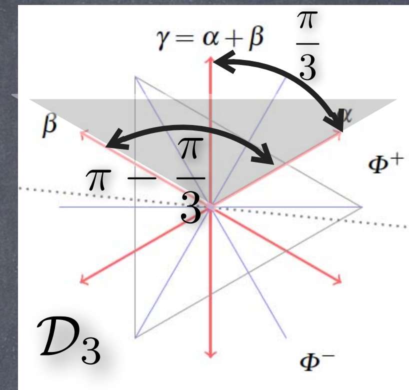
Initial section of reflection orders and KL-polynomials (M. Dyer): combinatorial formulas for KL-polynomials (F. Brenti, M. Dyer).

A combinatorial model for cambrian lattices/generalized associahedra in infinite case, or twisted Bruhat order and KL-polynomials (M. Dyer)? Examples suggest to «enlarge» Coxeter groups to have a weak order that is a complete lattice. Is it possible?

Geometric representations

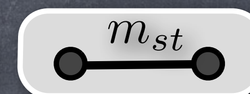
Many geometric representations of (W,S)

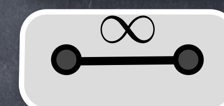
(Tits, Vinberg)



- (V,B) real quadratic space and $\Delta \subseteq V$ s.t.
  - $\mathrm{cone}(\Delta) \cap \mathrm{cone}(-\Delta) = \{0\}$;
  - $\Delta = \{\alpha_s \,|\, s \in S\}$ s.t.

$$B(\alpha_s, \alpha_t) = \begin{cases} -\cos\left(\frac{\pi}{m_{st}}\right) & \text{if } m_{st} < \infty \\ a \leq -1 & \text{if } m_{st} = \infty \end{cases}$$





- $W \leq \mathrm{O}_B(V)$ "B-isometry":

$$s(v) = v - 2B(v,\alpha)\alpha, \;\; s \in S$$

Root system: $\Phi = W(\Delta), \;\; \Phi^+ = \mathrm{cone}(\Delta) \cap \Phi = -\Phi^-$

Proposition. Let $s \in S$ and $w \in W$, then:

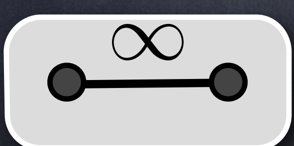
$$\ell(ws) = \ell(w) + 1 \iff w(\alpha_s) \in \Phi^+$$

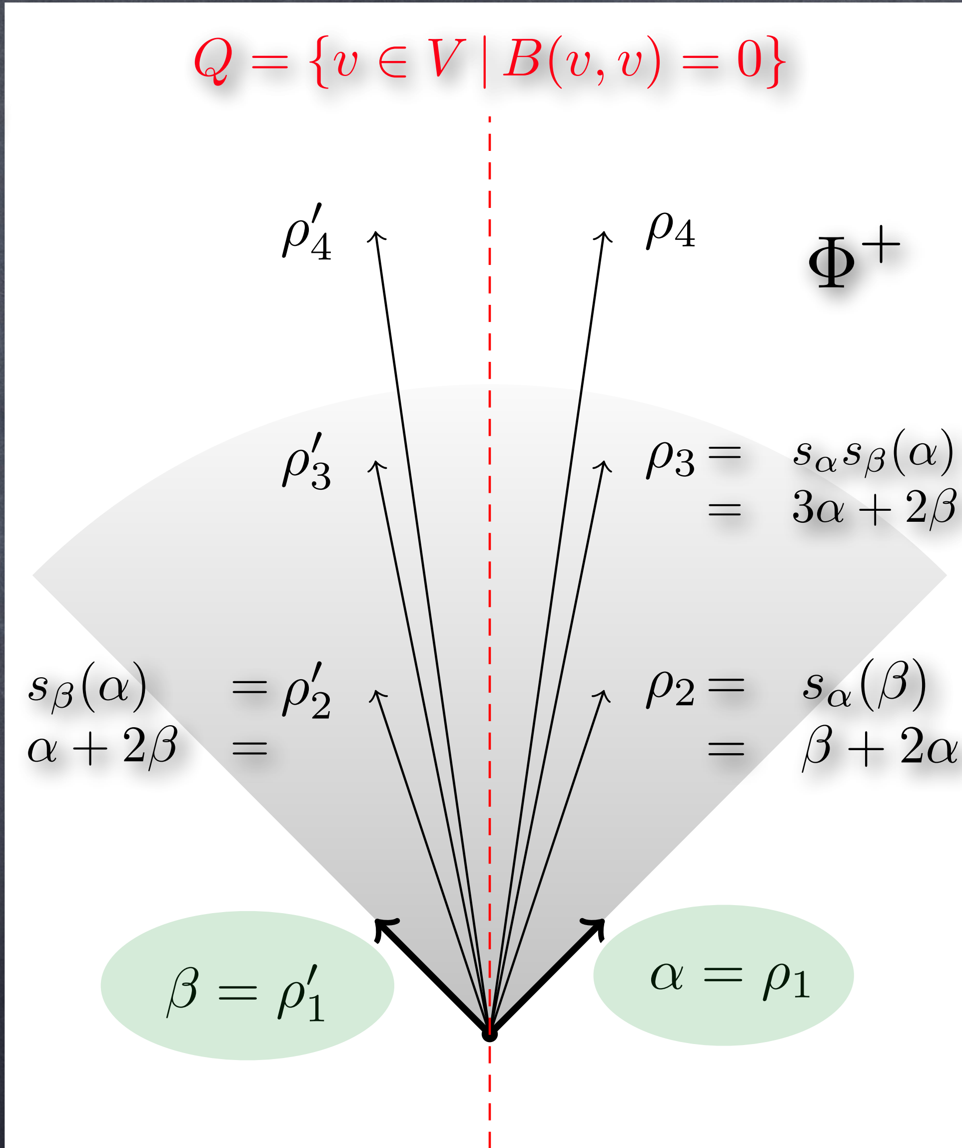
Geometric representations

$$\rho'_n = n\alpha + (n+1)\beta$$

$$\rho_n = (n+1)\alpha + n\beta$$

Infinite dihedral group I



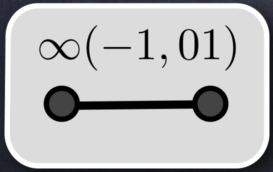


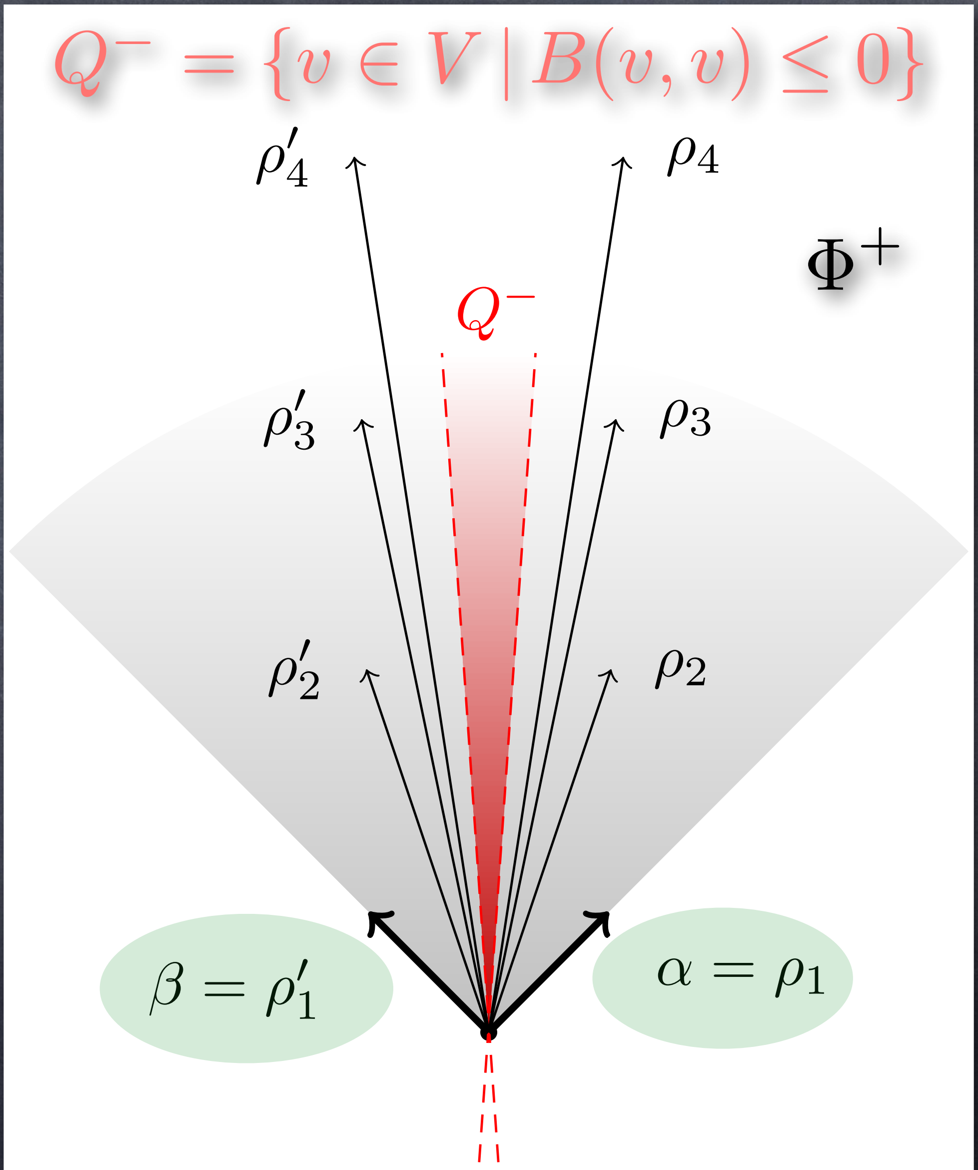
(a) $B(\alpha,\beta) = -1$

$$s_\alpha(v) = v - 2B(v,\alpha)\alpha.$$

Geometric representations

Infinite dihedral group II





(b) $B(\alpha,\beta) = -1.01 < -1$

$s_\alpha(v) = v - 2B(v,\alpha)\alpha.$

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A Projective view of root systems

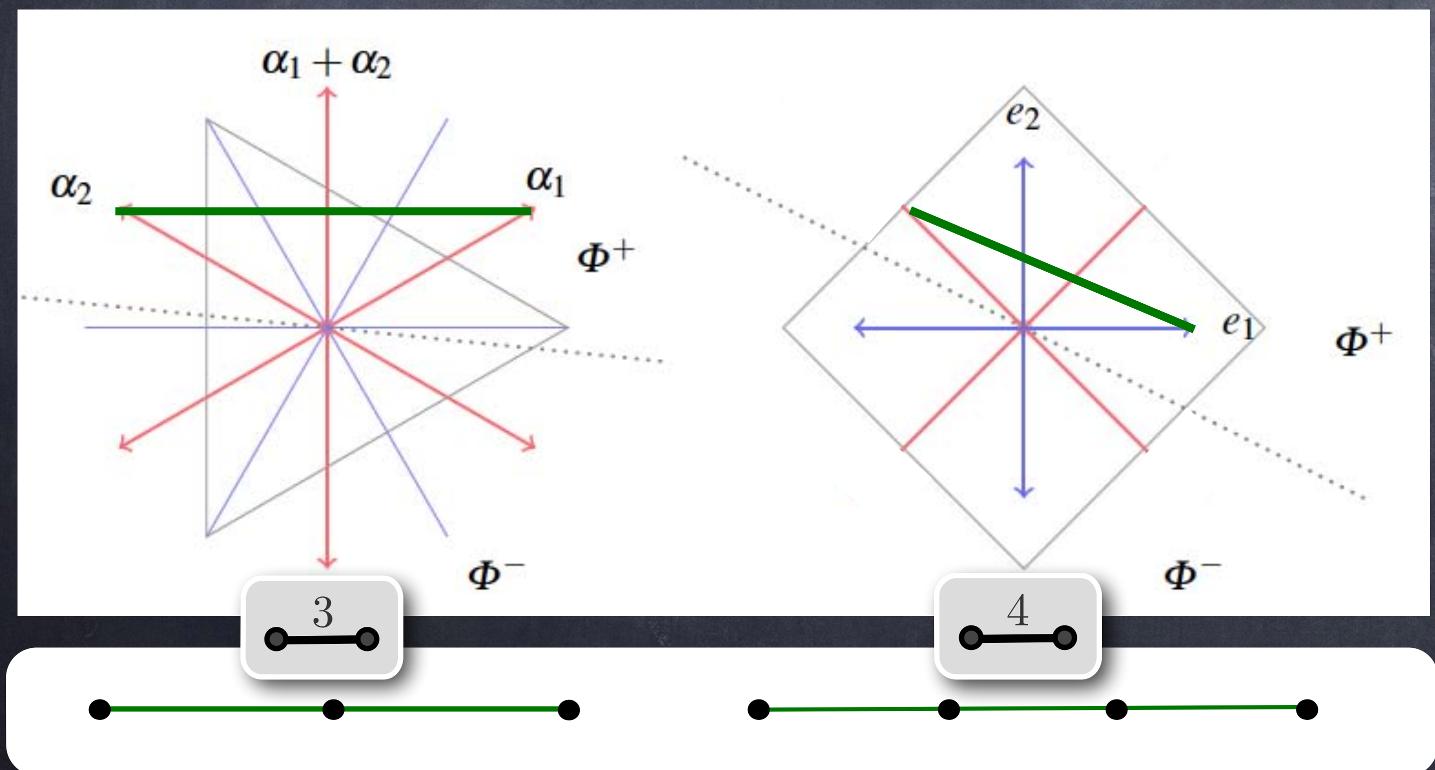
`Cut' $\mathrm{cone}(\Delta)$ by an affine hyperplane: $V_1 = \{v \in V \mid \sum_{\alpha\in\Delta} v_\alpha = 1\}$

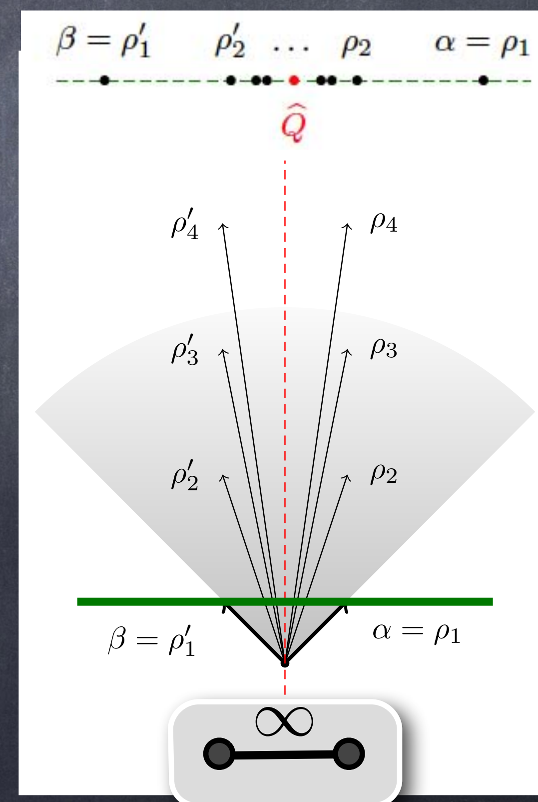
Normalized roots: $\widehat{\rho} := \rho / \sum_{\alpha\in\Delta} \rho_\alpha$ in $\widehat{\Phi} := \bigcup_{\rho\in\Phi} \mathbb{R}\rho \cap V_1$

Action of W on $\widehat{\Phi}$: $w \cdot \hat{\rho} = \widehat{w(\rho)}$

Normalized isotropic cone: $\widehat{Q} := Q \cap V_1$

- Rank 2 root systems





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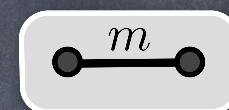
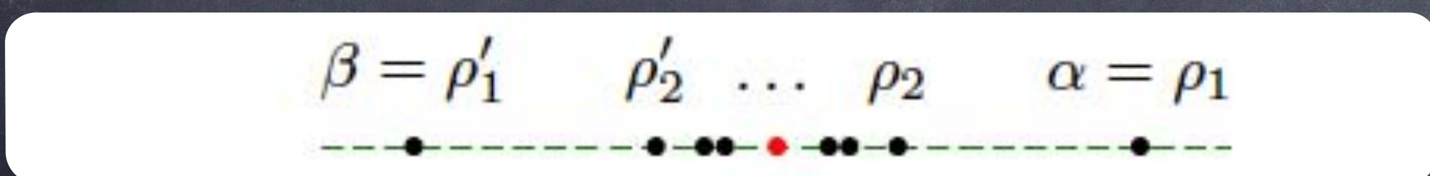
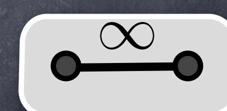
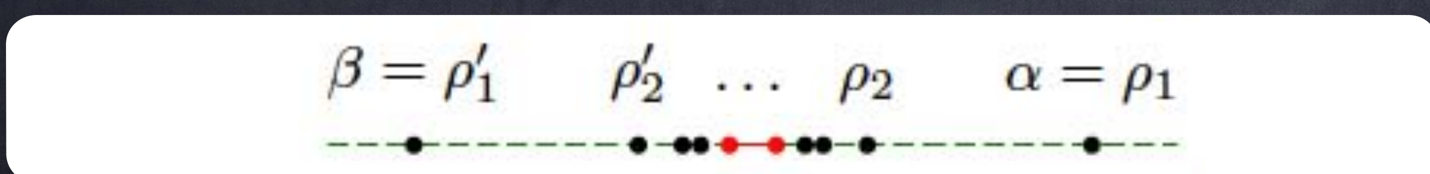
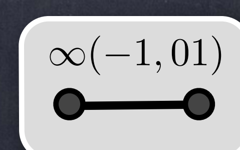






$\rho'_n = n\alpha + (n+1)\beta$ $\qquad$ $\rho_n = (n+1)\alpha + n\beta$



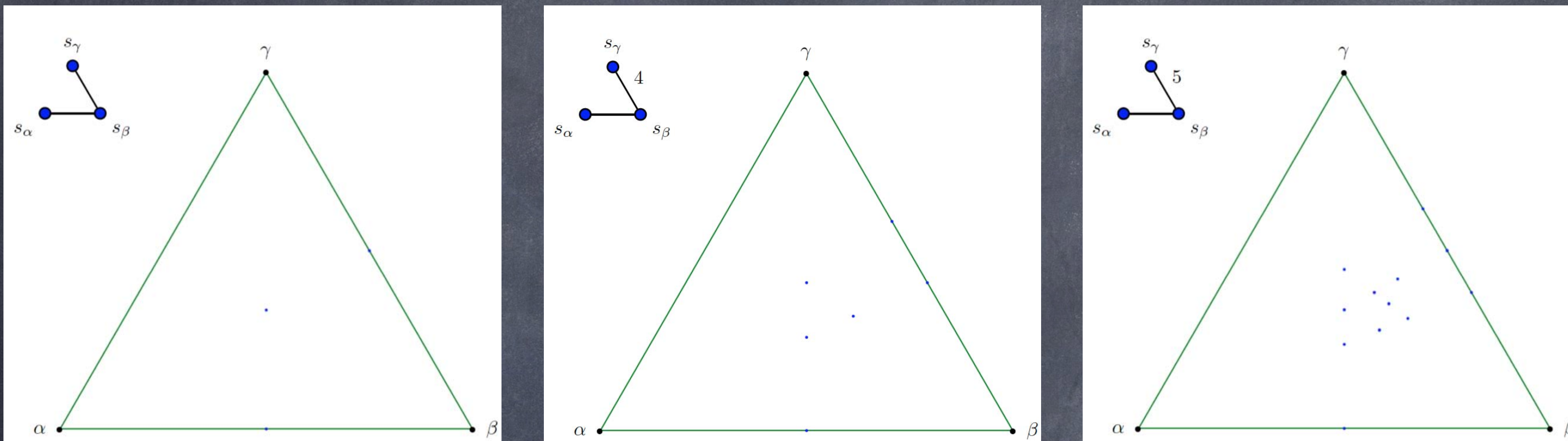


A Projective view of root systems

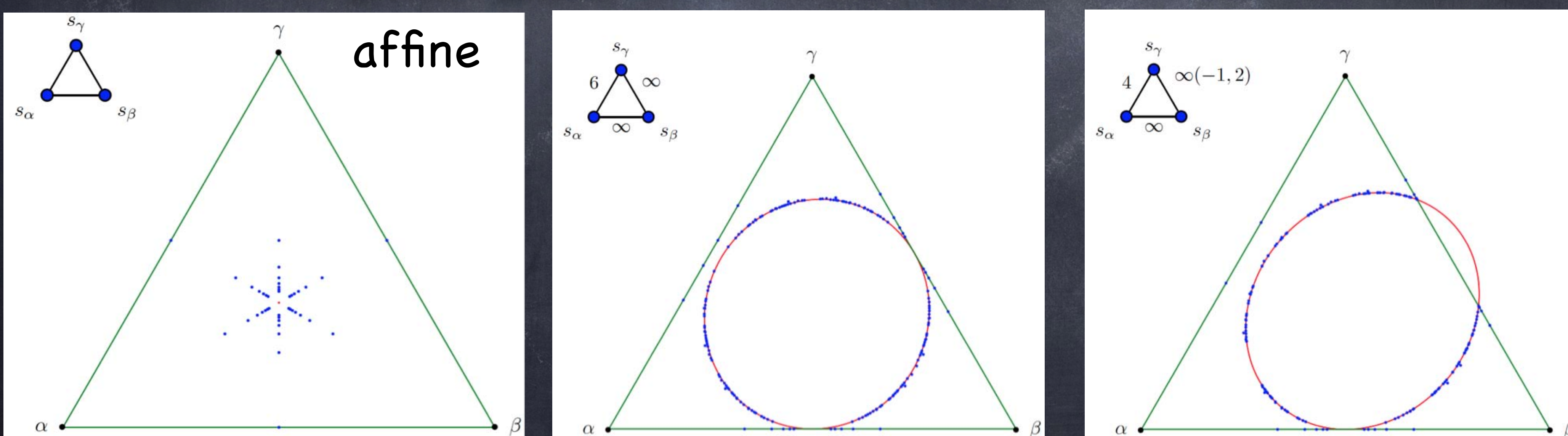
From joint works with:

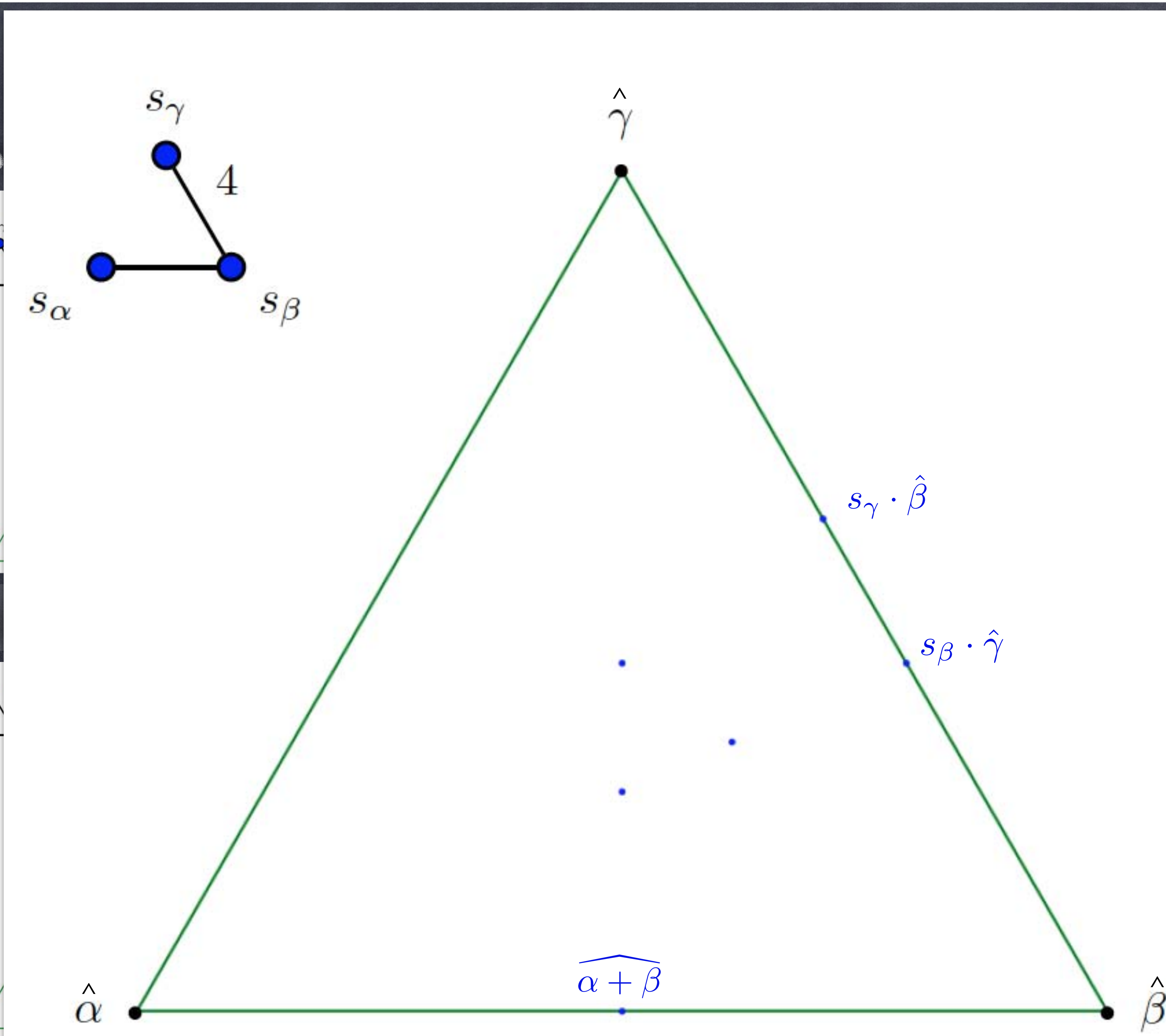
- J.P Labbé and V. Ripoll (2012)
- M. Dyer and V. Ripoll (2013)

- Rank 3 root systems

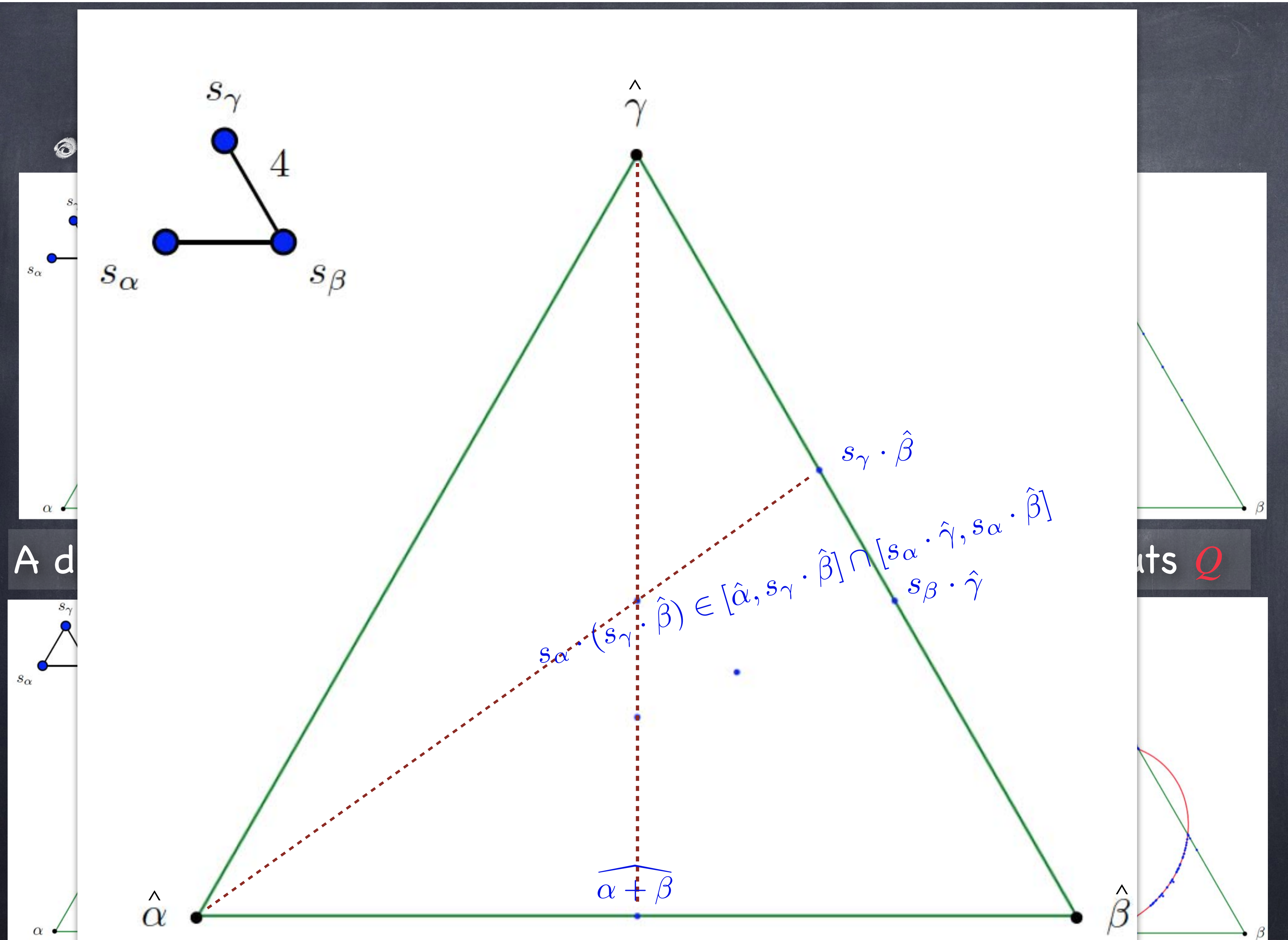


A dihedral subgroup group is infinite iff the associated line cuts Q

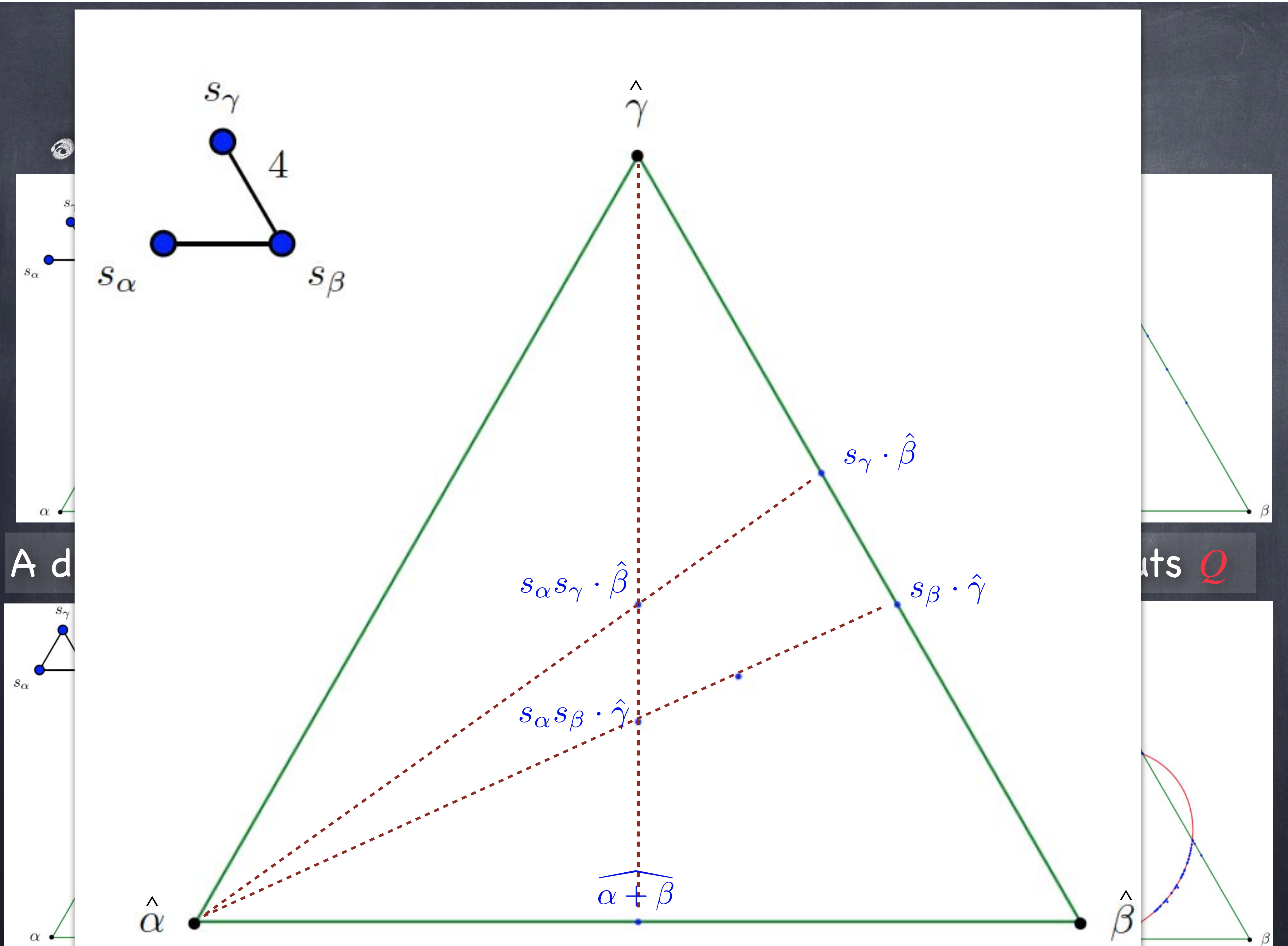




s_γ
4
s_α
s_β
$\hat{\gamma}$
$s_\gamma \cdot \hat{\beta}$
$s_\beta \cdot \hat{\gamma}$
$\hat{\alpha}$
$\widehat{\alpha+\beta}$
$\hat{\beta}$



s_γ
4
s_α
s_β
$\hat{\gamma}$
$s_\gamma \cdot \hat{\beta}$
$s_\alpha \cdot (s_\gamma \cdot \hat{\beta}) \in [\hat{\alpha}, s_\gamma \cdot \hat{\beta}] \cap [s_\alpha \cdot \hat{\gamma}, s_\alpha \cdot \hat{\beta}]$
$s_\beta \cdot \hat{\gamma}$
$\widehat{\alpha + \beta}$
$\hat{\alpha}$
$\hat{\beta}$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$\widehat{\alpha+\beta}$
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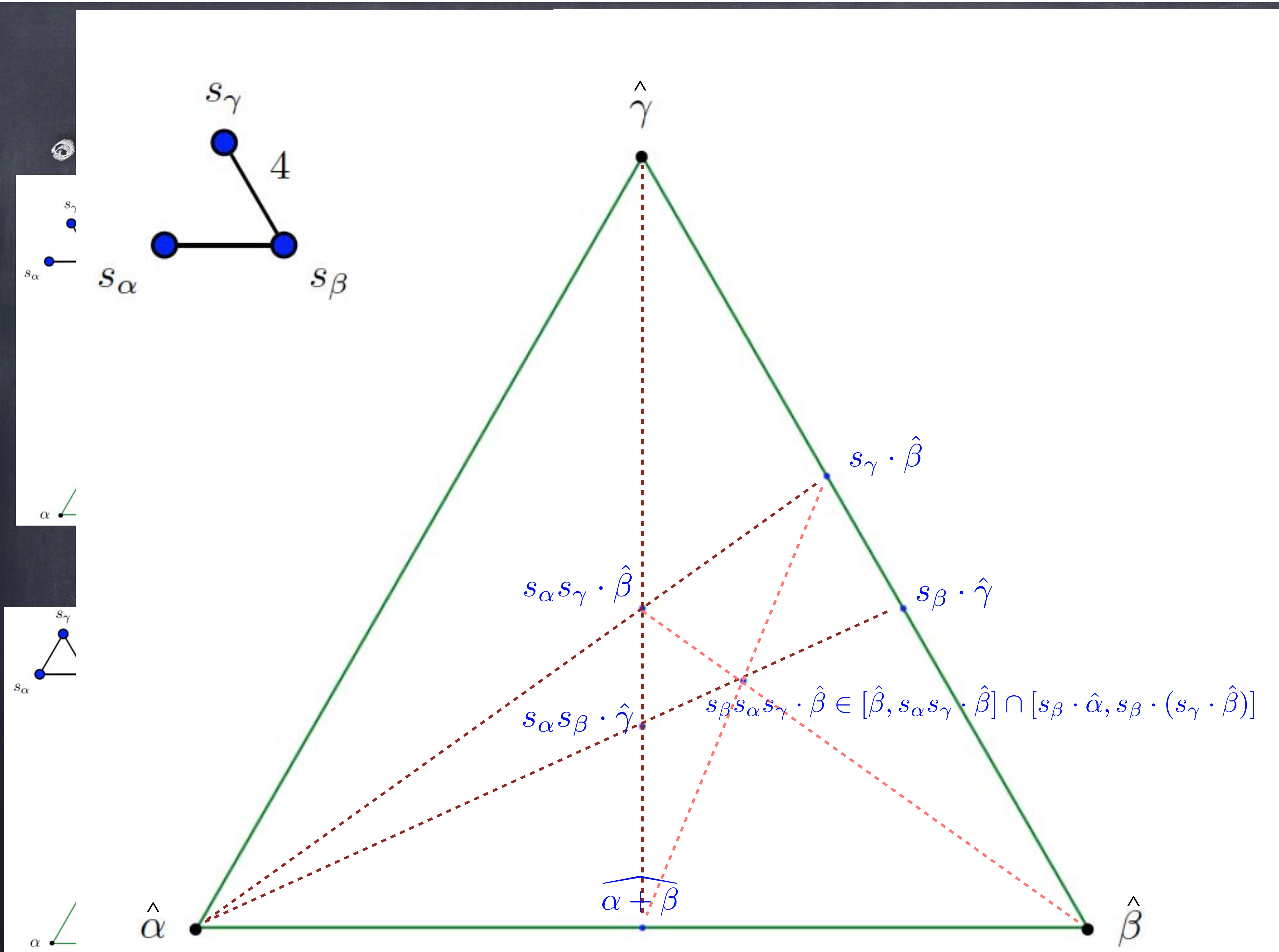
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$s_\gamma \cdot \hat{\beta}$
$s_\alpha s_\gamma \cdot \hat{\beta}$
$s_\beta \cdot \hat{\gamma}$
$s_\beta s_\alpha s_\gamma \cdot \hat{\beta} \in [\hat{\beta}, s_\alpha s_\gamma \cdot \hat{\beta}] \cap [s_\beta \cdot \hat{\alpha}, s_\beta \cdot (s_\gamma \cdot \hat{\beta})]$
$s_\alpha s_\beta \cdot \hat{\gamma}$
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$\hat{\alpha}$
$\hat{\beta}$

A Projective view of root systems

From joint works with:
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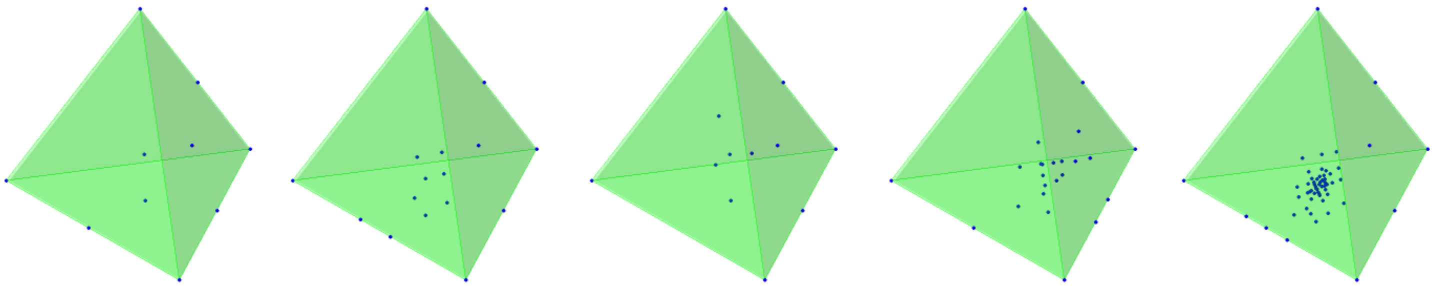
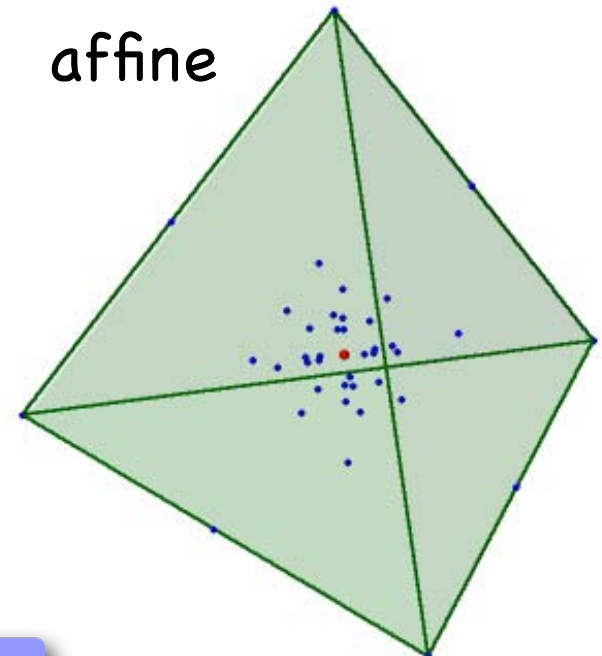
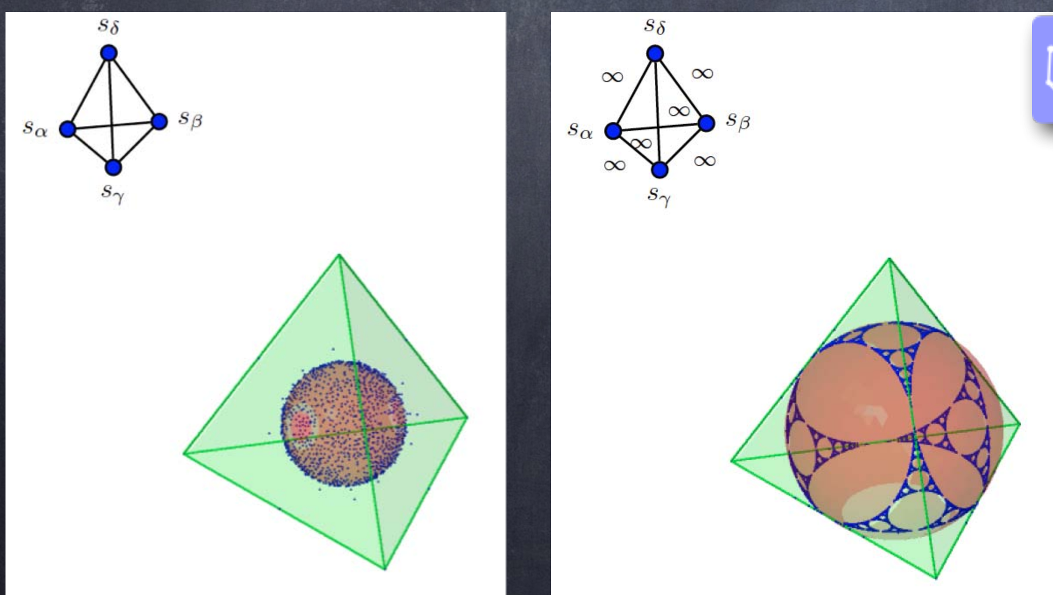
- Rank 4 root systems

finite



affine





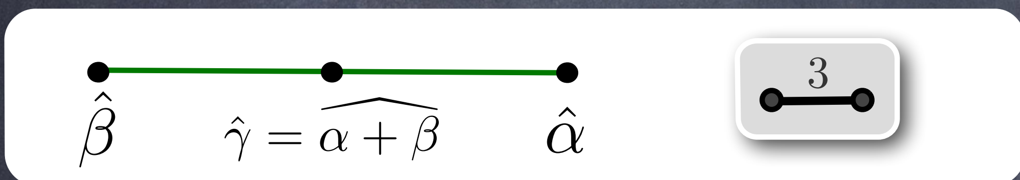
Sgn is $(2,2)$

(weakly) hyperbolic

Weak order and roots

Definition. The inversion set of a reduced word $w = s_1 s_2 \dots s_k$ is $N(w) := \{\alpha_1, s_1(\alpha_2), \cdots, s_1 \dots s_{k-1}(\alpha_k)\}$

	e	$s = s_\alpha$	$t = s_\beta$	st	ts	$sts = tst$
$\ell(w)$	0	1	1	2	2	3
$\widehat{N(w)}$	$\emptyset$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}; s_\alpha \cdot \hat{\beta} = \widehat{\gamma}$	$\hat{\beta}; \widehat{\gamma}$	$\hat{\Phi}$



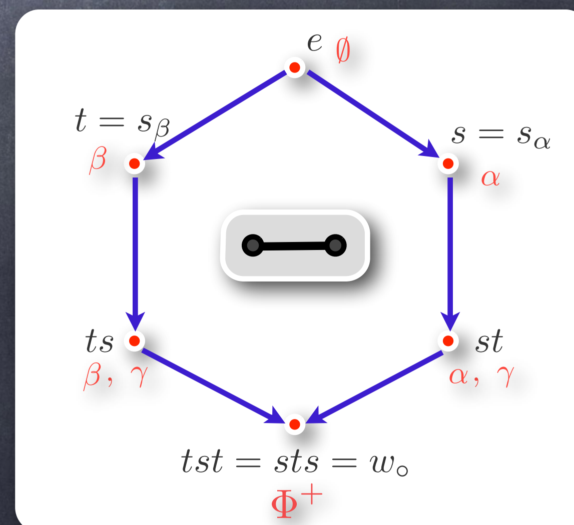
Proposition. $|N(w)| = \ell(w)$

Proposition. The map

$$N : (W, \leq) \to (\mathcal{P}(\Phi), \subseteq)$$

is an injective morphism of posets.

What is $\mathrm{Im}(N)$?



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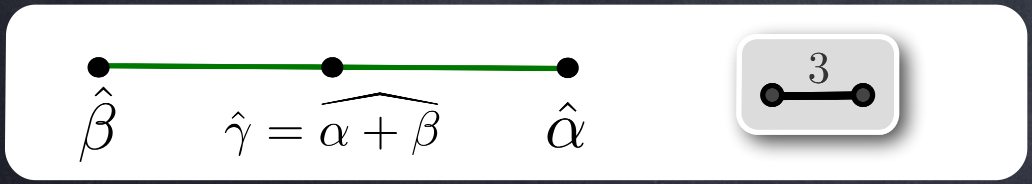
Weak order and roots

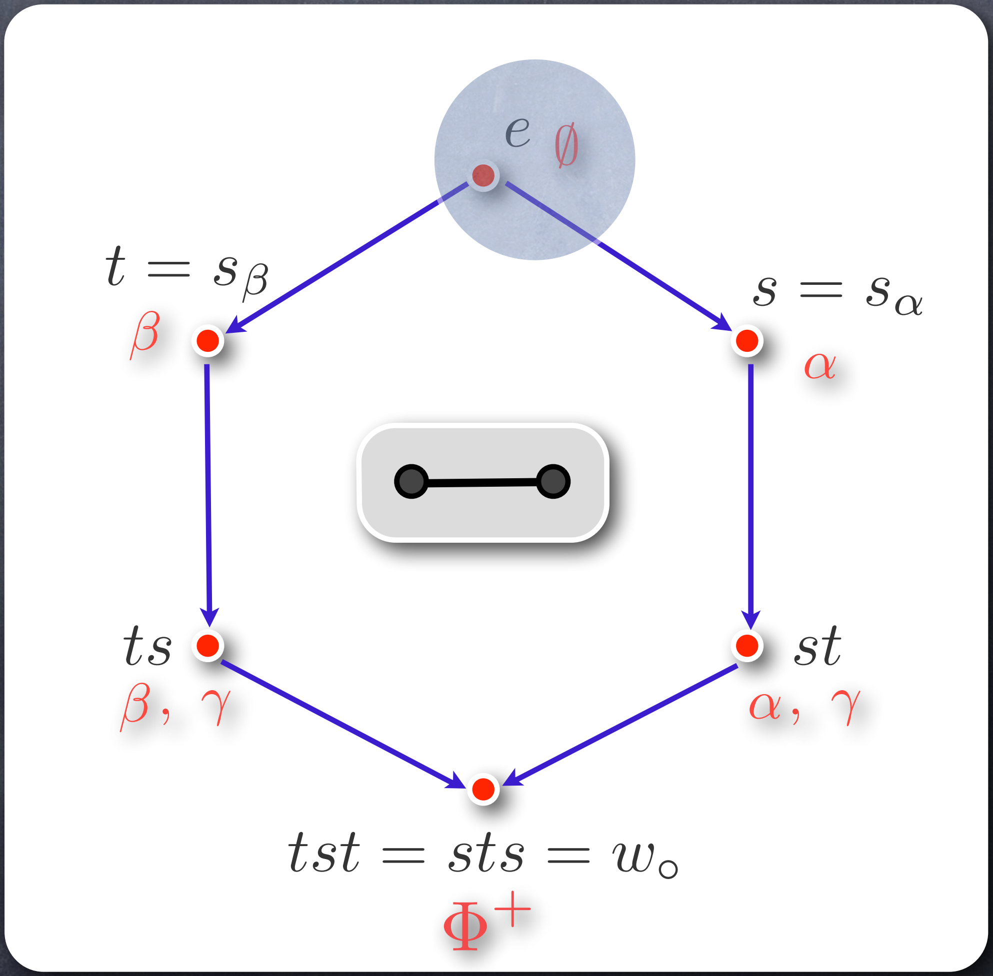
- $A \subseteq \Phi^+$ is closed if for all $[\hat{\alpha}, \hat{\beta}] \cap \hat{\Phi} \subseteq \hat{A},\ \forall \alpha, \beta \in A$;
- $A \subseteq \Phi^+$ is biclosed if $A,\ A^c := \Phi^+ \setminus A$ are closed.
- $\mathcal{B}(W) = \{\text{biclosed sets}\}$; $\mathcal{B}_0(W) = \{A \subseteq \mathcal{B}(W) \mid |A| < \infty\}$

Proposition. The map

$$N : (W, \leq) \to (\mathcal{B}_0(W), \subseteq)$$

is a poset isomorphism and $N(w_\circ) = \Phi^+$ if W is finite.





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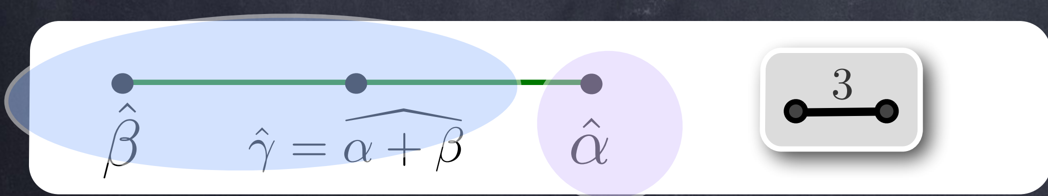

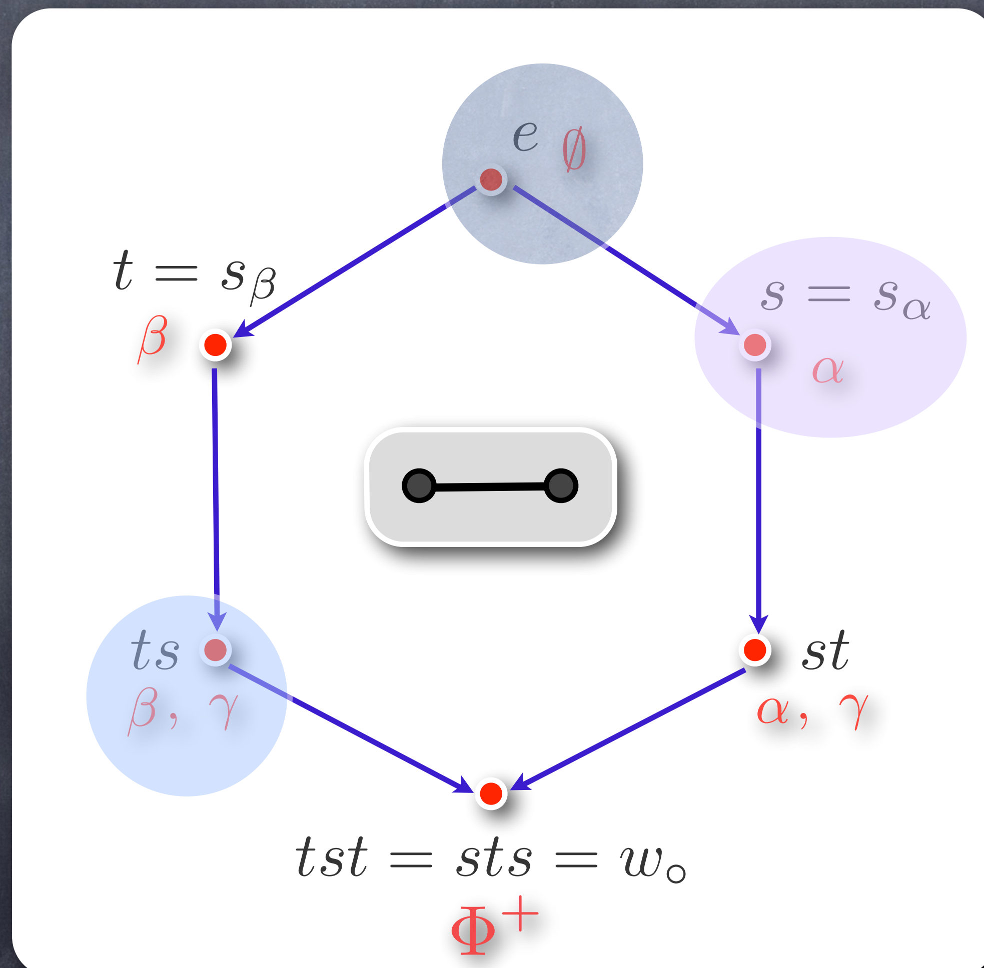

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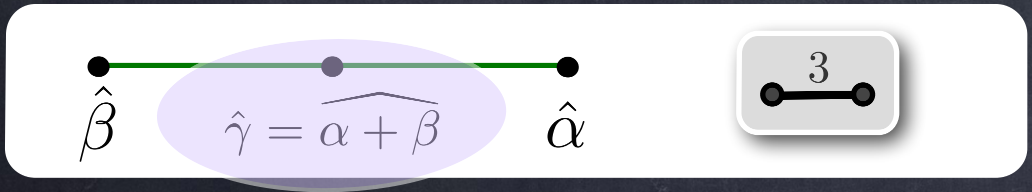
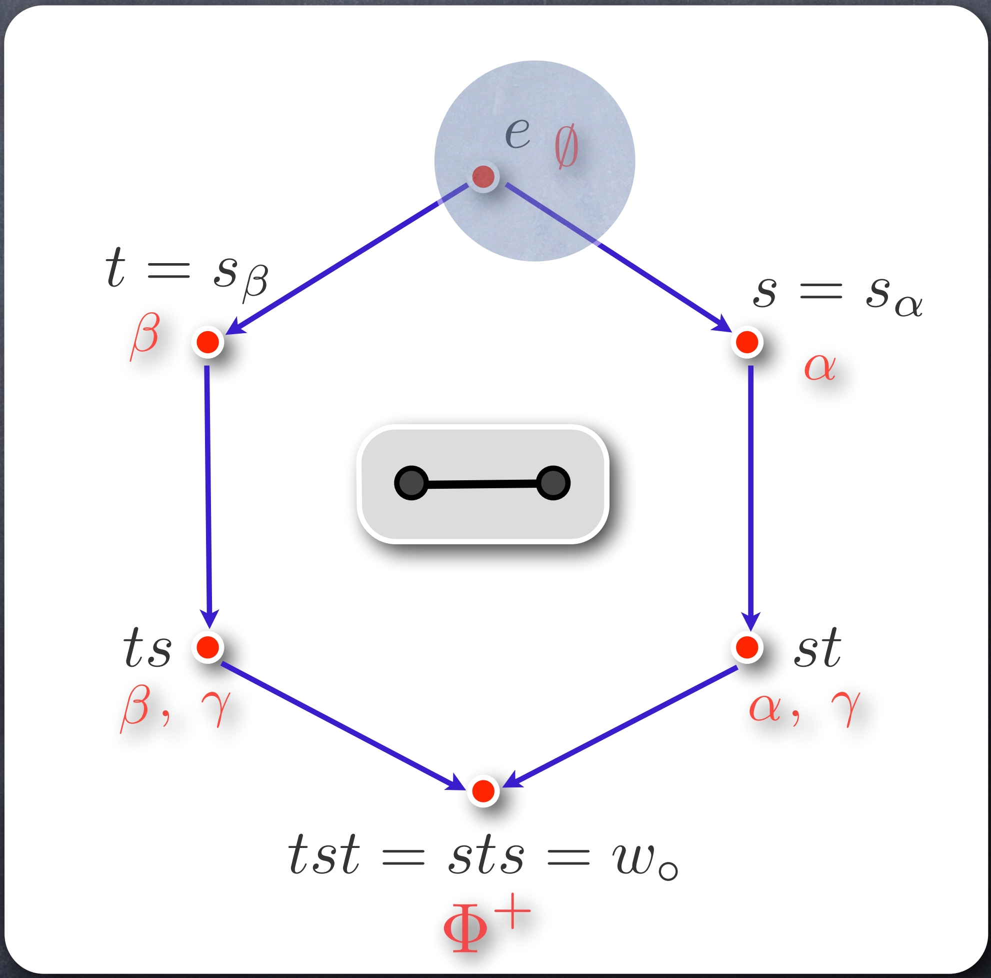


closed not bicolsed



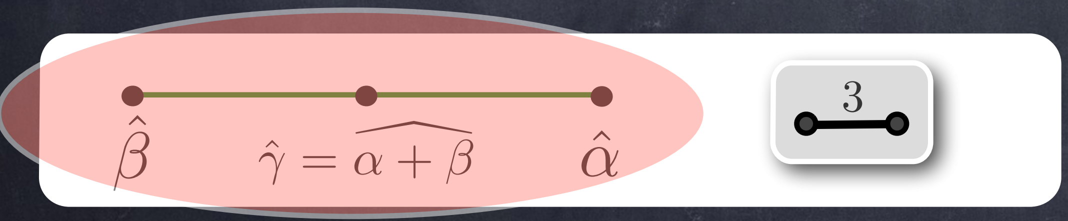
Weak order and roots

- $A \subseteq \Phi^+$ is closed if for all $[\hat{\alpha}, \hat{\beta}] \cap \hat{\Phi} \subseteq \hat{A},\ \forall \alpha, \beta \in A$;
- $A \subseteq \Phi^+$ is biclosed if $A,\ A^c := \Phi^+ \setminus A$ are closed.
- $\mathcal{B}(W) = \{\text{biclosed sets}\}$; $\mathcal{B}_0(W) = \{A \subseteq \mathcal{B}(W) \mid |A| < \infty\}$

Proposition. The map

$$N : (W, \leq) \to (\mathcal{B}_0(W), \subseteq)$$

is a poset isomorphism and $N(w_\circ) = \Phi^+$ if W is finite.



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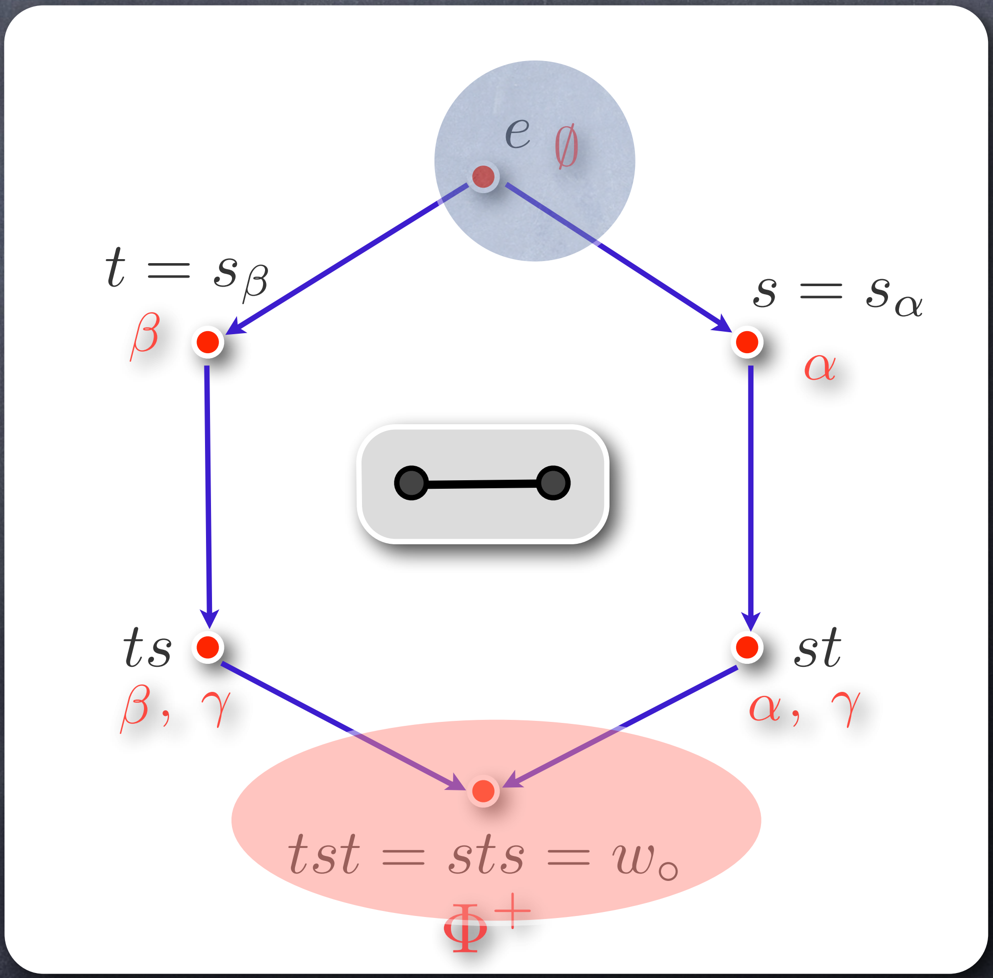

Weak order and roots

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- $\mathcal{B}(W) = \{\text{biclosed sets}\}$; $\mathcal{B}_0(W) = \{A \subseteq \mathcal{B}(W) \mid |A| < \infty\}$

Proposition. The map

Inverse map (recursive construction)

$\exists \alpha \in \Delta \cap A$, $s_\alpha(A \setminus \{\alpha\})$ is finite biclosed and

$$A = \{\alpha\} \sqcup s_\alpha(A \setminus \{\alpha\})$$

$$w_A = s_\alpha w_{s_\alpha(A\setminus\{\alpha\})}$$

Weak order and root system

world of words

If W is finite, then:

(i) a unique $w_\circ \in W$ s.t $u \leq w_\circ,\ \forall u \in W$

(ii) $w \mapsto ww_\circ$ is a poset antiautomorphism.

(iii) the weak order is a complete lattice.

(iv) $u \wedge v = (uw_\circ \vee vw_\circ)w_\circ$

$\overset{N}{\longleftrightarrow}$

world of roots

If W is finite, then:

(i) $N(w_\circ) = \Phi^+$ and $A \subseteq \Phi^+,\ \forall A \in \mathcal{B} = \mathcal{B}_0$

(ii) $A \mapsto A^c$ is a poset antiautomorphism.

(iii) the weak order is a complete lattice.

(iv) $A \wedge B = (A^c \vee B^c)^c$

Weak order and root system

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(iii) the weak order is a complete lattice.

(iv) $A \wedge B = (A^c \vee B^c)^c$

Conjecture (M. Dyer, 2011). $(\mathcal{B}, \subseteq)$ is a complete lattice (with minimal element $\emptyset$ and maximal element Φ^+).

- $\vee \neq \cup$; $\wedge \neq \cap$ so how to understand them geometrically?
- if the join $\vee$ exists then

$$A \wedge B = (A^c \vee B^c)^c$$

Weak order and Bruhat order

Bruhat order: transitive closure of $w \leq_B ws_\beta$ if $\ell(w) < \ell(ws_\beta)$

Bruhat graph of $W = \langle S \rangle$

- vertices W
- edges $w \xrightarrow{\beta} ws_\beta$

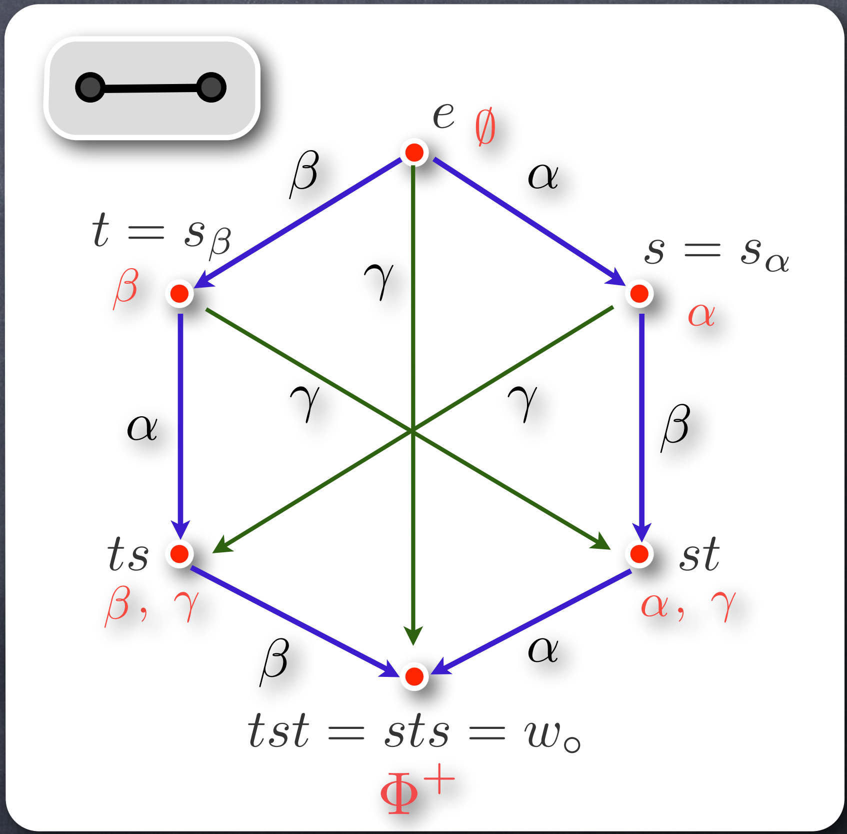
Weak order implies Bruhat order.

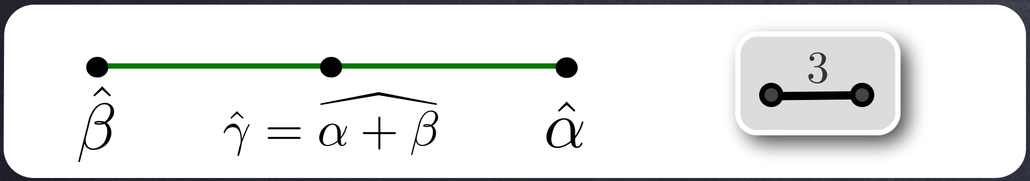
A- path: path starting with e in the Bruhat graph and indexed by elements in $A \cup B$.

Exemple. $A = \{\alpha, \gamma\}$:

$e \to w_\circ = s_\gamma$

$e \to s \to ts$





Weak order and Bruhat order

B-closure of $A \subseteq \Phi^+$: $\overline{A} = \{\beta \in \Phi^+ \mid s_\beta \text{ is in a } A - \text{path}\}$

Conjecture (M. Dyer).
Let A, B be biclosed sets, then

$$A \vee B = \overline{A \cup B}$$

This conjecture is open even in finite cases!

Exemple. $A = \{\alpha, \gamma\}$:

$$e \to w_\circ = s_\gamma$$
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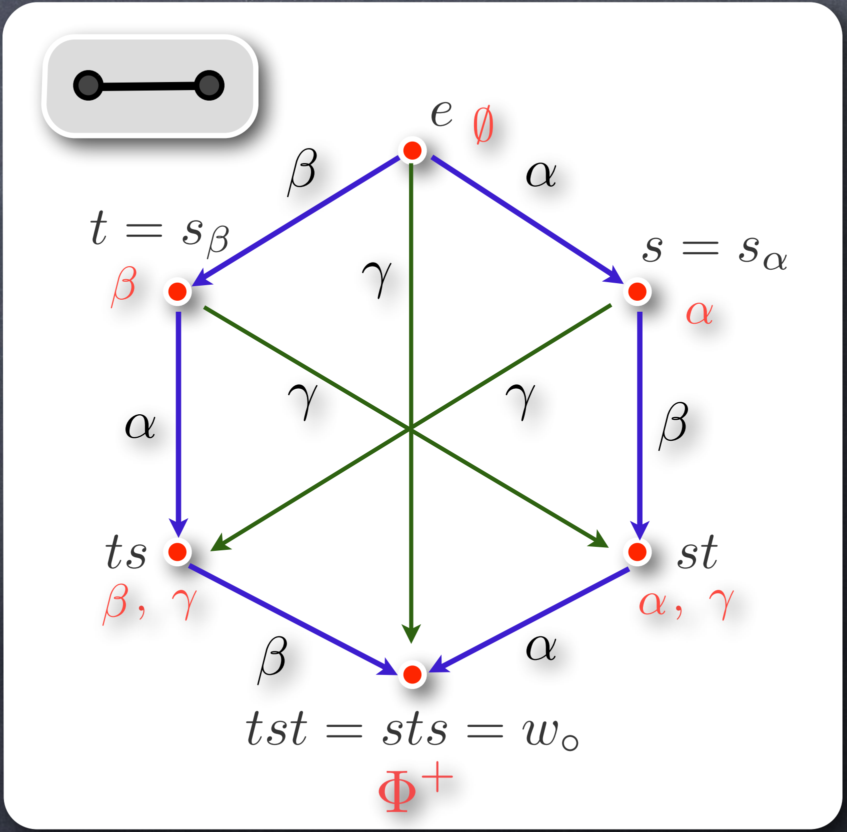
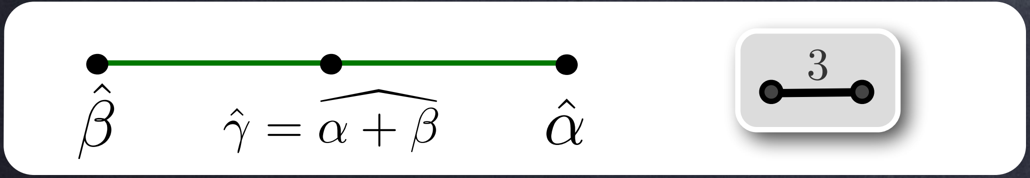
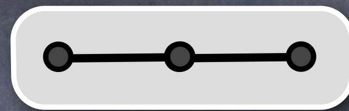




Weak order and Bruhat order

Another example: (W,S) is 

$\tau_1 \quad \tau_2 \quad \tau_3$

$A = N(\tau_1\tau_2) = \{\alpha_1, \tau_1(\alpha_2)\} = \{\alpha_1, \alpha_1+\alpha_2\}; \quad s_{\alpha_1+\alpha_2} = \tau_1\tau_2\tau_1$

$B = N(\tau_3) = \{\alpha_3\}$

$A \cup B = \{\alpha_1, \alpha_3, \alpha_1+\alpha_2\}$

$$\tau_1\tau_2 \vee \tau_3 = \tau_1\tau_3\tau_2\tau_3$$

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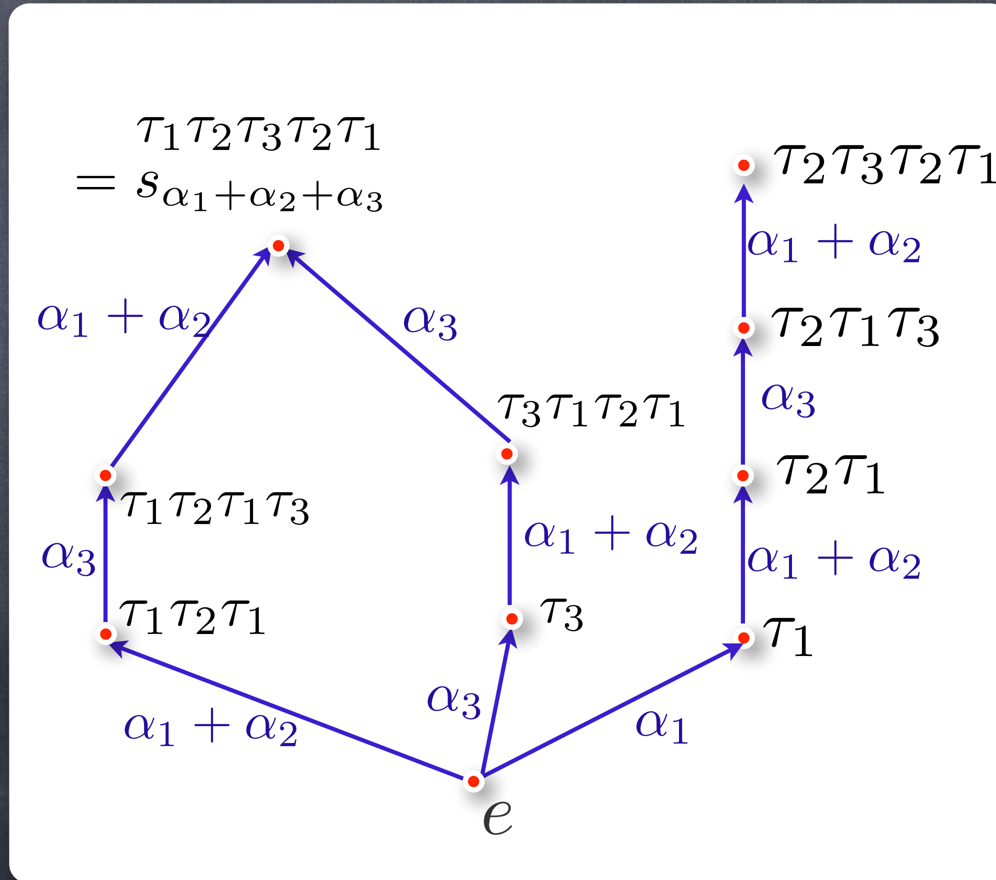
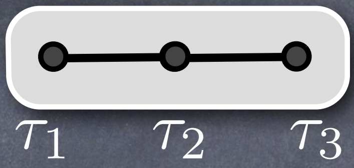


Graph of $A \cup B$ paths

Weak order and Bruhat order

Another example: (W,S) is



$$A = N(\tau_1\tau_2) = \{\alpha_1, \tau_1(\alpha_2)\} = \{\alpha_1, \alpha_1+\alpha_2\}; \quad s_{\alpha_1+\alpha_2} = \tau_1\tau_2\tau_1$$

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Conjecture (M. Dyer).
Let A, B be biclosed sets, then

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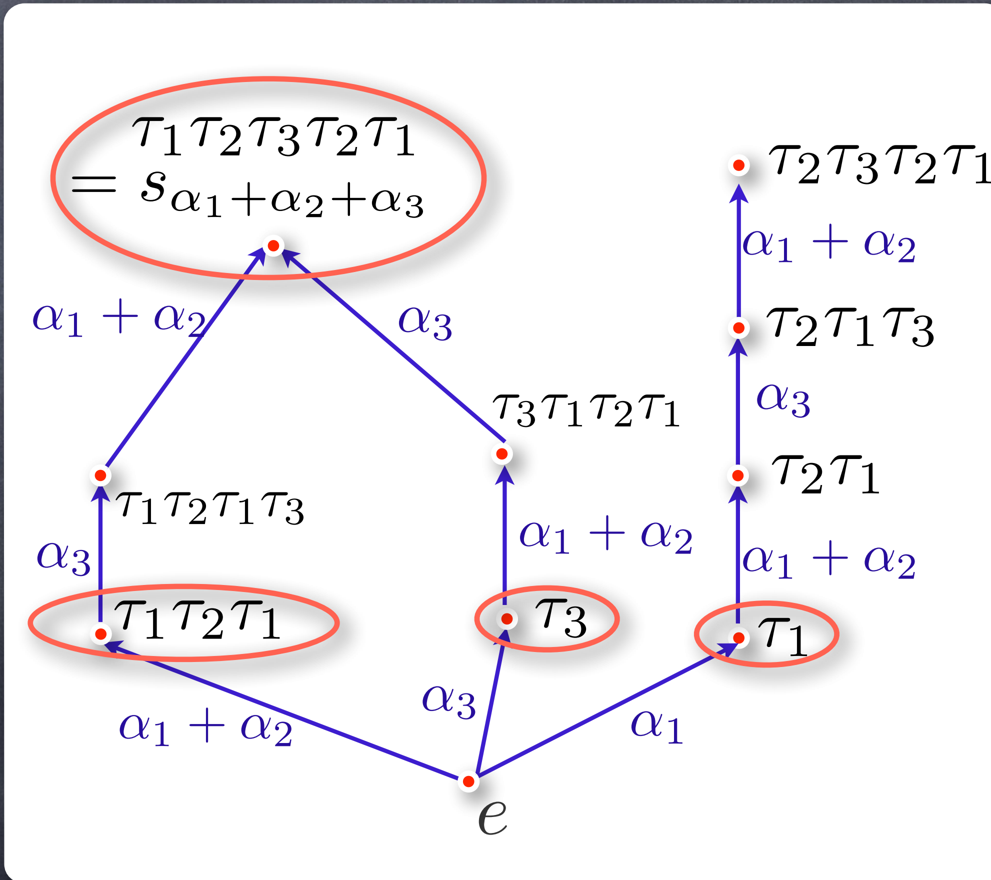


$$\overline{A \cup B} = \{\alpha_1, \alpha_3, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3\} = N(\tau_1\tau_3\tau_2\tau_3)$$

Ar

A

B

$A \cup$

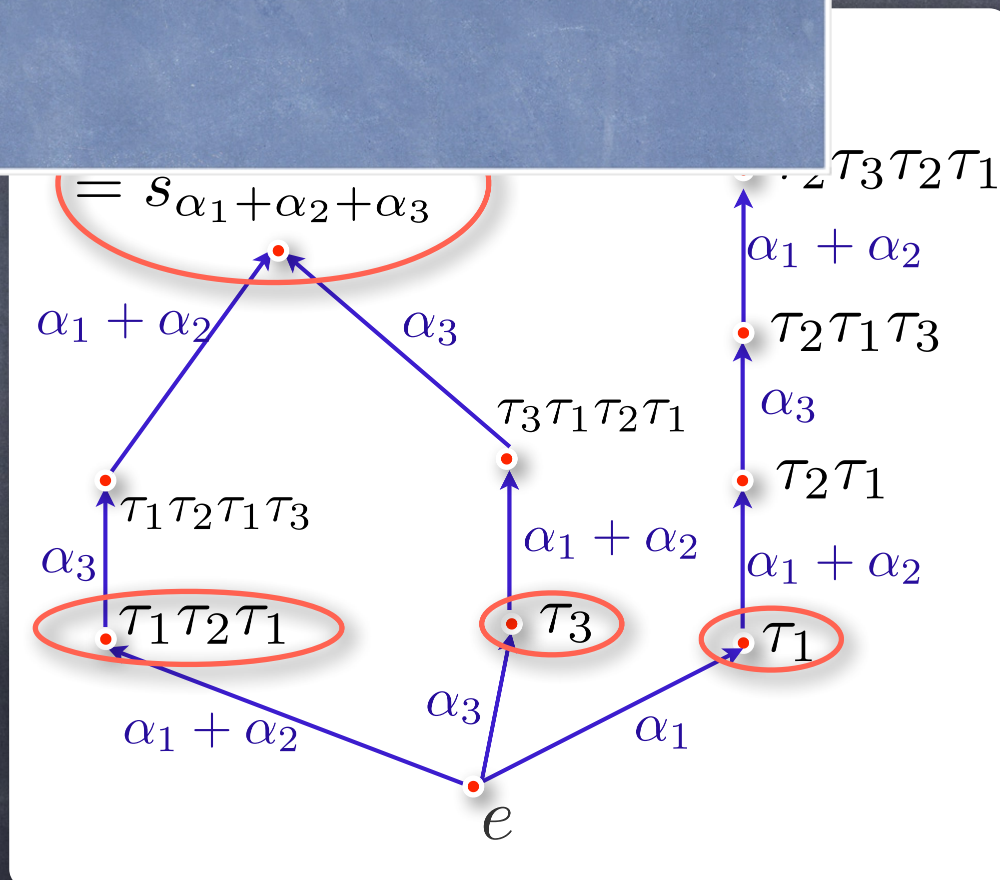
Another way to interpret the join?

$\tau_1\tau_2 \vee \tau_3 = \tau_1\tau_3\tau_2\tau_3$

Conjecture (M. Dyer).
Let A, B be biclosed sets, then

$$A \vee B = \overline{A \cup B}$$

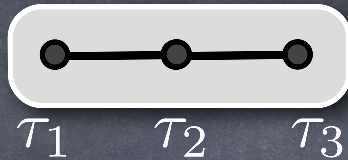
This conjecture is open even in finite cases!



$$\overline{A \cup B} = \{\alpha_1, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\} = N(\tau_1\tau_3\tau_2\tau_3)$$

Join in finite Coxeter groups

Example: (W, S) is



$A = N(\tau_1\tau_2) = \{\alpha_1, \tau_1(\alpha_2)\} = \{\alpha_1, \alpha_1 + \alpha_2\}$; $B = N(\tau_3) = \{\alpha_3\}$

$\tau_1\tau_2 \vee \tau_3 = \tau_1\tau_3\tau_2\tau_3$; $N(\tau_1\tau_3\tau_2\tau_3) = \{\alpha_1, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\}$

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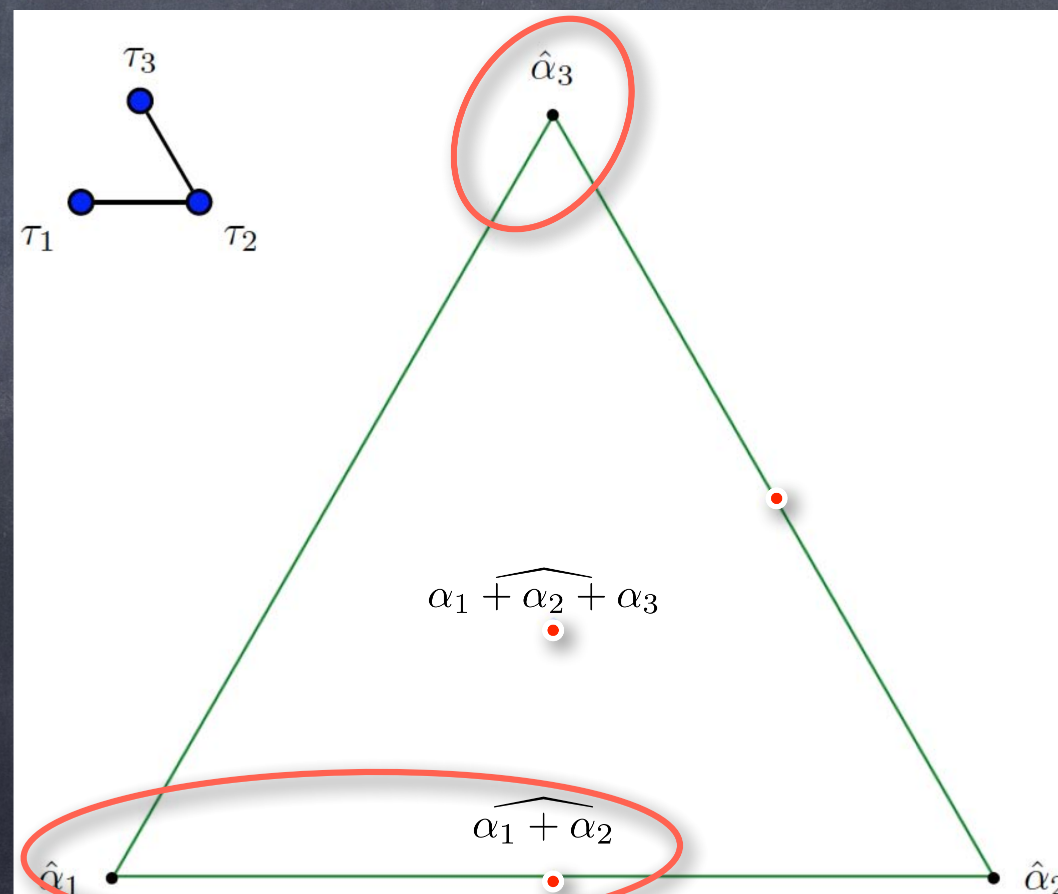

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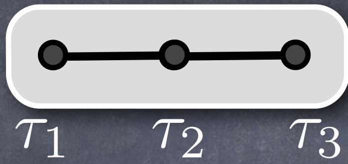
Example: (W,S) is 

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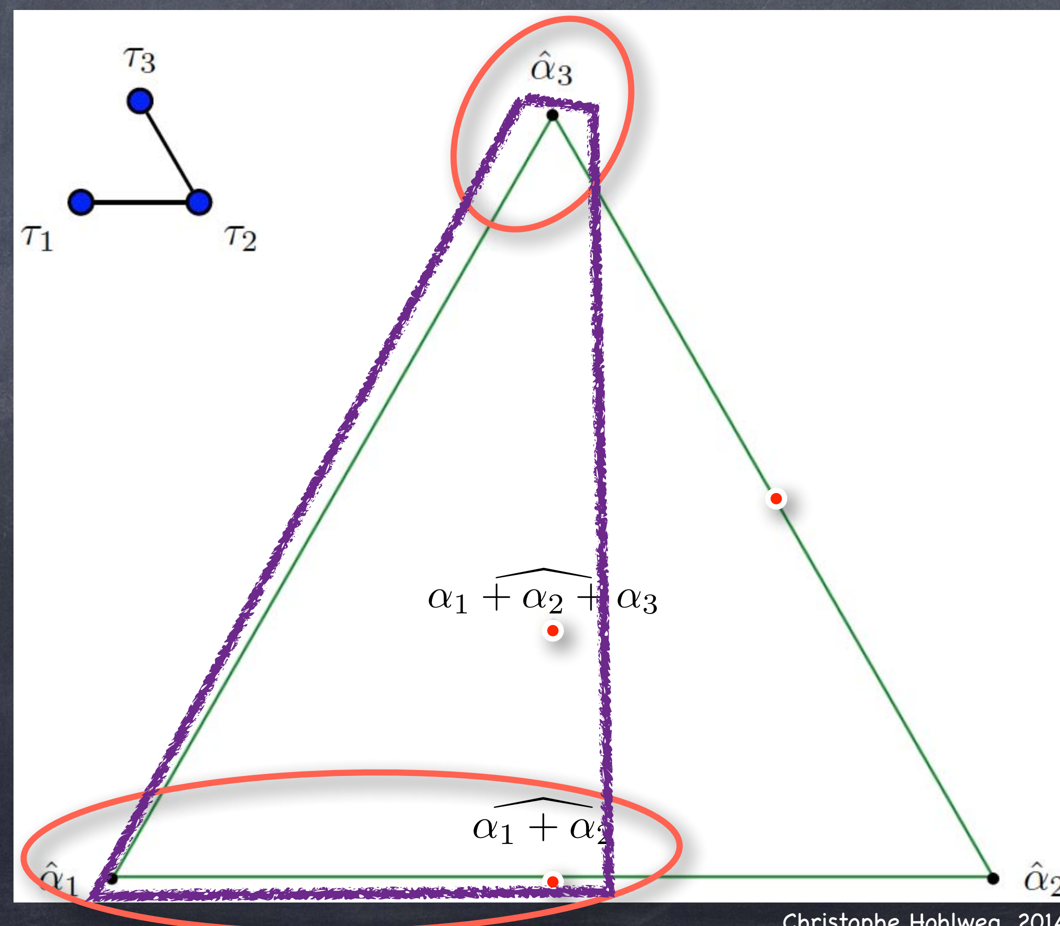
$$A \cup B = \{\alpha_1, \alpha_3, \alpha_1+\alpha_2\}$$

$$\hat{A} \vee \hat{B} = \mathrm{conv}(\hat{A} \cup \hat{B}) \cap \hat{\Phi}$$



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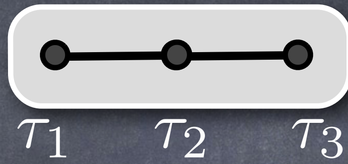
Example: (W,S) is 

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Proposition (CH, Labbé).
Let A, B be biclosed sets in a finite Coxeter group, then

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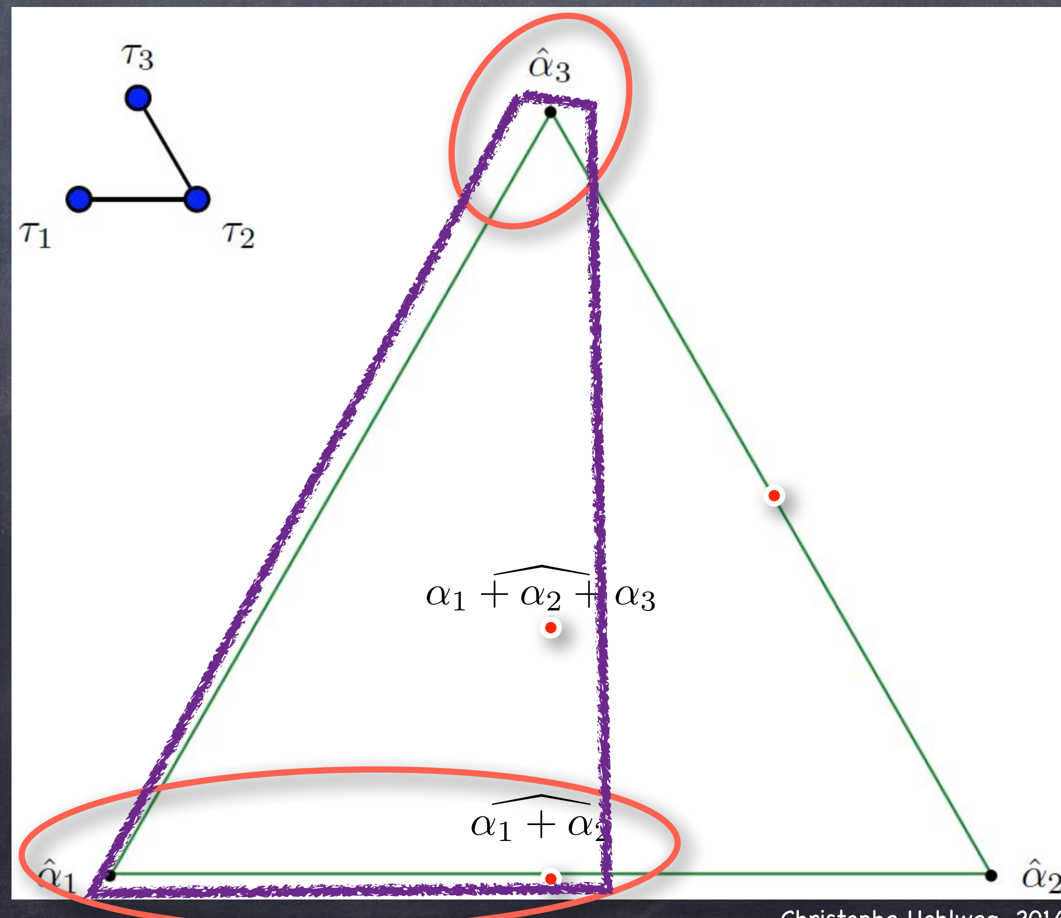

Ex

$A =$ … $_3\}$

$\tau_1\tau_2$ … $\alpha_3\}$

$A \cup$

No true in general: the convex hull of the union of biclosed is not biclosed in general

(counterexample in rank 4).

Proposition (CH, Labbé).
Let A, B be biclosed sets in a finite Coxeter group, then

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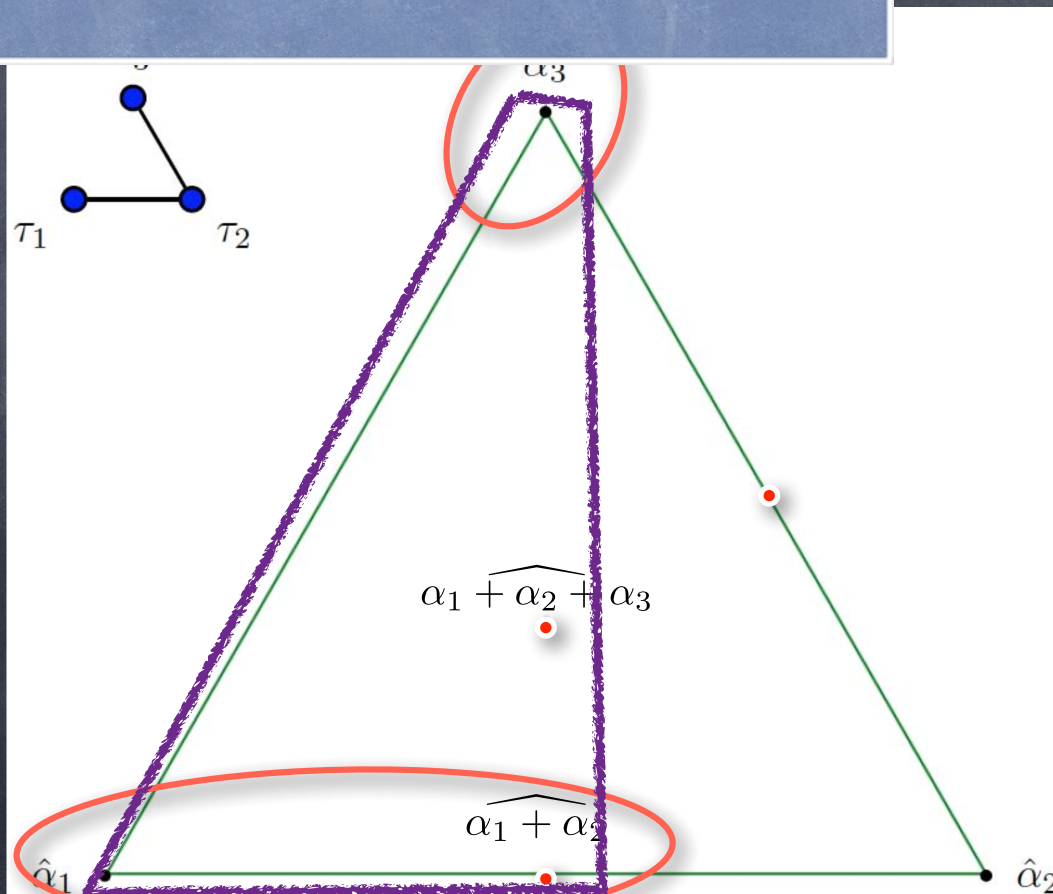


Ex N l: the convex

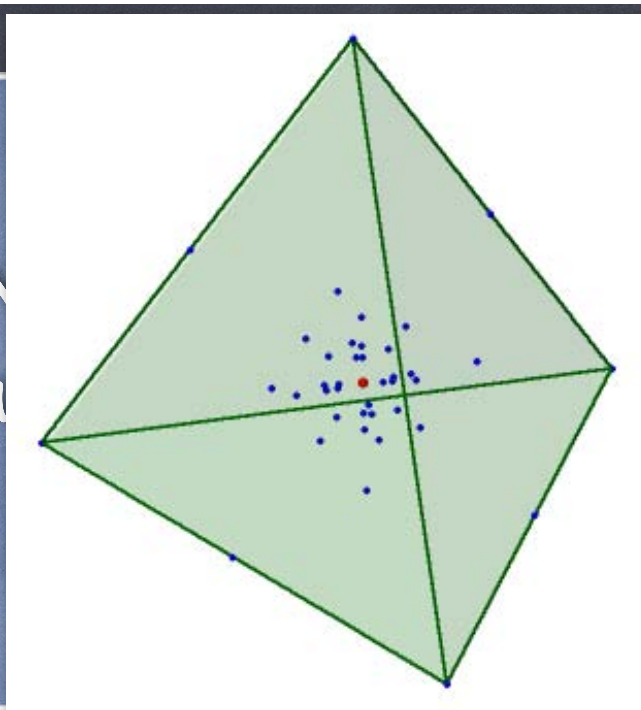
u is not biclose

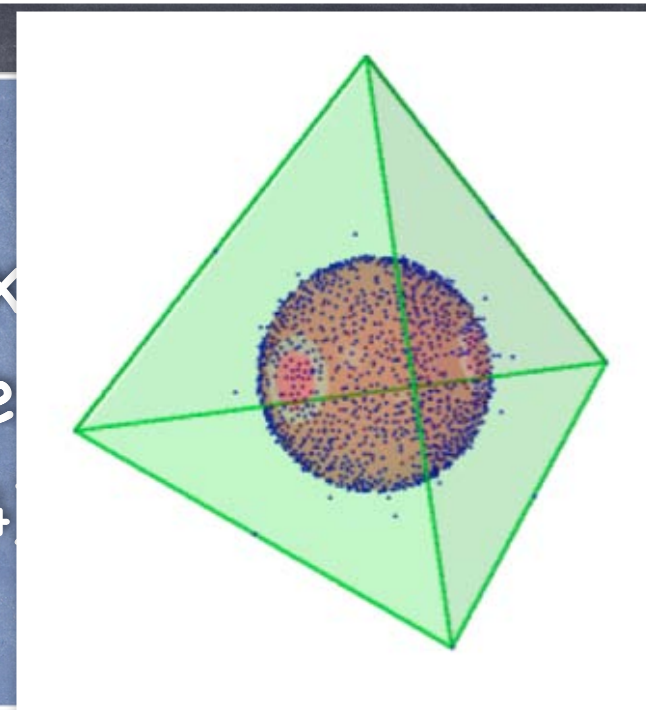
example in rank 4

$A =$

$\tau_1\tau_2$

$A \cup$





Question:

Is it possible to recognize biclosed sets??

Depth of a root is $\mathrm{dp}(\rho) = 1 + \min\{k \mid \rho = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}(\alpha_{k+1}),\ \alpha_1, \dots, \alpha_k, \alpha_{k+1} \in \Delta\}.$

If A is biclosed, properties of

$$\sum_{\beta \in A} q^{\mathrm{dp}(\beta)} \quad ?$$

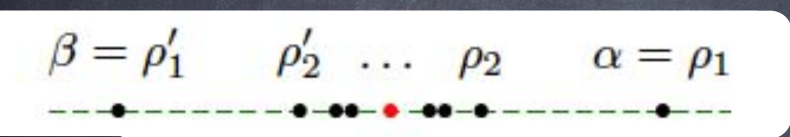
Limit roots

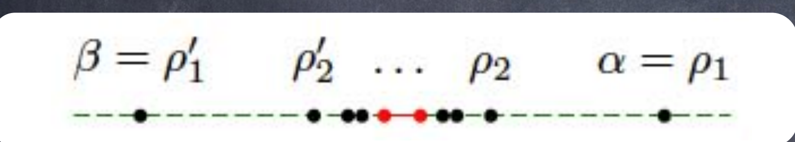
Limit roots (CH, Labbé, Ripoll 2013): the set of limit roots is:

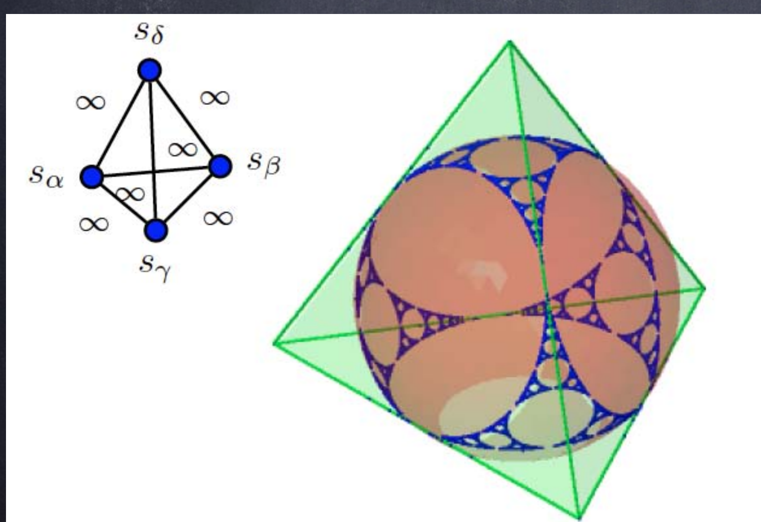
$$E(\Phi) = \mathrm{Acc}(\widehat{\Phi}) \subseteq Q \cap \mathrm{conv}(\Delta)$$

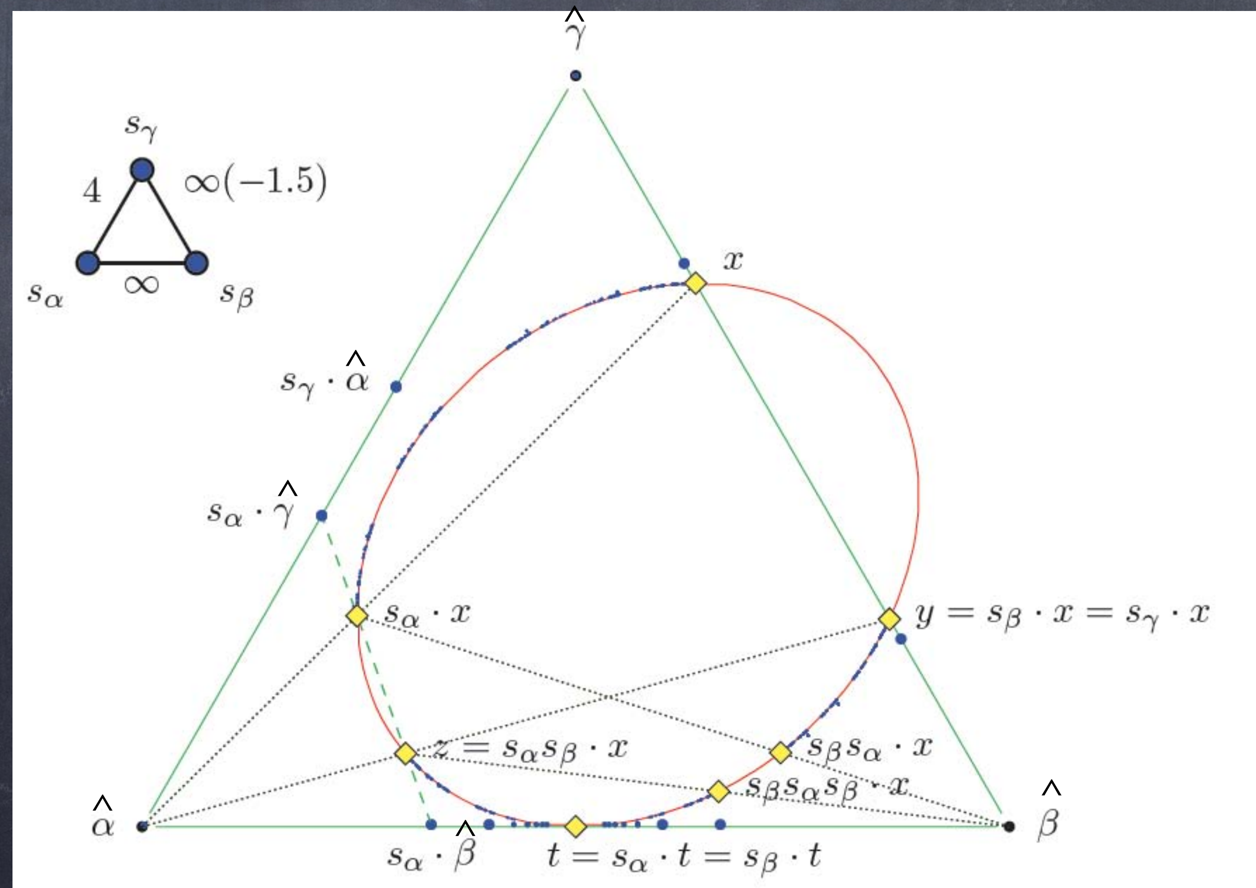
- Action of W on $\widehat{\Phi} \sqcup E$: given on E by $\widehat{Q} \cap L(\alpha, x) = \{x, s_\alpha \cdot x\}$

Remark. $E = \hat{Q}$ is a singleton in the case of affine root system.









Limit roots

Theorem (Dyer, CH, Ripoll, 2013)

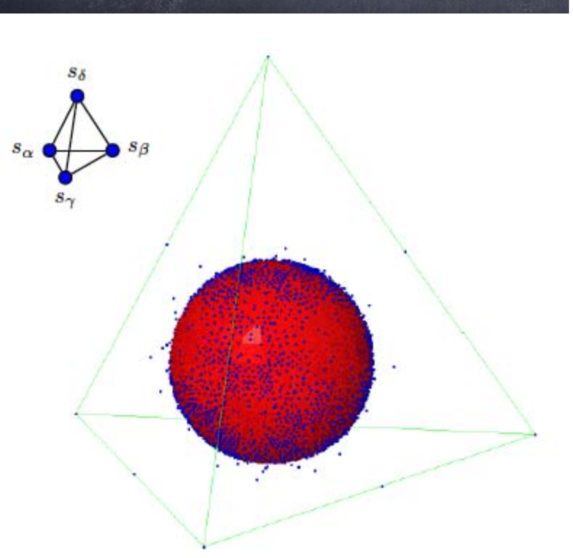
$$E = \hat{Q} \iff \hat{Q} \subseteq \mathrm{conv}(\Delta)$$

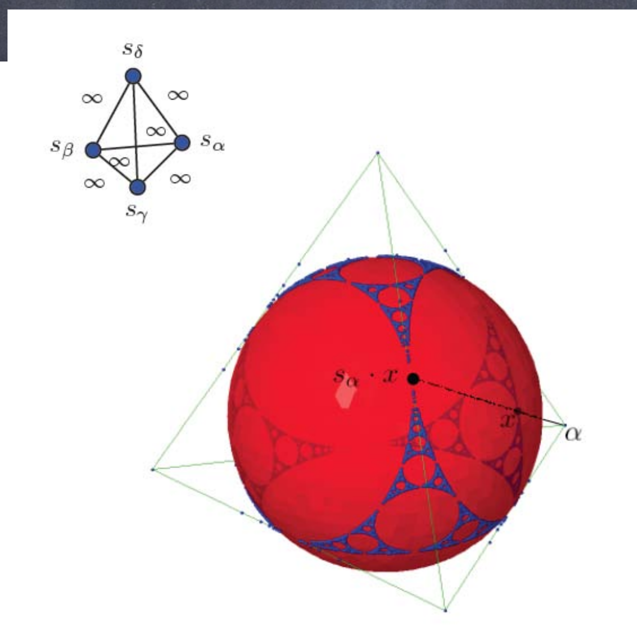
Morever, in this case,

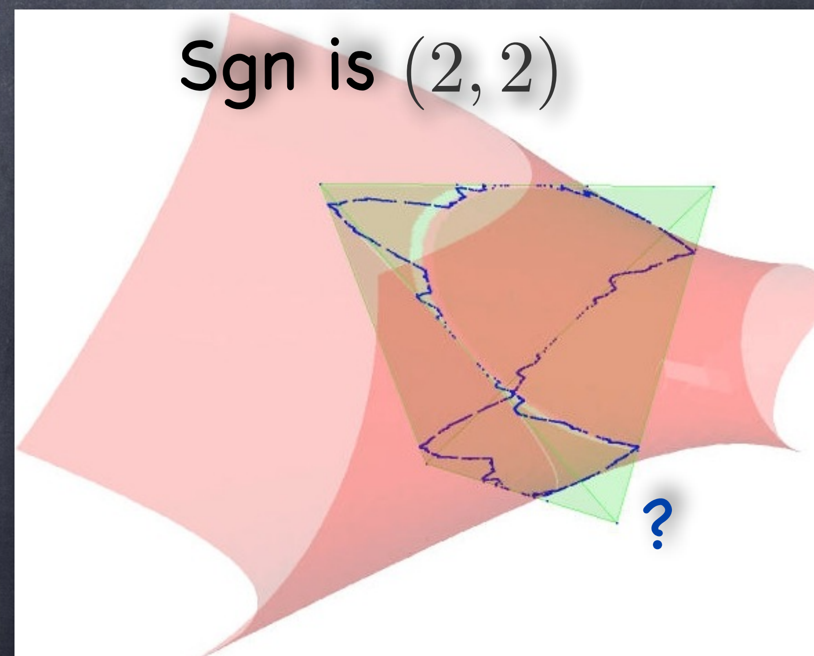
$$\mathrm{sgn}(B) = (n, 1, 0)$$

Problems: is it true for other indefinite types? Classification of Coxeter graphs for a given signature?

Theorem (Dyer, CH, Ripoll 2013) For irreducible root of signature $(n, 1, 0)$ we have: $E = \mathrm{conv}(E) \cap Q$



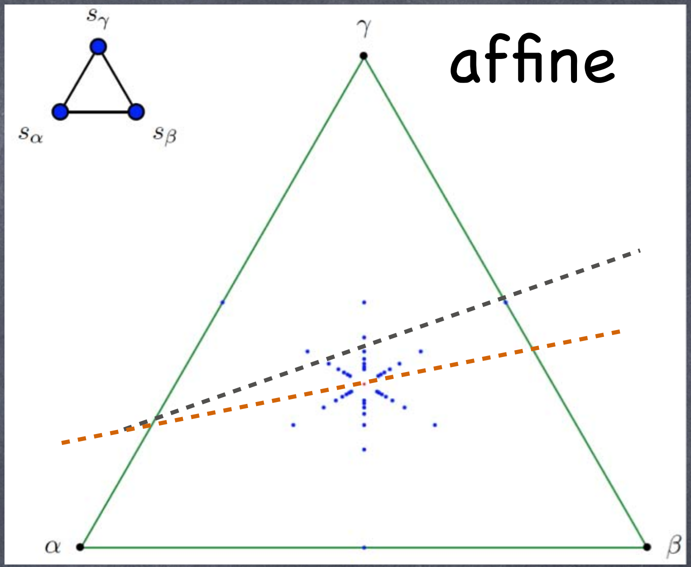


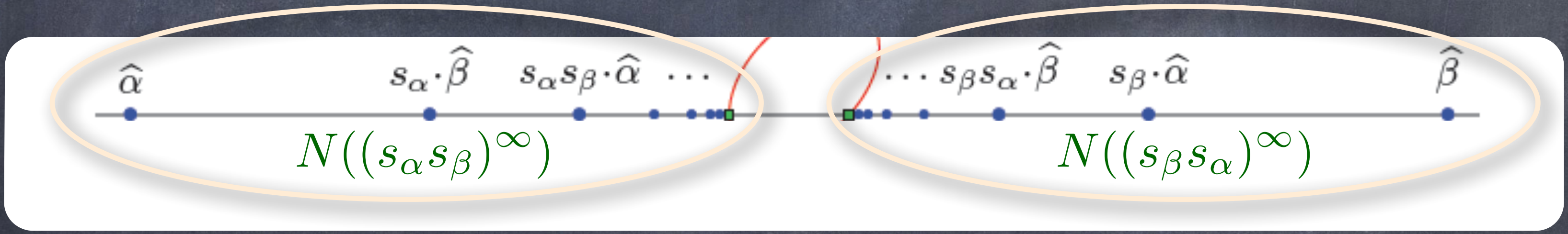


Inversion sets of infinite words

Infinite reduced words on S. For an infinite word $w = s_1 s_2 s_3 \ldots, \; s_i \in S$, write:

- $w_i = s_1 s_2 s_3 \cdots s_i$;
- $\beta_0 = \alpha_{s_1}$ and $\beta_i = w_i(\alpha_{s_{i+1}}) \in \Phi^+$.
- w is reduced if the w_i's are.
- Inversion set: $N(w) = \{\beta_i \,|\, i \in \mathbb{N}\}$.





Theorem (Cellini & Papi, 1998). Let the root system be affine, i.e., E is a singleton. Let $A \subseteq \Phi^+$, then $A = N(w)$, with w finite or infinite iff A biclosed and $\mathrm{conv}(\hat{A}) \cap E = \emptyset$ iff A is $\mathrm{conv}(\hat{A}) \cap \mathrm{conv}(\hat{A}^c) = \emptyset$ and $\mathrm{conv}(\hat{A}) \cap E = \emptyset$

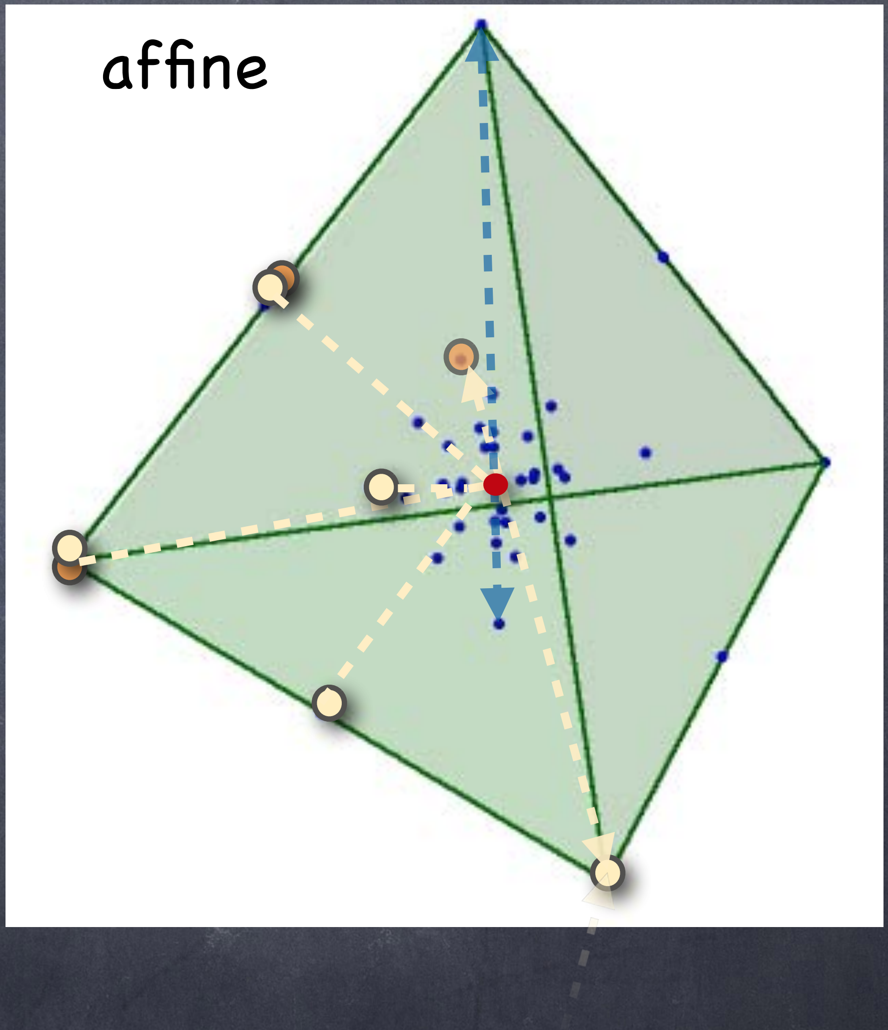
Inversion sets of infinite words

Remark. The class of $A \subseteq \Phi^+$ s.t. A or A^c verify $\mathrm{conv}(\hat{A}) \cap E = \emptyset$ is not satisfying (negative answer to a question asked by Lam & Pylyavskyy; Baumann, Kamnitzer & Tingley)

$$\hat{N}(21321) \vee \hat{N}(214) = \bullet \vee \bullet$$

$$= \mathrm{conv}(\bullet \cup \bullet) \cap \hat{\Phi}$$

does not arise as an inversion set of a word (finite or infinite)



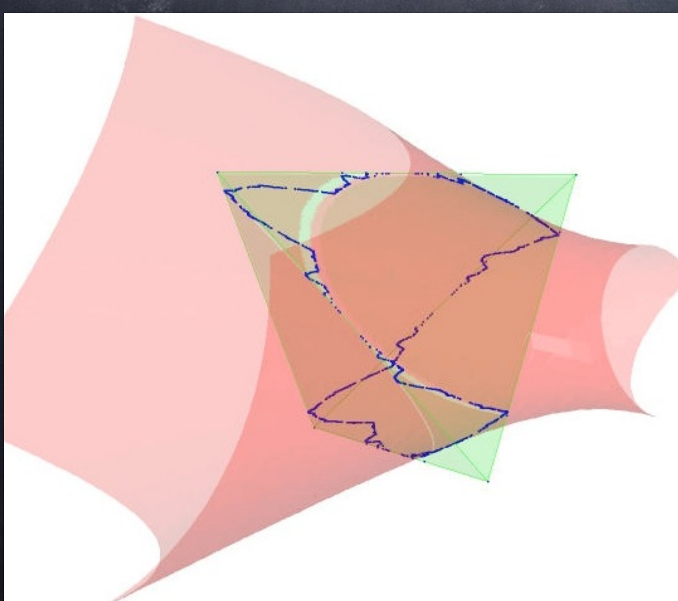
Inversion sets of infinite words (CH & JP Labbé)

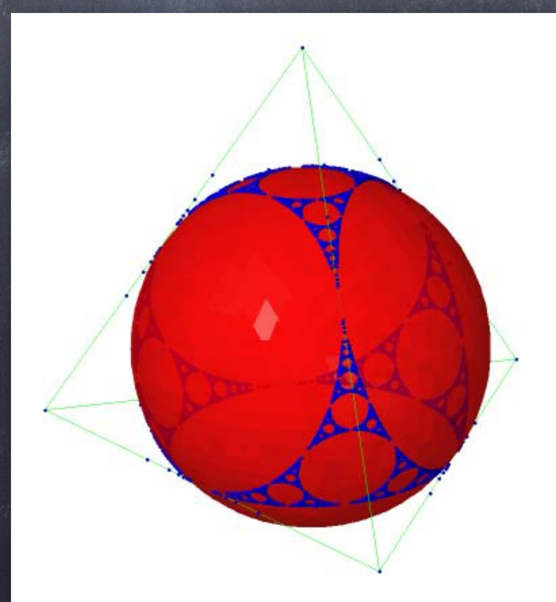
Proposition. Let the root system be arbitrary.
If $A = N(w)$ with w reduced infinite or finite word, then A is $\mathrm{conv}(\hat{A}) \cap \mathrm{conv}(\hat{A}^c) = \emptyset$ and $\mathrm{conv}(\hat{A}) \cap E = \emptyset$.

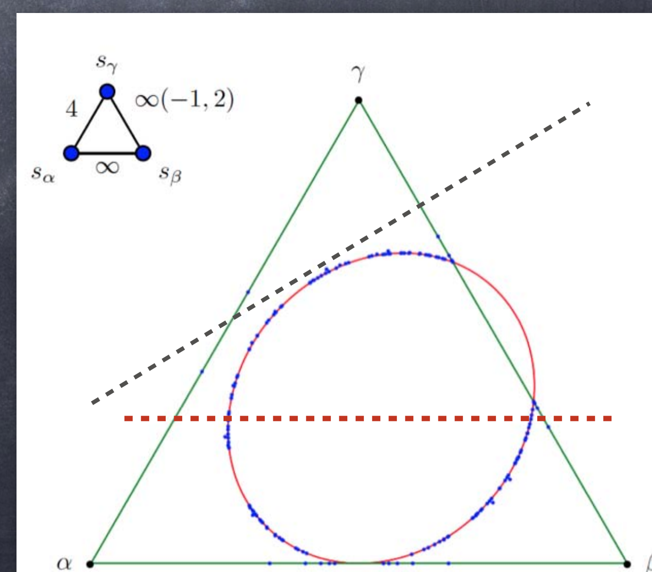
Questions:

i) Is the converse true? (true for affine by Cellini & Papi);

ii) $|\mathrm{Acc}(N(w)| \leq 1$?; obviously true for finite and affine; true for weakly hyperbolic (H. Chen & JP Labbé, 2014)







Limit roots and imaginary cone

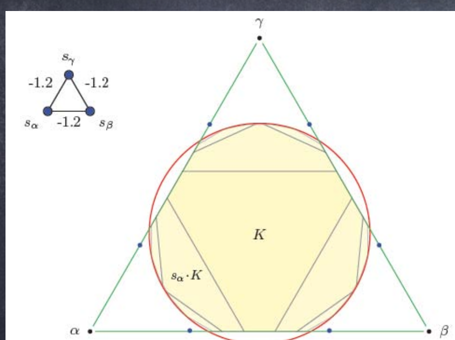
Tiling of $\mathrm{conv}(E)$

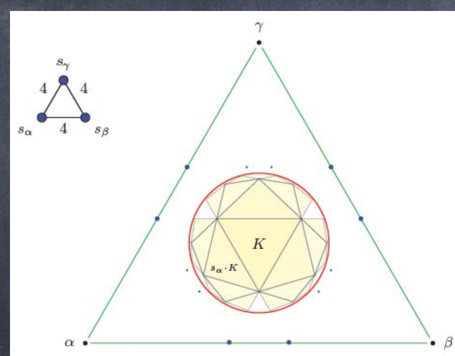
Assume the root system to be not finite nor affine

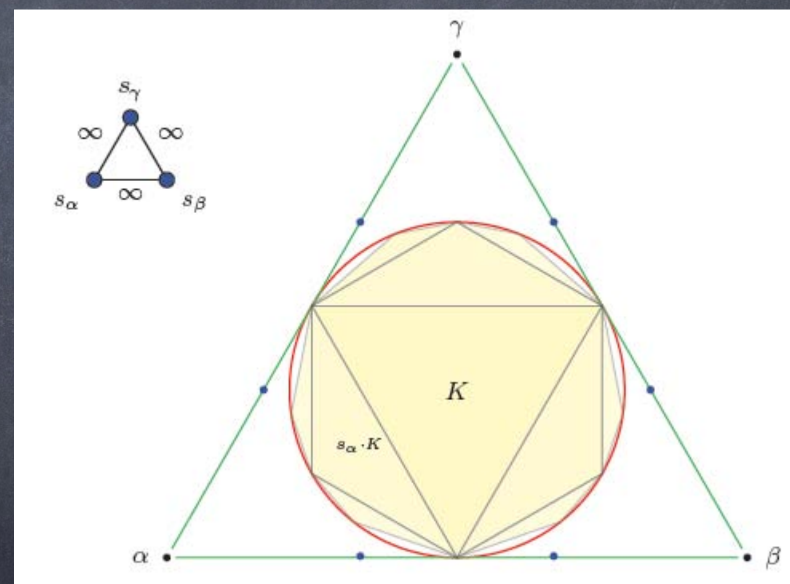
- Imaginary convex set $\mathcal{I}$ is the $W-$orbit of the polytope

$$K = \{v \in \mathrm{conv}(\Delta) \mid B(v,\alpha) \leq 0, \forall \alpha \in \Delta\}$$

Theorem (Dyer, 2012). $\overline{\mathcal{I}} = \mathrm{conv}(E)$







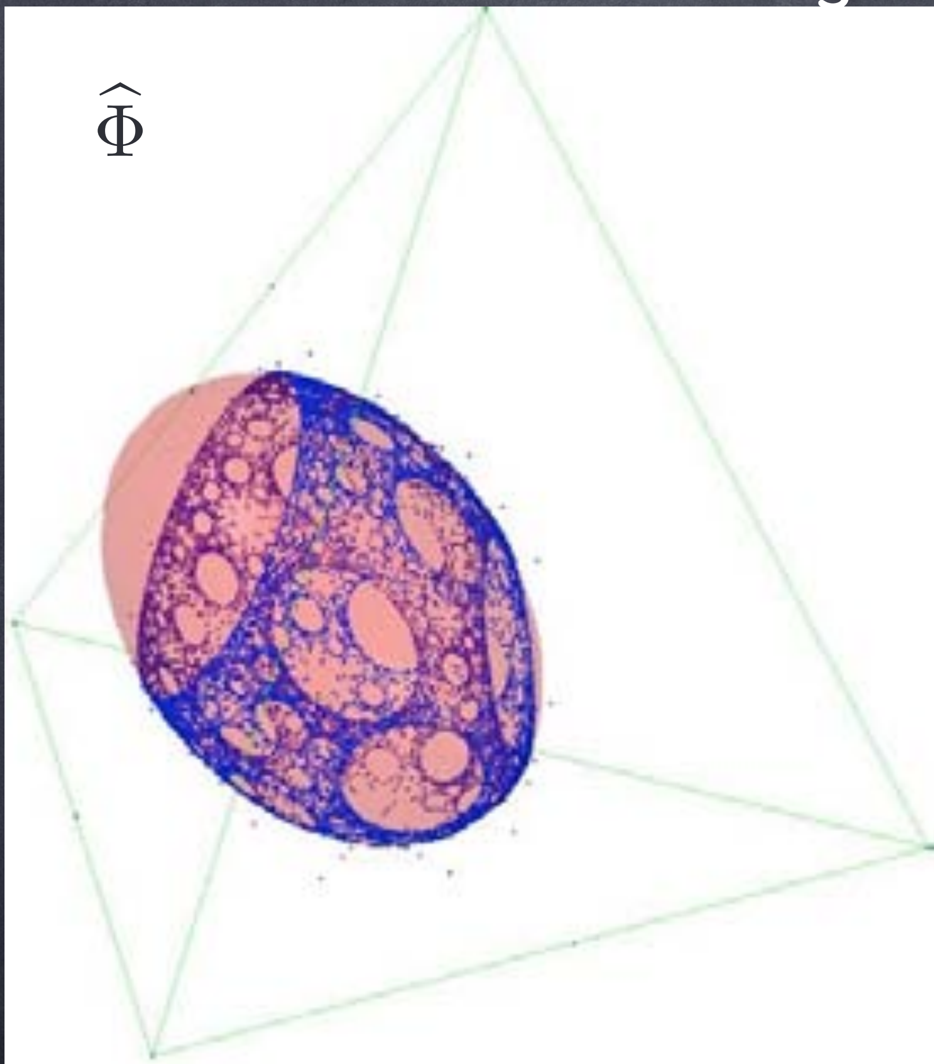
Proposition (Dyer, CH, Ripoll 2013). The action of W on E extends to an action of W on $\mathrm{conv}(E)$. So W acts on $\widehat{\Phi} \sqcup \mathrm{conv}(E)$

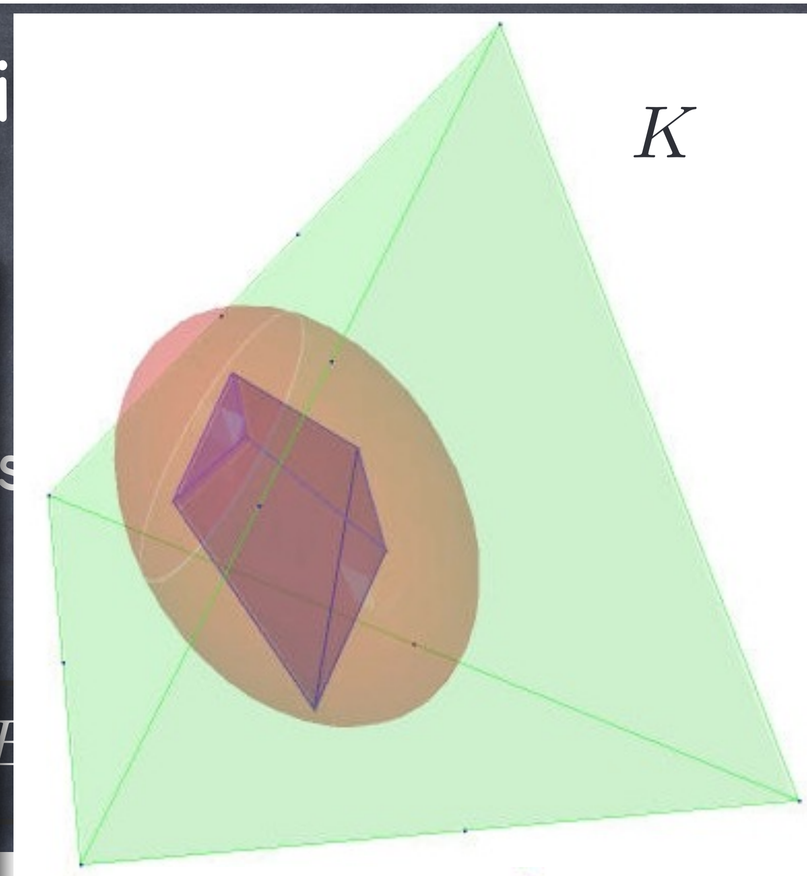
Limit roots and i…

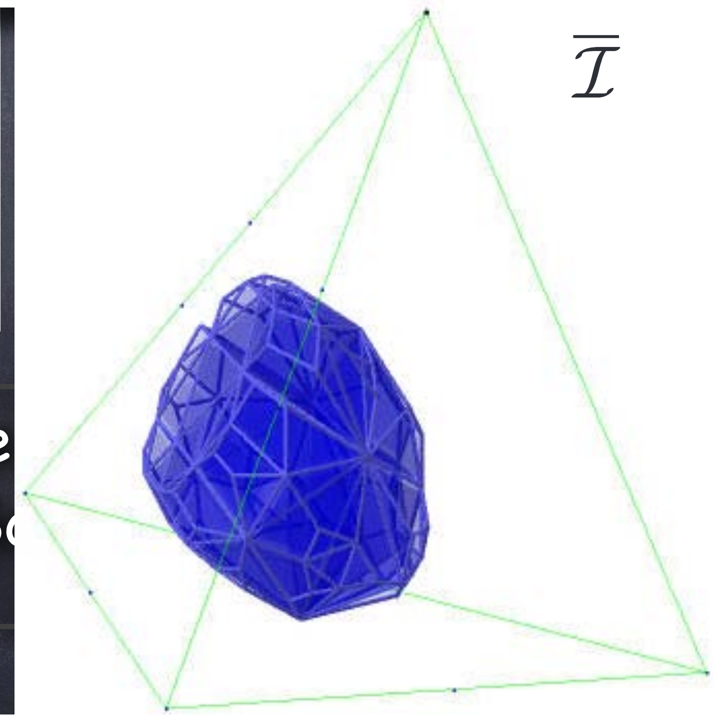
Tiling of …

…is… …ope

…E…







Proposition (Dyer, CH, Ripoll 2013). The … nds to an action of W on $\mathrm{conv}(E)$. So … (E)

Inversion sets of infinite words (CH & JP Labbé) and $\mathrm{conv}(E)$

Assume the root system to be not finite nor affine

For a reduced $w = s_1 s_2 s_3 \ldots,\ s_i \in S$, recall that:

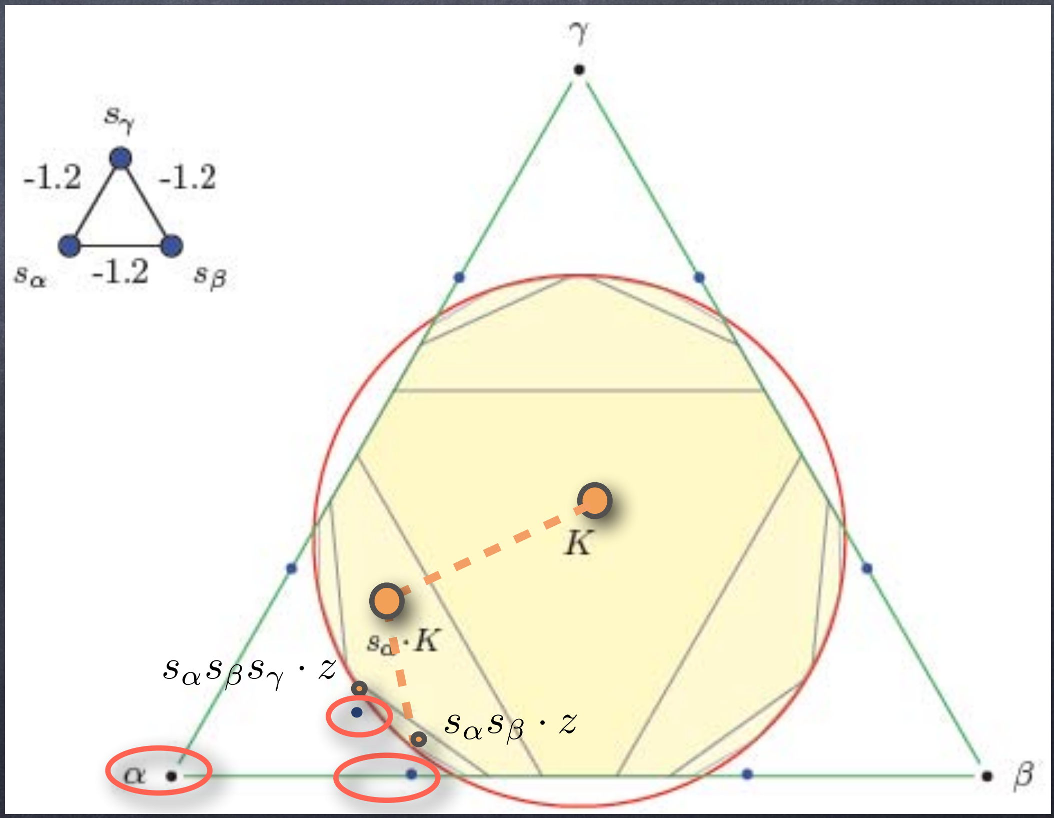
- $w_i = s_1 s_2 s_3 \cdots s_i$; reduced; $\beta_0 = \alpha_{s_1}$ and $\beta_i = w_i(\alpha_{s_{i+1}}) \in \Phi^+$.
- Inversion set: $N(w) = \{\beta_i \mid i \in \mathbb{N}\}$.

Representation in $\mathrm{conv}(E)$:

$z \in \mathrm{relint}(K)$ and $\{w_i \cdot z,\ i \in \mathbb{N}\}$

Conjecture.

$$\mathrm{Acc}(\hat{N}(w)) = \mathrm{Acc}(\{w_i \cdot z,\ i \in \mathbb{N}\})$$

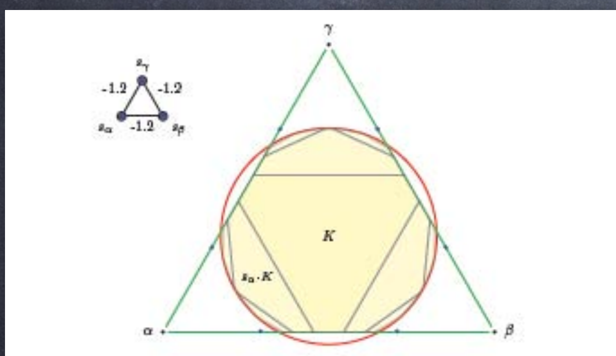


Ordre faible et cone imaginaire dans les groupes de Coxeter infinis

- Séminaire de combinatoire Philippe Flajolet -

IHP, Paris, 3 avril 2014

Christophe Hohlweg, LaCIM, UQAM
(en sabbatique à l'IRMA, Strasbourg)



FIN