Ergodicité de certains automates cellulaires bruités

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Séminaire Flajolet, Jeudi 1er juin 2017

The XOR Cellular Automaton

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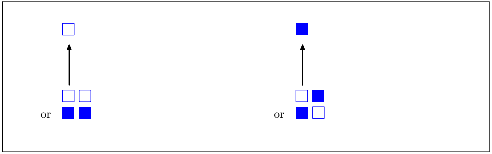

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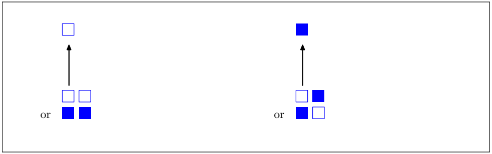

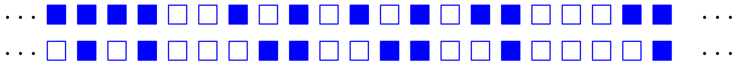

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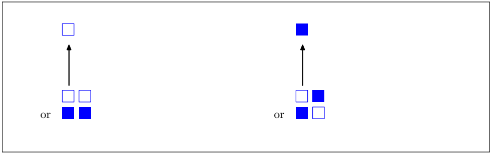

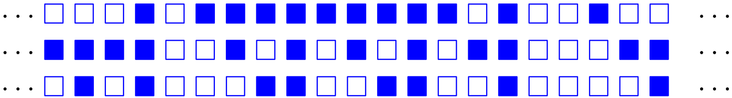

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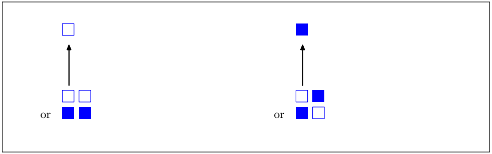
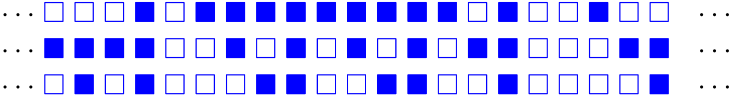


Additive CA (XOR)



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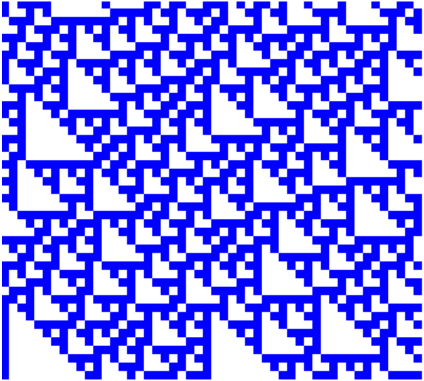

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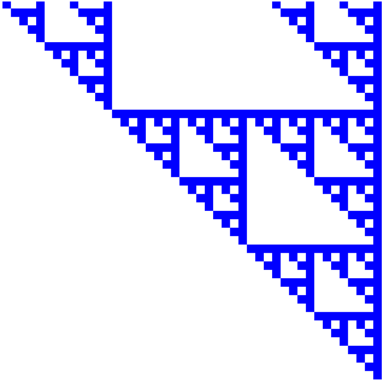

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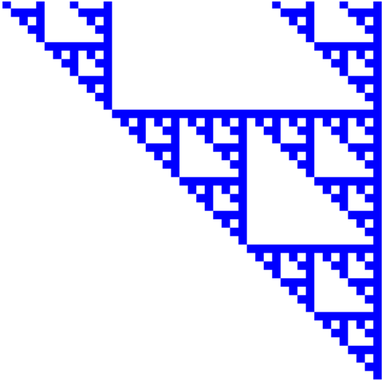

The system keeps some information on its initial configuration.

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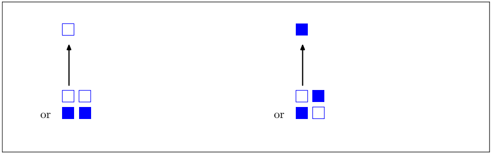

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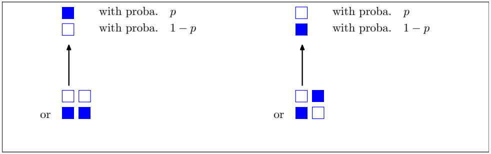

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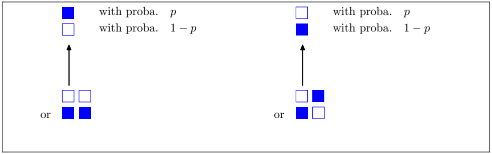
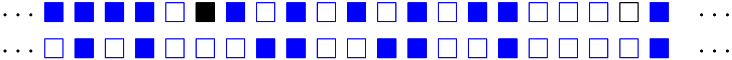
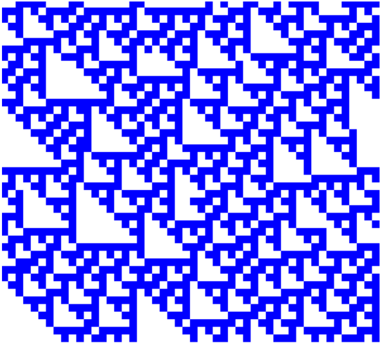




The XOR Cellular Automaton

$p = 0.01$



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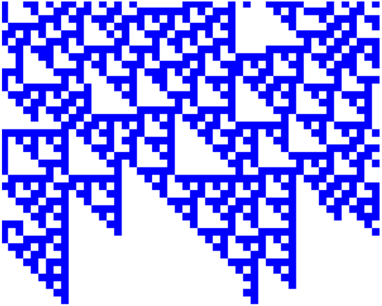

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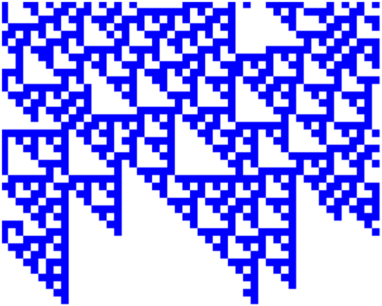
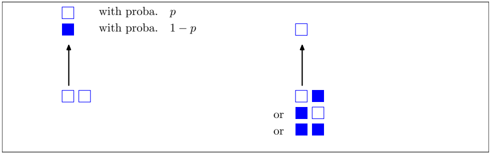


The system **forgets** its initial configuration.

The noisy hardcore Cellular Automaton



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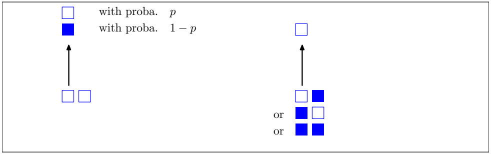

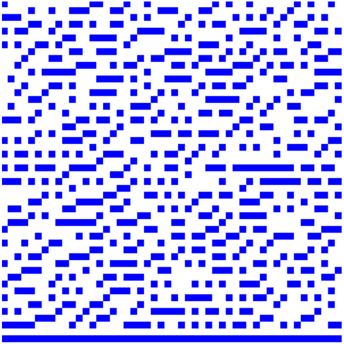

$p = 0.5$

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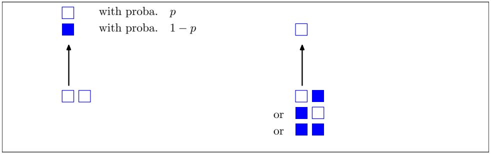

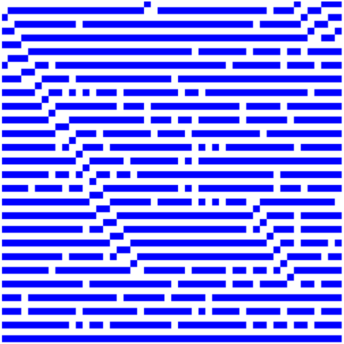

$p = 0.1$

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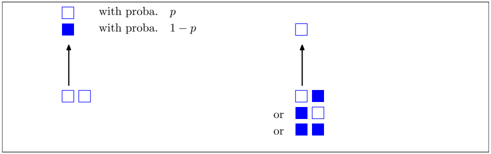
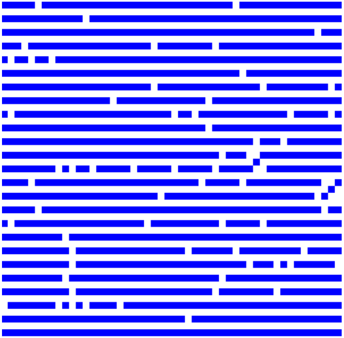




$p = 0.05$

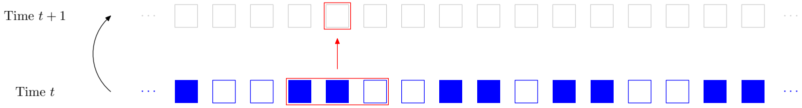
Definition of Cellular Automata

A CA F on $E = \mathbb{Z}^d$ is defined by:

- a finite alphabet $\mathcal{A}$ (set of symbols)
- a finite neighbourhood $\mathcal{N} \subset \mathbb{Z}^d$,
- a local function $f : \mathcal{A}^{\mathcal{N}} \to \mathcal{A}$.

From configuration $(x_k)_{k \in \mathbb{Z}^d} \in \mathcal{A}^{\mathbb{Z}^d}$, cell k is updated by symbol

$$f((x_{k+v})_{v \in \mathcal{N}})$$



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A PCA F on $E = \mathbb{Z}^d$ is defined by:

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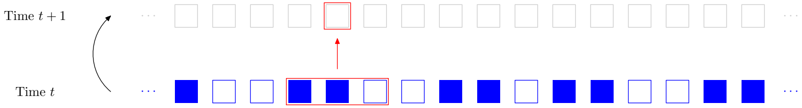
- a local function $f : \mathcal{A}^{\mathcal{N}} \to \mathcal{M}(\mathcal{A})$.

From configuration $(x_k)_{k \in \mathbb{Z}^d} \in \mathcal{A}^{\mathbb{Z}^d}$, cell k is updated by symbol y with probability:

$$f((x_{k+v})_{v \in \mathcal{N}})(y),$$

simultaneously and independently of the other cells.



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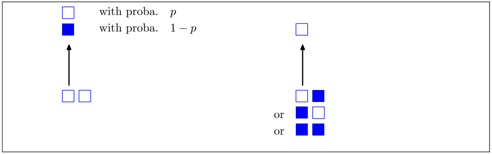

Set of cells: $E = \mathbb{Z}$
Alphabet: $\mathcal{A} = \{\blacksquare, \square\}$
Neighbourhood: $\mathcal{N} = \{0, 1\}$

Local function:

$$f : \{\blacksquare, \square\}^2 \to \mathcal{M}(\{\blacksquare, \square\})$$

defined by:

$$f(\square\square) = (1-p)\delta_{\blacksquare} + p\delta_{\square}$$

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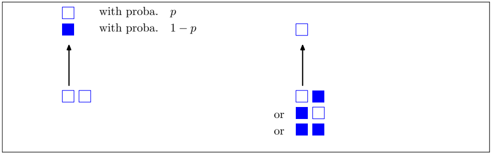

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Global function:

$$F : \mathcal{M}(\{\blacksquare, \square\}^{\mathbb{Z}}) \to \mathcal{M}(\{\blacksquare, \square\}^{\mathbb{Z}})$$

$$\mu \mapsto \mu F$$

Cellular automata

- $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a CA $\iff$ F continuous and $F \circ \sigma = \sigma \circ F$
- Model of parallel computing
- Simple definition and wide variety of behaviour!

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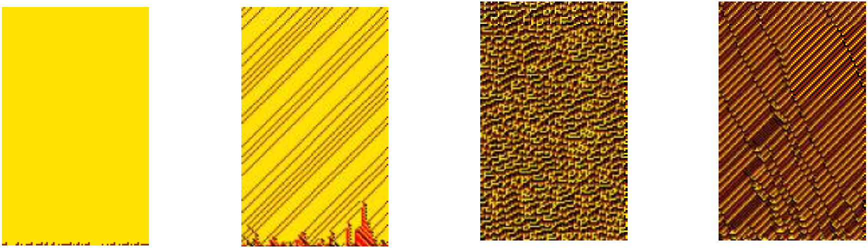
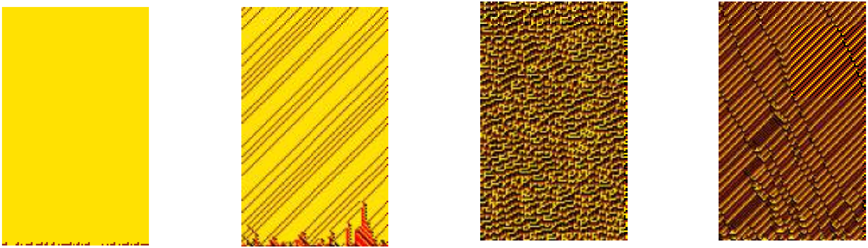


- $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a CA $\iff$ F continuous and $F \circ \sigma = \sigma \circ F$
- Model of parallel computing **with noise**
- Simple definition and wide variety of behaviour!



Notion of ergodicity

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If a PCA forgets every bit of information on the initial configuration, we say it is ergodic.

Ergodicity

A PCA F on $\mathcal{A}^{\mathbb{Z}^d}$ is ergodic if:

- it has a **unique invariant probability distribution** $\pi \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}^d})$, such that $F\pi = \pi$,
- for any initial measure $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}^d})$, the sequence of iterates $(F^n\mu)_{n\geq 0}$ **converges** weakly to π.

Positive rates conjecture

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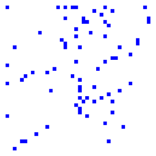
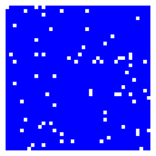
- Remark: **false** for $d \geq 2$

Counter-example for $d = 2$: Toom CA

Majority rule: $(\mathcal{T}(x))_{i,j} = \text{maj}(x_{i,j}, x_{i,j+1}, x_{i+1,j})$





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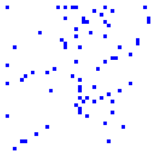

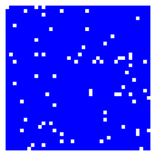

Counter-example for $d = 1$: very complicated! [Gács 2001]

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Elementary PCA: $\mathcal{A} = \{0,1\}, \mathcal{N} = \{0,1\}, f(01) = f(10)$
Is the positive rate conjecture true for elementary PCA?
It has been shown that for 90% of the cube of parameters, the PCA is ergodic [Toom et al. 1990].

Content

- The case of deterministic CA

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- The **envelope PCA** and a general criterion of ergodicity for the high-noise regime

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- The **envelope PCA** and a general criterion of ergodicity for the high-noise regime

- Some proofs of ergodicity for PCA **close to being deterministic**
  1. The noisy hardcore CA
  2. Noisy nilpotent CA
  3. Noisy spreading CA
  4. Noisy permutive CA

Ergodicity of deterministic CA

A deterministic CA $F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ is nilpotent if there exists $\alpha \in \mathcal{A}$ such that: $\forall x \in \mathcal{A}^{\mathbb{Z}^d}, \exists n \in \mathbb{N}, F^n(x) = \alpha^{\mathbb{Z}^d}$.

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Proposition [Bušić-Mairesse-M., STACS 2011 & Adv. Appl. Proba]

For deterministic CA, ergodicity $\Leftrightarrow$ nilpotency.

Proof. $\Leftarrow$ is easy ; $\Rightarrow$ in two steps:

1. the unique invariant measure has to be a measure concentrated on a monochromatic configuration $\alpha^{\mathbb{Z}^d}$,
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Corollary (with [Kari 1992])

The ergodicity of one-dimensional deterministic CA (and hence of PCA) is undecidable.

Update function of a PCA

A way to run a PCA (on $\mathcal{A} = \{0, 1\}$) from configuration $x \in \mathcal{A}^{\mathbb{Z}}$:

- generate for each cell k independently and uniformly a random number r_k in $[0, 1]$,
- choose the new state of the cell k to be

$$\mathbf{0} \text{ if } r_k < f((x_{k+v})_{v \in \mathcal{N}})(0), \text{ and } \mathbf{1} \text{ otherwise.}$$

$$\ldots \quad x_{-3} \quad x_{-2} \quad x_{-1} \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad \ldots$$

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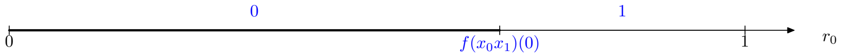

$$\mathbf{y_0}$$

$$\dots \quad x_{-3} \quad x_{-2} \quad x_{-1} \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad \dots$$

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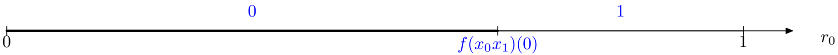


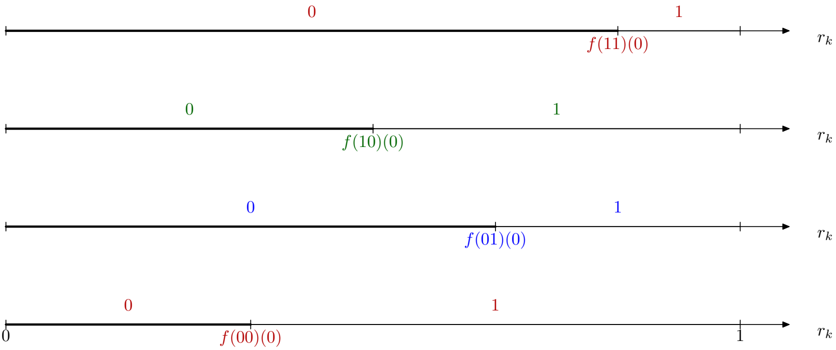
It defines an update function for F, given by:

$$\phi : \mathcal{A}^{\mathbb{Z}} \times [0,1]^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$$

$$\phi(x,r)_k = \begin{cases} \mathbf{0} \text{ if } r_k < f((x_i)_{i\in k+\mathcal{N}})(0) \\ \mathbf{1} \text{ otherwise.} \end{cases}$$

Update function of a PCA

Example: $\mathcal{A} = \{0,1\}$, neighbourhood $\mathcal{N} = \{0,1\}$



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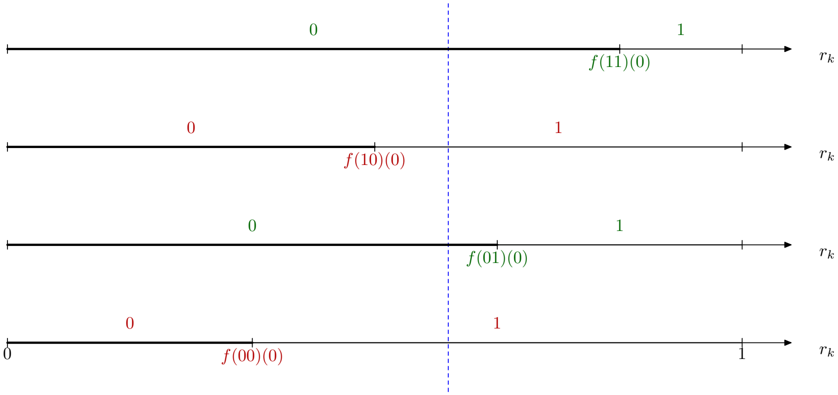

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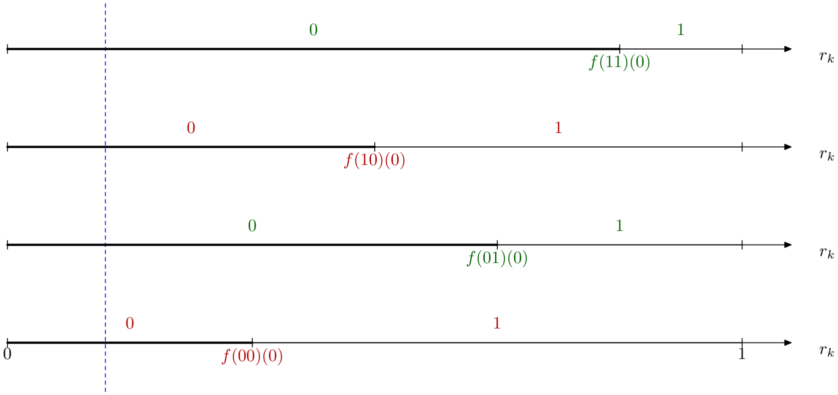

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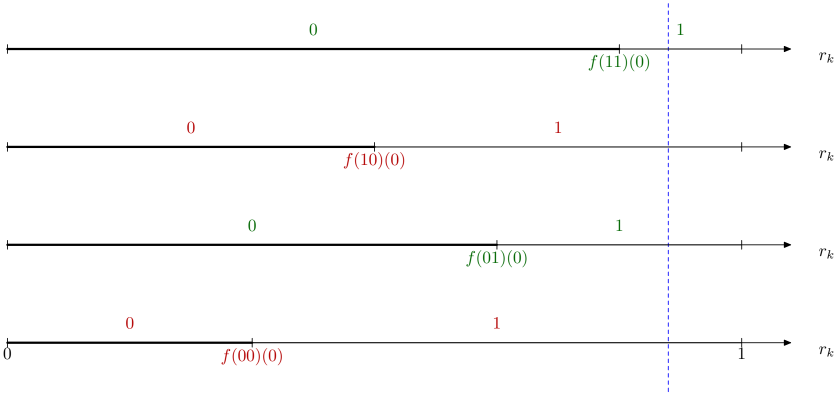
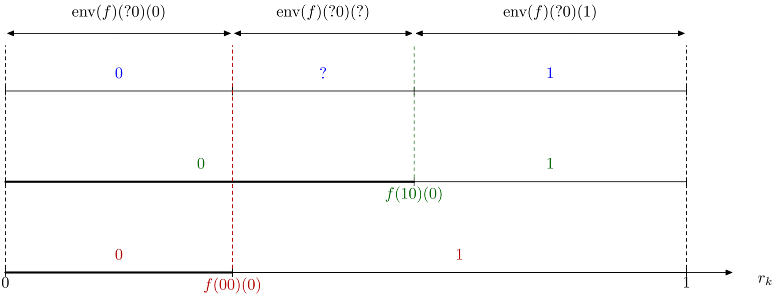


Envelope PCA

Introduction of an envelope PCA defined on the alphabet

$$\mathcal{B} = \{0 = \{0\}, 1 = \{1\}, ? = \{0,1\}\},$$

to handle configurations partially known.



The update function $\tilde{\phi}$ of env(P) satisfies for $x \in \mathcal{A}^{\mathbb{Z}}$ and $y \in \mathcal{B}^{\mathbb{Z}}$,

$$x \in y \Rightarrow \forall r \in [0,1]^{\mathbb{Z}}, \phi(x,r) \in \tilde{\phi}(y,r).$$

Coupling from the past algorithm

Let F be a PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$.

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$a \quad b$

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? ?

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? ?

? ? ? $(r_i^1)_{0 \leq i \leq 2}$

Coupling from the past algorithm

Let F be a PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0,1\}$, with $\mathcal{N} = \{0,1\}$.

?	**1**		
?	?	?	$(r_i^1)_{0 \leq i \leq 2}$

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$$\begin{array}{cccc} ? & 1 & & \\ ? & ? & ? & \\ ? & ? & ? & ? \end{array}$$

$(r_i^1)_{0 \leq i \leq 2}$
$(r_i^2)_{0 \leq i \leq 3}$

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?	1			
?	0	1		$(r_i^1)_{0\leq i\leq 2}$
?	?	?	?	$(r_i^2)_{0\leq i\leq 3}$

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?	?	?	?		$(r_i^2)_{0 \le i \le 3}$
?	?	?	?	?	$(r_i^3)_{0 \le i \le 4}$

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1	?	?	0			$(r_i^2)_{0\leq i\leq 3}$
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?	?	?	?	?	?	$(r_i^4)_{0\leq i\leq 5}$

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?	1					
?	0	1				$(r_i^1)_{0\leq i\leq 2}$
1	?	?	0			$(r_i^2)_{0\leq i\leq 3}$
?	0	?	1	?		$(r_i^3)_{0\leq i\leq 4}$
?	?	?	?	?	?	$(r_i^4)_{0\leq i\leq 5}$

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1	1	?	0			$(r_i^2)_{0\leq i\leq 3}$
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?	?	?	?	?	?	$(r_i^4)_{0\leq i\leq 5}$

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Proposition

If this algorithm stops a.s. then the PCA is ergodic, and the algorithm samples perfectly its unique invariant distribution.

Proposition

The algorithm stops a.s. if and only if the envelope PCA is ergodic.

Perfect sampling using EPCA

Proposition

The algorithm stops a.s. if and only if the envelope PCA is ergodic.

But the ergodicity of the envelope PCA is not equivalent to the ergodicity of the original PCA!

Ergodicity criterion

Let F be a PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$.

?

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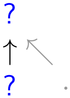

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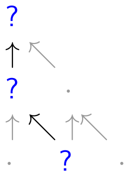

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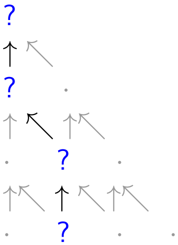

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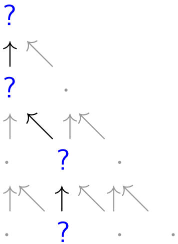

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Let
$$p_? = 1 - \min_{x \in \mathcal{A}^{\mathcal{N}}} f(x)(0) - \min_{x \in \mathcal{A}^{\mathcal{N}}} f(x)(1)$$
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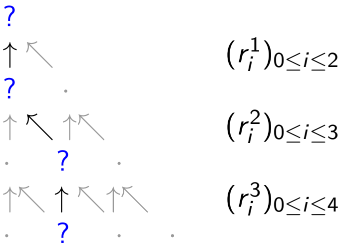

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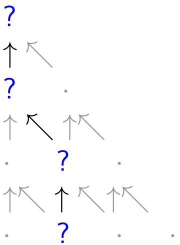

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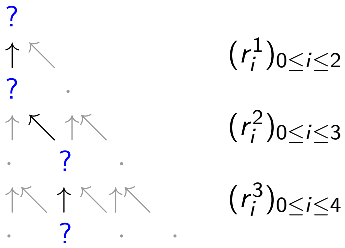

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Remark: $\alpha^* \geq 1/|\mathcal{N}|$.

The noisy Hardcore Cellular Automaton

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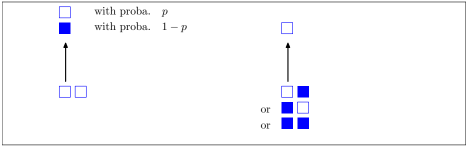
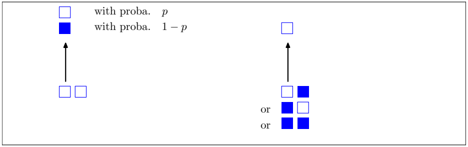


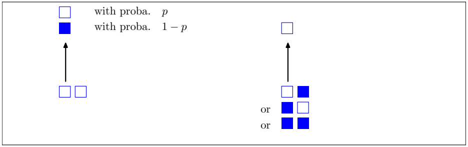
PCA F

The noisy Hardcore Cellular Automaton



PCA F

$p_{?} = 1 - p$



PCA F

$p_{?} = 1 - p$

By the ergodicity criterion, F is ergodic for $p > 1 - \alpha^* \approx 0.295$.

What can we say for small values of p?

The noisy Hardcore Cellular Automaton

Theorem [Holroyd-M.-Martin, arXiv 2015]

For any parameter $p \in (0,1)$, the noisy Hardcore CA is ergodic.

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Right-weight of a symbol "?" $=$

- 3 if it is followed by a 0, then by a 1,
- 2 if it is followed by a 0, then by something else than a 1,
- 1 otherwise.

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Total weight $=$left-weight $+$ right-weight.

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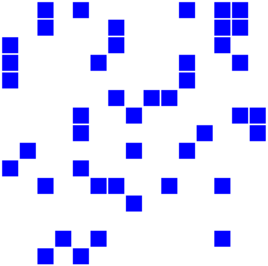
Total weight $=$left-weight $+$ right-weight.

Example: in 10??10, the first "?" has a weight 3+1=4 and the second one a weight $1 + 1 = 2$.

Definition of the percolation game

Grid $\mathbb{N} \times \mathbb{N}$, with each site colored in blue independently with probability p (here, $p = 0.2$).



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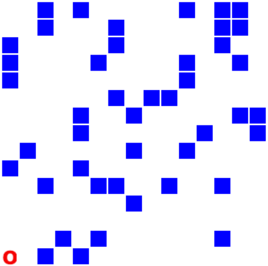

One token, that **two** players move alternatively, from position x to a white position among $x + (0, 1)$ or $x + (1, 0)$.

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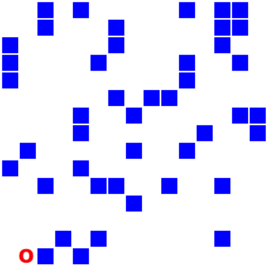

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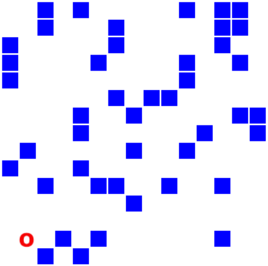

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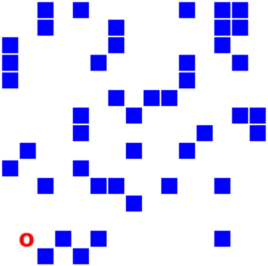

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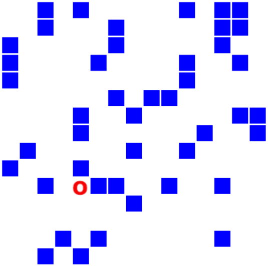

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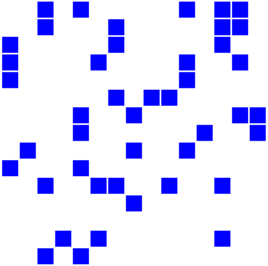
Winning positions, loss and draws

A position is:

- a win (**W**) if from this position, the player whose turn it is to play has a winning strategy,
- a loss (**L**) if from this position, the other player has a winning strategy,
- a draw (**D**) if neither player has a winning strategy, so that with "best play", the game will continue for ever.



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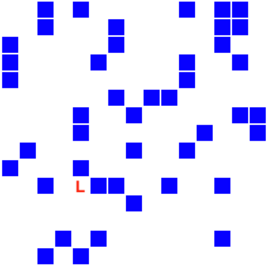

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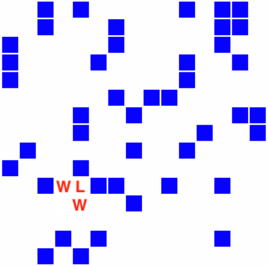

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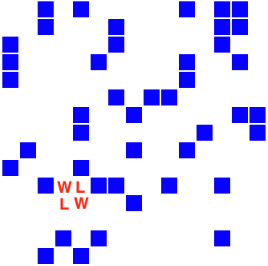

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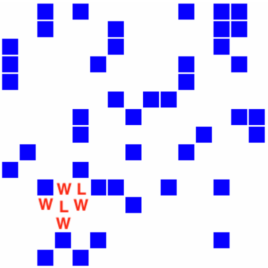

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Question: what can we say for the percolation game on $\mathbb{N}^3$?

Noisy Nilpotent CA

Nilpotent CA with noise

Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a nilpotent CA.
There exists $\alpha \in \mathcal{A}$ and $s \geq 1$ such that: $\forall x \in \mathcal{A}^{\mathbb{Z}}, F^s(x) = \alpha^{\mathbb{Z}}$.

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Let R be any probabilistic cellular automaton on $\mathcal{A}^{\mathbb{Z}}$.

We denote by F_ε the PCA that consists in, for each cell independently, choosing to apply

- the local rule of F with probability $1 - \varepsilon$,
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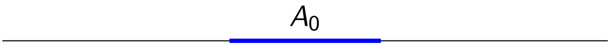
Proposition [M.-Sablik-Taati]

If ε is small enough, then the PCA F_ε is ergodic.

Nilpotent CA with noise

$t = 0$



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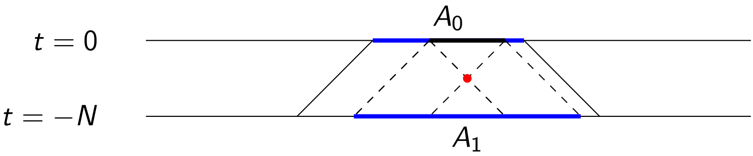

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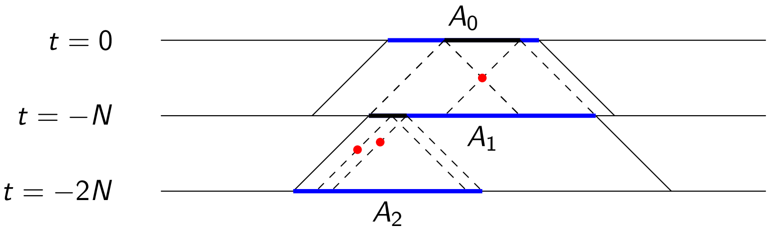

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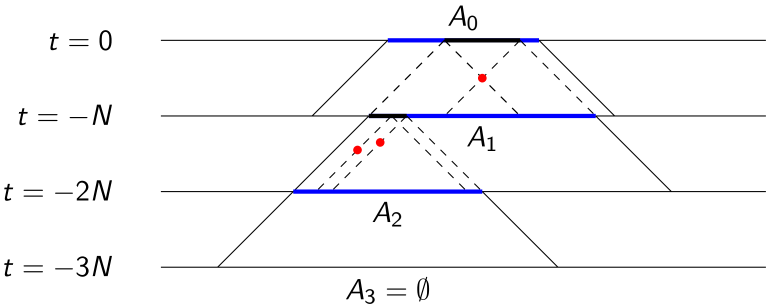

Spreading CA with noise

Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a deterministic CA. We say that the letter $\alpha \in \mathcal{A}$ is a spreading symbol for F if

$$(x_{k+n} = \alpha \;\text{ for some }\; n \in \mathcal{N}) \quad \Longrightarrow \quad F(x)_k = \alpha.$$

Spreading CA with noise

Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a deterministic CA. We say that the letter $\alpha \in \mathcal{A}$ is a spreading symbol for F if

$$(x_{k+n} = \alpha \text{ for some } n \in \mathcal{N}) \implies F(x)_k = \alpha.$$

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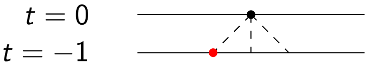
Proposition [M.-Sablik-Taati]

If $\varepsilon > 0$ and $p(\alpha) > 0$, then the PCA F_ε is ergodic.

Spreading CA with noise

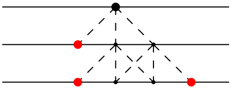
$t = 0$

Spreading CA with noise



Spreading CA with noise

$t = 0$
$t = -1$
$t = -2$



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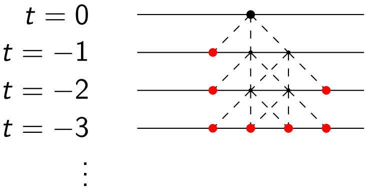

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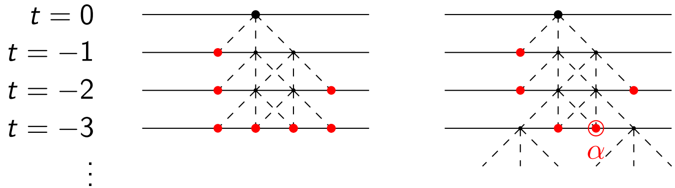

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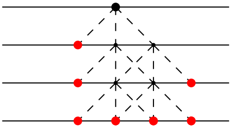
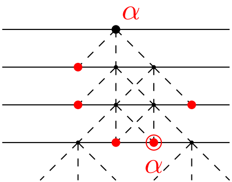
$\vdots$





Spreading CA with noise

Remark: the hypothesis on the noise is not the same as for nilpotent CA.
Small ε-perturbation *versus* memoryless zero-range noise given by the distribution p.

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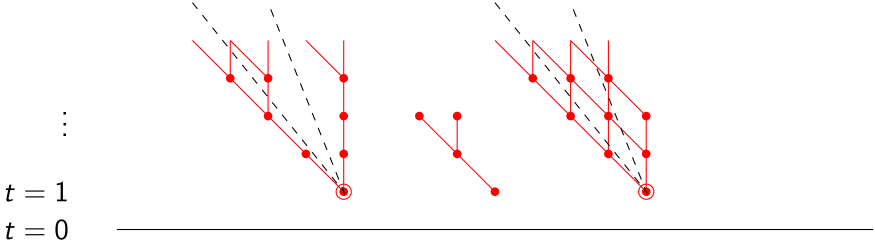
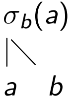


Noisy Permutive CA

Permutive CA

A deterministic CA F of neighb. $\mathcal{N} = \{0, 1\}$ is left-permutive if:

$$\forall b \in \mathcal{A},\ \exists \text{ permutation } \sigma_b \text{ of } \mathcal{A} \text{ such that } f(ab) = \sigma_b(a).$$



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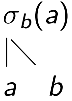

Example

Additive CA: $\mathcal{A} = \mathbb{Z}/n\mathbb{Z}$ and $f(ab) = a + b$.

Permutive CA with noise

Let F be a left-permutive CA.

We denote by F_ε the PCA that consists in, for each cell independently,

- applying the local rule of F with probability $1 - \varepsilon$,
- choosing uniformly a symbol in $\mathcal{A}$ with probability ε.

Proposition [M.-Sablik-Taati]

For any $\varepsilon \in (0, 1)$, the PCA F_ε is ergodic.

Permutive CA with noise

Each $a \in \mathcal{A}$ induces a permutation σ_a on $\mathcal{A}^n$:

$$y_1 \dots y_n = \sigma_a(x_1 \dots x_n).$$

$$\begin{array}{llllll} t=1 & y_1 & y_2 & \dots & y_n & \\ t=0 & x_1 & x_2 & \dots & x_n & a \end{array}$$

When adding the noise R, each $a \in \mathcal{A}$ induces a Markov chain P_a on $\mathcal{A}^n$:

$$\tilde{y}_1 \dots \tilde{y}_n \sim P_a\,(x_1 \dots x_n,\ \bullet\ ).$$

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P_a is irreducible and aperiodic for any $a \in \mathcal{A}$.
The uniform measure on $\mathcal{A}^n$ is invariant under P_a for any $a \in \mathcal{A}$.

Permutive CA with noise

For each $a \in \mathcal{A}$, there exists $\theta_a < 1$ such that:

$$||P_a\mu - P_a\nu||_1 \leq \theta_a||\mu - \nu||_1.$$

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In particular, for $\nu = \lambda_n$ (uniform measure on $\mathcal{A}^n$), we obtain that for any distribution μ on $\mathcal{A}^n$ and any $a_1, \ldots, a_t \in \mathcal{A}$,

$$||P_{a_t} \ldots P_{a_2}P_{a_1}\mu - \lambda_n||_1 \leq \theta^t||\mu - \lambda_n||_1 \leq 2\theta^t.$$

Interacting gliders

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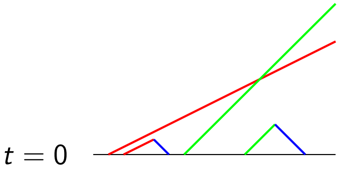
- Gliders with annihilation rules



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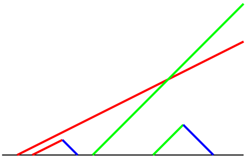
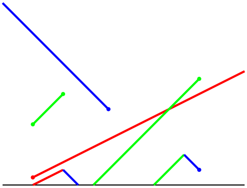
$t = 0$  

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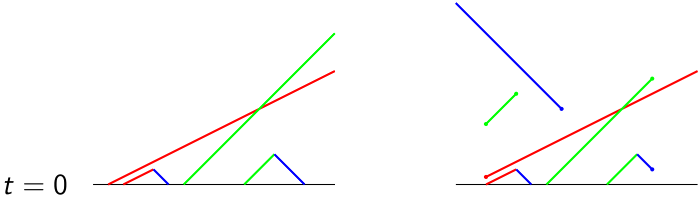
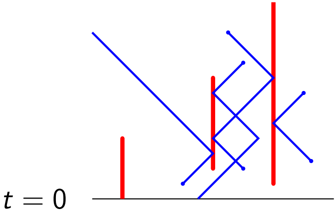


- Gliders with reflecting walls



Perspectives

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Thank you for your attention!