Law of large numbers for matchings, extensions and applications

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INRIA & ENS

Joint works with Charles Bordenave (CNRS & Uni. Toulouse), Justin Salez (Paris 7), Mathieu Leconte (Technicolor-INRIA), Laurent Massoulié (MSR-INRIA) and Hang Zhou (ENS).

HASHING

CONTENT PLACEMENT

HASH TABLE

- m balls and n bins
- each ball chooses a bin uniformly at random
- Goal: avoid collisions.

This is known as the Birthday problem. The probability of no collision is given by

$$\begin{aligned} p(n,m) &= \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-m+1}{n}\right) \\ &\approx \exp\left(-\frac{1+2+\cdots+m-1}{n}\right) \\ &\approx \exp\left(-\frac{m^2}{2n}\right) \end{aligned}$$

To avoid collision we must have

$$p(n,m) \approx 1 \quad \Leftrightarrow \quad m << \sqrt{n}.$$

Load factor $\rho = \frac{m}{n} \to 0$ as $n \to \infty$.

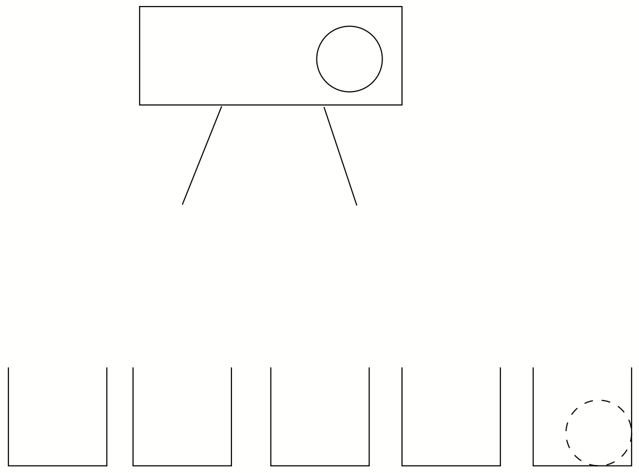
CUCKOO HASHING

Introduced by Pagh & Rodler, ESA'01:

- two bins are assigned at random to each ball

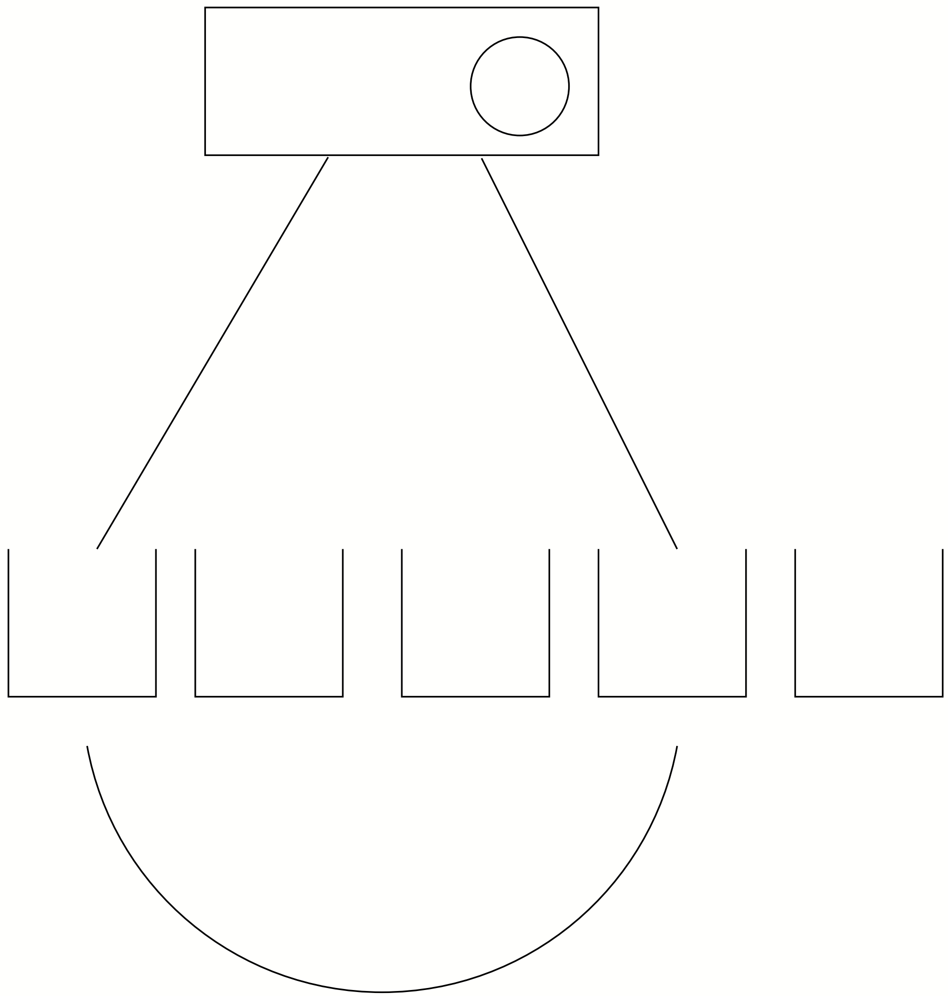
- each ball is placed in one of these two bins

- bins have capacity one, i.e. no collision allowed



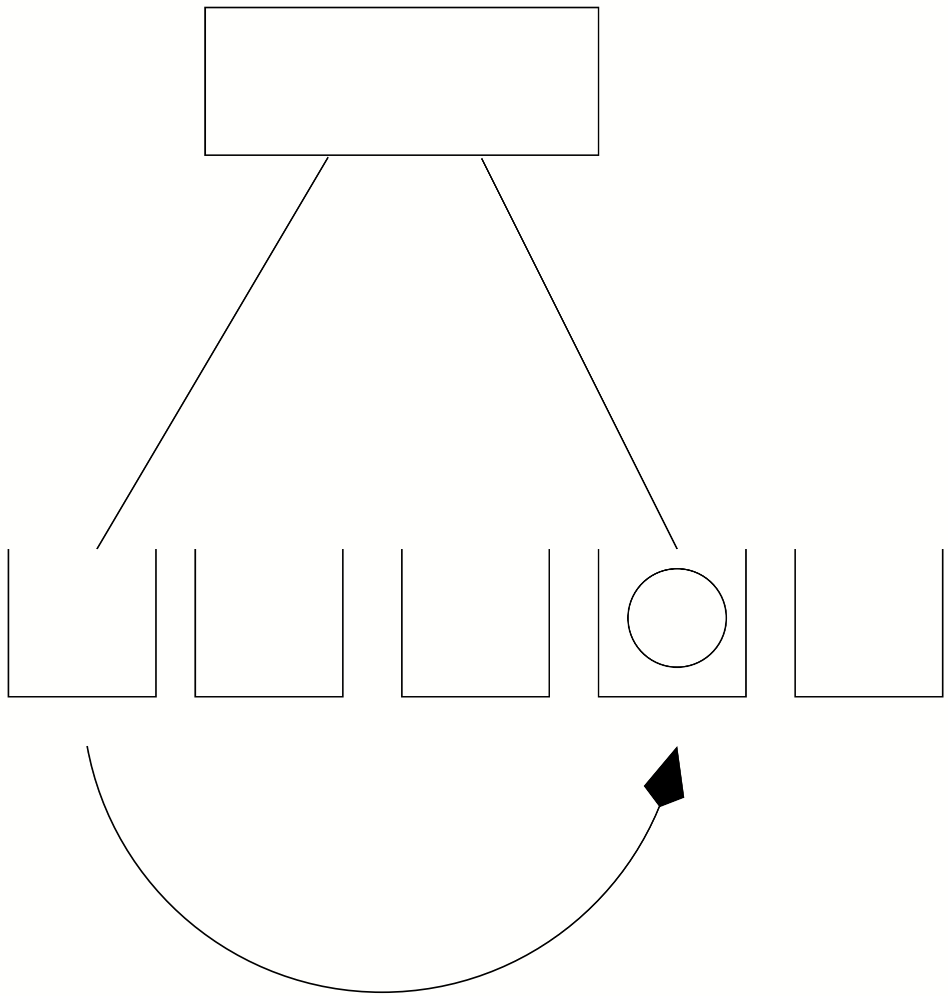
Q: How many balls m can you put into n bins with these constraints?

RANDOM GRAPH ORIENTATION



Random graph $G(n, m)$.

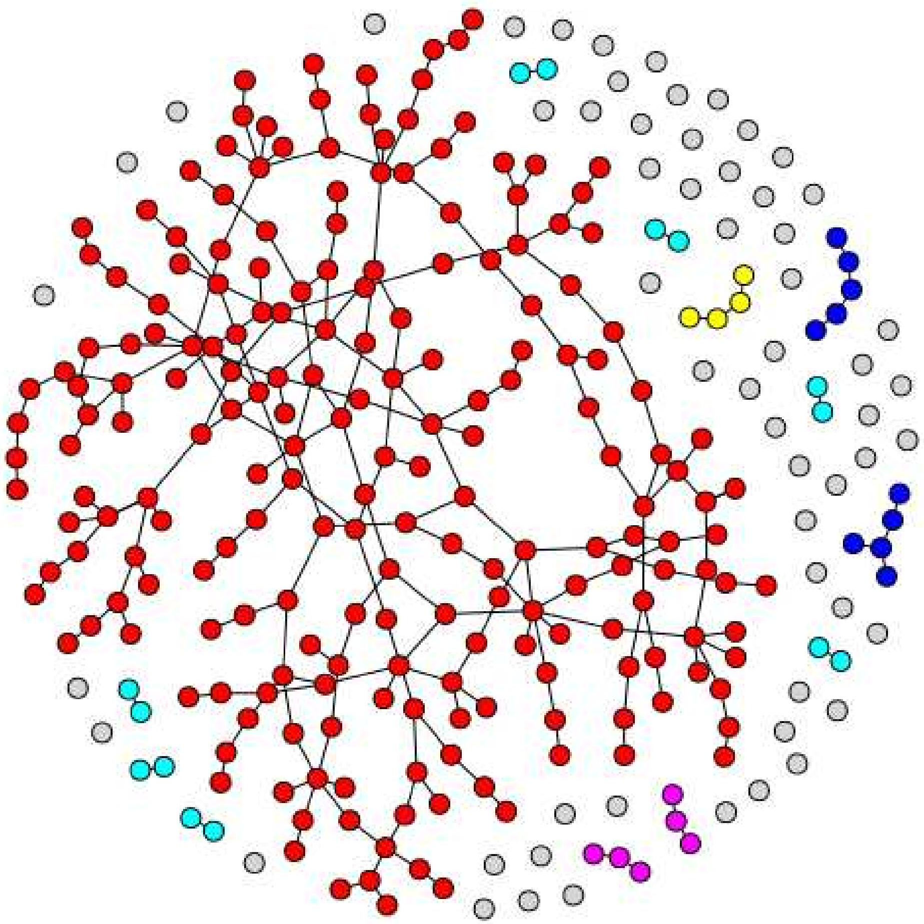
RANDOM GRAPH ORIENTATION



Q: How large can m be so that $G(n, m)$ is still orientable?

POSITIVE LOAD FACTOR

Recall that the degree is a Bin $\left(m, \frac{n-1}{\binom{n}{2}}\right)$ random variable with mean $\frac{2m}{n}$ so that if $2m > n$, there is a giant component:



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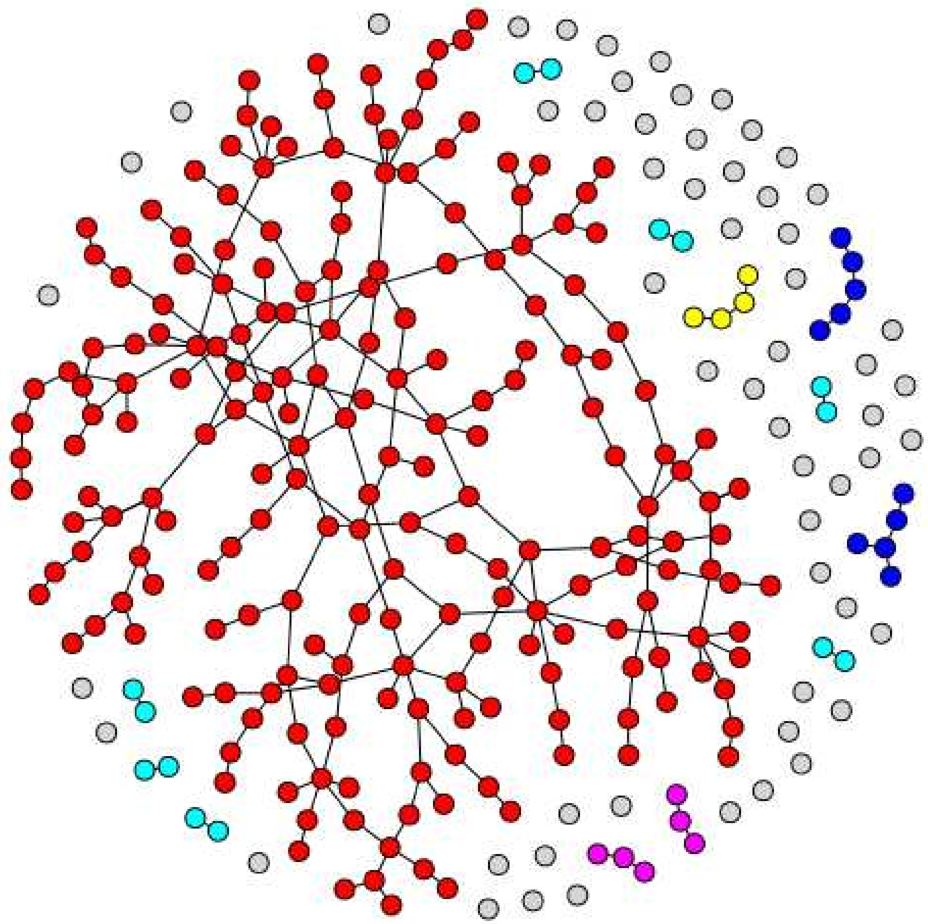
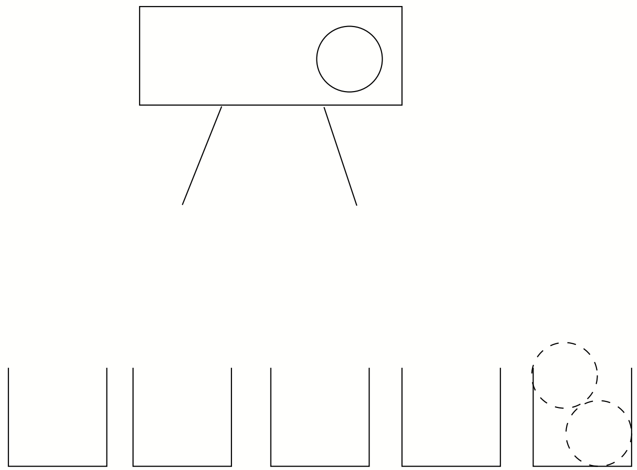


For cuckoo hashing with two choices, the critical load factor is $\rho = \frac{1}{2}$.

GENERALIZATIONS

Adding capacities to the bins $k \geq 1$:

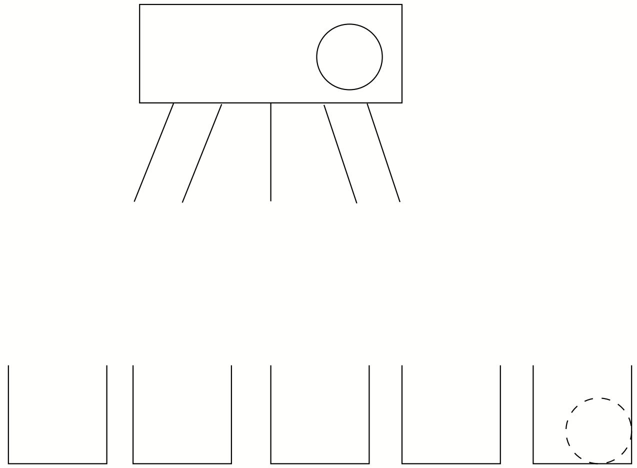


Q: k-orientation of the random graph $G(n, m)$?

Cain, Sanders, Wormald, Fernholz, Ramachandran SODA'07

GENERALIZATIONS

Adding choices for each ball $h \geq 1$:



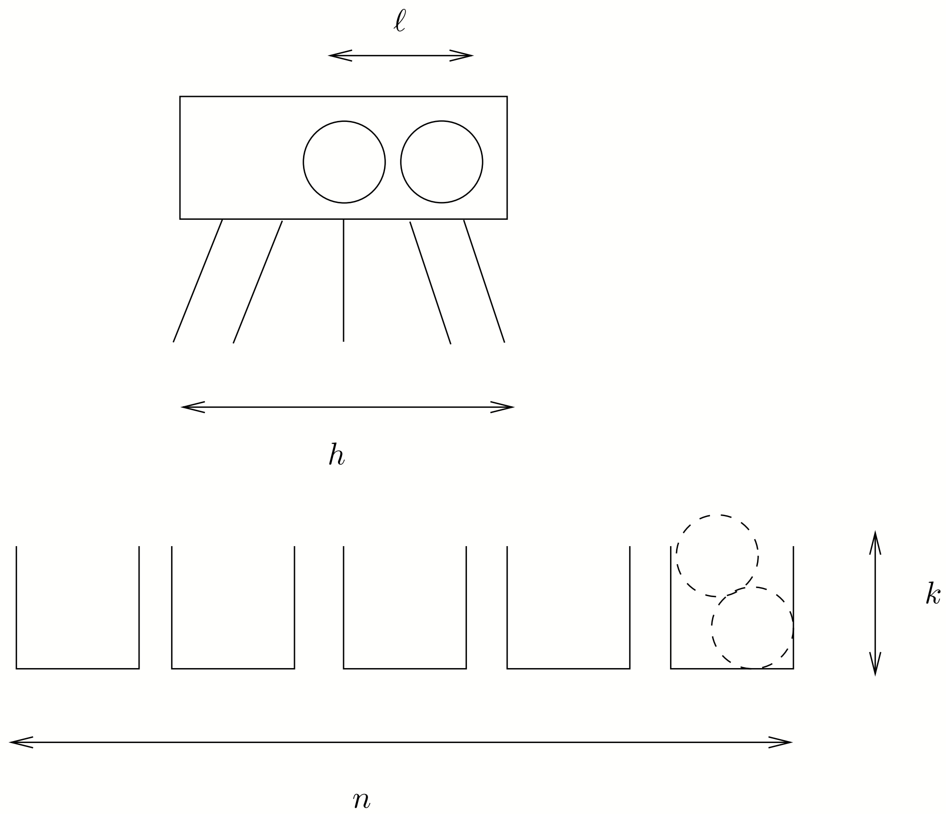
Q: 1-orientation of the random hypergraph $H(n, m, h)$?

Dietzfelbinger, Goerdt, Mitzenmacher, Montanari, Fountoulakis, Panagiotou ICALP'10

Frieze, Melsted, Bordenave, Lelarge, Salez

GENERALIZATIONS

Adding balls $h > \ell \geq 1$ proposed by Gao, Wormald STOC'10:



Case $\ell = 1$ solved by Fountoulakis, Kosha, Panagiotou SODA'11

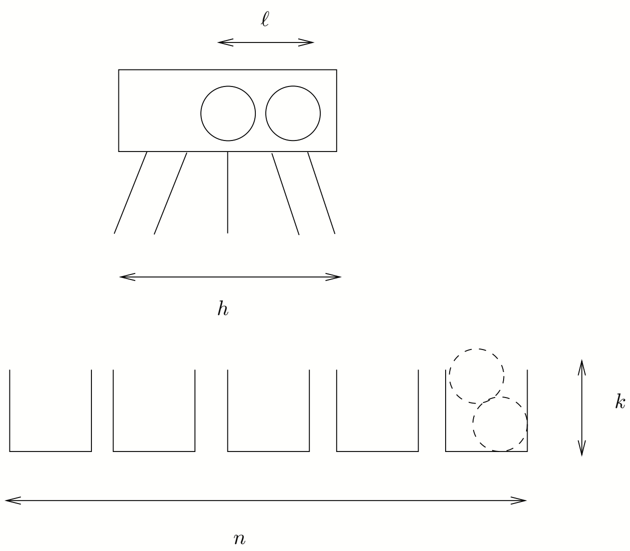
For large k, Gao, Wormald STOC'10: "The full definition of [the critical load factor] is rather complicated, involving the solution of a differential equation system given in (3.4-3.14)."

$$
\begin{aligned}
z'_{L,h-j}(x) &= \frac{z_{L,h-j}}{z_L}\left(-1-\frac{(h-j-1)z_{L,h-j}}{z_{B,h-j}}\right) \\
&\quad +\frac{z_{L,h-w+1}}{z_L}\left(\frac{(h-w)z_{H,h-w+1}}{z_{B,h-w+1}}\cdot\frac{(k+1)z_A}{z_B-z_L}\cdot k\cdot\frac{z_{H,h-j}}{z_B-z_L}\right) \\
&\quad +\frac{z_{L,h-j+1}}{z_L}\frac{(h-j)z_{L,h-j+1}}{z_{B,h-j+1}},\quad j=1,\ldots,w-1, && (3.4)\\
z'_{H,h-j}(x) &= \frac{z_{L,h-j}}{z_L}\left(-\frac{(h-j-1)z_{H,h-j}}{z_{B,h-j}}\right) \\
&\quad -\frac{z_{L,h-w+1}}{z_L}\left(\frac{(h-w)z_{H,h-w+1}}{z_{B,h-w+1}}\cdot\frac{(k+1)z_A}{z_B-z_L}\cdot k\cdot\frac{z_{H,h-j}}{z_B-z_L}\right) \\
&\quad +\frac{z_{L,h-j+1}}{z_L}\frac{(h-j)z_{H,h-j+1}}{z_{B,h-j+1}},\quad j=1,\ldots,w-1, && (3.5)\\
z'_L(x) &= -1+\frac{z_{L,h-w+1}}{z_L}\left(-\frac{(h-w)z_{L,h-w+1}}{z_{B,h-w+1}}+(h-w)k\cdot\frac{z_{H,h-w+1}}{z_{B,h-w+1}}\cdot\frac{(k+1)z_A}{z_B-z_L}\right) && (3.6)\\
z'_B(x) &= -1-\frac{(h-w)z_{L,h-w+1}}{z_L} && (3.7)\\
z'_{HV}(x) &= -\frac{z_{L,h-w+1}}{z_L}\frac{(h-w)z_{H,h-w+1}}{z_{B,h-w+1}}\cdot\frac{(k+1)z_A}{z_B-z_L} && (3.8)\\
\lambda'(x) &= \frac{((z'_B-z'_L)z_{HV}-(z_B-z_L)z'_{HV})f_{k+1}(\lambda)}{z_{HV}^2(f_k(\lambda)+\lambda e^{-\lambda}\cdot\frac{\lambda^{k-1}}{(k-1)!}-\frac{z_B-z_L}{z_{HV}}\cdot e^{-\lambda}\cdot\frac{\lambda^k}{k!})} && (3.9)\\
z_{L,h}(x) &= z_L(x)-\sum_{i=1}^{w-1}z_{L,h-j}(x),\quad z_{H,h}(x)=z_B(x)-z_L(x)-\sum_{i=1}^{w-1}z_{H,h-j}(x), && (3.10)\\
z_{B,h-j}(x) &= z_{L,h-j}(x)+z_{H,h-j}(x),\quad \text{for every } 0\le j\le w-1, && (3.11)\\
z_A(x) &= \frac{\lambda(x)^{k+1}}{e^{\lambda(x)}(k+1)!f_{k+1}(\lambda(x))}z_{HV}(x), && (3.12)
\end{aligned}
$$

where $f_k(\lambda)$ was defined in (3.1). The initial conditions are

$$
\begin{aligned}
z_B(0) &= \bar{\mu},\ z_{L,h-j}(0)=0,\ z_{H,h-j}(0)=0,\ \text{for all } 1\le j\le w-1, && (3.13)\\
z_L(0) &= \bar{\mu}(1-f_k(\bar{\mu})),\ z_{HV}(0)=1-\exp(-\bar{\mu})\sum_{i=0}^{k}\bar{\mu}^i/i!,\ \lambda(0)=\bar{\mu}. && (3.14)
\end{aligned}
$$

A SIMPLE RESULT



Allocation is possible (in the large n limit w.h.p.) only if $m = cn$ with $c < c_{h,\ell,k}$ and

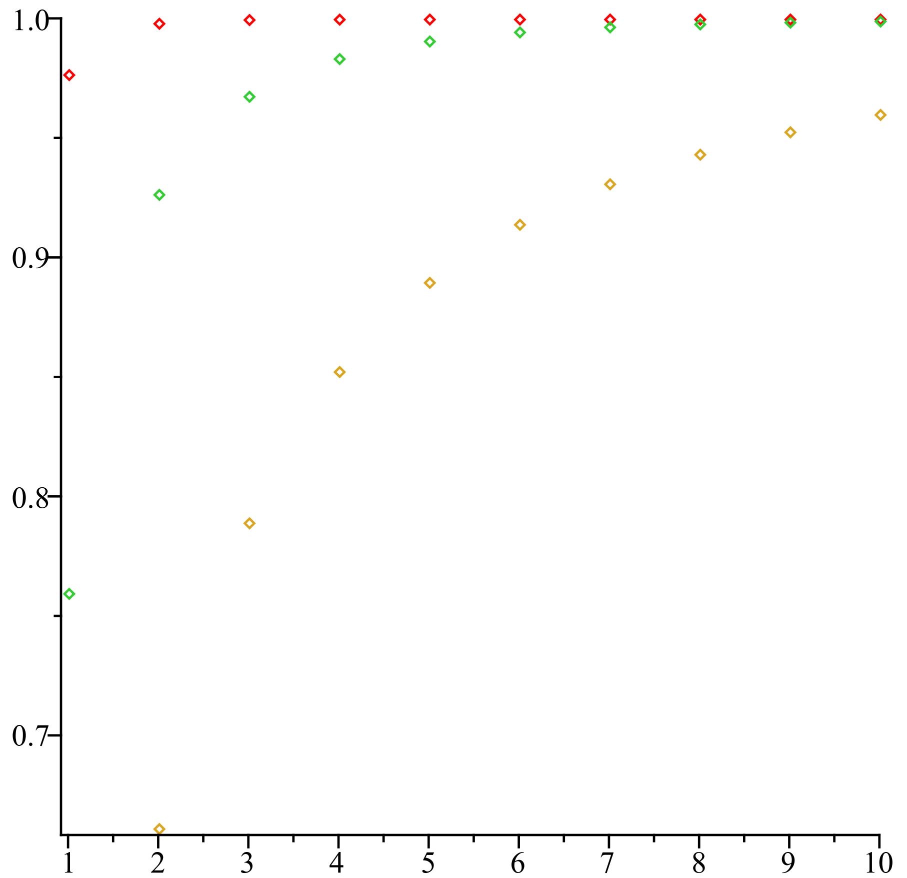
$$c_{h,\ell,k} = \frac{\xi^*}{h\mathbb{P}\left(\text{Bin}(h-1, 1-Q(\xi^*, k) < \ell)\right)},$$

where $Q(x,y) = e^{-x}\sum_{j\geq y}\frac{x^j}{j!}$ and ξ^* is the unique solution to:

$$hk = \xi^* \frac{\mathbb{E}\left[(\ell - \text{Bin}(h, 1-Q(\xi^*, k)))^+\right]}{Q(\xi^*, k+1)\mathbb{P}\left(\text{Bin}(h-1, 1-Q(\xi^*, k)) < \ell\right)}.$$

Lelarge SODA'12

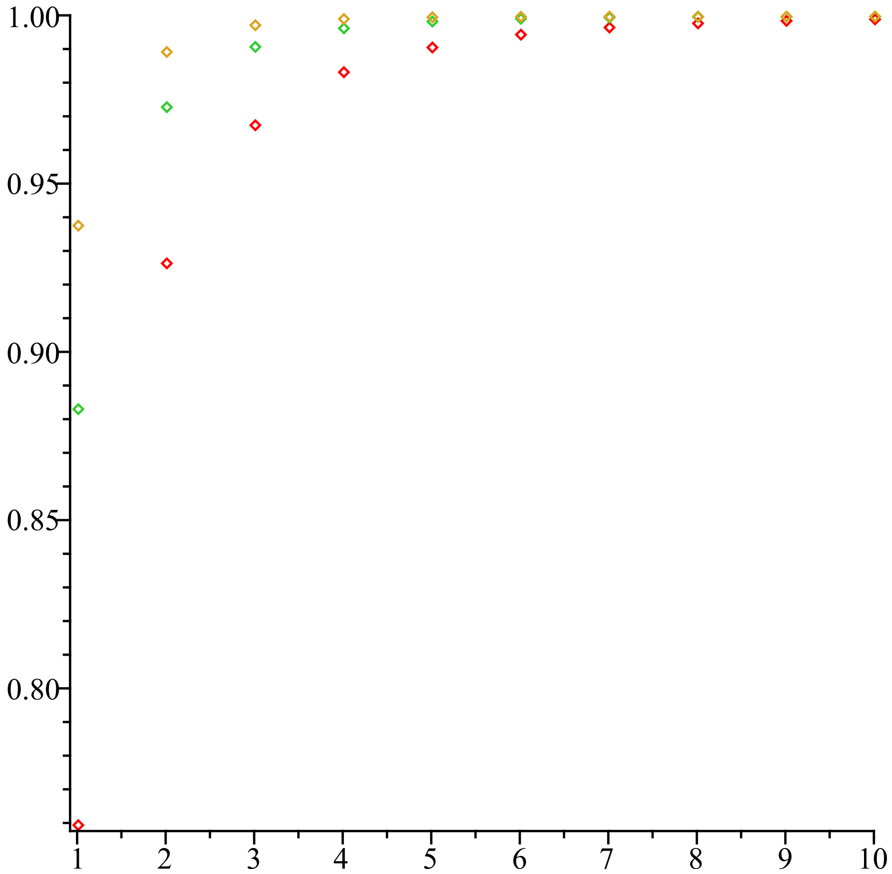
SOME RESULTS



Critical load $\frac{\ell c_{h,\ell,k}}{k}$ as a function of $k = 1 \ldots 10$ capacity of each bin with:

- $h = 4$ choices per batch
- $\ell = 1, 2, 3$ balls per batch

SOME RESULTS



Critical load $\frac{\ell c_{h,\ell,k}}{k}$ as a function of $k = 1 \ldots 10$ capacity of each bin with:

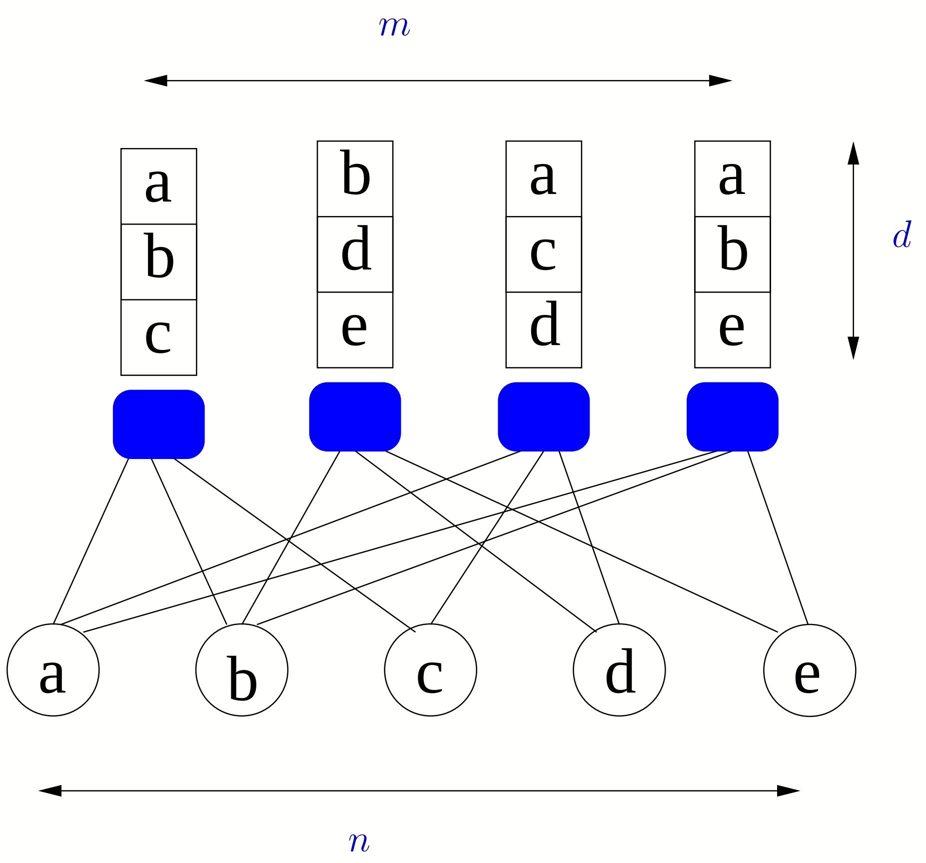
- $h = 4, 5, 6$ choices per batch

- $\ell = 2$ balls per batch

HASHING

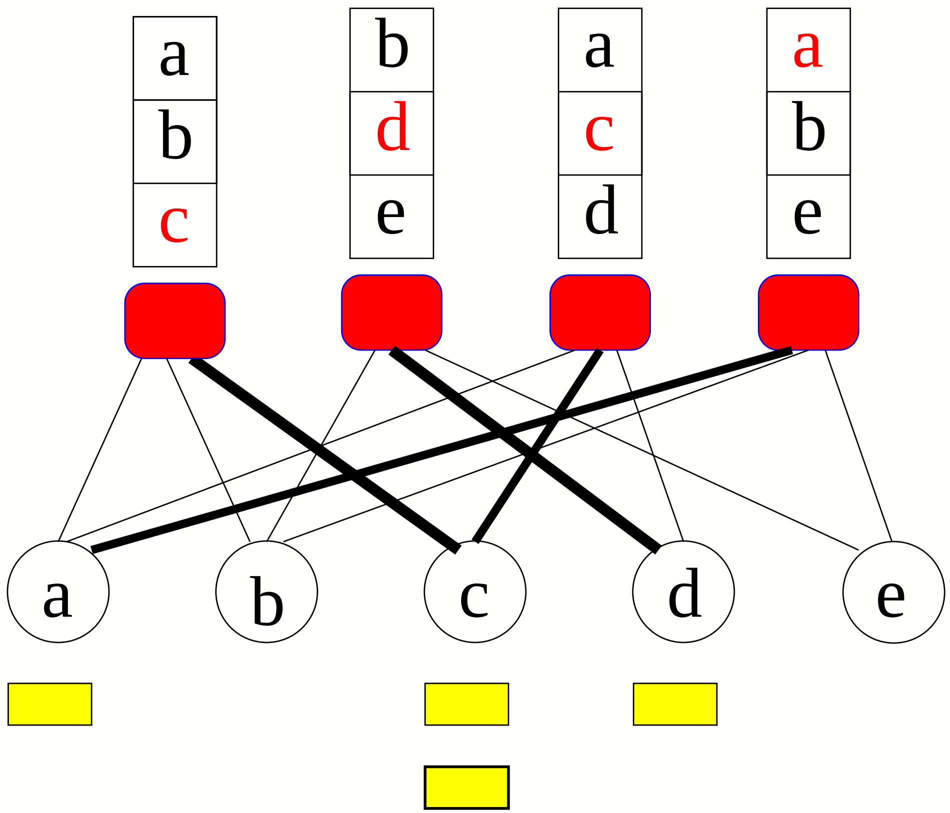
CONTENT PLACEMENT

BIPARTITE GRAPH REPRESENTATION



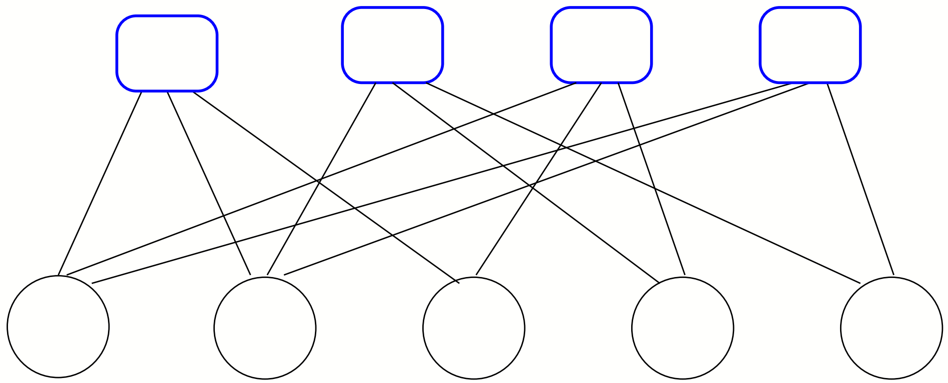
- n contents
- m servers, each storing d contents sampled independently (but not uniformly).
- the degree of a content is the number of replicas for this content in the system.

OPTIMAL ALLOCATION



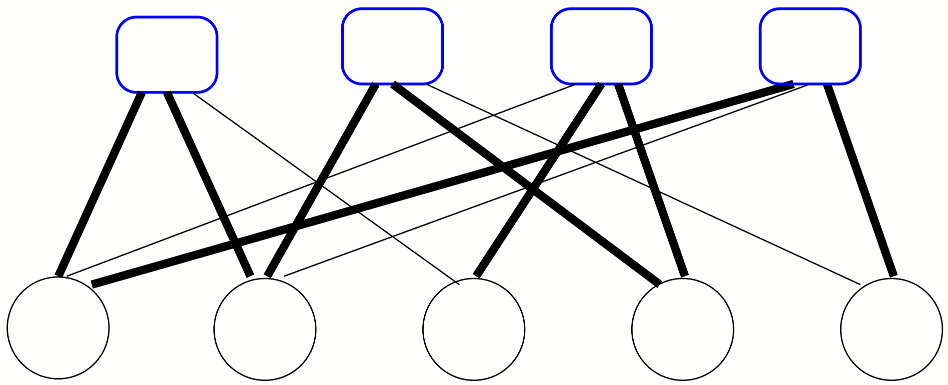
SPANNING SUBGRAPHS OF BIPARTITE RANDOM GRAPHS

- Black nodes = n bins
- Blue nodes = m batches of ℓ balls
- Edge = possible choice for the balls of the batch. Each blue node has degree $h > \ell$.



SPANNING SUBGRAPHS OF BIPARTITE RANDOM GRAPHS

- n black nodes
- m blue nodes of degree h
- Allocation = for each blue node, select ℓ edges such that in the spanning subgraph, all black nodes have degree less than k.



Example with $k = \ell = 2$.

A COMBINATORIAL DETOUR

A simple identity:

$$\Omega(G,\boldsymbol{\lambda},\mathbf{x}) = \prod_{vew\in E}(1+\lambda_e x_v x_w) = \sum_{H\subseteq E}\boldsymbol{\lambda}^H\mathbf{x}^{\deg(H)},$$

with $\boldsymbol{\lambda}^H = \prod_{e\in H}\lambda_e$ and $\mathbf{x}^{\deg(H)} = \prod_{v\in V} x_v^{\deg(v,H)}$.

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with $\boldsymbol{\lambda}^H = \prod_{e \in H} \lambda_e$ and $\mathbf{x}^{\deg(H)} = \prod_{v \in V} x_v^{\deg(v,H)}$.

We are interested in:

$$Z(G, \boldsymbol{\lambda}, \mathbf{x}) = \sum_{H \subseteq E} \boldsymbol{\lambda}^H \mathbf{x}^{\deg(H)} \mathbb{1}(H \text{ is a matching})$$

SCHUR-SZEGÖ COMPOSITION

If $P(z) = \sum_{j=0}^{d} c_j z^j$ is nonvanishing in the open right half-plane and $K(z) = \sum_{j=0}^{d} \binom{d}{j} u_j z^j$ has only real nonpositive zeros, then $Q(z) = \sum_{j=0}^{d} u_j c_j z^j$ is nonvanishing in the open right half-plane.

APPLYING SCHUR-SZEGÖ COMPOSITION

Consider the case $u_0 = u_1 = 1$ and $u_k = 0$ for $k \geq 2$ and define $K_v(z) = 1 + \deg(v) z$.

Let $F_0(\mathbf{x}) = \Omega(G, \boldsymbol{\lambda}, \mathbf{x})$ and define $F_v(\mathbf{x})$ as the Schur-Szegö composition of $F_{v-1}(x_v)$ and $K_v(x_v)$. (Wagner 2009)

$$\begin{aligned}
F_0(\mathbf{x}) &= \sum_{H \subseteq E} \boldsymbol{\lambda}^H \mathbf{x}^{\deg(H)} \\
F_1(\mathbf{x}) &= \sum_{H \subseteq E} \boldsymbol{\lambda}^H \mathbb{I}(\deg(v, H) \leq 1) \mathbf{x}^{\deg(H)} \\
&\vdots \\
F_n(\mathbf{x}) &= \sum_{H \subseteq E} \boldsymbol{\lambda}^H \prod_{v=1}^{n} \mathbb{I}(\deg(v, H) \leq 1) \mathbf{x}^{\deg(H)} \\
&= \sum_{H \subseteq E} \boldsymbol{\lambda}^H \mathbf{x}^{\deg(H)} \mathbb{I}(H \text{ is a matching}) = Z(G, \boldsymbol{\lambda}, \mathbf{x}).
\end{aligned}$$

ANALOGY WITH STATISTICAL PHYSICS

$Z(G, \mathbf{1}, z^{1/2}\mathbf{1}) = \sum_M z^{|M|} = \sum_k m_k(G) z^k = P_G(z)$, where $m_k(G)$ is the number of k-edge matchings of G.

The fact that $P_G(z)$ has its zeros on the negative real axis allows to define the Gibbs measure

$$\mu_G^z(M) = \frac{z^{|M|}}{P_G(z)}$$

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(Heilmann Lieb 1972)

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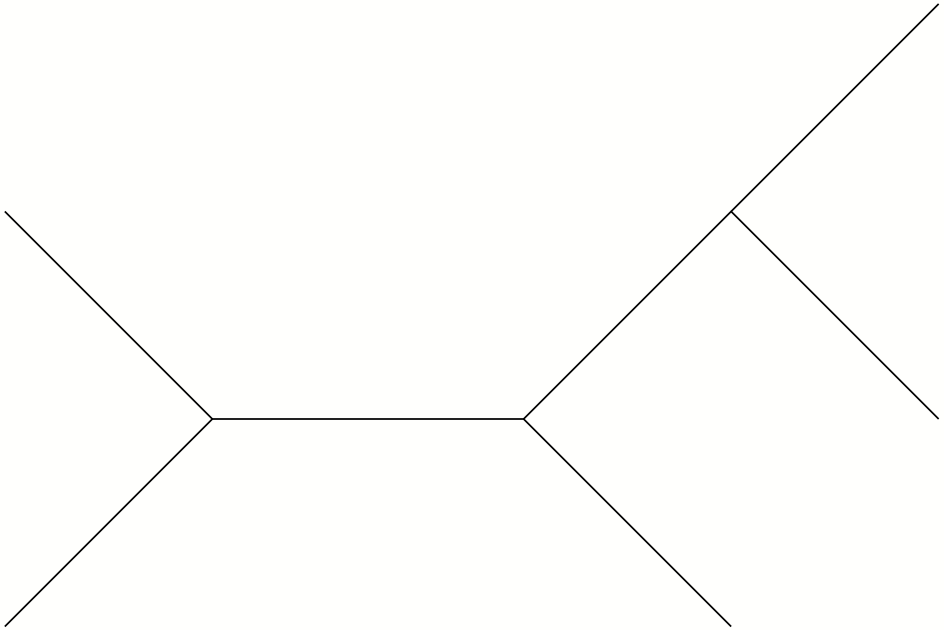
This technique can be used as a step towards computations BUT it fails for more general spanning subgraphs, i.e. for degree constraints larger than 3.

A SIMPLE GREEDY ALGORITHM ON TREES



For simplicity, spanning subgraph H with $\deg(v, H) \leq 2 = w$.

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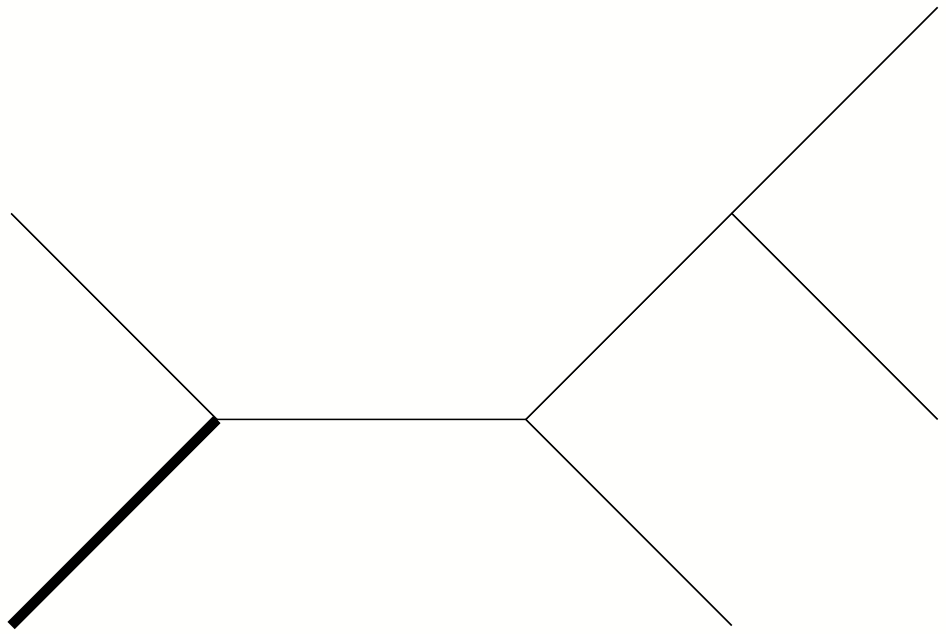

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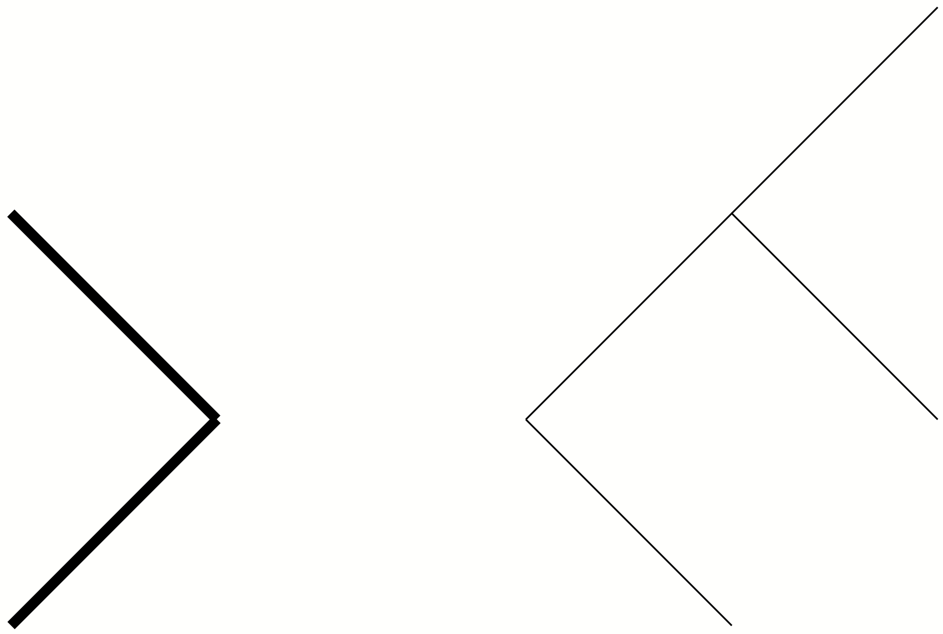

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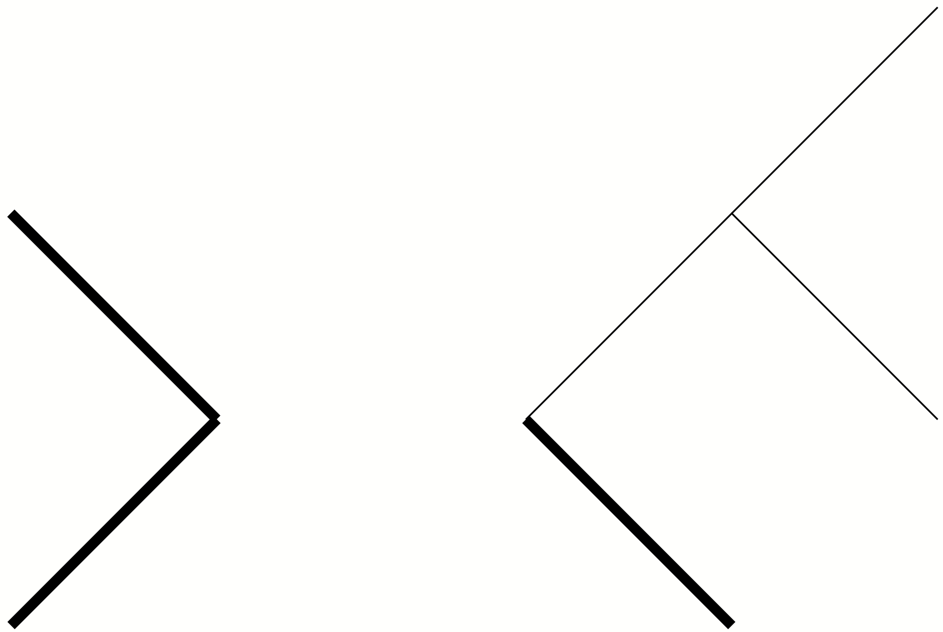

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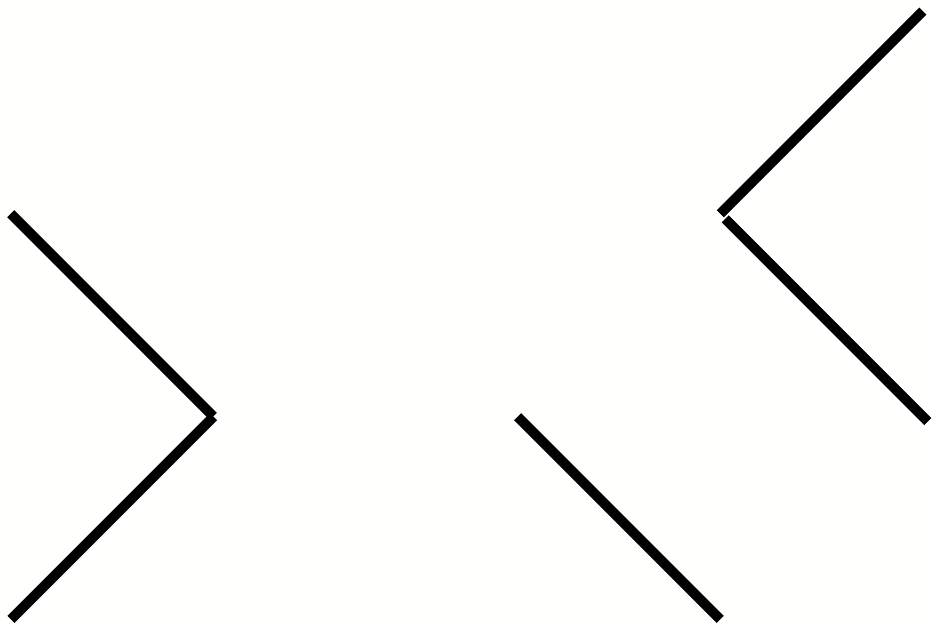

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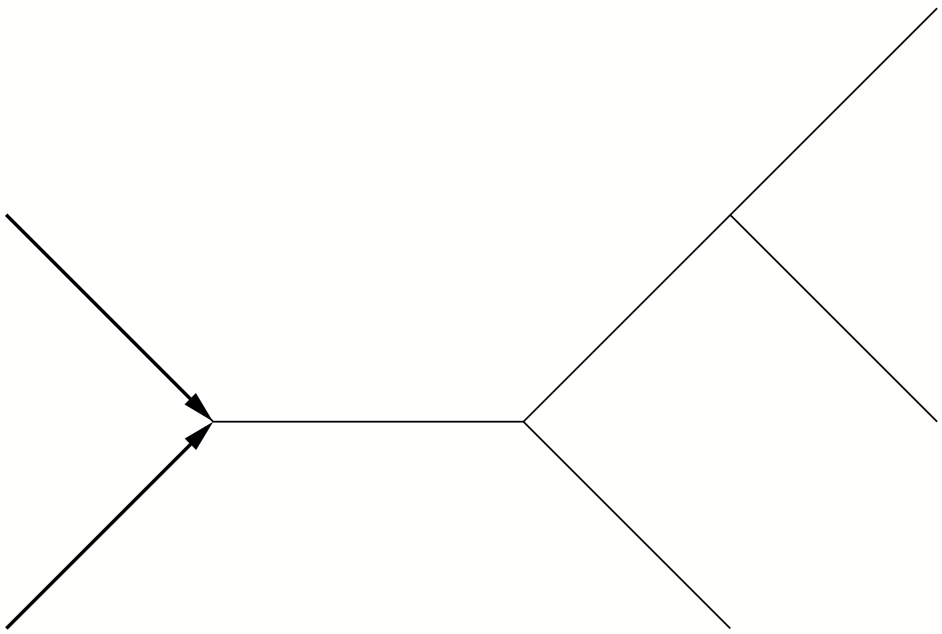
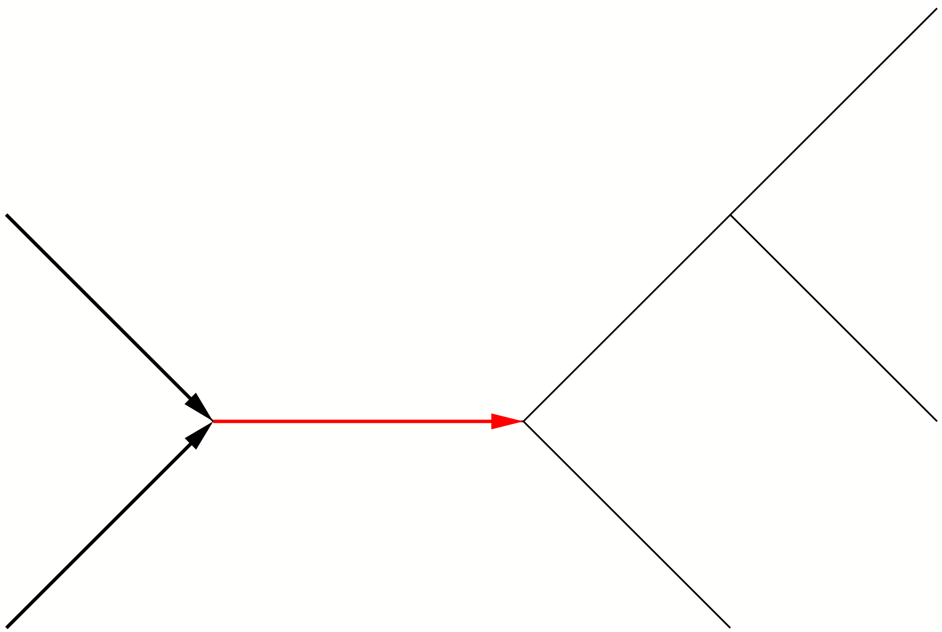
A MESSAGE PASSING VERSION OF THE GREEDY ALGORITHM



Black arrow: 'I want to match you'

A MESSAGE PASSING VERSION OF THE GREEDY ALGORITHM



Black arrow: ’I want to match you’

Red arrow: ’Sorry, I am saturated’

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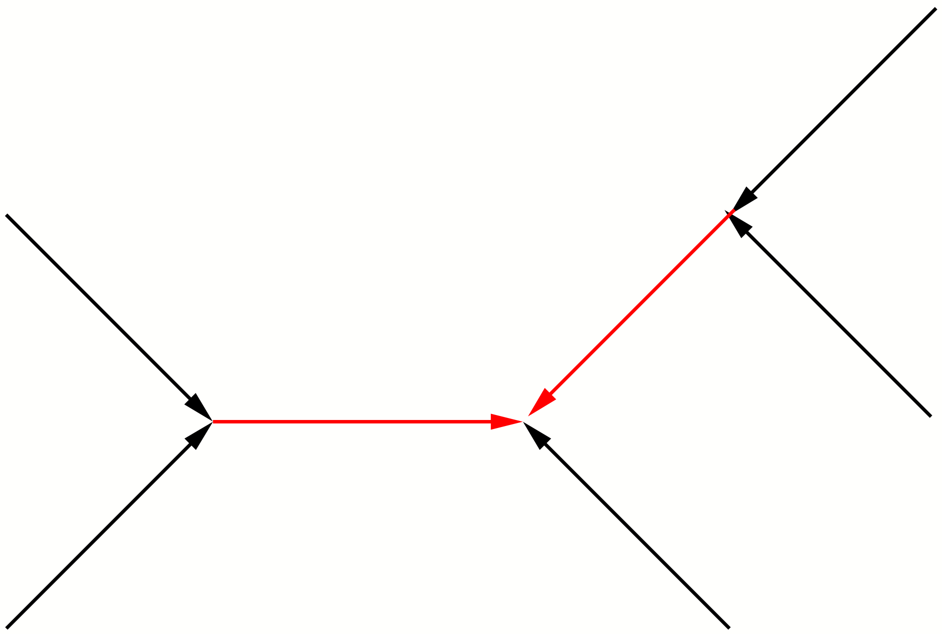

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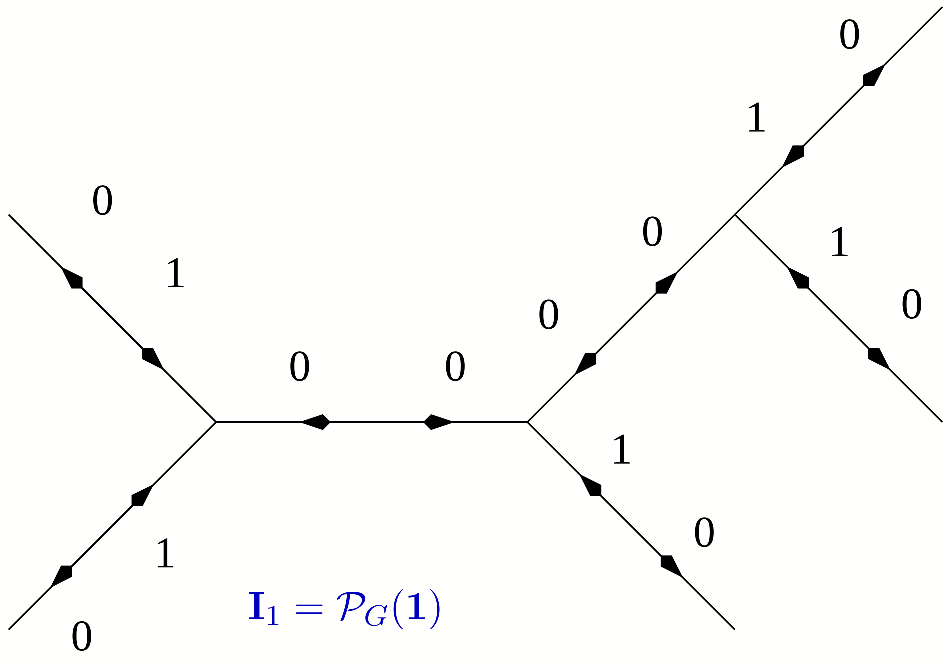
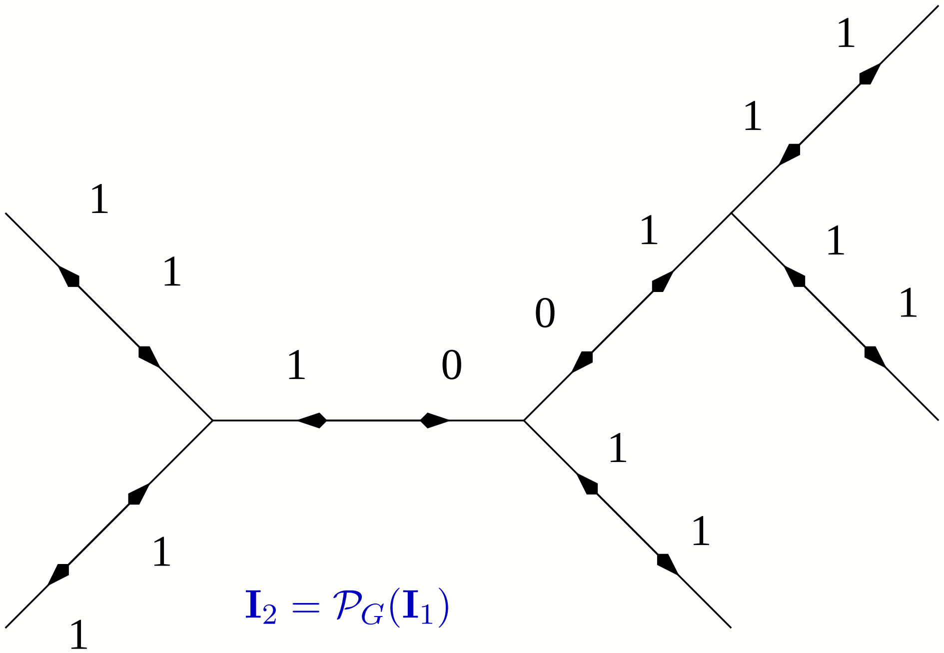


Replace black arrows by 1 messages and red arrows by 0 messages and run simultaneously.

For any directed edge, sum the incoming messages from the other edges. If this sum is larger than $w = 2$ then $\mathcal{P}_G$ returns 0, otherwise returns 1 on this directed edge.

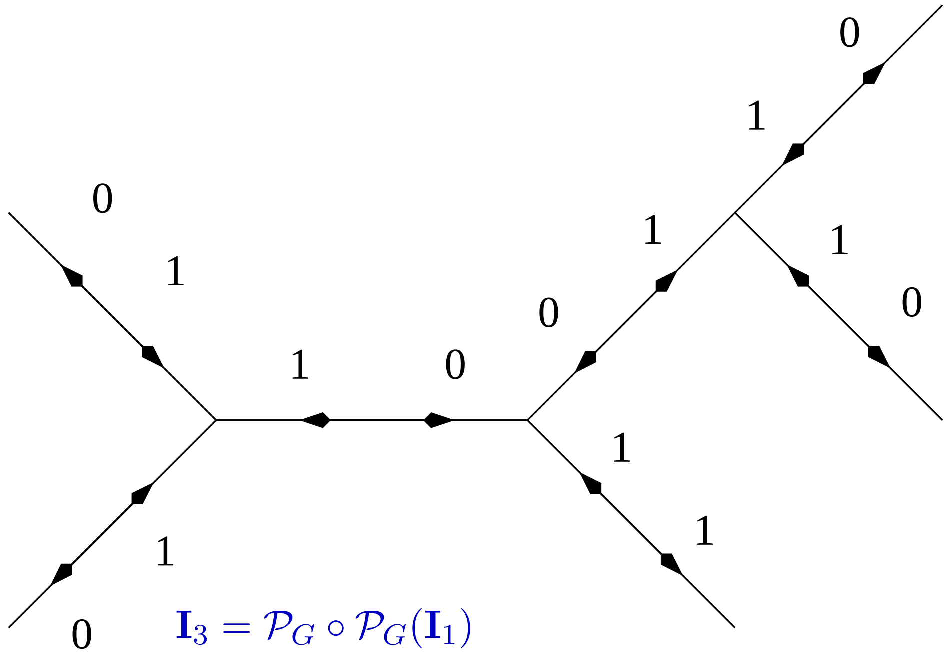
A MESSAGE PASSING VERSION OF THE GREEDY ALGORITHM



$$\mathbf{I}_2 = \mathcal{P}_G(\mathbf{I}_1)$$

Iterate...

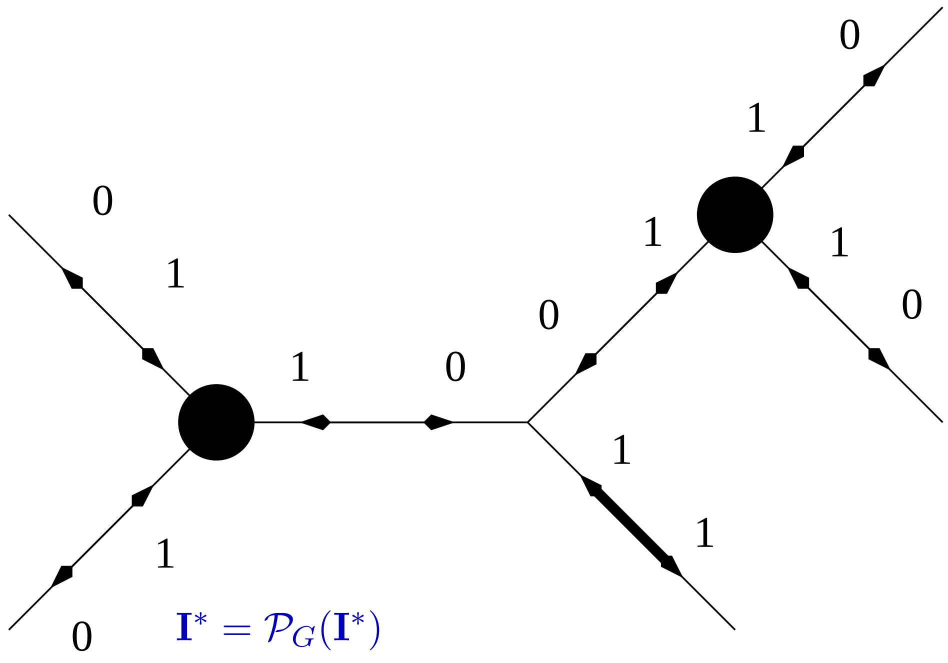
A MESSAGE PASSING VERSION OF THE GREEDY ALGORITHM



$\mathbf{I}_3 = \mathcal{P}_G \circ \mathcal{P}_G(\mathbf{I}_1)$

... until you get a fixed point $\mathbf{I}^*$.

A MESSAGE PASSING VERSION OF THE GREEDY ALGORITHM



On finite trees, the algorithm converges and $\mathbf{I}^*$ allows to get the size of a maximum spanning subgraph.

$$\sum_{v \in V} \left(w \mathbb{1} \left(\sum_{\vec{e} \in \partial v} I^*_{\vec{e}} \geq w + 1 \right) + \frac{1}{2} \mathbb{1} \left(\sum_{\vec{e} \in \partial v} I^*_{\vec{e}} \leq w \right) \sum_{\vec{e} \in \partial v} I^*_{\vec{e}} \right)$$

RUNNING THE ALGORITHM ON AN INFINITE TREE

Let simplify further $\ell = k = 1$ and Poisson Galton-Watson tree with mean offspring λ.

- Let p be the probability of sending a 1 message

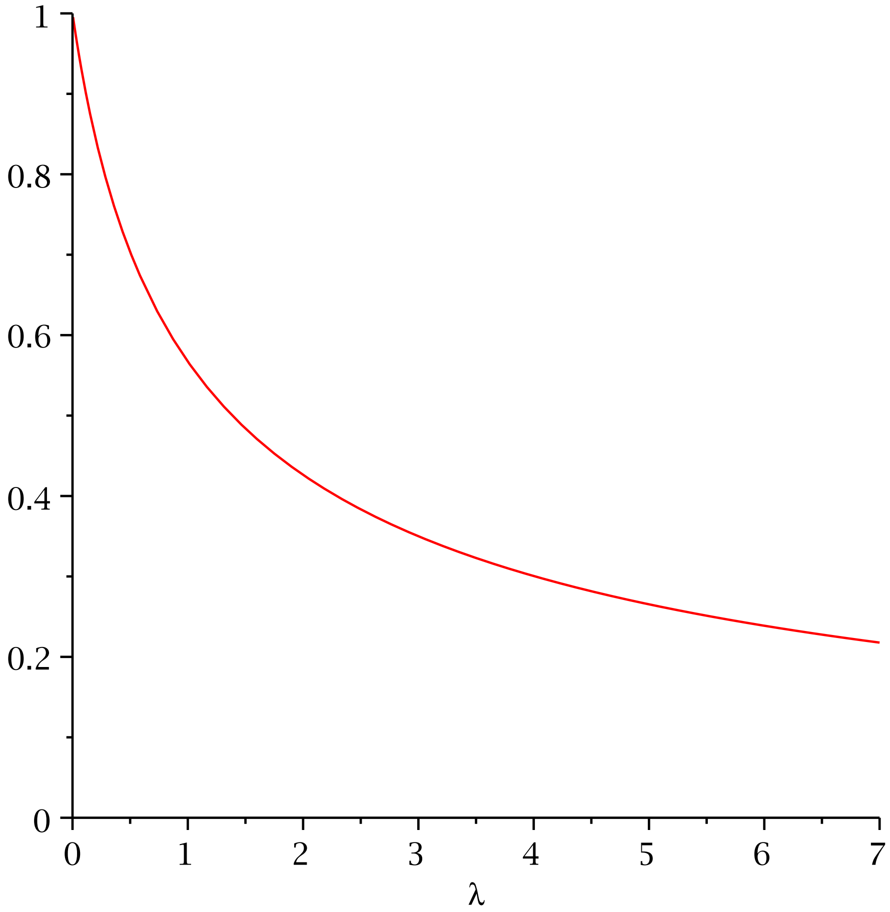
$$p = \mathbb{P}\left(I^*_{\vec{e}} = 1\right)$$

- Thanks to the branching property:

$$p = \mathbb{P}\left(\text{no children send a } 1 \text{ message}\right) = e^{-\lambda p}$$

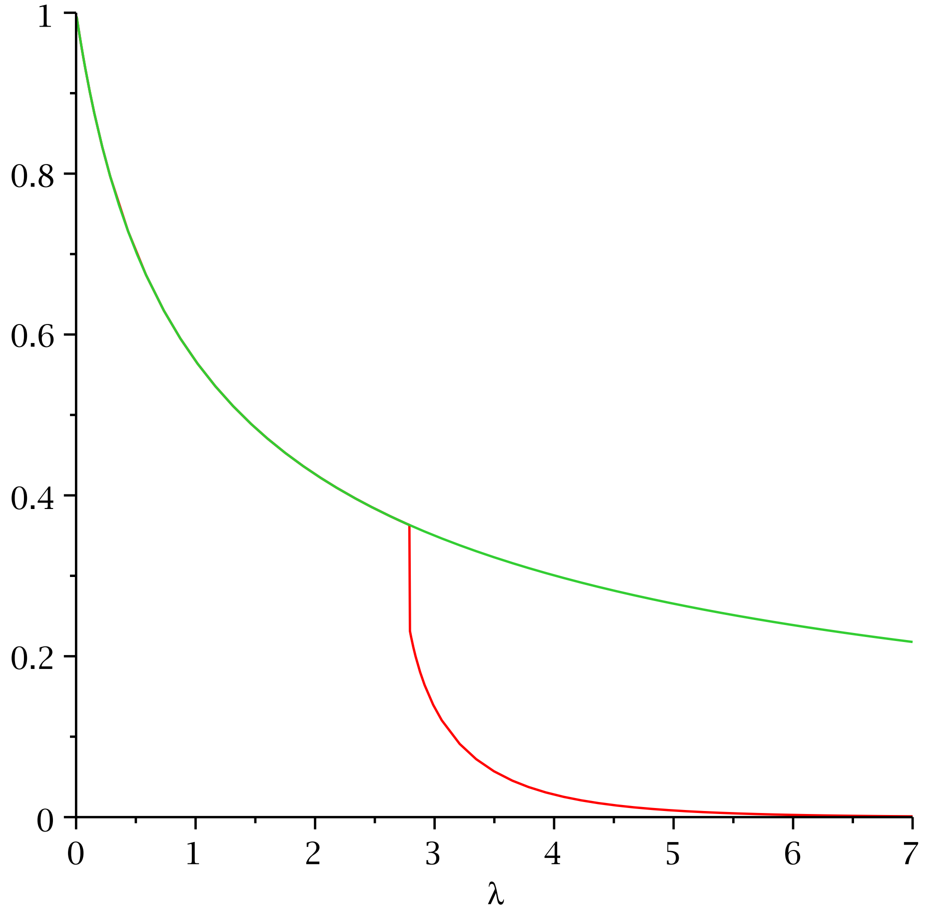
and so $p = \frac{W(\lambda)}{\lambda}$.

A NAIVE GUESS



The function $\frac{W(\lambda)}{\lambda}$ as a function of λ.

TRUTH



The true value of p as a function of λ.

WHAT HAPPENED?

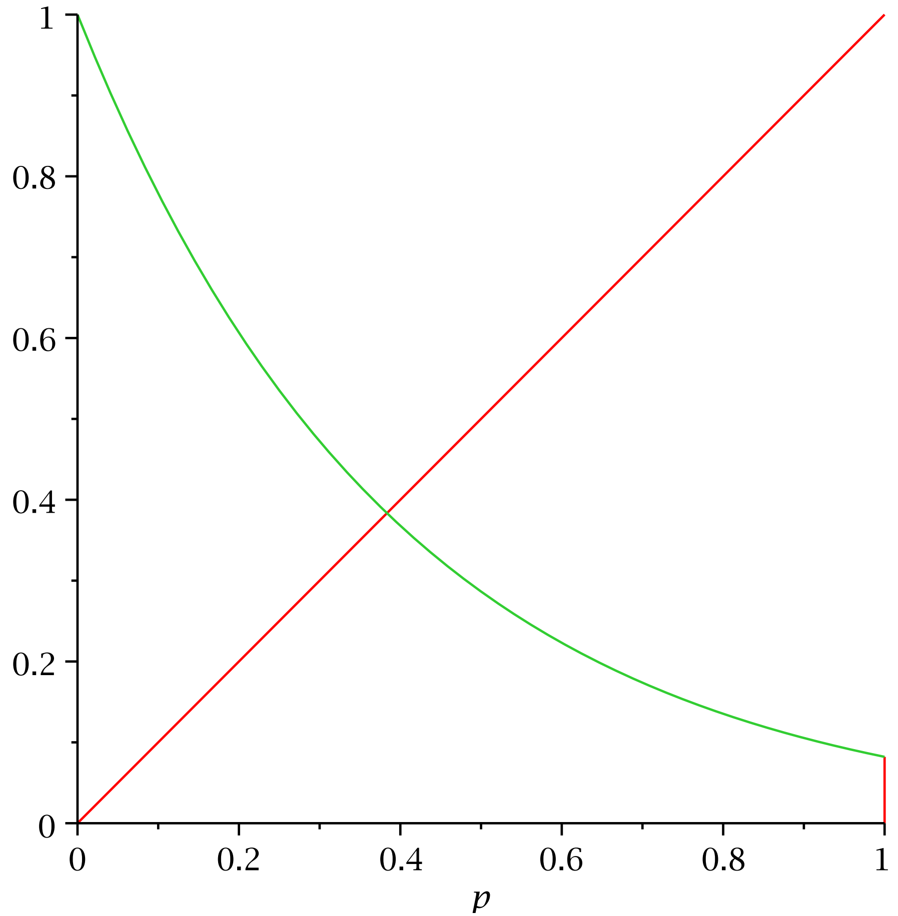
Let p_k be the probability of the root sending message 1 for the tree truncated at depth k.

- $p_0 = 1$
- $p_1 = e^{-\lambda}$
- then for $k \geq 0$

$$p_{k+1} = e^{-\lambda p_k}$$

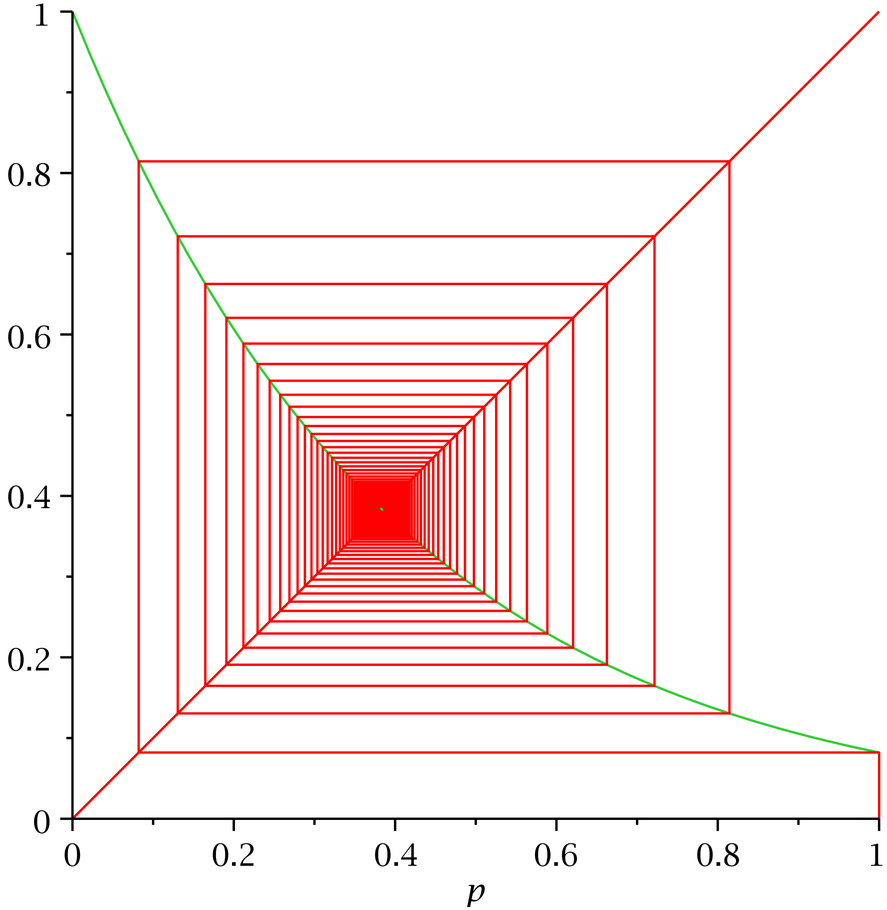
We computed the fixed point of the map $p \mapsto e^{-\lambda p}$ but the truth is given by iterating it...

ITERATING



$$\lambda = 2.5$$

ITERATING



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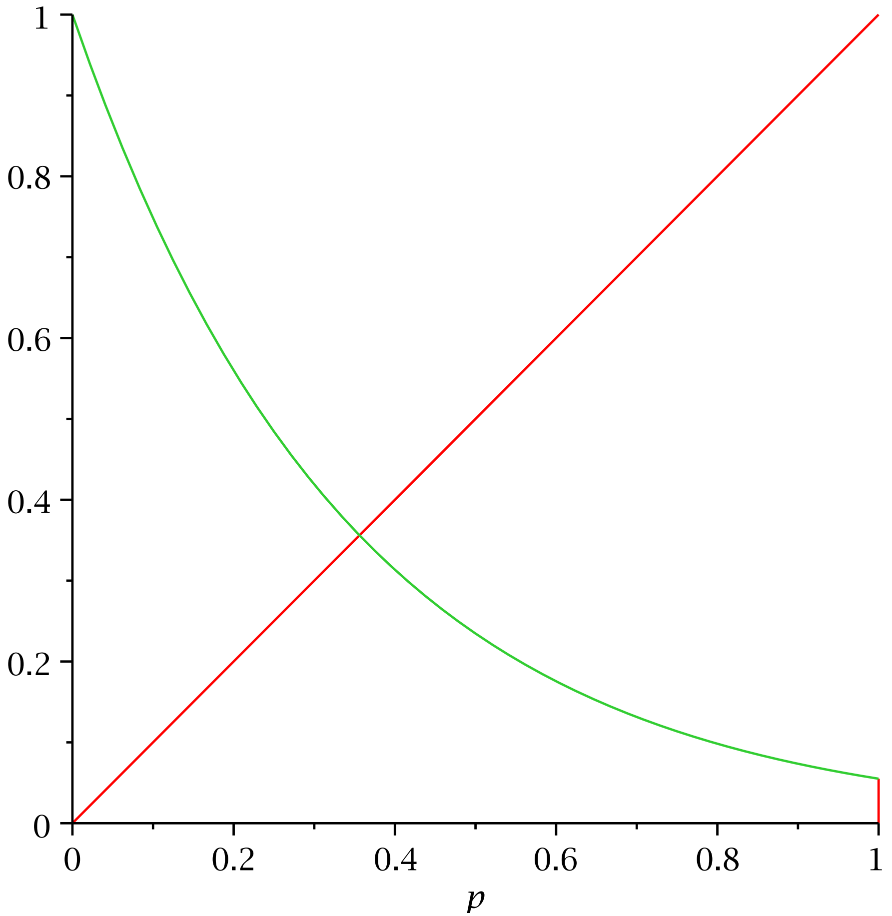
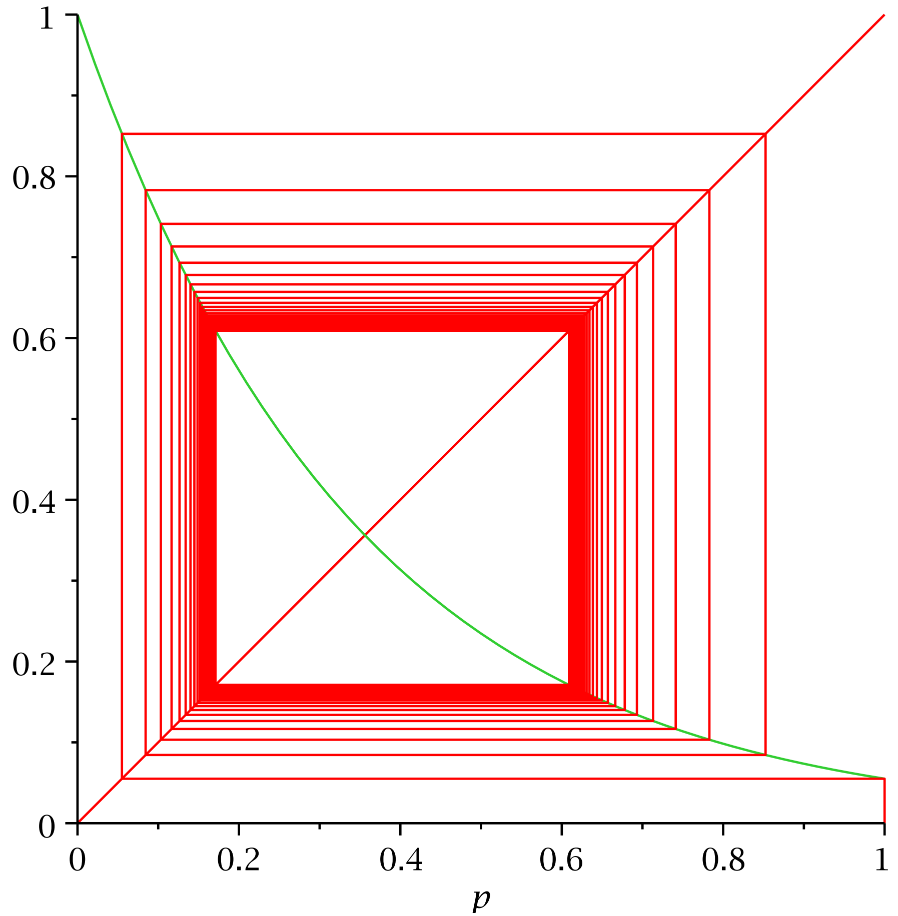


$$\lambda = 2.9$$

ITERATING



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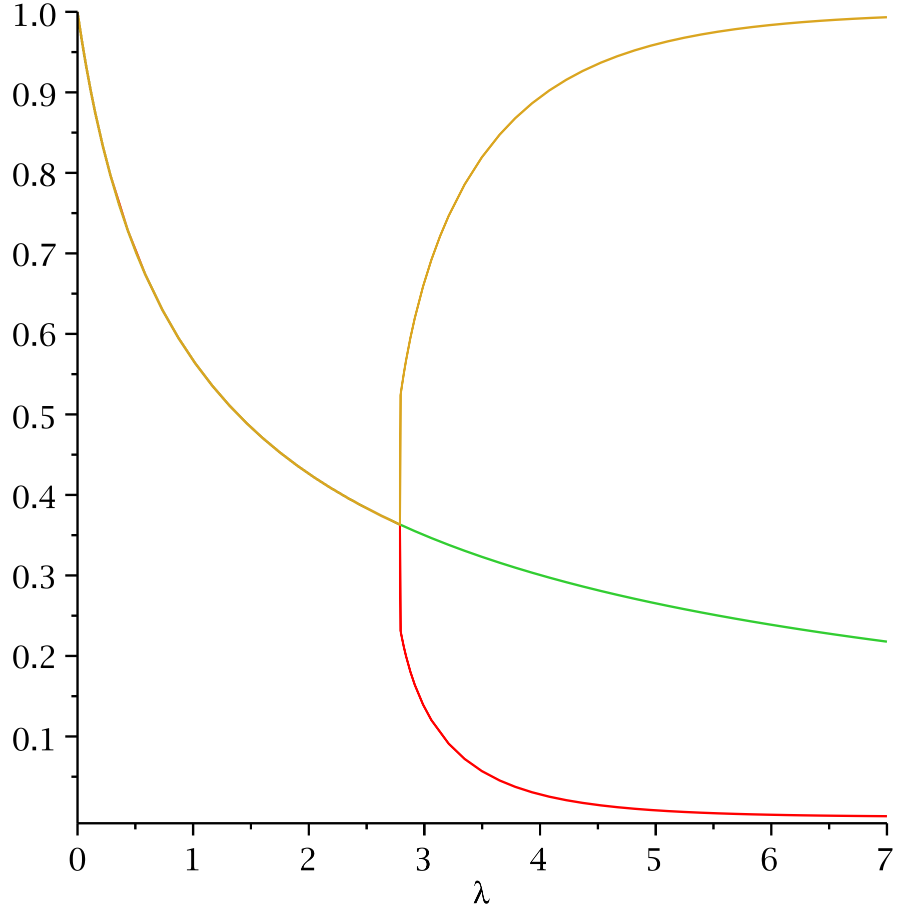
ABSENCE OF CORRELATION DECAY



Influence of the boundary conditions remains positive.

BYPASSING CORRELATION DECAY

- Introduce the Gibbs measure on allocations:

$$\mu_G^z(\mathbf{B}) = \frac{z^{\sum_e B_e}}{P_G(z)}$$

so that the size of a maximum allocation of the graph $G = (V, E)$ is given by

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MESSAGE PASSING ALGORITHM

Define $Y_e(z) \in \mathbb{R}$ by $\mu^z_{G,e}(B_e = 1) = \frac{Y_e(z)}{1+Y_e(z)}$. Then the recursion is

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with

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In the case of matchings, $w = 1$ so that

$$\mathcal{R}_e(\mathbf{Y}) = \frac{1}{1 + \sum_{f \prec e} Y_f}.$$

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- Using a convexity argument, invert the limits in n and z.

RESULT ON INFINITE UNIMODULAR TREES

Assumption: G_n has random weak limit $\rho\left([G,\circ]\right)$, a unimodular probability measure concentrated on trees.

For any $\mathbf{I} \in \{0,1\}^{\overrightarrow{E}}$,

$$F_\circ(\mathbf{I}) = w_\circ \mathbb{I}\Big(\sum_{x\in\partial\circ} \mathcal{P}_{x\to\circ}(\mathbf{I}) \geq w_\circ + 1\Big) + w_\circ \wedge \sum_{x\in\partial\circ} I_{x\to\circ}.$$

Then

$$\lim_{n\to\infty} \frac{1}{n} M(G_n) = \frac{1}{2} \inf\left\{\int F_\circ(\mathbf{I}) d\rho([G,\circ])\right\},$$

where the infimum is over all spatially invariant solutions of $\mathbf{I} = \mathcal{P}_G \circ \mathcal{P}_G(\mathbf{I})$.

For matchings, the Recursive Distributional Equation (RDE) becomes:

$$Y(z) \stackrel{d}{=} \frac{z}{1 + \sum_{i=1}^{N} Y_i(z)}$$

where $N \sim$ the standard size biased degree distribution of the random graph.

By iterating once

$$\frac{Y(z)}{z} \stackrel{d}{=} \frac{1}{1 + \sum_{i=1}^{N} \frac{1}{\frac{1}{z} + \sum_{j=1}^{N_{ij}} \frac{Y_{ij}(z)}{z}}}$$

so that we obtain for $X = \lim_{z \to \infty} \frac{Y(z)}{z} \in [0, 1]$ the simple RDE:

$$X \stackrel{d}{=} \frac{1}{1 + \sum_{i=1}^{N} \frac{1}{\sum_{j=1}^{N_{ij}} X_{ij}}}$$

SOLVING THE RDE AT $z = \infty$

If φ is the generating function of the asymptotic degree distribution, let

$$G(x) = \varphi'(1)x\overline{x} + \varphi(1-x) + \varphi(1-\overline{x}) - 1,$$

where $\overline{x} = \varphi'(1-x)/\varphi'(1)$.

G admits an historical record at x if $x = \overline{\overline{x}}$ and $G(x) > G(y)$ for any $0 \leq y < x$.

Theorem 1. *If $p_1 < \ldots < p_r$ are the locations of the historical records of G, then the RDE admits exactly r solutions, say $0 \leq X_1 <_{st} \ldots <_{st} X_r \leq 1$, and for any $i \in \{1, \ldots, r\}$, $\mathbb{E}[X_i] = G(p_i)$ and $\mathbb{P}(X_i > 0) = p_i$.*

From the values $p_1 < \ldots < p_r$, we can compute the limit of the matching number (rescaled by n) when $n \to \infty$.

CONCLUSION

- General method to compute law of large numbers for combinatorial structures on sparse (random) graphs.

  (a) to bypass the correlation decay, add a (small) noise parameter.

  (b) crucially use monotonicity of the recursions

- Our method works for matchings, spanning subgraphs with degree constraints and b-matchings.

- The absence of phase transition has also algorithmic implications: sublinear algorithms to approximate the number of matchings.

- Open problem: Counting of other large subgraphs: long cycles (Marinari & Semerjian 2006).

THANK YOU!