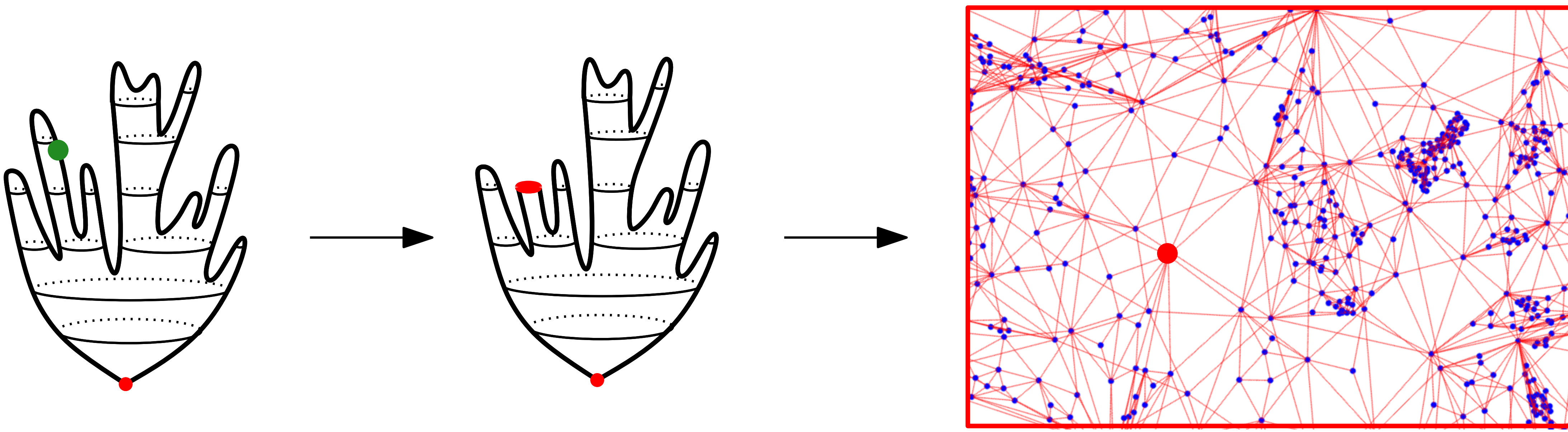
Triangulations with spins: algebraicity and local limit

Laurent Ménard (Paris Nanterre)

joint work with **Marie Albenque** and **Gilles Schaeffer** (CNRS and LIX)



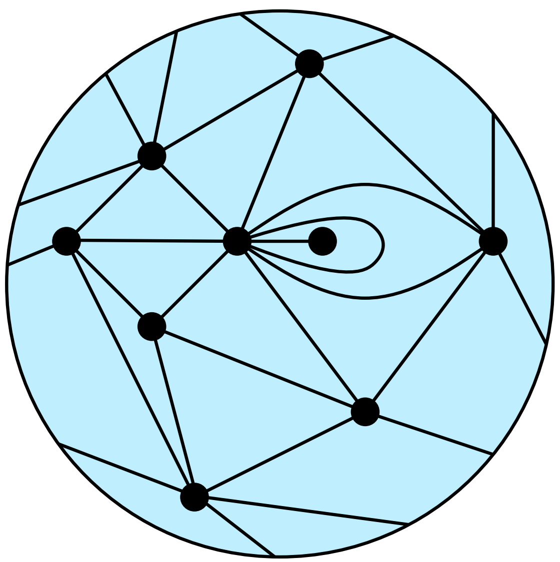
Séminaire Philippe Flajolet, septembre 2017

Outline

1. Motivation (is Watabiki right?)
2. Local weak topology
3. Combinatorics of triangulations with spins
4. Local limit of triangulations with spins

Planar Maps as discrete planar metric spaces

Definition:
A **planar map** is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).



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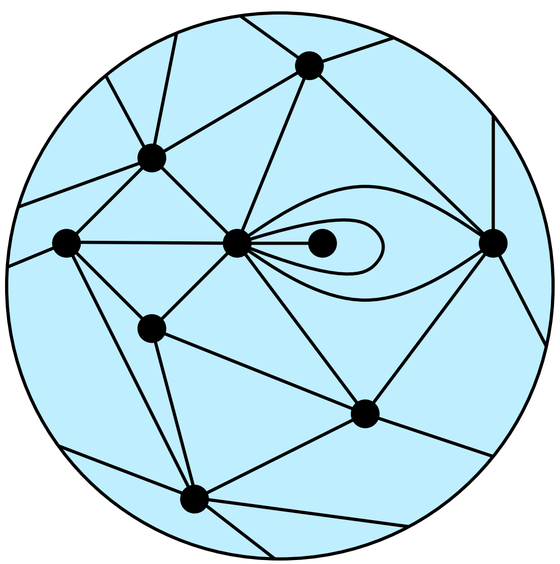

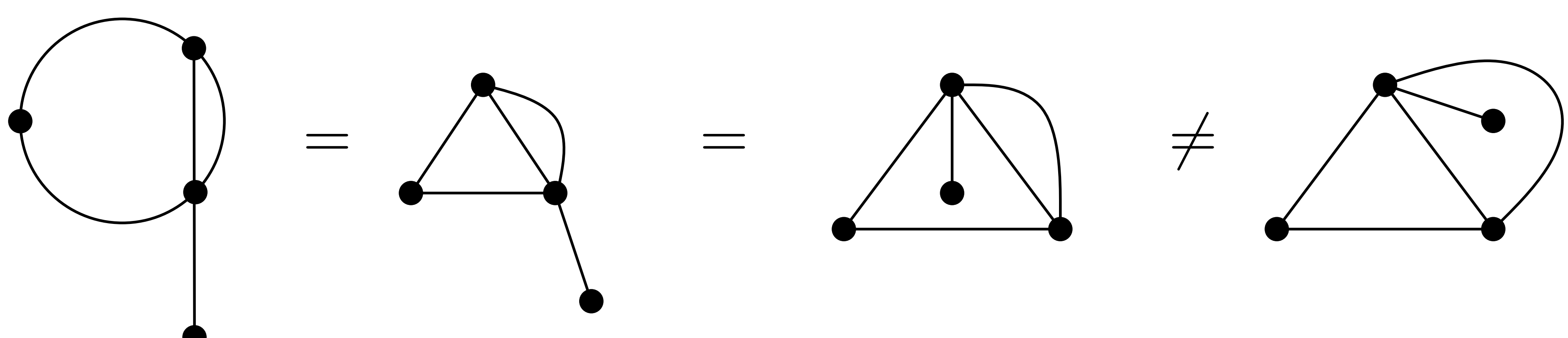

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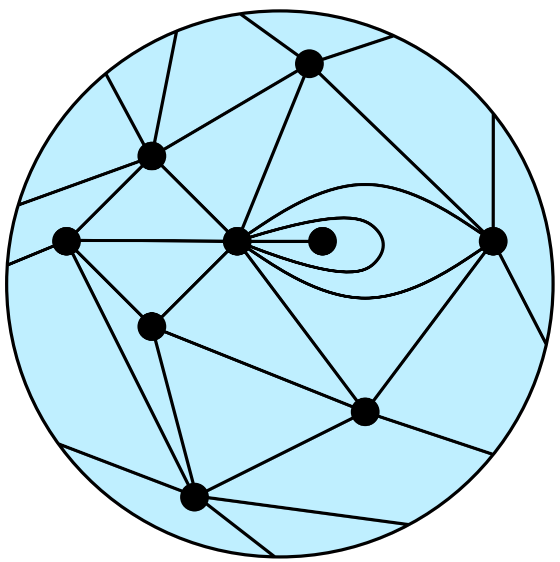
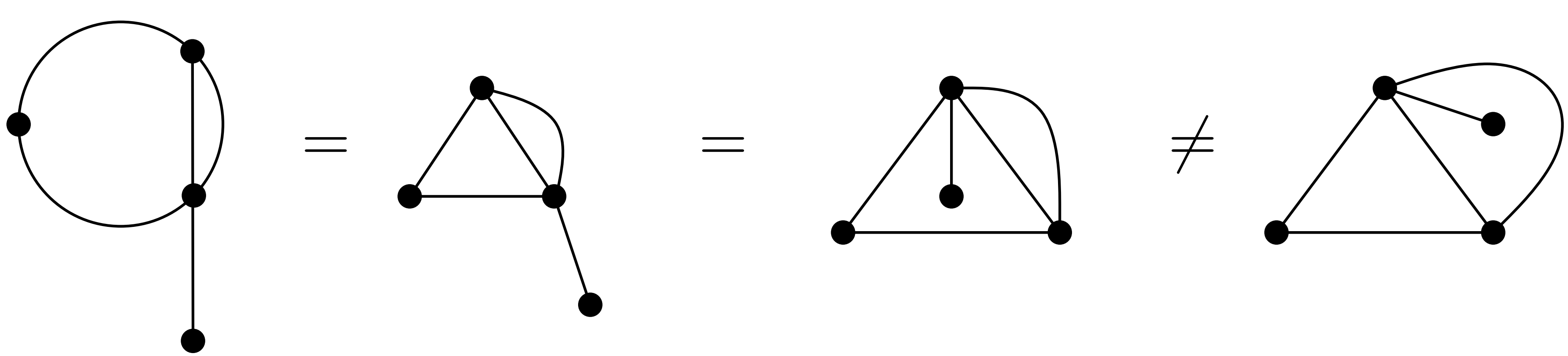


faces: connected components of the complement of edges

p-angulation: each face is bounded by p edges



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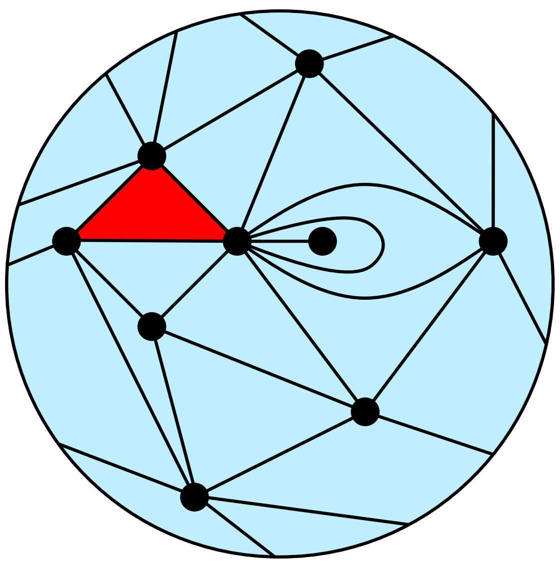

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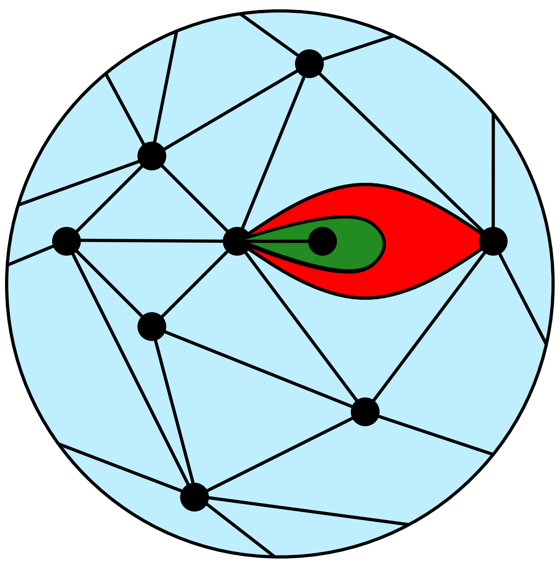

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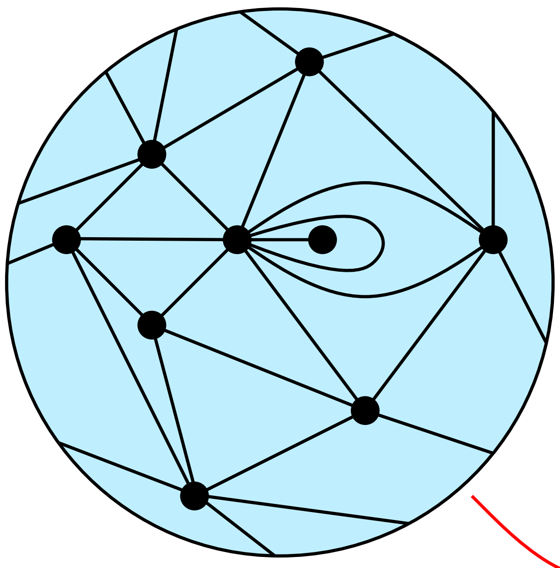

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This is a triangulation

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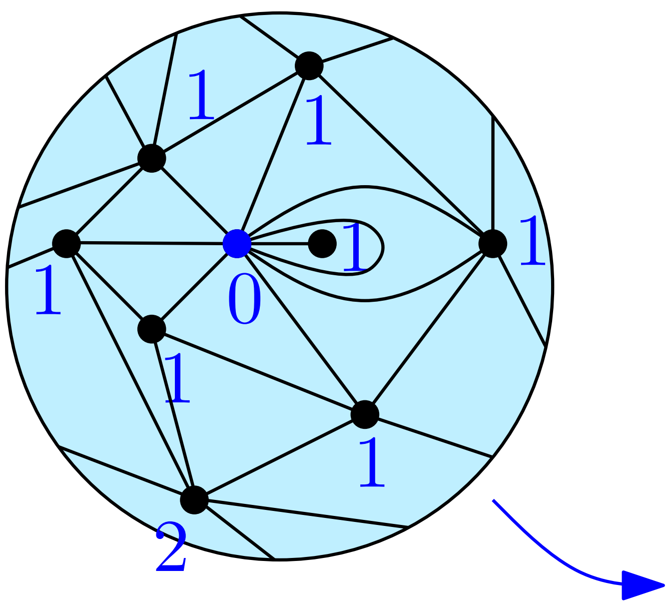

In blue, distances from •

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- $V(M) :=$ set of vertices of M
- $d_{gr} :=$ graph distance on $V(M)$
- $(V(M), d_{gr})$ is a (finite) metric space

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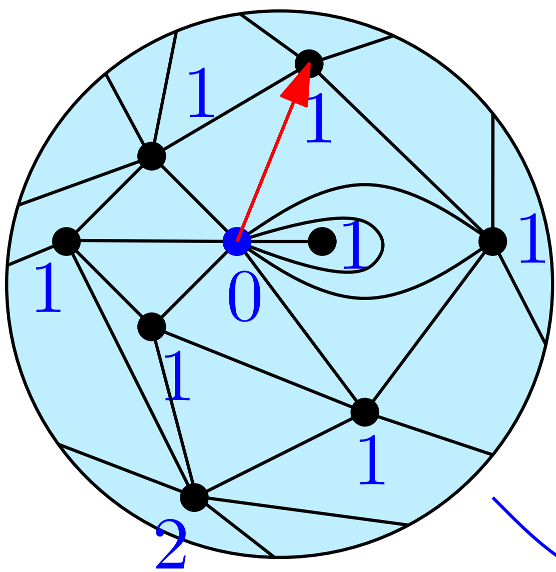

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Rooted map: mark an oriented edge of the map →

"Classical" large random triangulations

Take a triangulation with n edges uniformly at random. What does it look like if n is large ?

Two points of view: global/local, continuous/discrete

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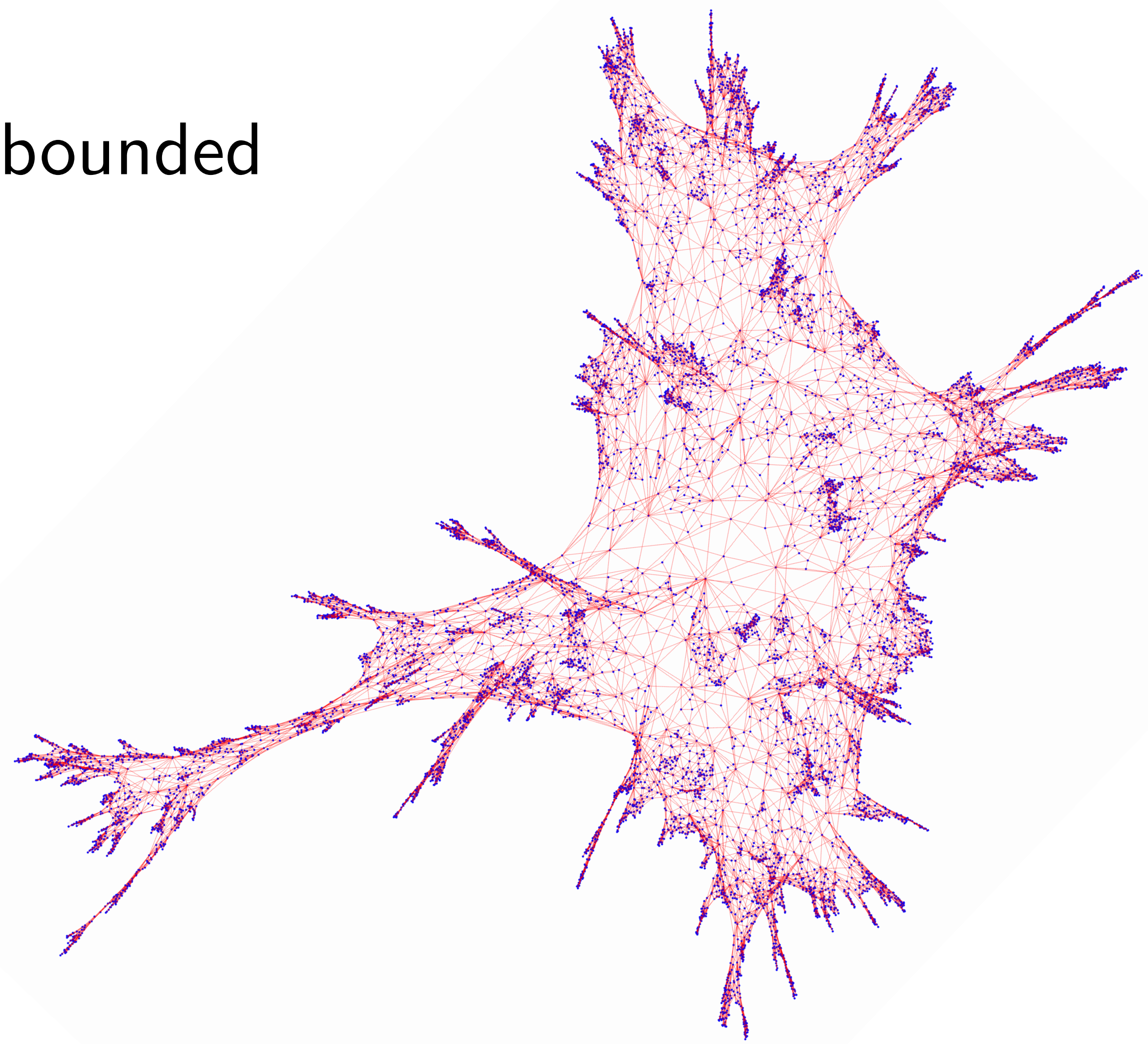
Global :
Rescale distances to keep diameter bounded

[Le Gall 13, Miermont 13]:
converges to the **Brownian map**.

- Gromov-Hausdorff topology
- Continuous metric space
- Homeomorphic to the sphere
- Hausdorff dimension 4
- **Universality**



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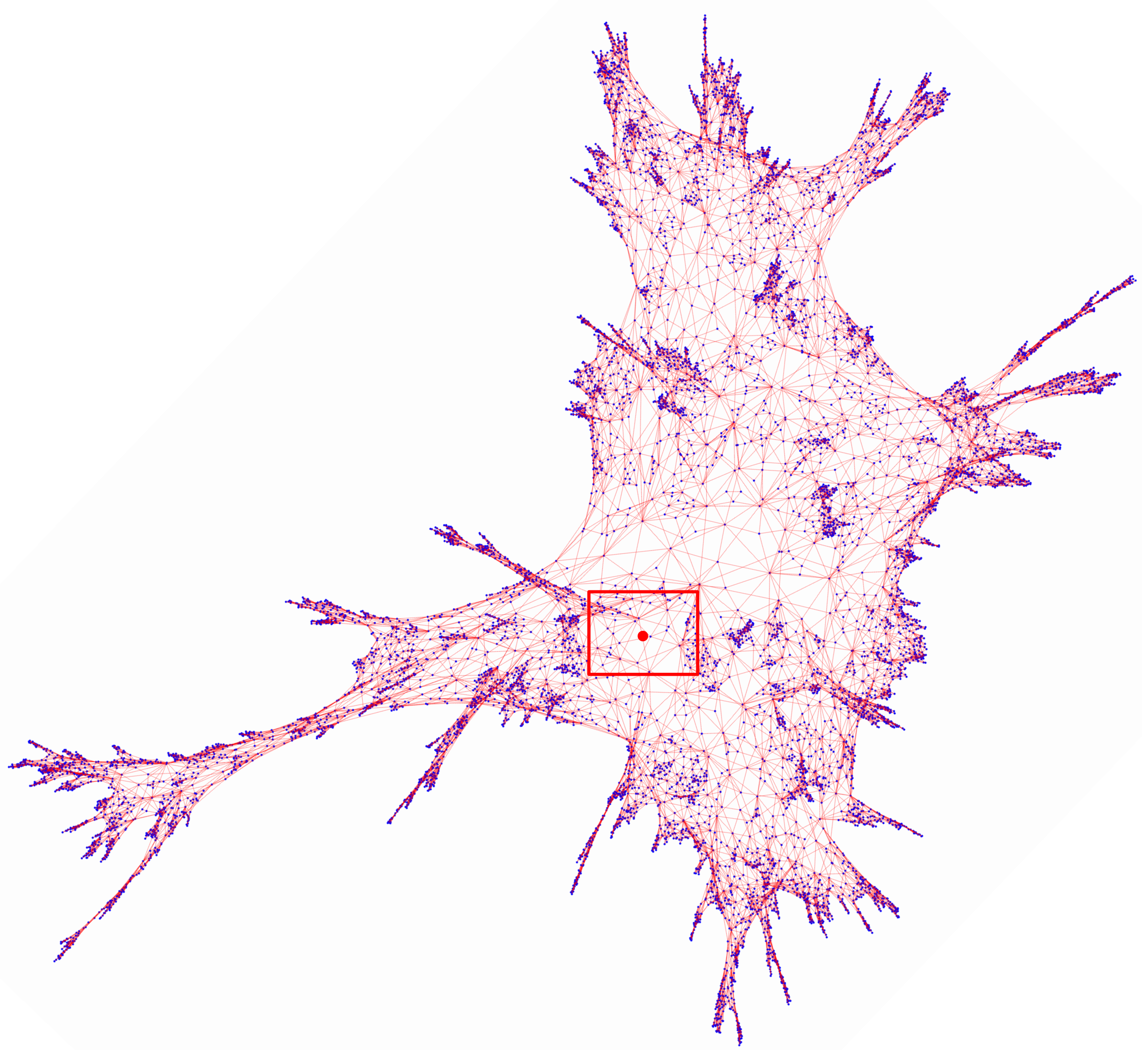
Local :
Don't rescale distances and look at neighborhoods of the root



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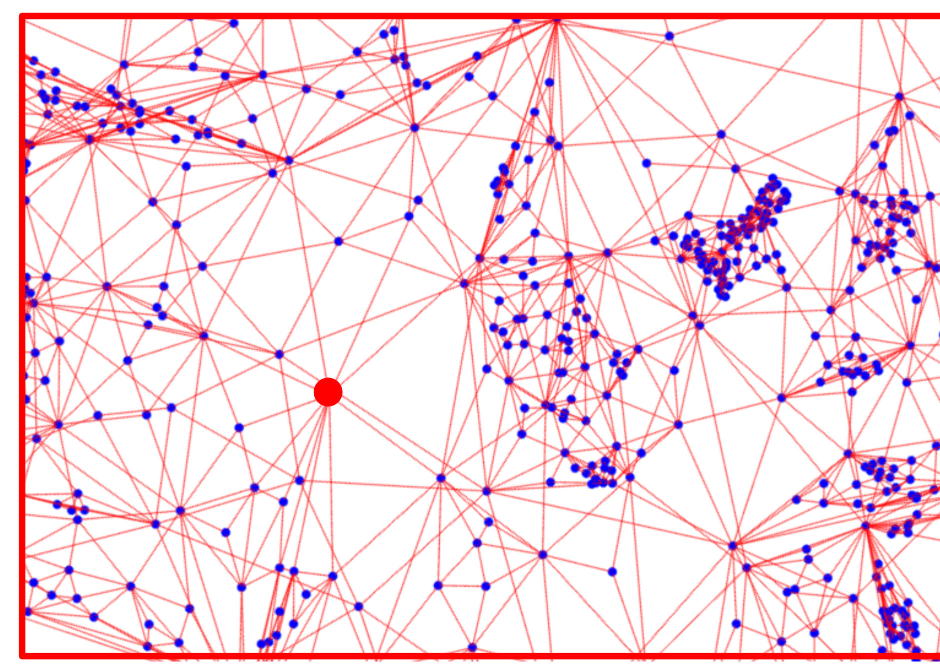
[Angel – Schramm 03, Krikun 05]:
Converges to the **Uniform Infinite Planar Triangulation**



- Local topology
- Metric balls of radius R grow like R^4
- **"Universality"** of the exponent 4.

Adding matter: Ising model on triangulations

How does Ising model influence the underlying map?

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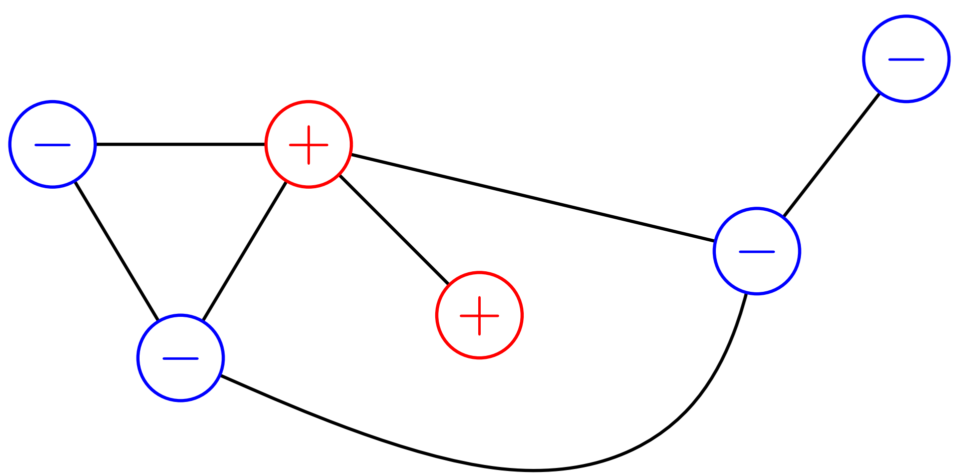
$G = (V, E)$ finite graph



Spin configuration on G:

$$\sigma : V \to \{-1, +1\}.$$

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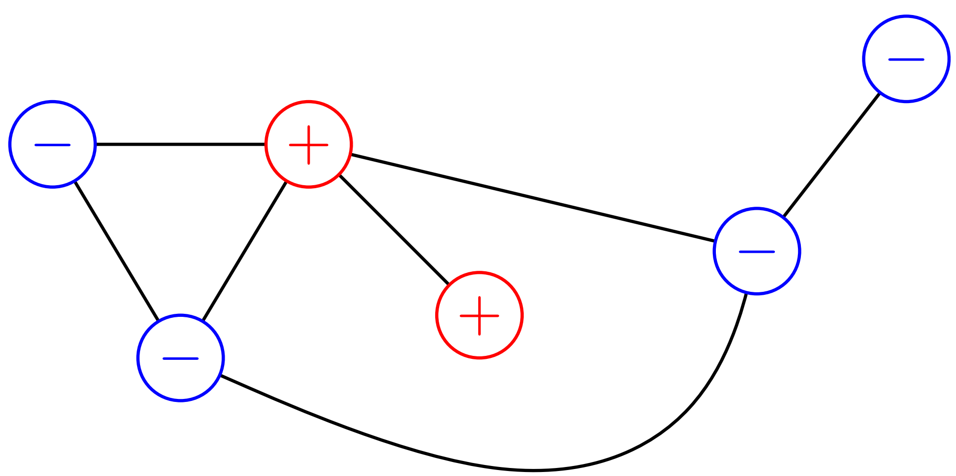

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Ising model on G: take a random spin configuration with probability

$$P(\sigma) \propto e^{-\frac{\beta}{2} \sum_{v \sim v'} \mathbf{1}_{\{\sigma(v) \neq \sigma(v')\}}}$$

$\beta > 0$: inverse temperature.

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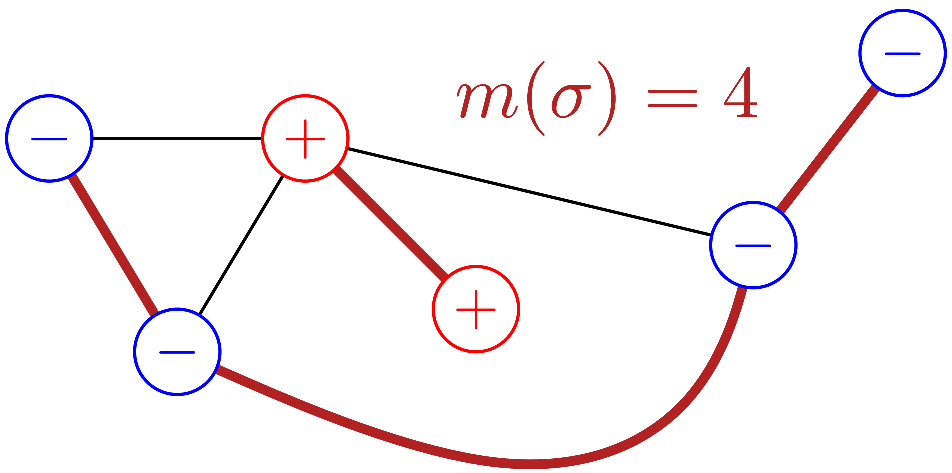

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Combinatorial formulation: $P(\sigma) \propto \nu^{m(\sigma)}$
with $m(\sigma)$ = number of monochromatic edges and $\nu = e^{\beta}$.

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$\mathcal{T}_n = \{$rooted planar triangulations with $3n$ edges$\}$.

Random triangulation in $\mathcal{T}_n$ with probability $\propto \nu^{m(T,\sigma)}$?

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Generating series of **Ising-weighted triangulations**:

$$Q(\nu, t) = \sum_{T \in \mathcal{T}_f} \sum_{\sigma : V(T) \to \{-1,+1\}} \nu^{m(T,\sigma)} t^{e(T)}.$$

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Theorem [Bernardi – Bousquet-Mélou 11]
For every ν the series $Q(\nu,t)$ is algebraic, has $\rho_\nu > 0$ as unique dominant singularity and satisfies

$$[t^{3n}]Q(\nu,t) \underset{n\to\infty}{\sim} \begin{cases} \kappa\, \rho_{\nu_c}^{-n}\, n^{-7/3} & \text{if } \nu = \nu_c = 1 + \frac{1}{\sqrt{7}}, \\ \kappa\, \rho_{\nu}^{-n}\, n^{-5/2} & \text{if } \nu \neq \nu_c. \end{cases}$$

This suggests an unusual behavior of the underlying maps for $\nu = \nu_c$.
See also [Boulatov – Kazakov 1987], [Bousquet-Mélou – Schaeffer 03] and [Bouttier – Di Francesco – Guitter 04].

Adding matter: Watabiki's (controversial?) predictions

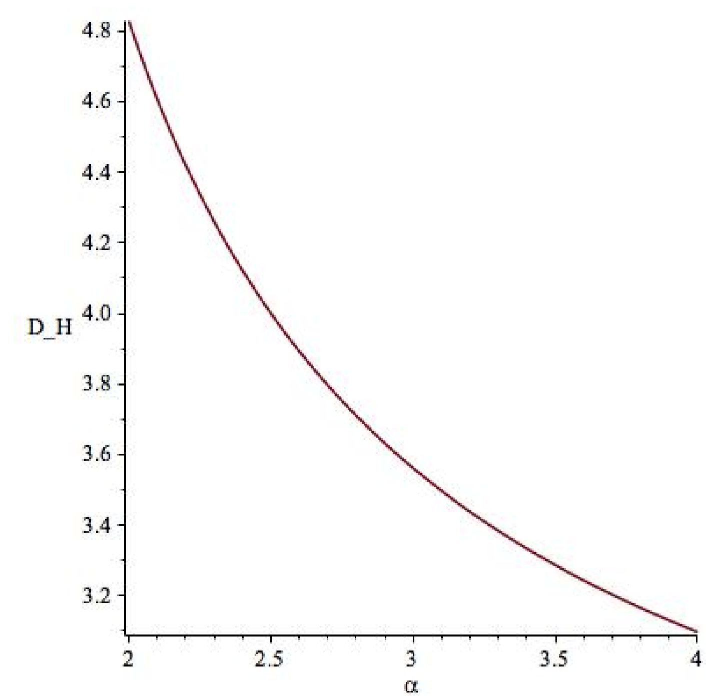
Counting exponent:
coeff $[t^n]$ of generating series of (decorated) maps $\sim \kappa \rho^{-n} n^{-\alpha}$

Central charge c:

$$\alpha = \frac{25 - c + \sqrt{(1-c)(25-c)}}{12}$$

Hausdorff dimension: [Watabiki 93]

$$D_H = 2\frac{\sqrt{25-c} + \sqrt{49-c}}{\sqrt{25-c} + \sqrt{1-c}}$$



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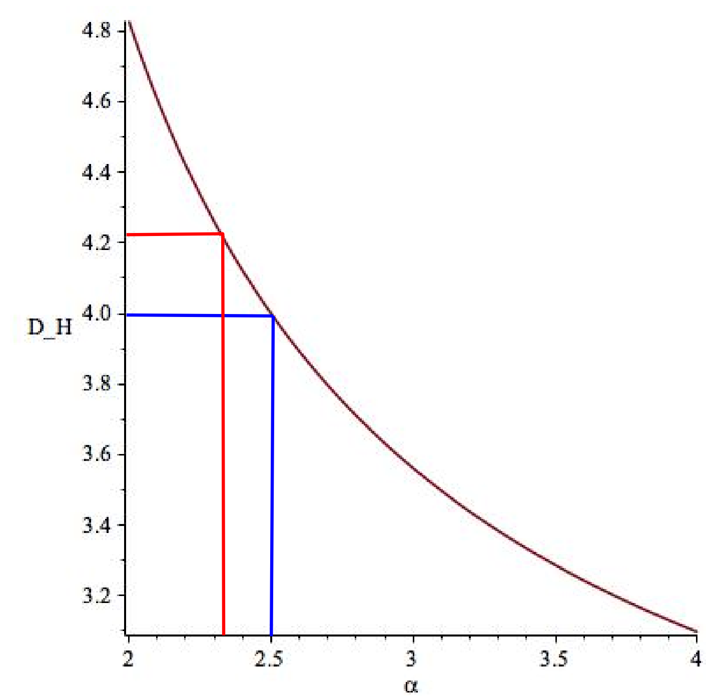
- $\alpha = 5/2$ gives $D_H = 4$
- $\alpha = 7/3$ gives $D_H = \frac{7+\sqrt{97}}{4} \approx 4.21$



Local convergence of triangulations with spins

Probability measure on triangulations of $\mathcal{T}_n$ with a spin configuration:

$$\mathbb{P}_n^{\nu}\Big(\{(T,\sigma)\}\Big) = \frac{\nu^{m(T,\sigma)}}{[t^{3n}]Q(\nu,t)}.$$

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Theorem [Albenque – M. – Schaeffer]
As $n \to \infty$, the sequence $\mathbb{P}_n^\nu$ converges **weakly** to a probability measure $\mathbb{P}^\nu$ for the **local topology**.
The measure $\mathbb{P}^\nu$ is supported on infinite triangulations with **one end**.

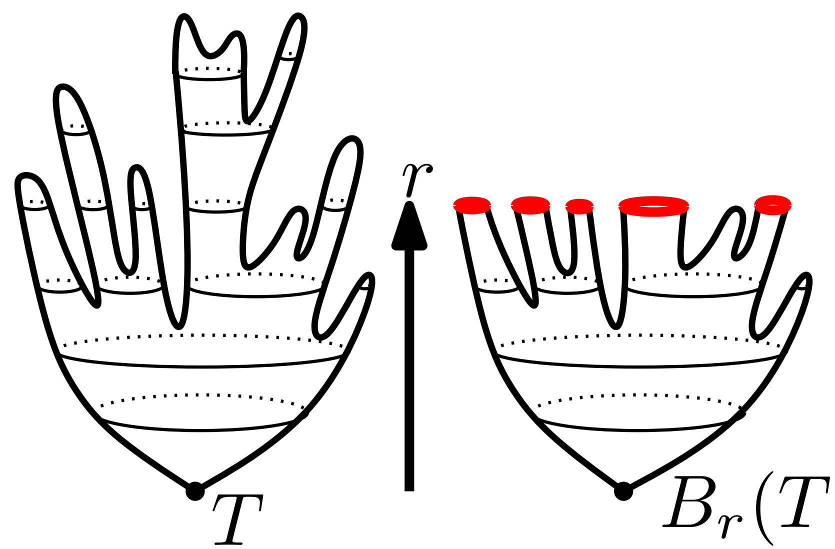
Local topology

$$\mathcal{T}_f := \{\text{finite rooted planar triangulations with spins}\}.$$

Definition:
The **local topology** on $\mathcal{T}_f$ is induced by the distance:

$$d_{loc}(T,T') := (1 + \max\{r \geq 0 : B_r(T) = B_r(T')\})^{-1}$$

where $B_r(T)$ is the submap (with spins) of T composed by the faces of T with a vertex at distance $< r$ from the root.



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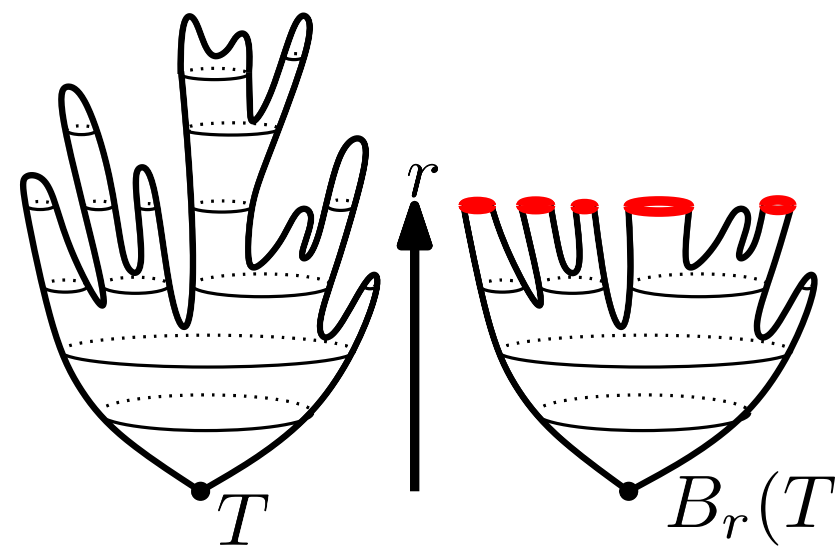
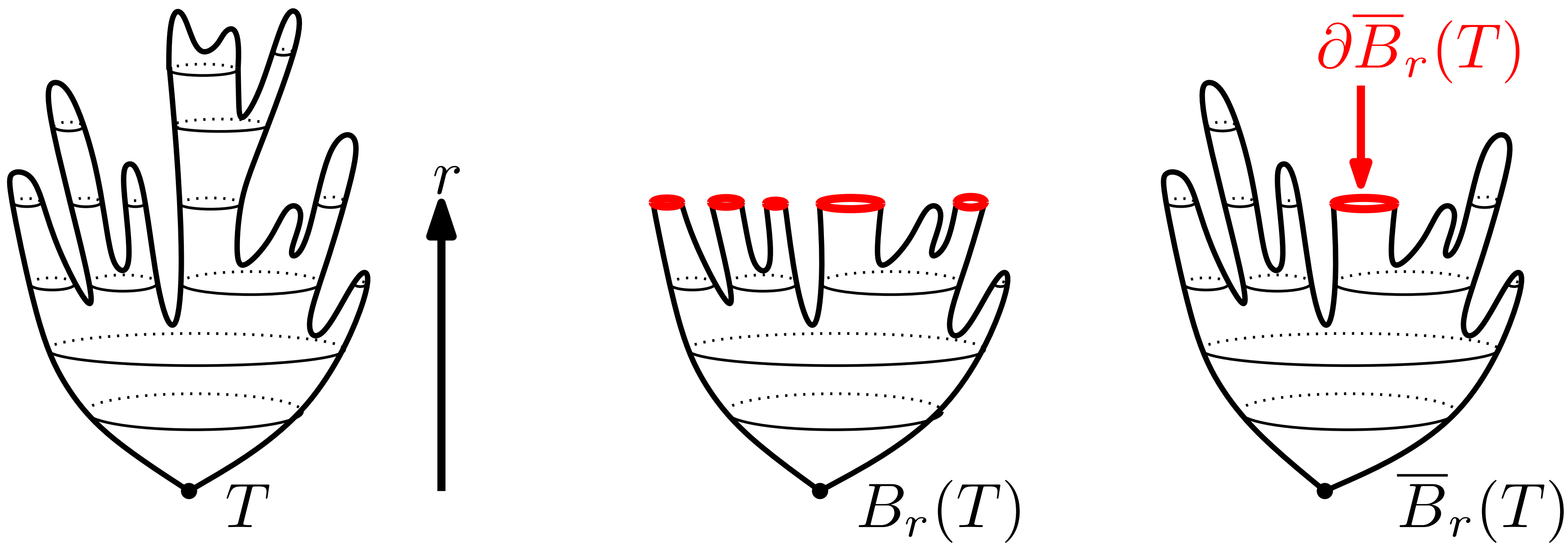


- $(\mathcal{T}, d_{loc})$: closure of $(\mathcal{T}_f, d_{loc})$. It is a **Polish** space.
- $\mathcal{T}_\infty := \mathcal{T} \setminus \mathcal{T}_f$ set of **infinite** planar triangulations with spins.

Local topology: Hulls

Balls $B_r(T)$ not practical (multiple holes). Take **hulls** instead:

$$\overline{B}_r(T) := \begin{array}{l}\text{everything not in the largest connected}\\ \text{component of } T \setminus B_r(T)\end{array}$$



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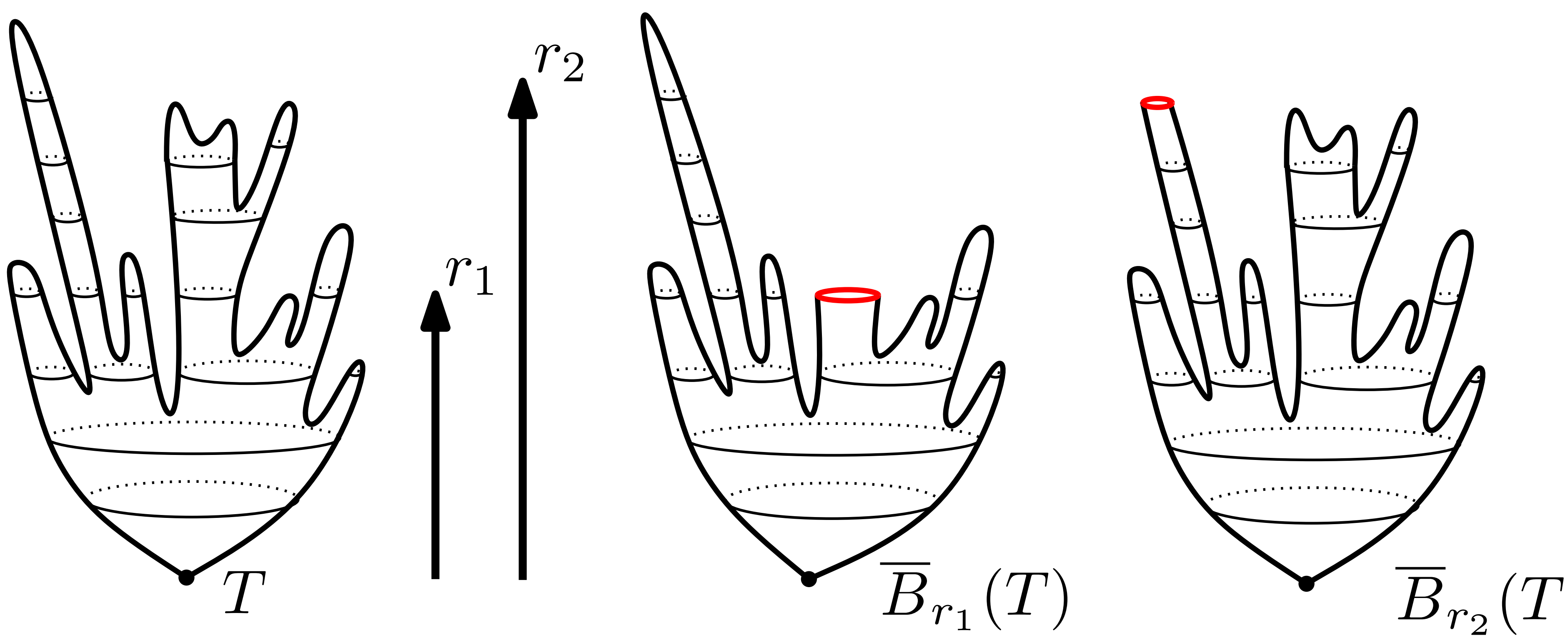
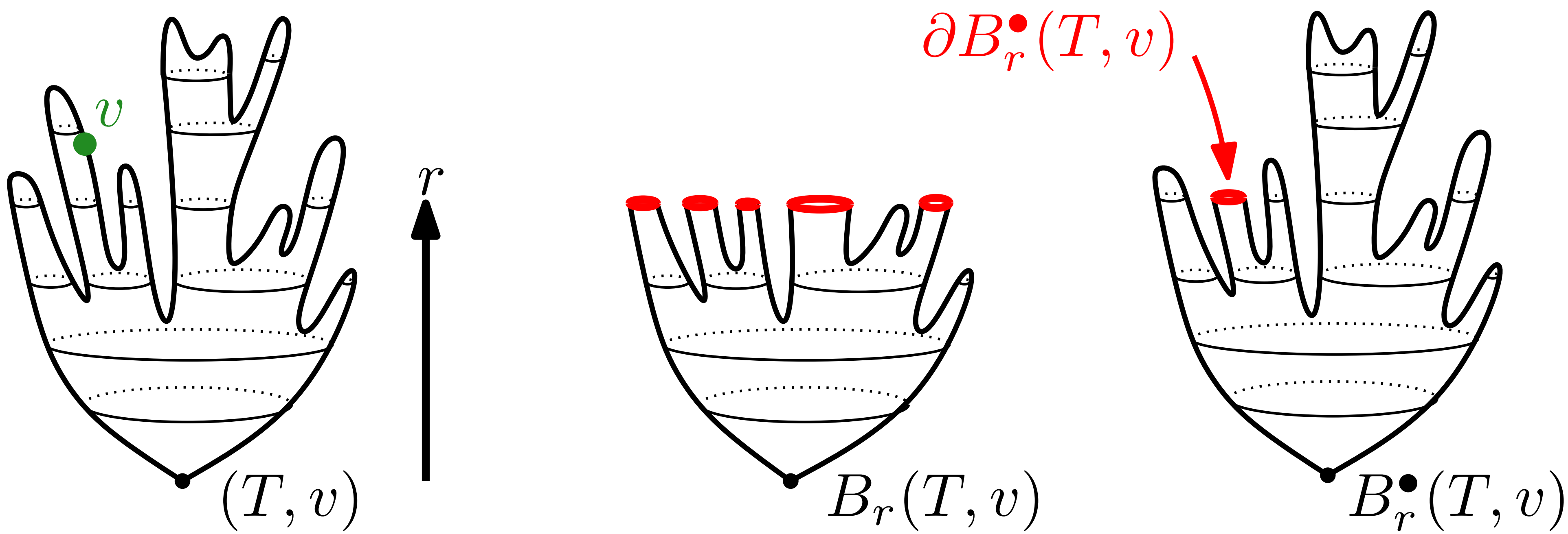


Problem: Hulls are not **nested** !

Local topology: Pointed hulls

For $(T,v) \in \mathcal{T}_f^\bullet :=$ { finite rooted triangulations with pointed vertex }

$$B_r^\bullet(T,v) = \begin{cases} (T,v) & \text{if } v \in B_r(T); \\ B_r(T) \text{ and the connected components of } T \setminus B_r(T) \text{ that do not contain } v & \text{if } v \notin B_r(T). \end{cases}$$



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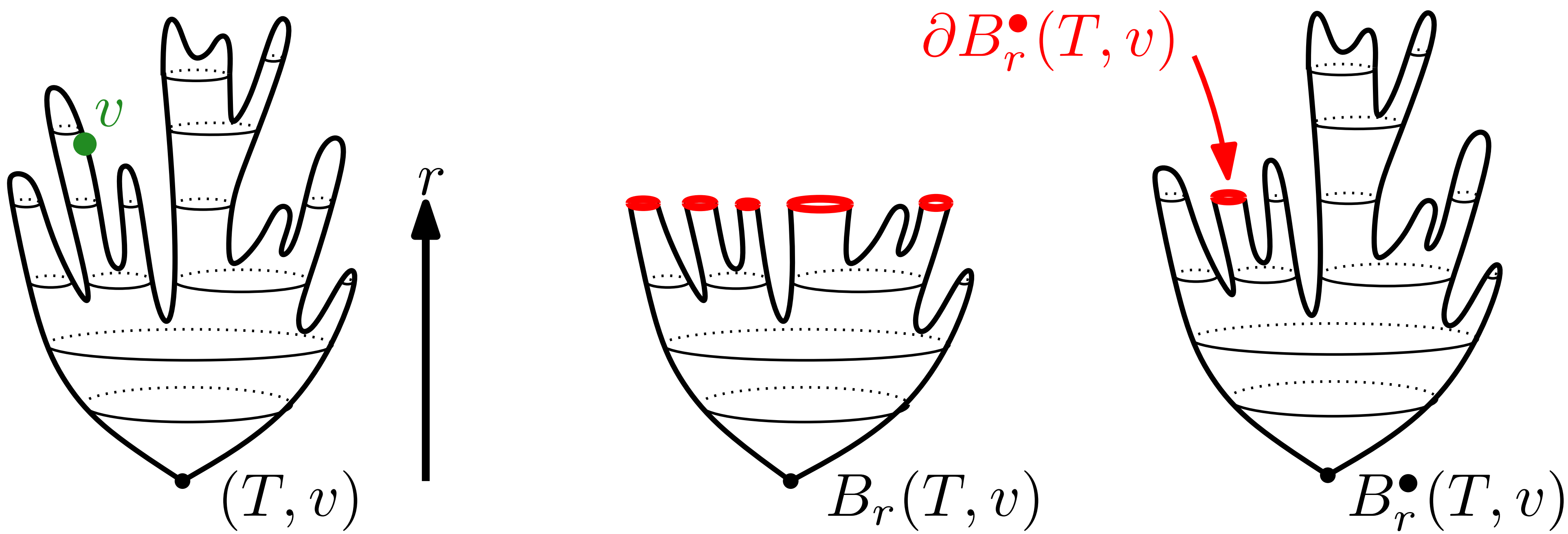

Convergence for $d_{loc}^\bullet \Rightarrow$ convergence for d_{loc} with the same limit.

Weak convergence for the local topology

Portemanteau theorem + Levy – Prokhorov metric:

The measures $\mathbb{P}_n^\bullet$ converge weakly to $\mathbb{P}^\nu$ if

1. For every $r > 0$ and every possible hull Δ

$$\mathbb{P}_n^\bullet\Big(\{(T,v) \in \mathcal{T}_n \,:\, B_r^\bullet(T,v) = \Delta\} \Big) \underset{n\to\infty}{\longrightarrow} \mathbb{P}^\nu\Big(\{T \in \mathcal{T}_\infty \,:\, B_r^\bullet(T) = \Delta\} \Big).$$

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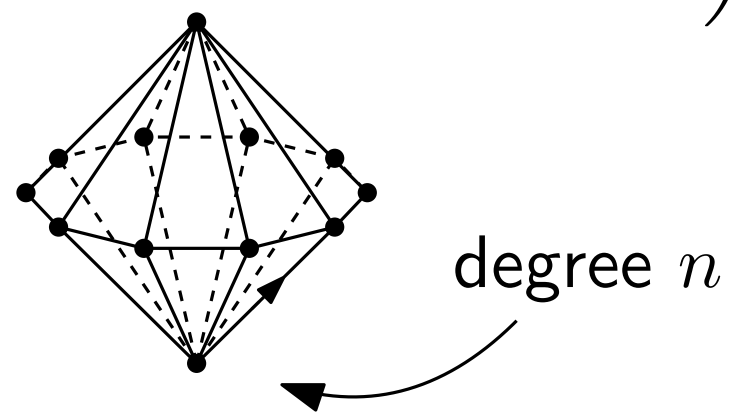

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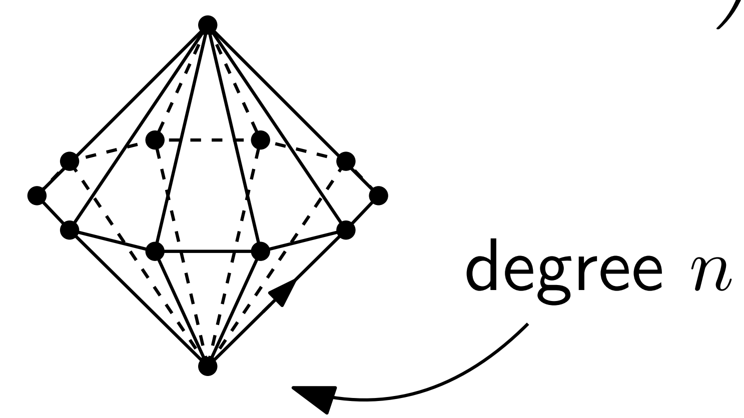

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The measure $\mathbb{P}^\nu$ defined by the limits in 1. **is a probability measure**.

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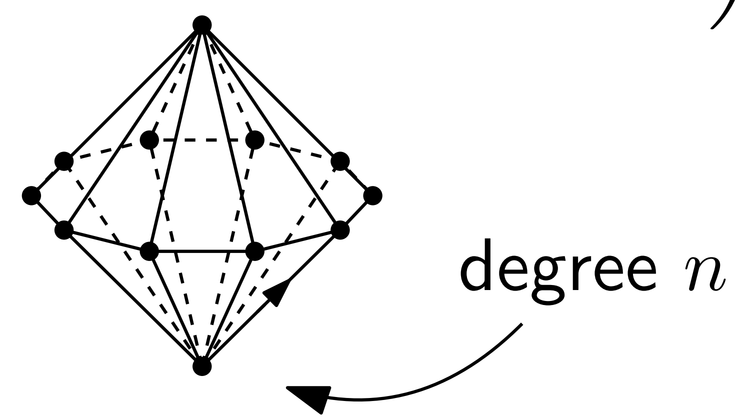

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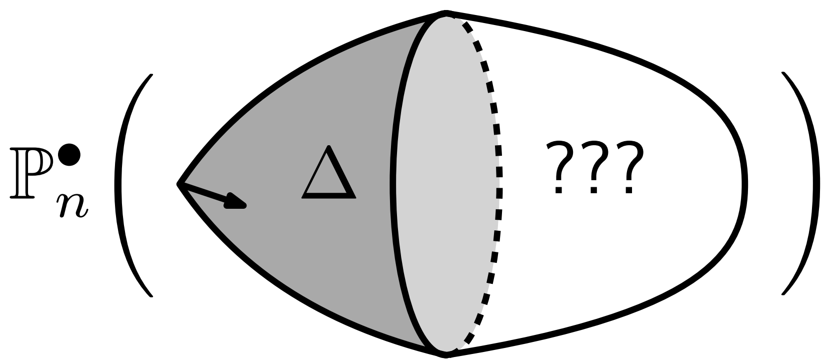
The measure $\mathbb{P}^\nu$ defined by the limits in 1. **is a probability measure**.

True if $\forall r \geq 0, \quad \sum_{r-\text{hulls}\,\Delta} \mathbb{P}^\nu\Big(\{T \in \mathcal{T}_\infty \,:\, B_r^\bullet(T) = \Delta\} \Big) = 1.$

Local convergence and generating series

Need to evaluate, for every possible hull Δ



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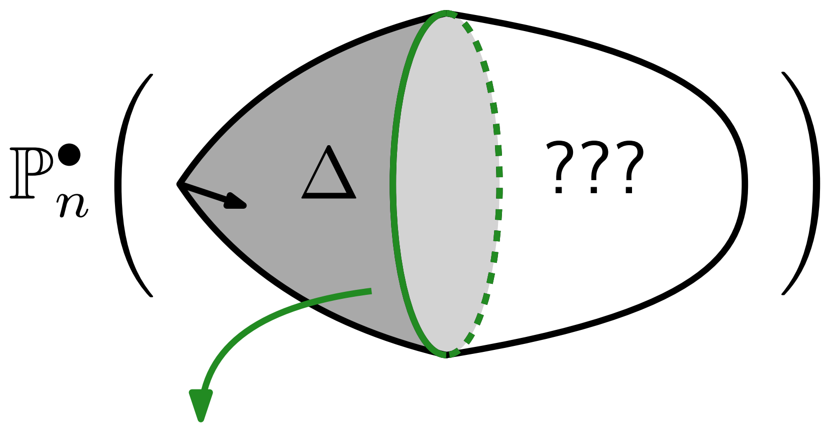

Simple (rooted) cycle,
spins given by a word ω

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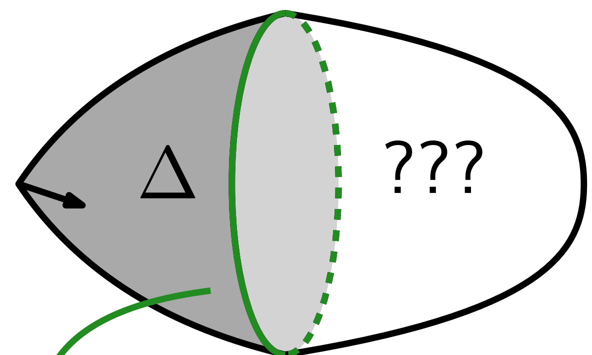

$$\mathbb{P}_n^\bullet\left(\cdots \right) = \frac{\nu^{m(\Delta)-m(\omega)}\,[t^{3n-e(\Delta)+|\omega|}]\mathbf{Z}_\omega^\bullet(\nu,t)}{[t^{3n}]Q^\bullet(\nu,t)}$$

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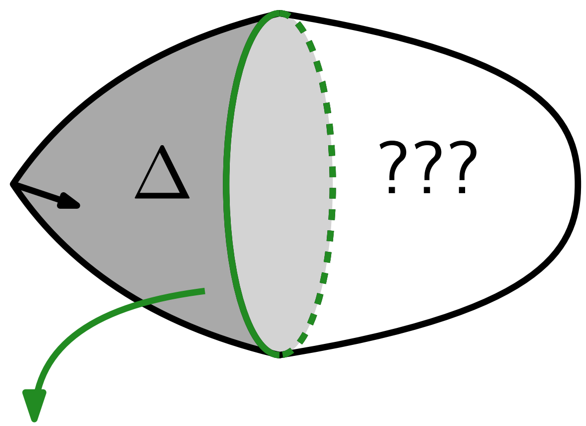

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Simple (rooted) cycle, spins given by a word ω

$\mathbf{Z}_\omega(\nu,t) :=$ generating series of triangulations with simple boundary ω

Theorem [Albenque – M. – Schaeffer]

For every ω, the series $t^{|\omega|}Z_\omega(\nu,t)$ is algebraic, has ρ_c as unique dominant singularity and satisfies

$$[t^{3n}]t^{|\omega|}Z_\omega(\nu,t) \underset{n\to\infty}{\sim} \begin{cases} \kappa_\omega(\nu_c)\,\rho_{\nu_c}^{-n}\,n^{-7/3} & \text{if } \nu=\nu_c=1+\frac{1}{\sqrt{7}}, \\ \kappa_\omega(\nu)\,\rho_\nu^{-n}\,n^{-5/2} & \text{if } \nu\neq\nu_c. \end{cases}$$

Triangulations with simple boundary

Fix a word ω, with injections from and into triangulations of the sphere:

$$[t^{3n}]t^{|\omega|}Z_\omega = \Theta\left(\rho_\nu^{-n} n^{-\alpha}\right), \text{ with } \alpha = 5/2 \text{ of } 7/3 \text{ depending on } \nu.$$

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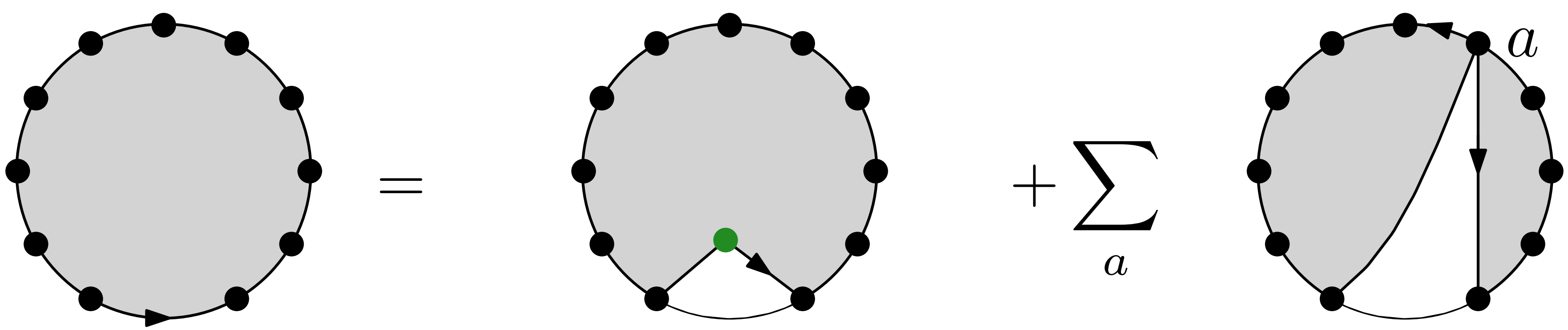
Tutte's equation (or peeling equation, or loop equation...):



$$Z_\omega = \left(Z_{\oplus\omega} + Z_{\ominus\omega} + \sum_{\omega=\omega_1 a \omega_2} Z_{a\omega_1} \cdot Z_{a\omega_2} \right) \times \nu^{\mathbf{1}_{\overleftarrow{\omega}=\overrightarrow{\omega}}}\, t$$

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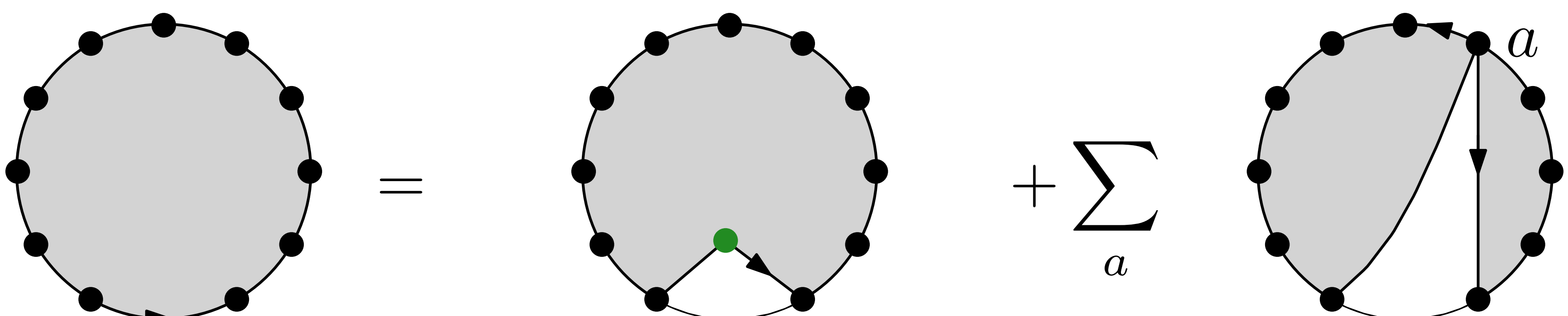

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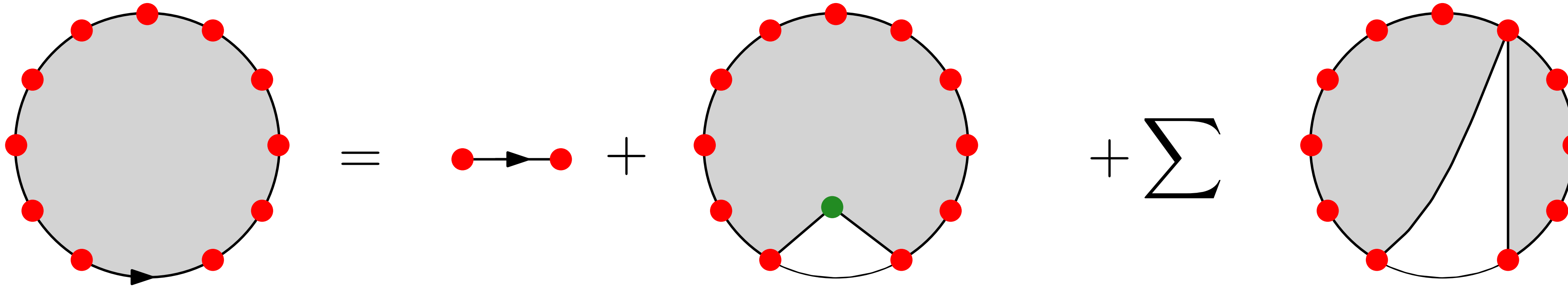
Double recursion on $|\omega|$ and number of $\ominus$'s:
enough to prove 1. and 2. for the $t^p Z_{\oplus^p}$'s

Positive boundary conditions: two catalytic variables



$$A(x) := \sum_{p\geq 1} Z_{\oplus^p} x^p = \nu t x^2 + \qquad + \frac{\nu t}{x}(A(x))^2$$

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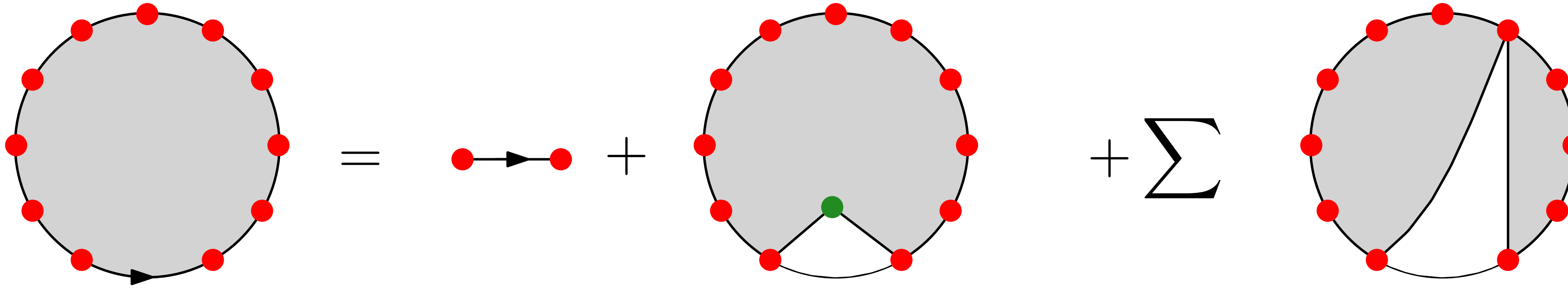

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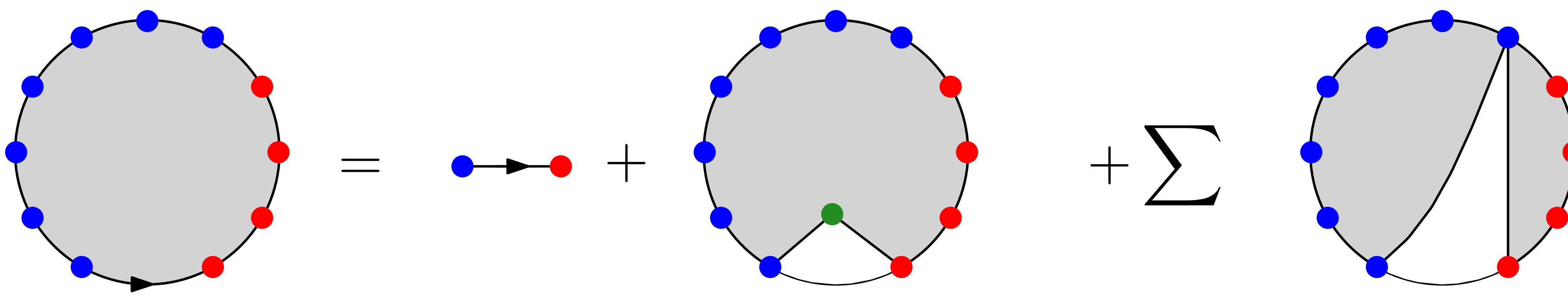
Peeling equation **at interface** $\ominus$–$\oplus$:



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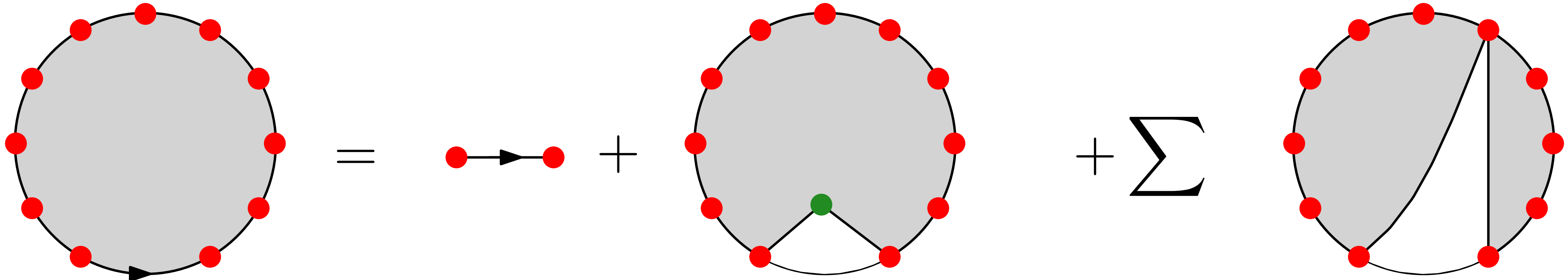

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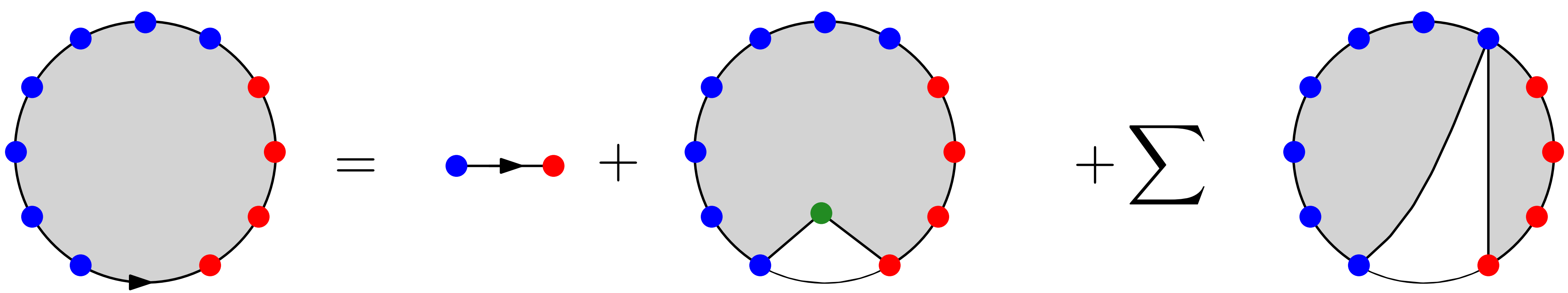

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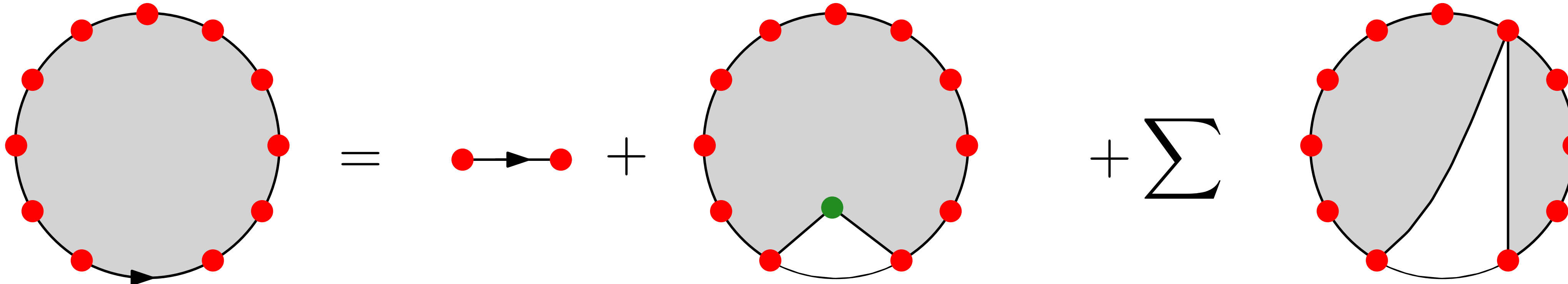

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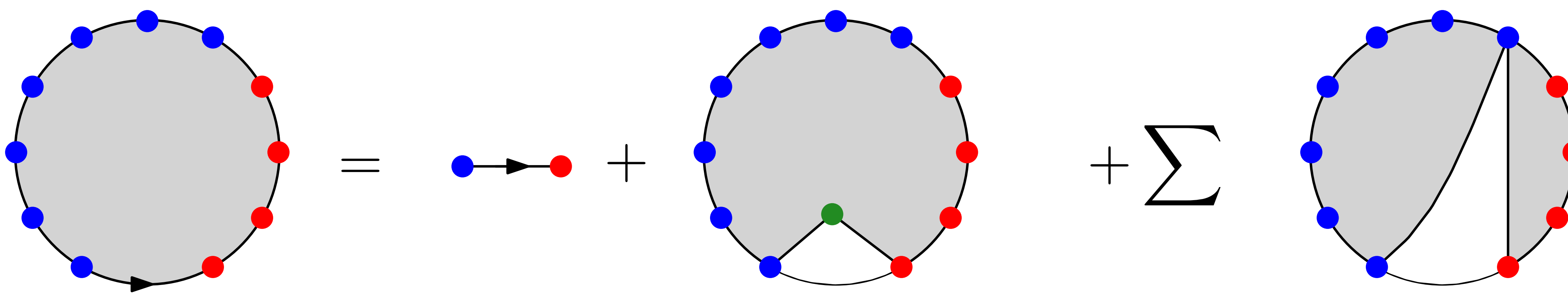

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$$= txy + \frac{t}{x}\left(S(x,y) - x[x]S(x,y)\right) + \frac{t}{y}\left(S(x,y) - y[y]S(x,y)\right)$$

$$+ \frac{t}{x}S(x,y)A(x) + \frac{t}{y}S(x,y)A(y)$$

From two catalytic variables to one: Tutte's invariants

Kernel method: equation for S reads

$$K(x,y)\cdot S(x,y)=R(x,y)$$

where $$K(x,y)=1-\frac{t}{x}-\frac{t}{y}-\frac{t}{x}A(x)-\frac{t}{y}A(y).$$

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3. Prove that $J(y)=C_0(t)+C_1(t)I(y)+C_2(t)I^2(y)$ with C_i's explicit polynomials in $t, Z_{\oplus}(t)$ and $Z_{\oplus^2}(t)$.

Equation with one catalytic variable for $A(y)$ with $Z_{\oplus}$ and $Z_{\oplus^2}$!

Explicit solution for positive boundary conditions

Equation with one catalytic variable reads:

$$2t^2\nu(1-\nu)\left(\frac{A(y)}{y} - Z_{\oplus}\right) = y \cdot \mathrm{Pol}\left(\nu, \frac{A(y)}{y}, Z_{\oplus}, Z_{\oplus^2}, t, y\right)$$

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Maple: **rational parametrization !**

$$t^3 = U\frac{P_1(\mu, U)}{4(1-2U)^2(1+\mu)^3}$$

$$ty = V\frac{P_2(\mu, U, V)}{(1-2U)(1+\mu)^2(1-V)^2}$$

$$t^3A(t, ty) = \frac{VP_3(\mu, U, V)}{4(1-2U)^2(1+\mu)^3(1-V)^3}$$

with $\nu = \frac{1+\mu}{1-\mu}$ and P_i's explicit polynomials.

Going back to local convergence

1. Fix $r \geq 0$ and take Δ a r-hull with boundary spins $\partial\Delta$:

$$\mathbb{P}_n^\bullet\left(B_r^\bullet(T,v) = \Delta\right) = \frac{\nu^{m(\Delta)-m(\partial\Delta)}\,[t^{3n-e(\Delta)+|\partial\Delta|}]Z_{\partial\Delta}^\bullet(\nu,t)}{[t^{3n}]Q^\bullet(\nu,t)}$$

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2. Remains to prove, for every r:

$$\sum_{r-\text{hulls}\,\Delta}\frac{\kappa_{\partial\Delta}}{\kappa}\nu^{m(\Delta)-m(\partial\Delta)}\,\rho^{(|\Delta|-2|\partial\Delta|)/3}=1.$$

No loss of mass at the limit

Decompose triangulations by hulls:

$$Q^\bullet(\nu,t) = Q^{\leq r}(\nu,t) + \sum_{r-\text{hulls } \Delta} \sum_{(T,v)\,:\,B_r^\bullet(T,v)=\Delta} \nu^{m(\Delta)+m(T\setminus\Delta)} t^{|\Delta|+|T\setminus\Delta|}$$

pointed at dist. $\leq r$ from the root

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Since $[t^{3n}]Q^\bullet(\nu,t) \gg [t^{3n}]Q^{\leq r}(\nu,t)$, exctracting $[t^{3n}]$ gives

$$[t^{3n}]Q^\bullet(\nu,t) \sim \sum_{r-\text{hulls } \Delta} \nu^{m(\Delta)-m(\partial\Delta)} [t^{3n-|\Delta|+|\partial\Delta|}] Z^\bullet_{\partial\Delta}(\nu,t)$$

$$\kappa\rho^{-n} n^{-\alpha+1} \sim \sum_{r-\text{hulls } \Delta} \nu^{m(\Delta)-m(\partial\Delta)} \kappa_{\partial\Delta}\, \rho^{-n+(|\Delta|-2|\partial\Delta|)/3}\, n^{-\alpha+1}$$

The story so far

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- Convergence in law for the local toplogy.
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- At least volume growth $\neq 4$ at ν_c?

Conference **Dynamics on random graphs and random planar maps**

October 23 to 27, 2017 in Marseille France

Org. LM, Pierre Nolin, Bruno Schapira and Arvind Singh

Thank you for your attention!