Random Tilings and Hurwitz Numbers

Jonathan Novak (MIT)

- Start with two things:

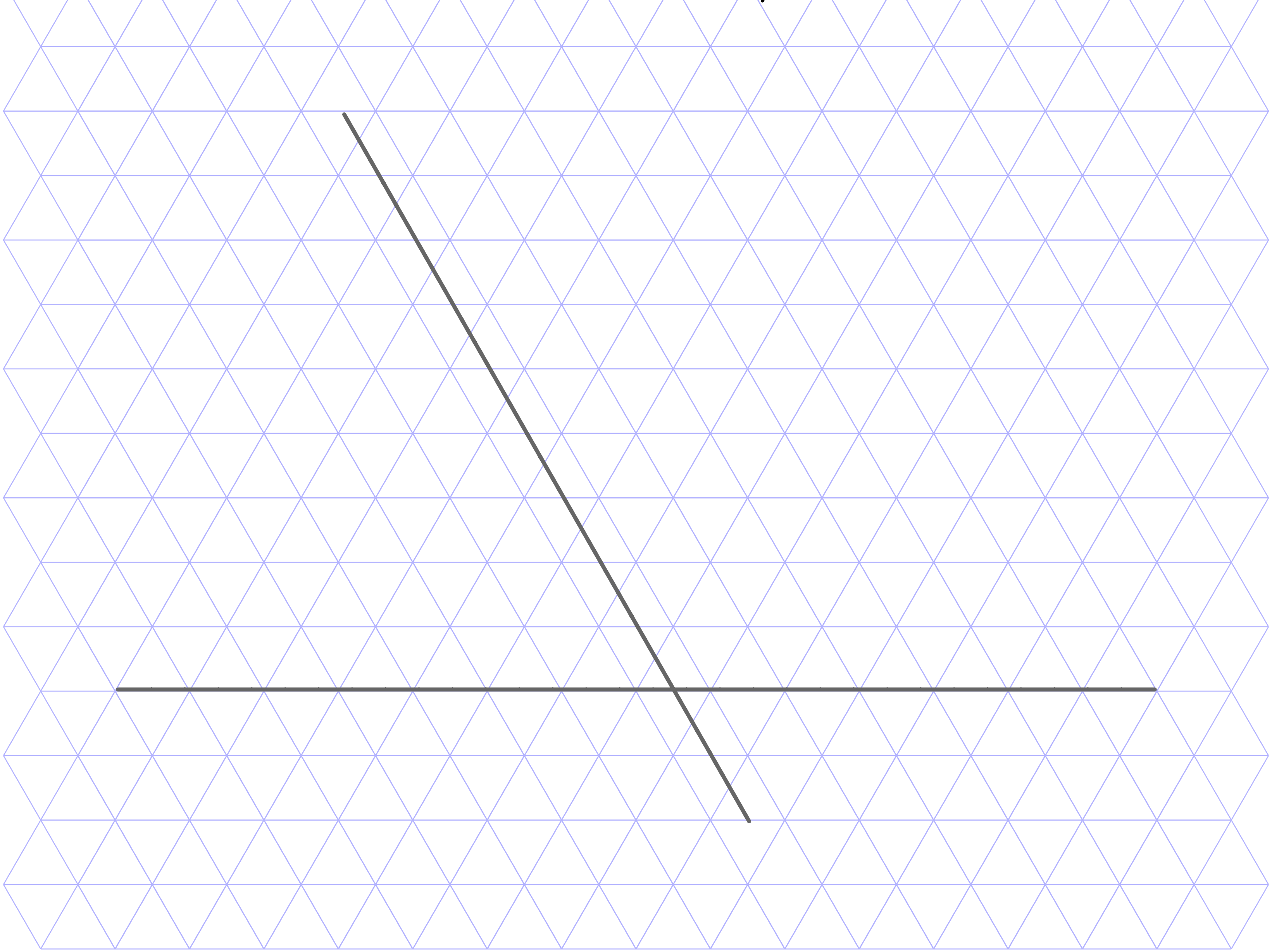
    1) A tiling of the plane by unit triangles;

    2) A triangular array of integers,

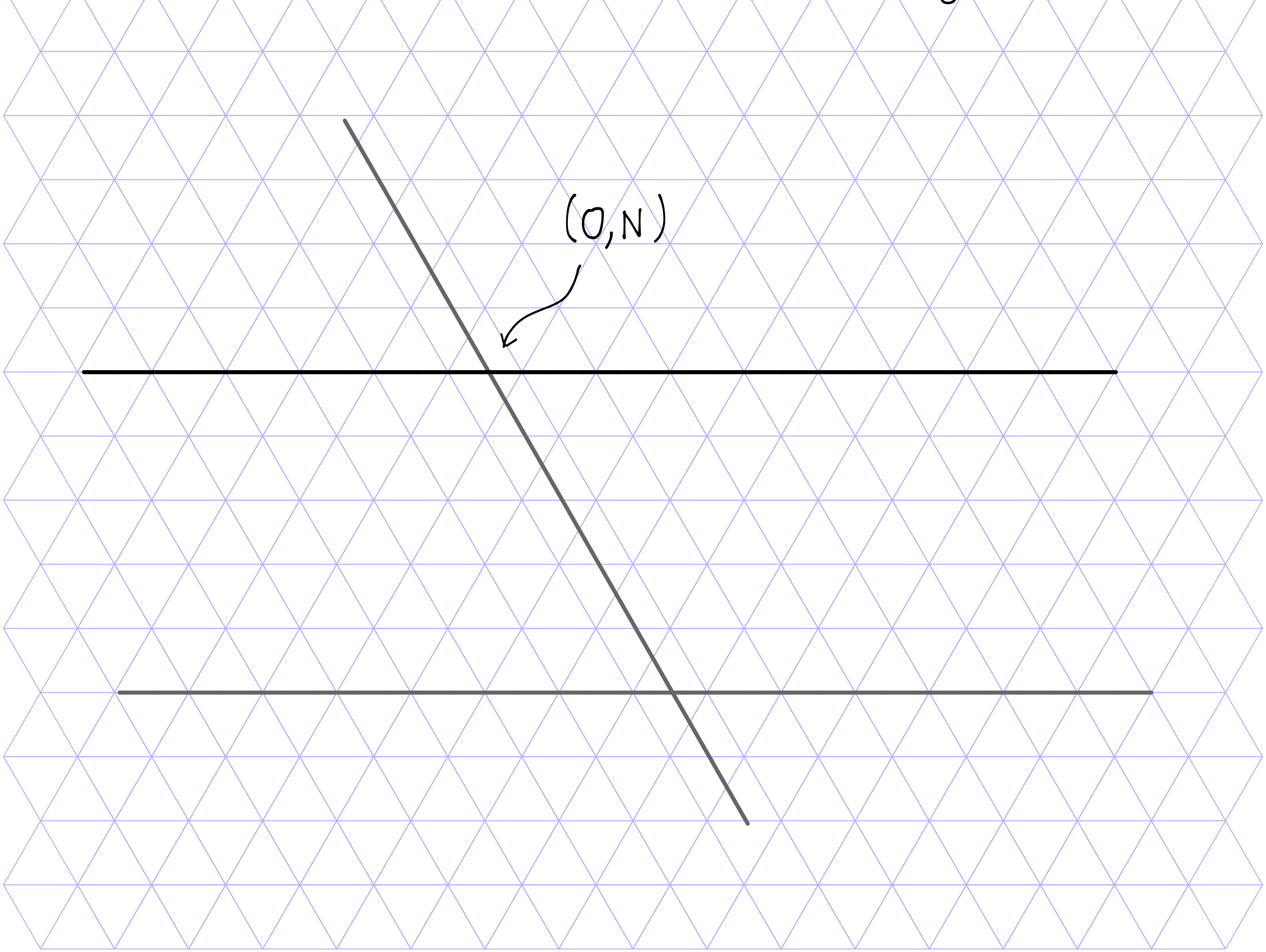
$$
\begin{array}{ccccc}
 & & b_1^{(1)} & & \\
 & b_1^{(2)} & & b_2^{(2)} & \\
b_1^{(3)} & & b_2^{(3)} & & b_3^{(3)} \\
\ddots & & & & \ddots
\end{array}
$$

- To this data, we associate a sequence $\Omega^{(N)}$ of planar domains.

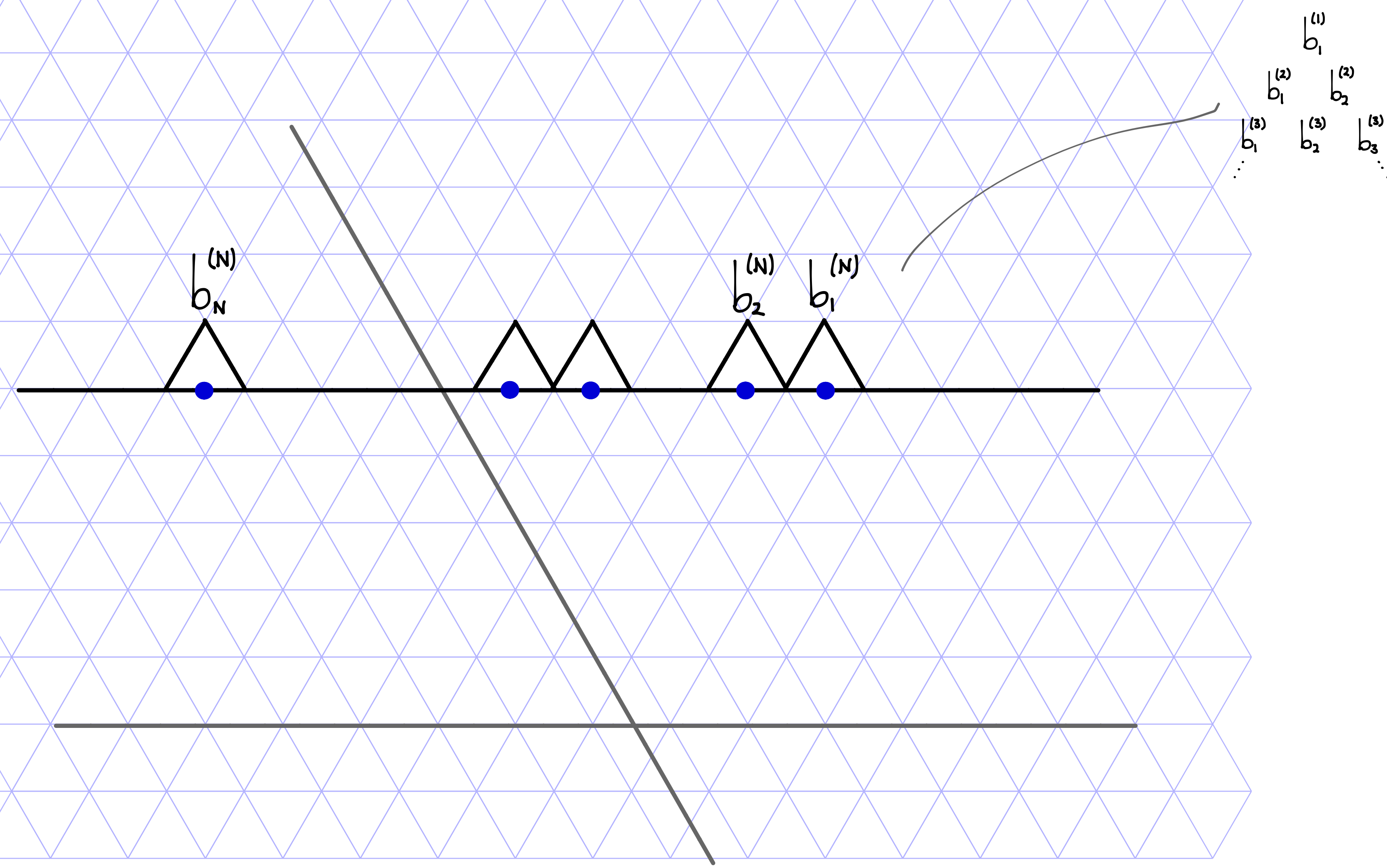
- STEP ZERO: Introduce a coordinate system.



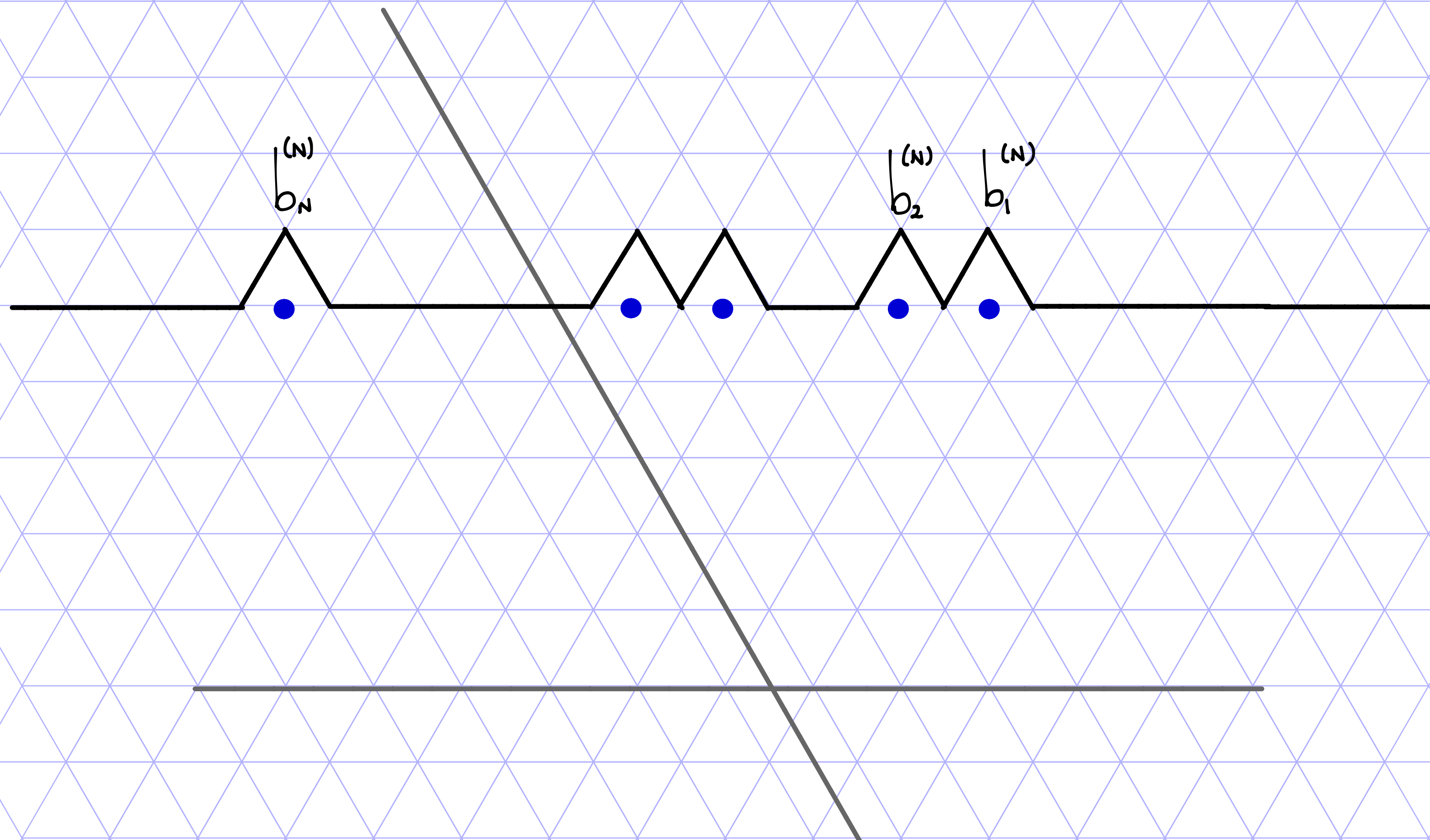
- STEP ONE: Construct the horizontal line through (O,N).



• STEP TWO: Construct N outward-facing unit triangles on the line, such that the midpoints of their bases have horizontal coordinates $b_1^{(N)} > \dots > b_N^{(N)}$.

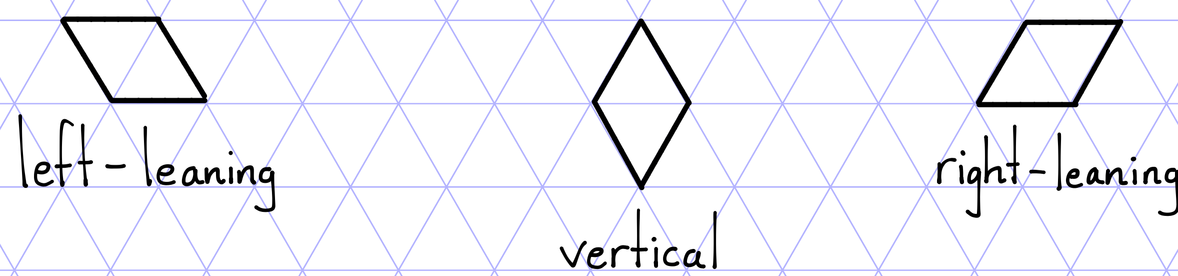


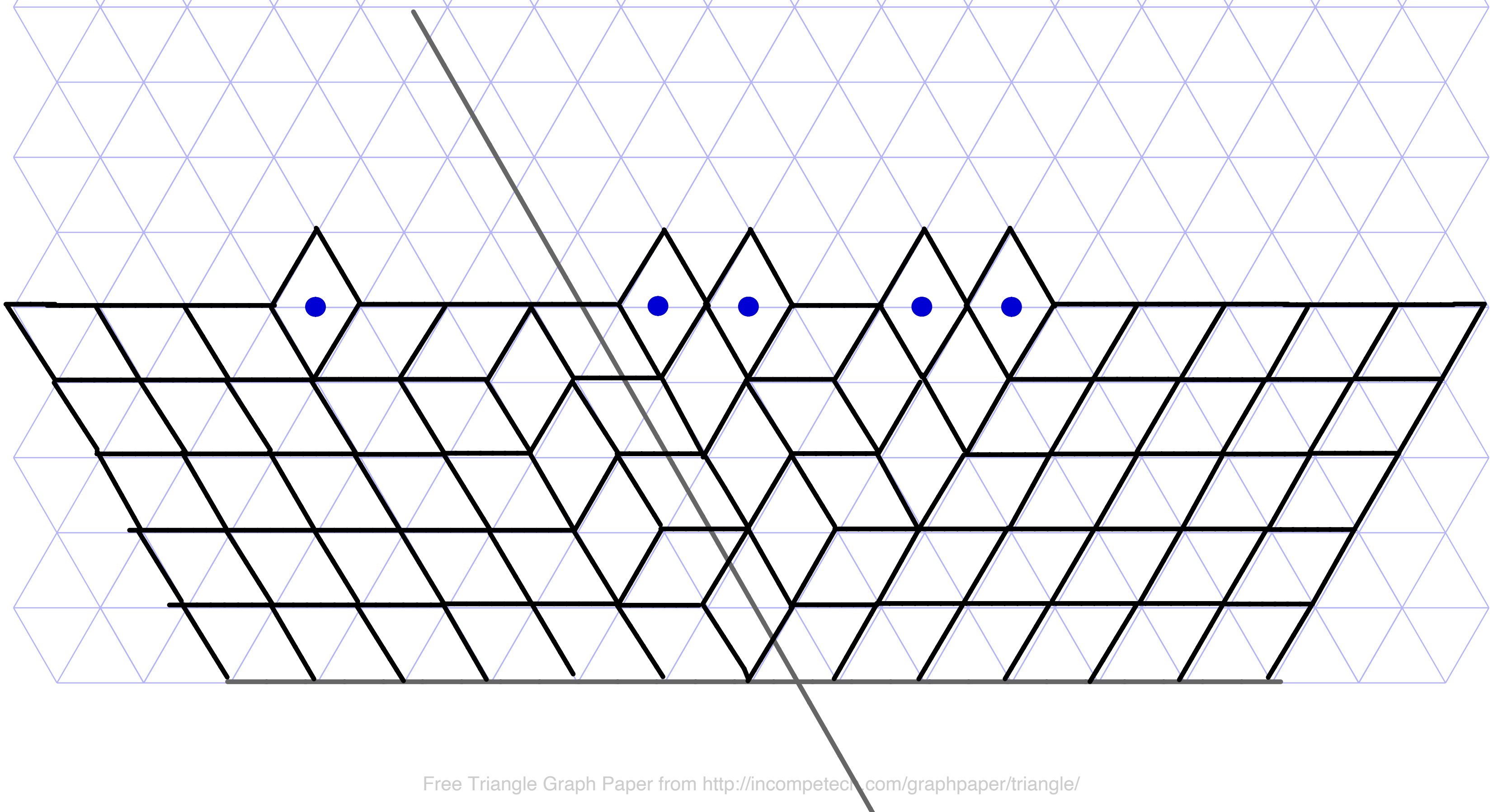
- STEP THREE: Erase the bases of the triangles.



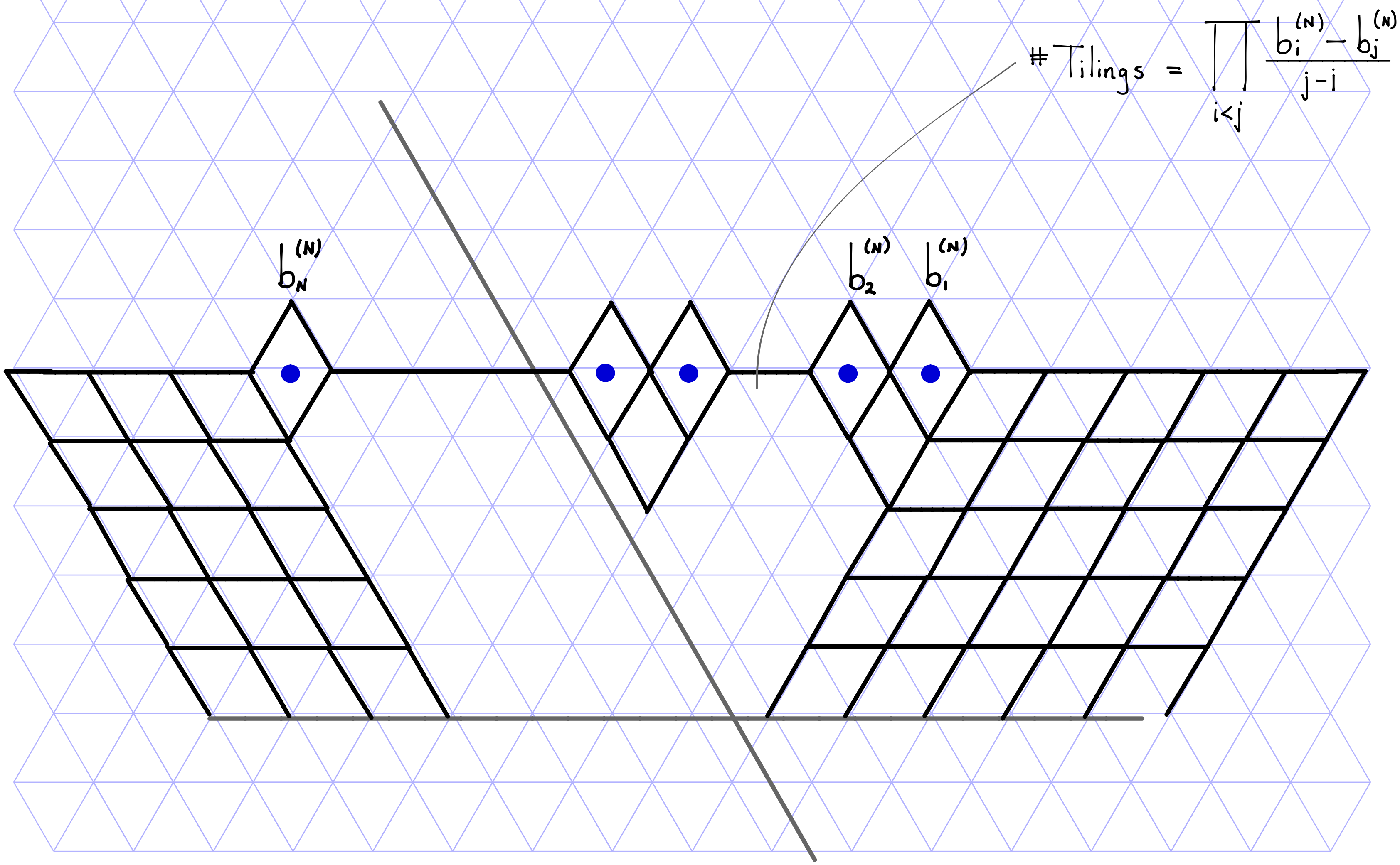
- This is the sawtooth domain $\Omega^{(N)}$ of rank N with boundary conditions $(b_1^{(N)}, \dots, b_N^{(N)})$.

• $\Omega^{(N)}$ can be tessellated using tiles of three types, called lozenges.

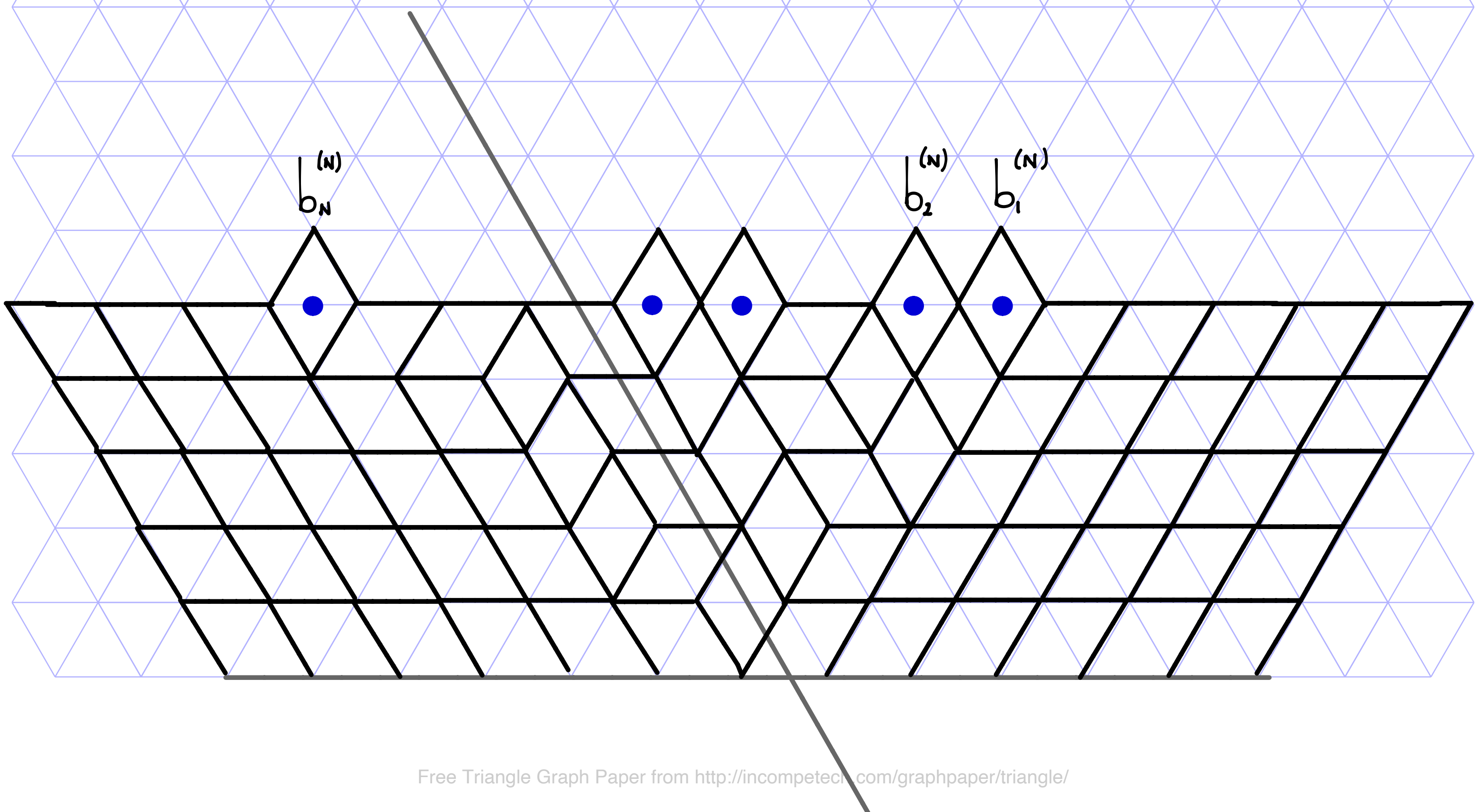




- Tilings of $\Omega^{(N)}$ are in bijection with tilings of a polygonal subdomain.



- Ensemble: $T^{(N)}$ is a uniformly random lozenge tiling of $\Omega^{(N)}$.
- What does $T^{(N)}$ look like as $N \to \infty$?
- Dependance on boundary conditions?

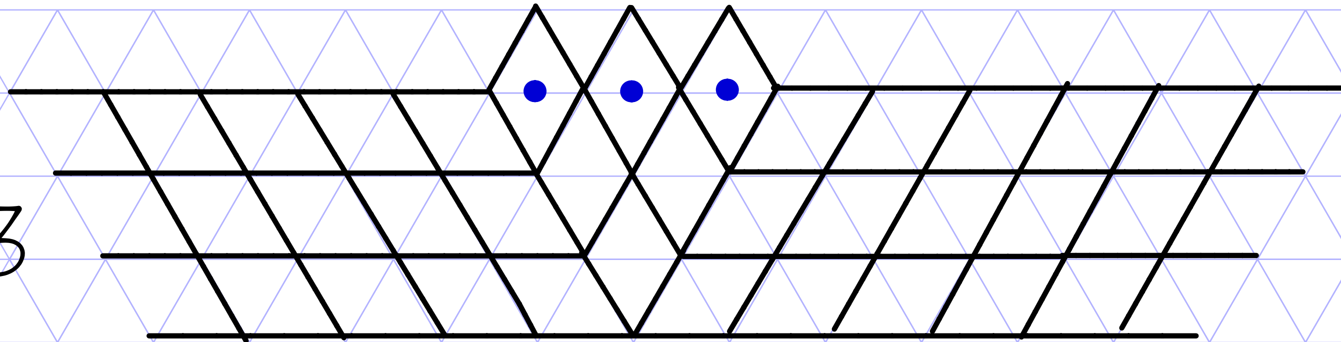


- Trivial example: sequence of boundary conditions is "fully packed"

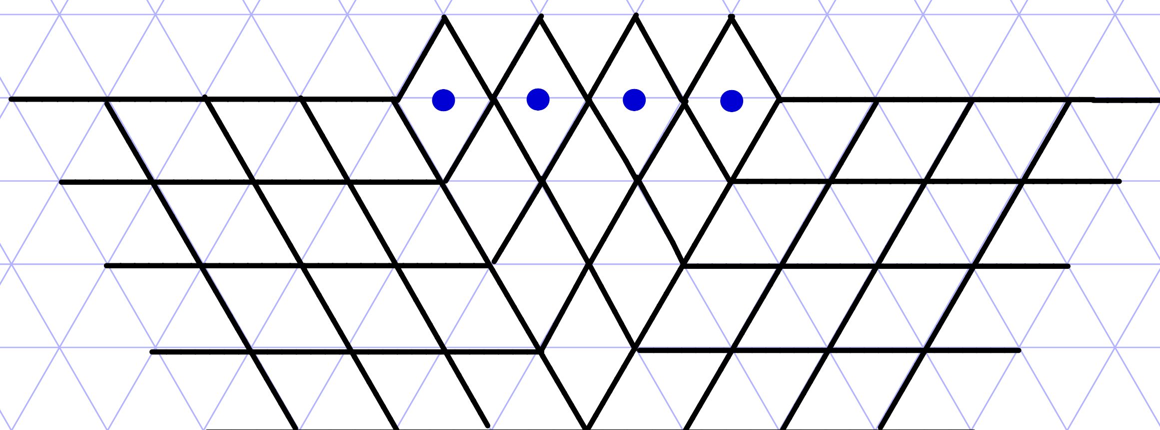
$$\begin{array}{ccccccccc} &&&&0&&&& \\ &&&1&&0&&& \\ &&2&&1&&0&& \\ &3&&2&&1&&0& \\ \iddots&&&&&&&&\ddots \end{array}$$

- $\Omega^{(N)}$ has all its teeth in one "clump."

N=3



N=4

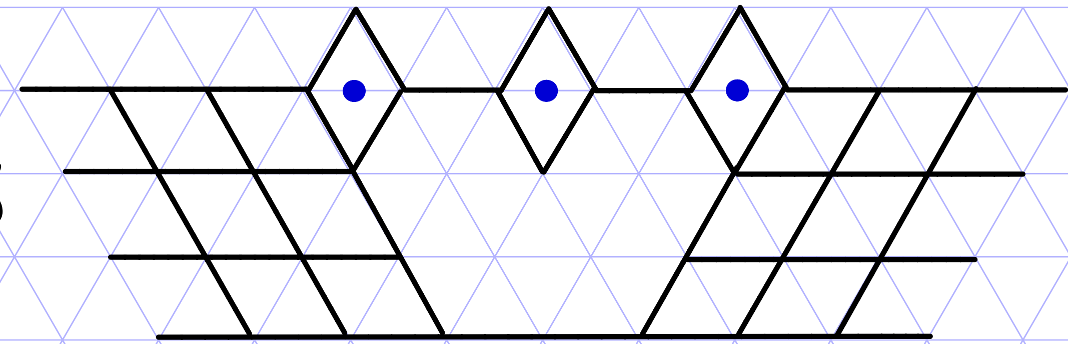


- Natural, non-trivial example: unit-spaced boundary conditions

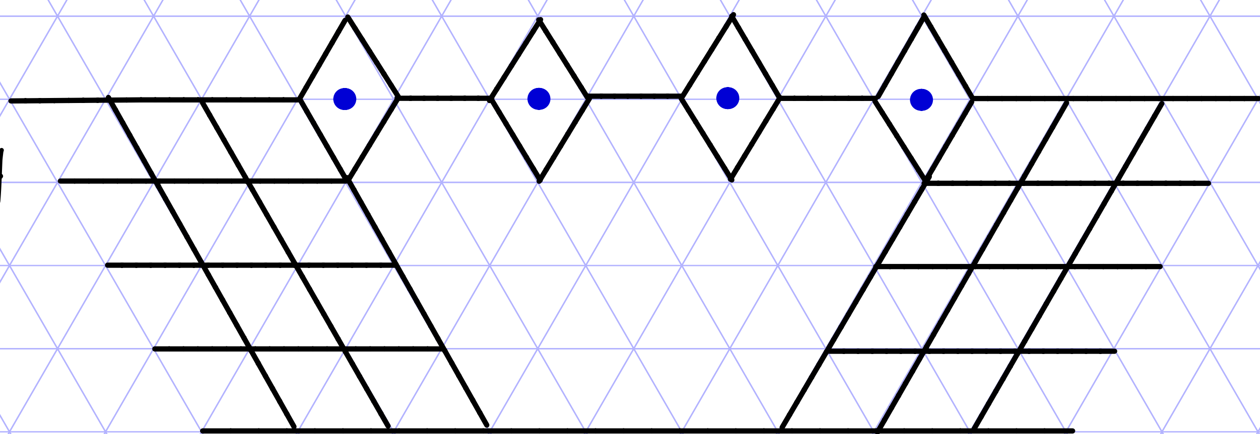
$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 2 & & 0 & & & \\
 & & 4 & & 2 & & 0 & & \\
 & 6 & & 4 & & 2 & & 0 & \\
\iddots & & & & & & & & \ddots
\end{array}$$

- $\Omega^{(N)}$ is the sawtooth domain of rank N with the maximal number of clumps, and minimal spread conditional on this.

N=3



N=4



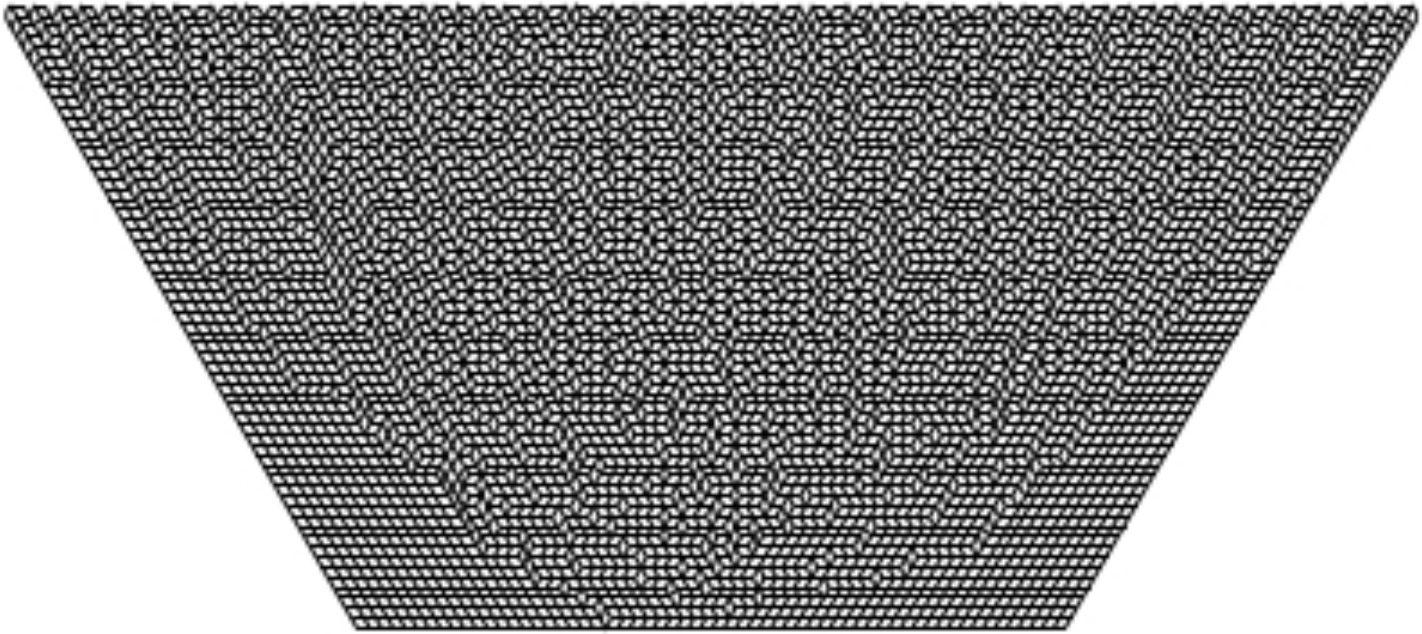


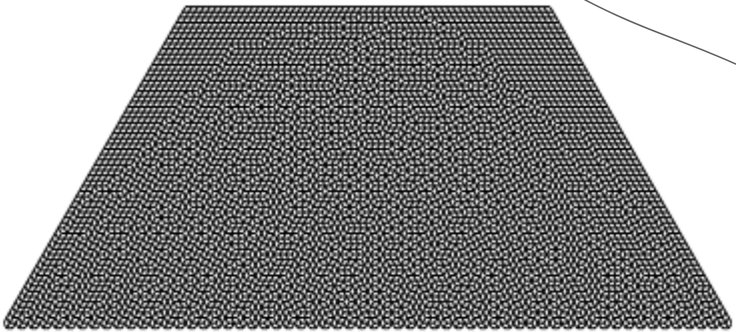
Fig. by Nordenstam and Young

The Arctic Parabola Theorem

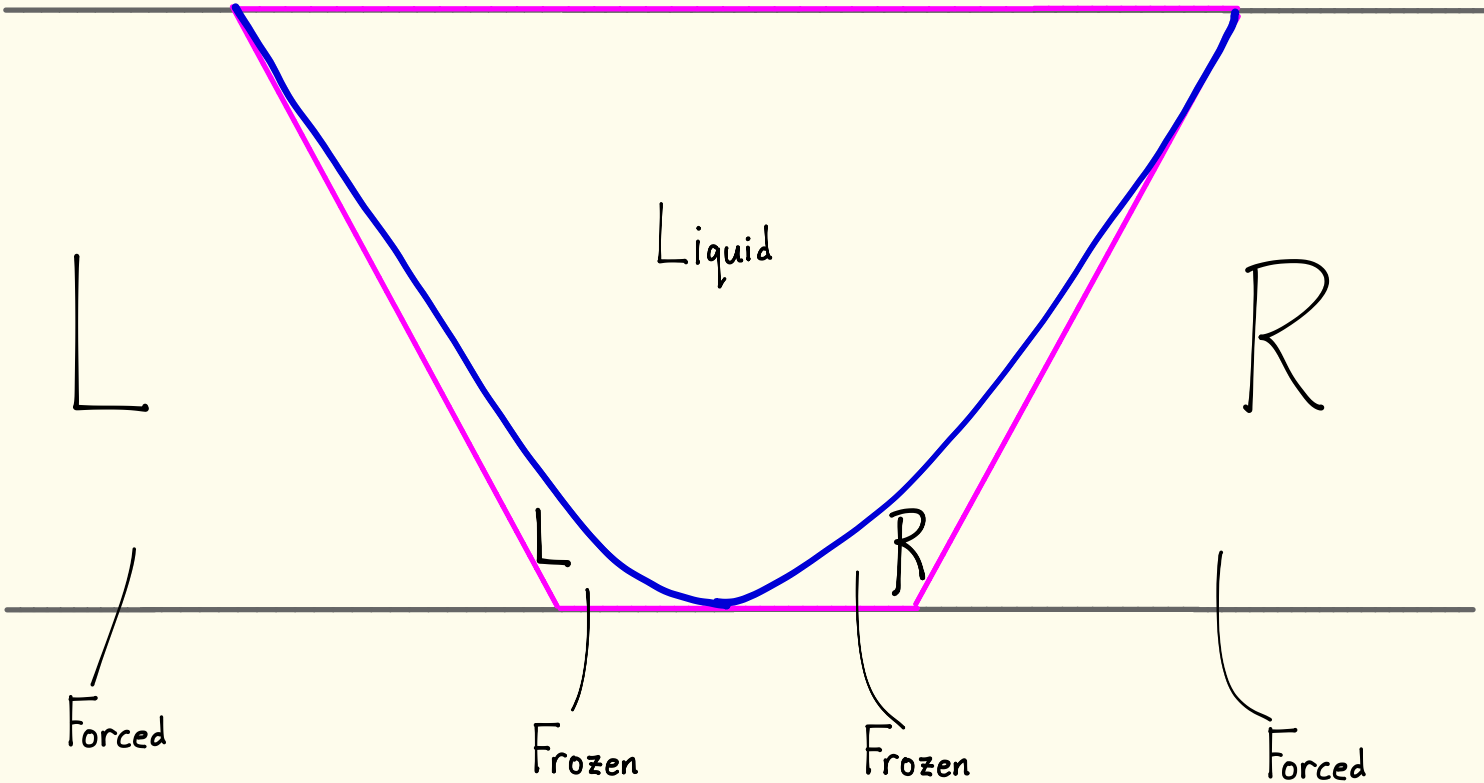
Nordenstam – Young

Theorem

Consider uniform measure on tilings of the Novak half-hexagon. The region in which the density of particles (i.e. vertical lozenges) is assymptotically non-zero is bounded by a parabola.



Spelling due to Nordenstam – Young

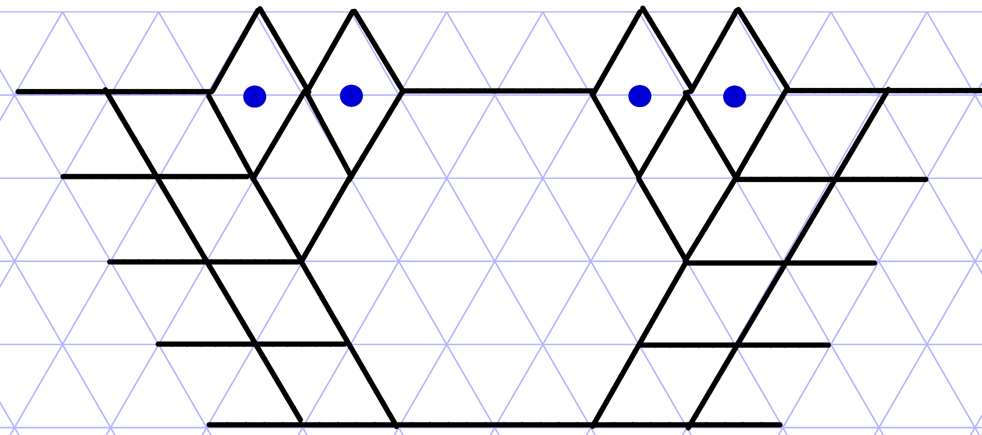

Liquid
L
R
L
R
Forced
Frozen
Frozen
Forced

- One more example:

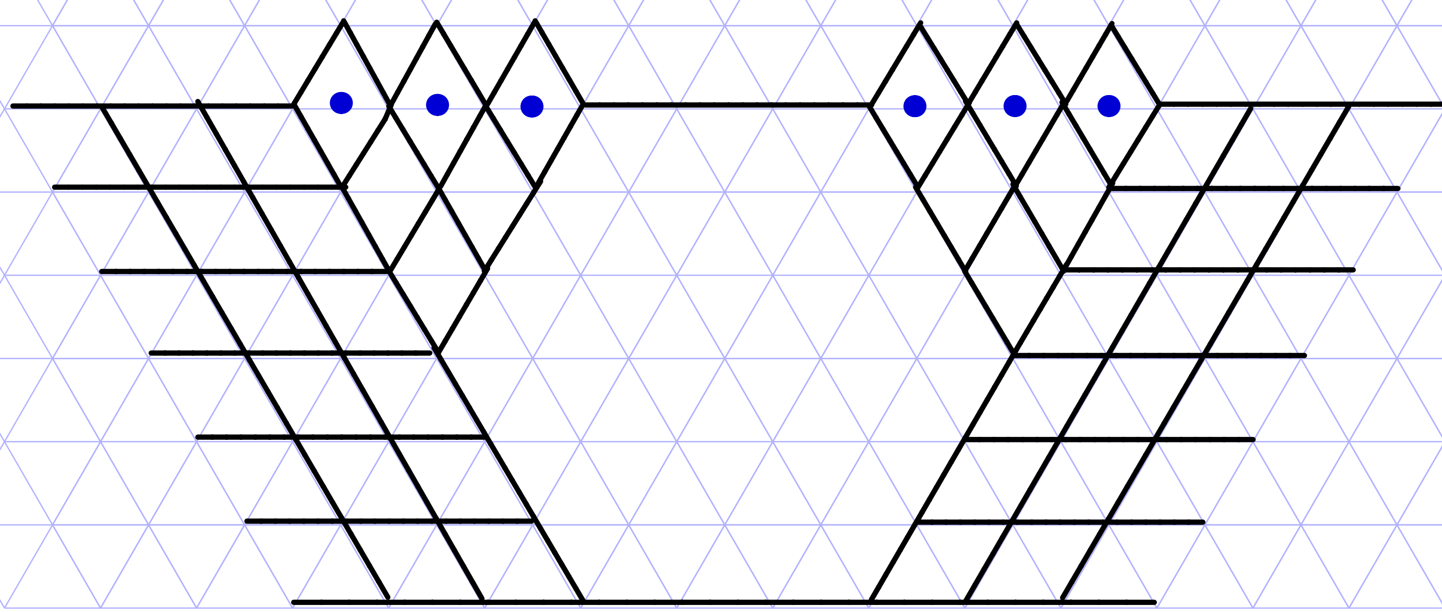
$$\begin{array}{c} 2 \quad 0 \\ 5 \quad 4 \quad 1 \quad 0 \\ 8 \quad 7 \quad 6 \quad 2 \quad 1 \quad 0 \\ ⋰ \qquad\qquad\qquad\qquad\quad ⋱ \end{array}$$

- $\Omega^{(N)}$ is a sawtooth domain of rank $N \equiv 0 \pmod{2}$ whose teeth form two clumps, each of size $N/2$. These clumps are separated by $N/2$ vacant sites.

$N=4$



$N=6$



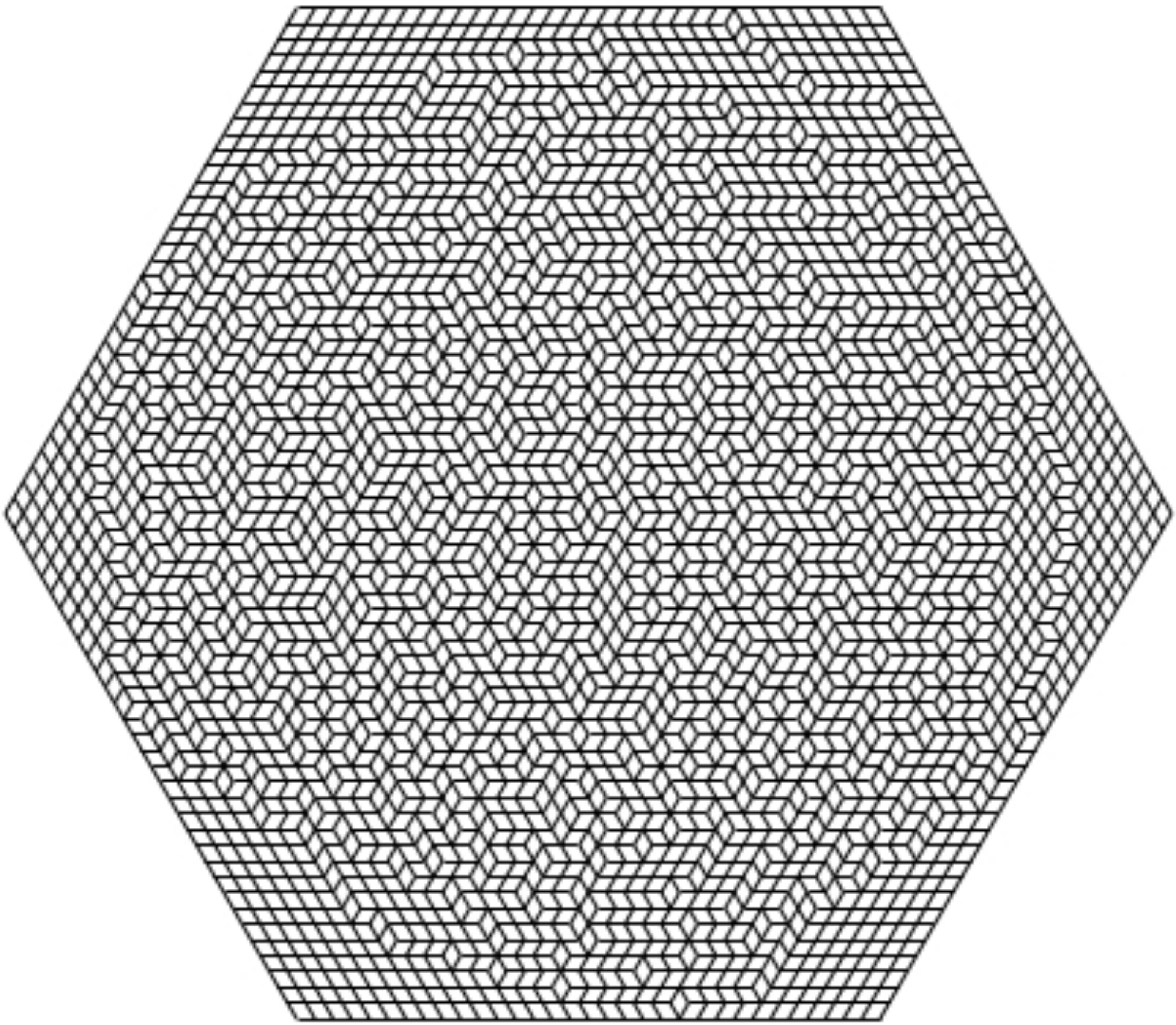


Figure by Cohn, Larsen, Propp

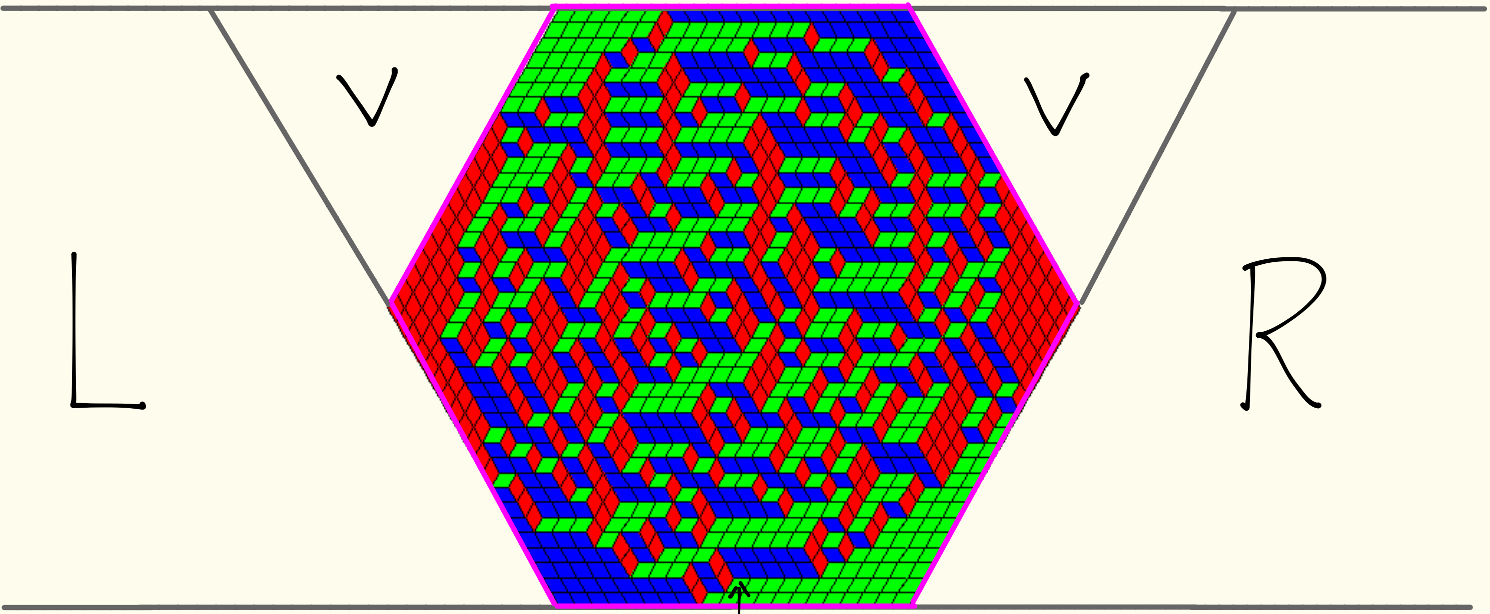
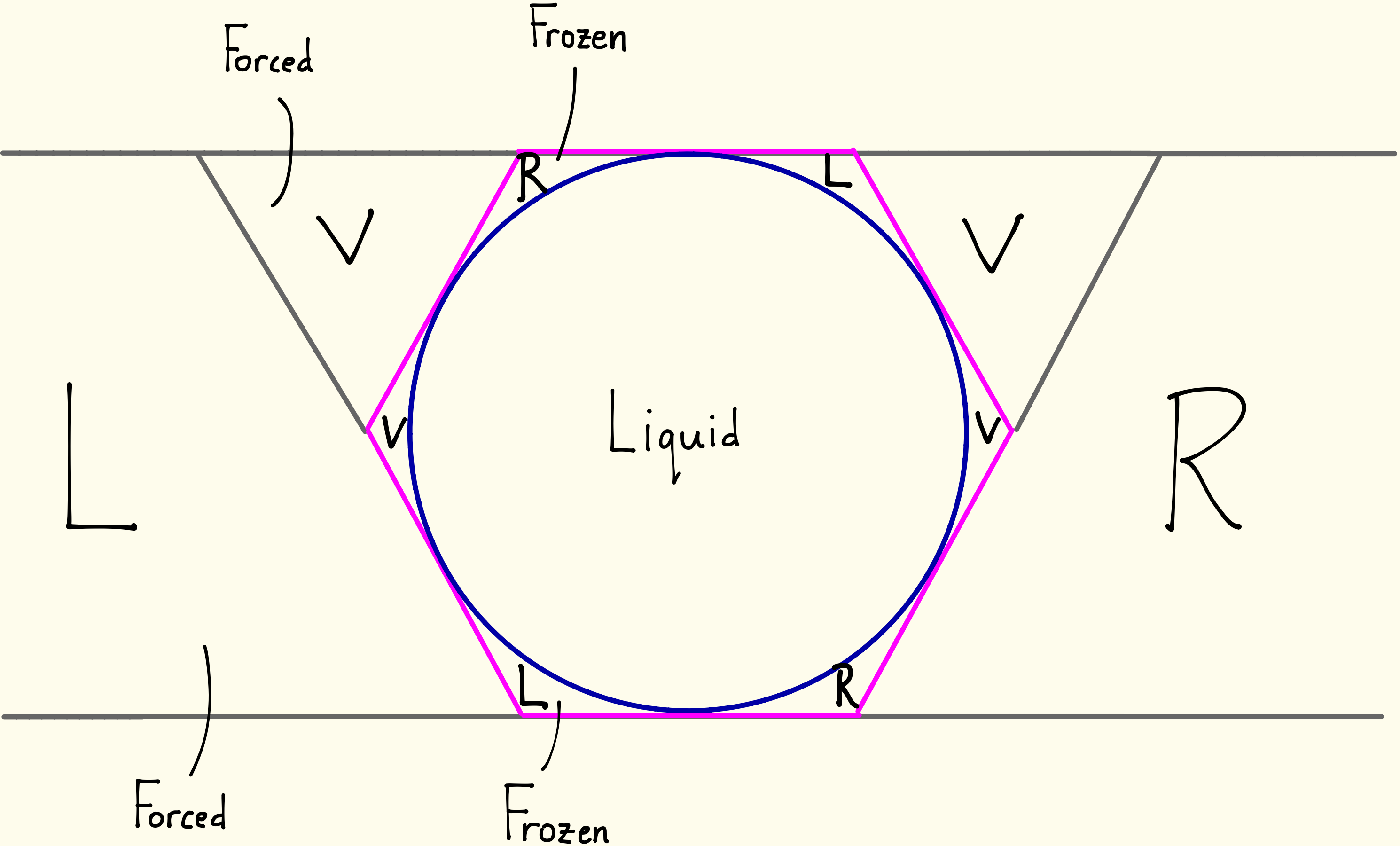
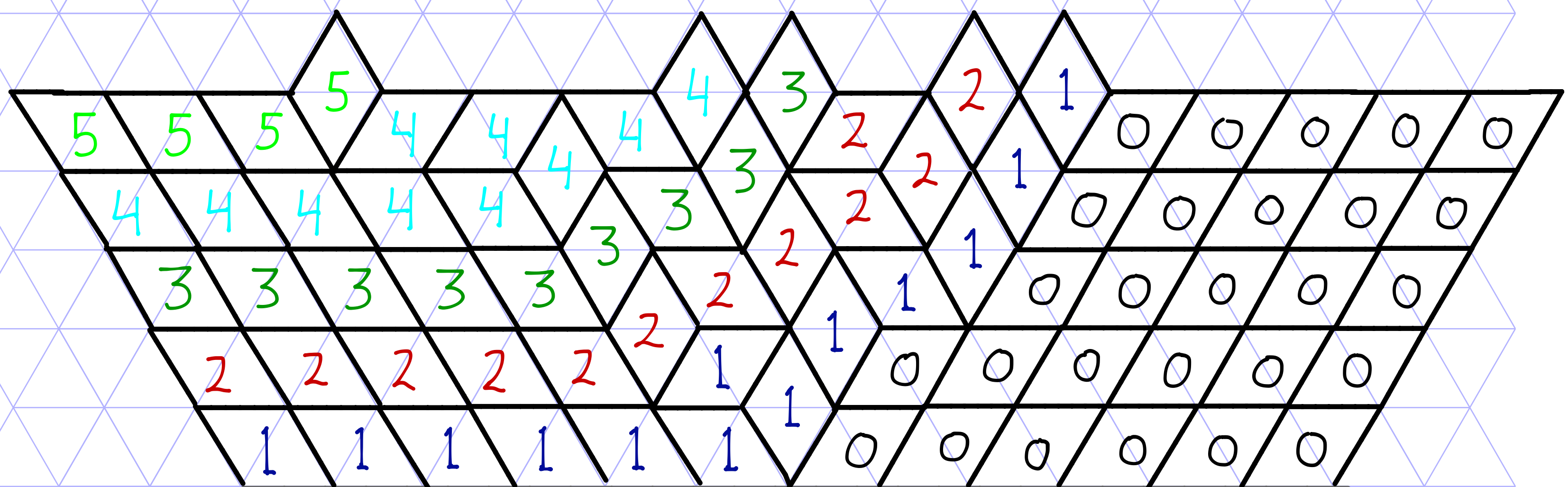


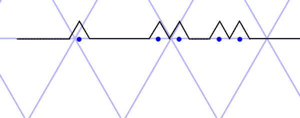
Figure by Propp


Forced
Frozen
R
L
V
V
L
v
Liquid
v
R
L
R
Forced
Frozen

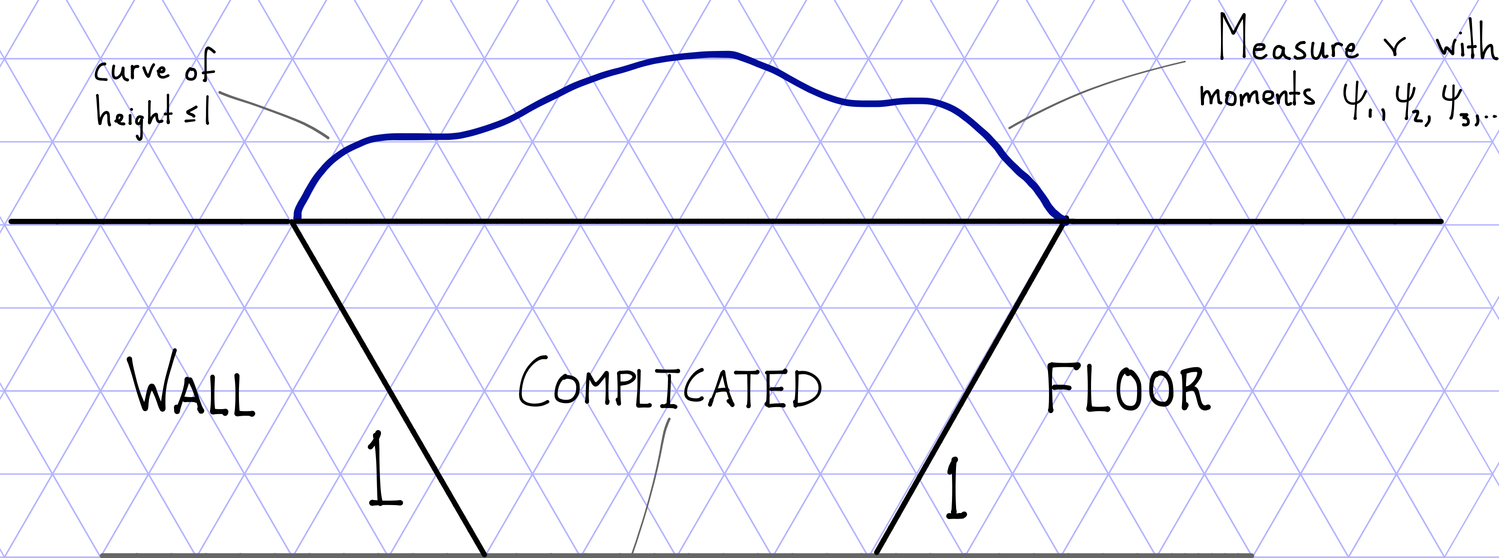
- These simulations illustrate the Law of Large Numbers for $T^{(N)}$.

- Height function $h_N : \Omega^{(N)} \longrightarrow \{0, 1, \dots, N\}$.



- Renormalized height function $\bar{h}_N$:  $\longrightarrow \{0, \frac{1}{N}, \dots, 1\}$.

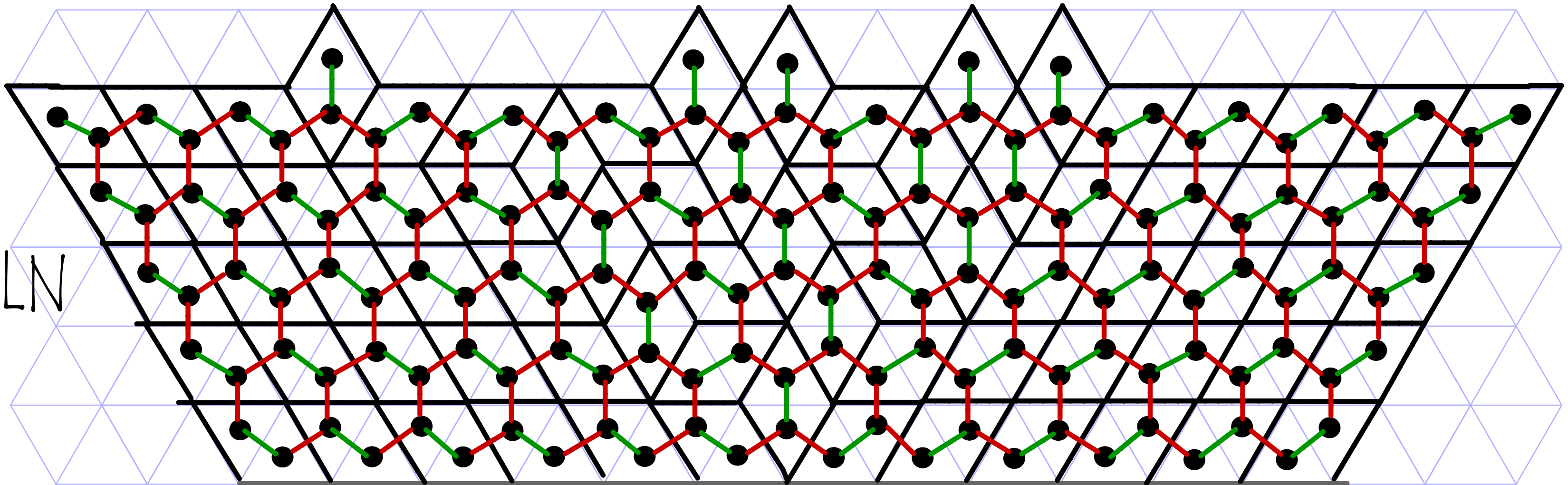
- LAW OF LARGE NUMBERS: The renormalized height function converges to a deterministic limit function on $\mathbb{R} \times [0,1]$.



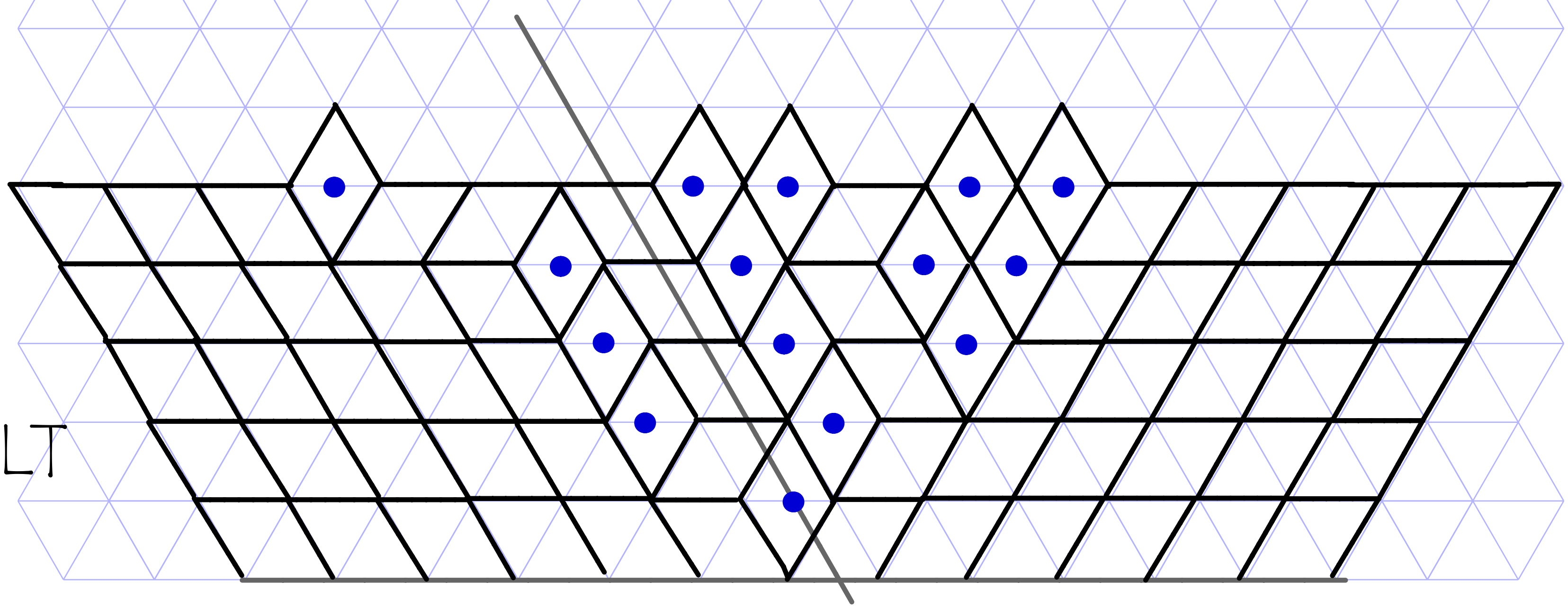
- In the "constant clump" case, extensive theory of limit shapes due to Kenyon–Okounkov.

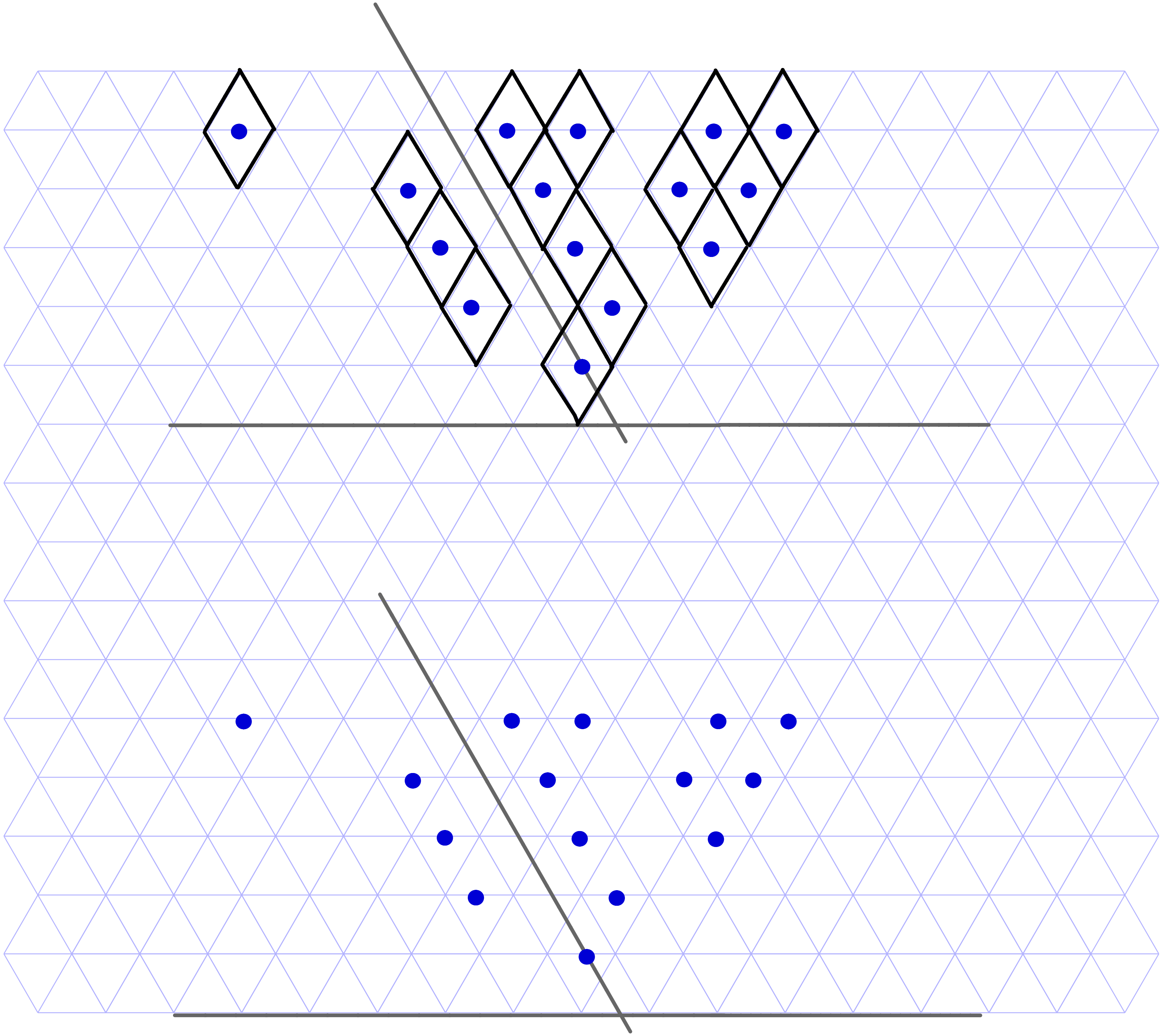
- When the number of "clumps" grows with N, limit shape theory is undergoing active development.

LLN

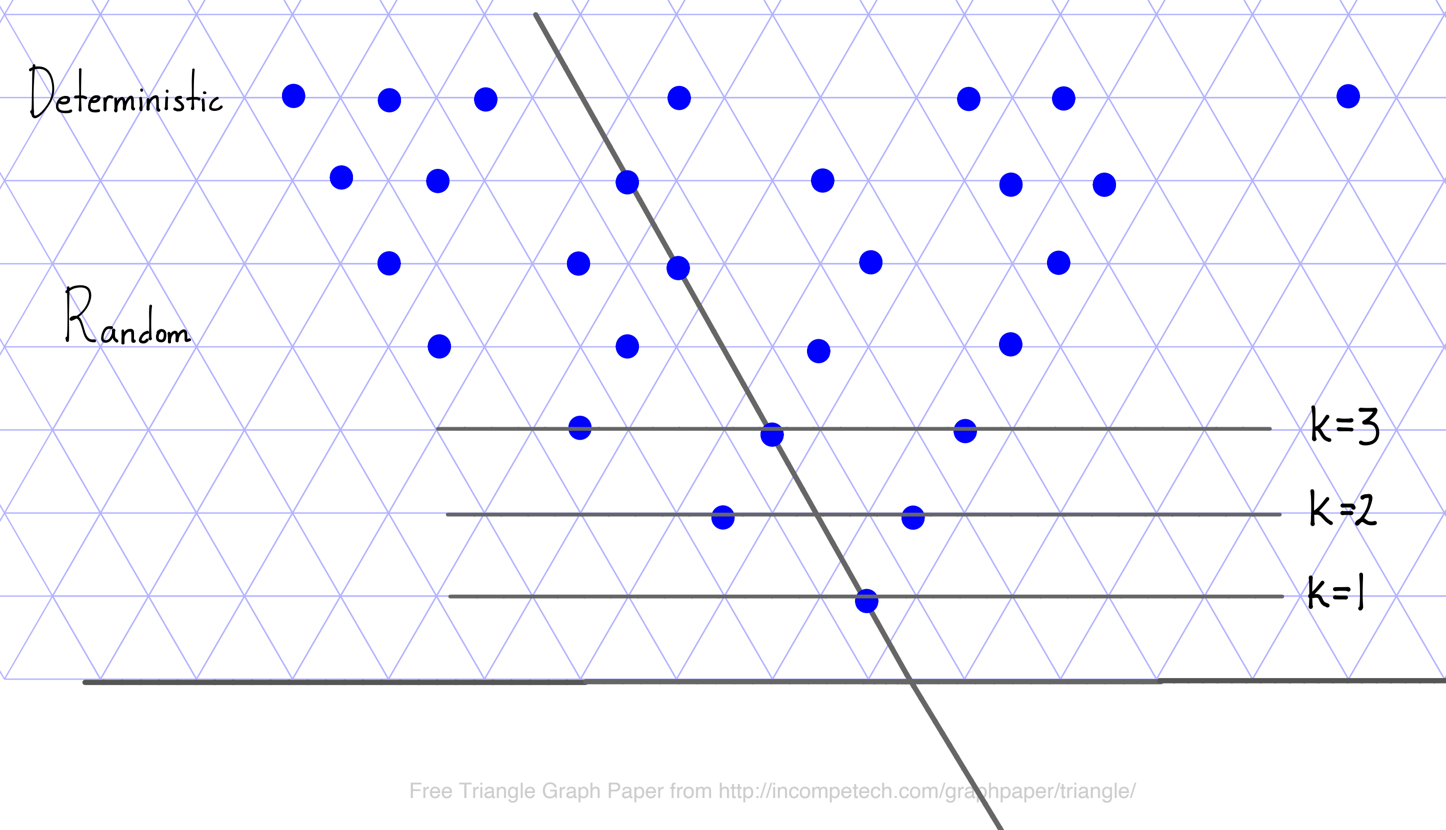


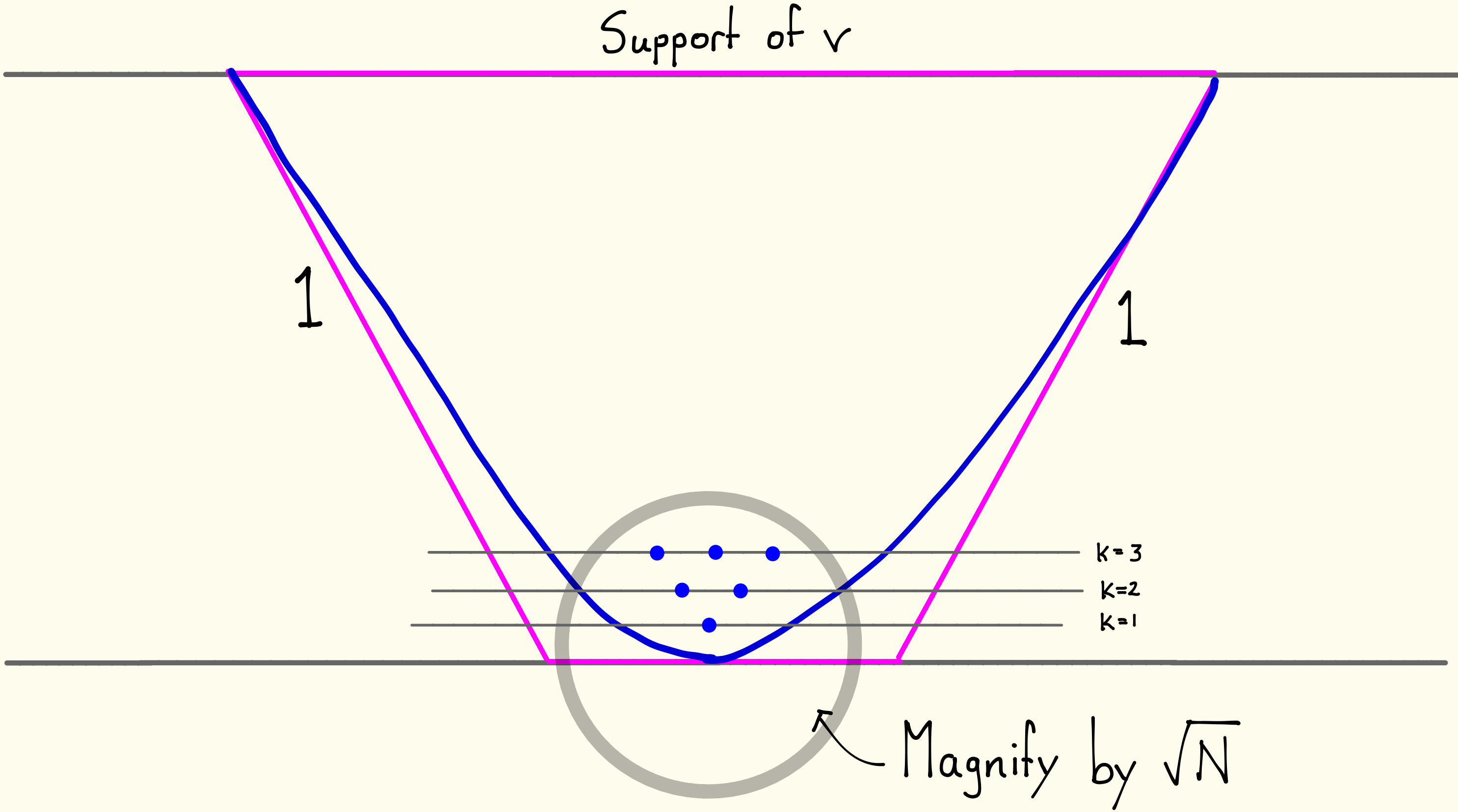
CLT





- The k^{th} line threads exactly k beads.
- Beads on consecutive threads interlace.




Support of v
1
1
k=3
k=2
k=1
Magnify by √N

- Okounkov–Reshetikhin: The random vector

$$\left(\frac{b_{k1}^{(N)}}{\sqrt{N}}, \dots, \frac{b_{kk}^{(N)}}{\sqrt{N}}\right)$$

converges weakly to the ordered list

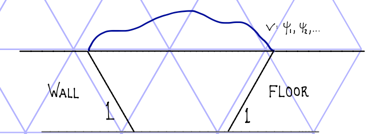
$$(x_1, \dots, x_k)$$

of eigenvalues of X_k, a random matrix drawn from the standard Gaussian measure on $k \times k$ Hermitian matrices.

- The joint density of eigenvalues is $e^{-\frac{1}{2}\sum x_i^2} \prod_{i<j} (x_i - x_j)^2$.

- Okounkov–Reshetikhin do not discuss centering and scaling.

- Gorin-Panova/Novak: $\left(\tilde{b}_{k1}^{(N)}, \dots, \tilde{b}_{kk}^{(N)}\right) \Rightarrow (x_1, \dots, x_k)$, where



$$\tilde{b}_{k\ell}^{(N)} = \frac{\frac{b_{k\ell}^{(N)}}{\sqrt{N}} - \left(\psi_1 - \frac{1}{2}\right)\sqrt{N}}{\sqrt{\psi_2 - \psi_1^2 - \frac{1}{12}}}$$

and ψ_1, ψ_2 are the first two moments of ν.

- Remark: ψ_1 and $\psi_2 - \psi_1^2$ are the mean and variance of ν, while $\frac{1}{2}$ and $\frac{1}{12}$ are the mean and variance of $U[0,1]$.

- Rest of the lecture: proof of this result.

- For each $1 \leq k \leq N$, replace the random vector $(b_{k1}^{(N)}, \ldots, b_{kk}^{(N)})$ with the random Hermitian matrix

$$B_k^{(N)} = U_k \begin{bmatrix} b_{k1}^{(N)} & & \\ & \ddots & \\ & & b_{kk}^{(N)} \end{bmatrix} U_k^{-1}, \qquad U_k = \text{Haar unitary.}$$

- Method: asymptotic analysis of the Laplace transform

$$A \mapsto \mathbb{E}\left[e^{\operatorname{Tr} AB_k^{(N)}}\right].$$

- Goal: as $N \to \infty$, $\mathbb{E}\left[e^{\frac{1}{\sqrt{N}}\operatorname{Tr} AB_k^{(N)}}\right] \sim e^{\ \operatorname{Tr} A + \ \frac{1}{2}\operatorname{Tr} A^2 + o(1)}.$

- View $A \mapsto \mathbb{E}\left[e^{\operatorname{Tr} AB_k^{(N)}}\right]$ as a mapping

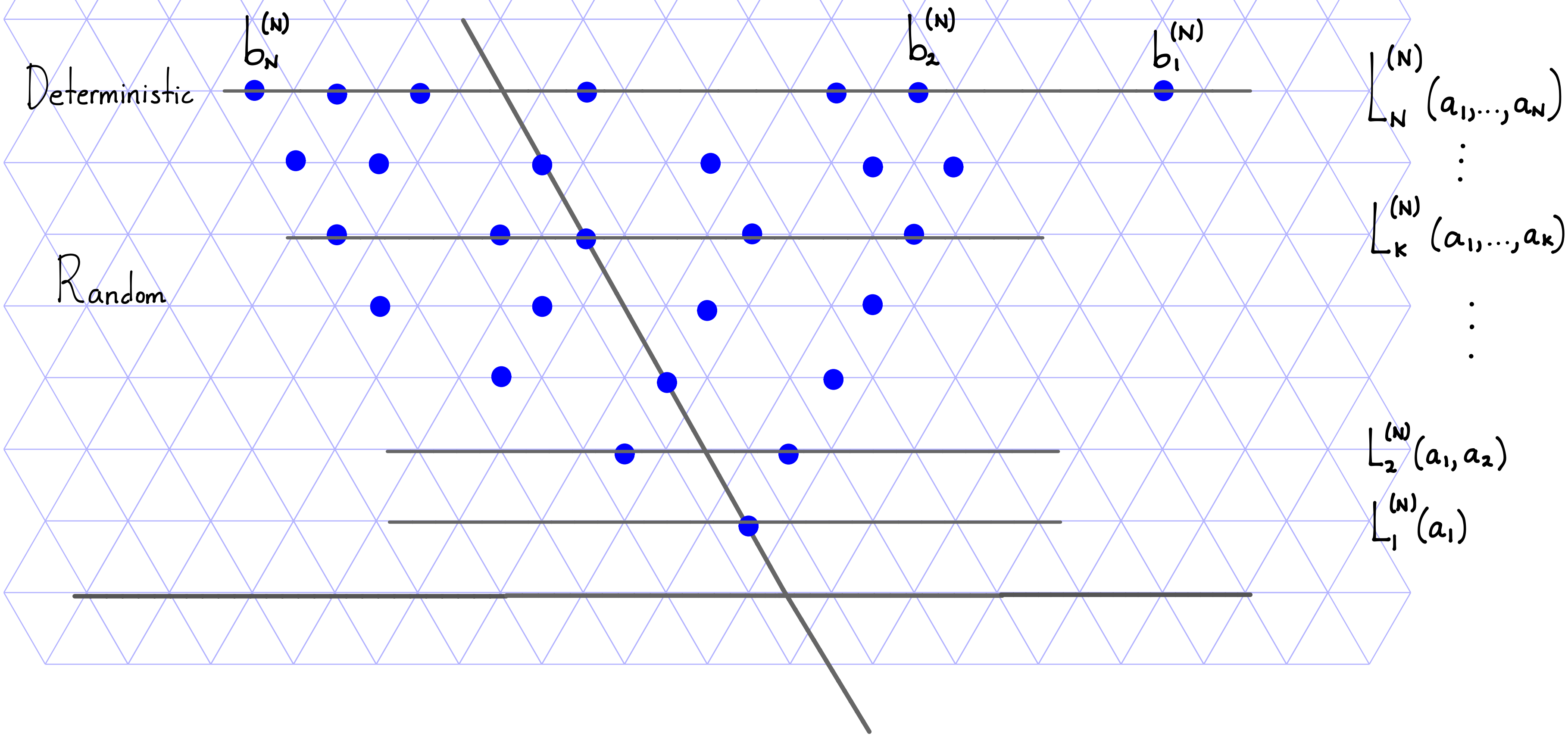
$$\{k \times k \text{ semisimple matrices}\} \longrightarrow \{\text{complex numbers}\}.$$

- By the Spectral Theorem,

$$\mathbb{E}\left[e^{\operatorname{Tr} AB_k^{(N)}}\right] = \sum_{\{b_1 > \dots > b_k\} \subset \mathbb{Z}} \Pr\left(b_{k1}^{(N)} = b_1, \dots, b_{kk}^{(N)} = b_k\right) \int_{U(k)} e^{\operatorname{Tr}\begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_k \end{bmatrix} U \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_k \end{bmatrix} U^{-1}} dU.$$

- This is an analytic function $L_k^{(N)}(a_1, \dots, a_k) := \mathbb{E}\left[e^{\operatorname{Tr} AB_k^{(N)}}\right]$ of k complex variables.

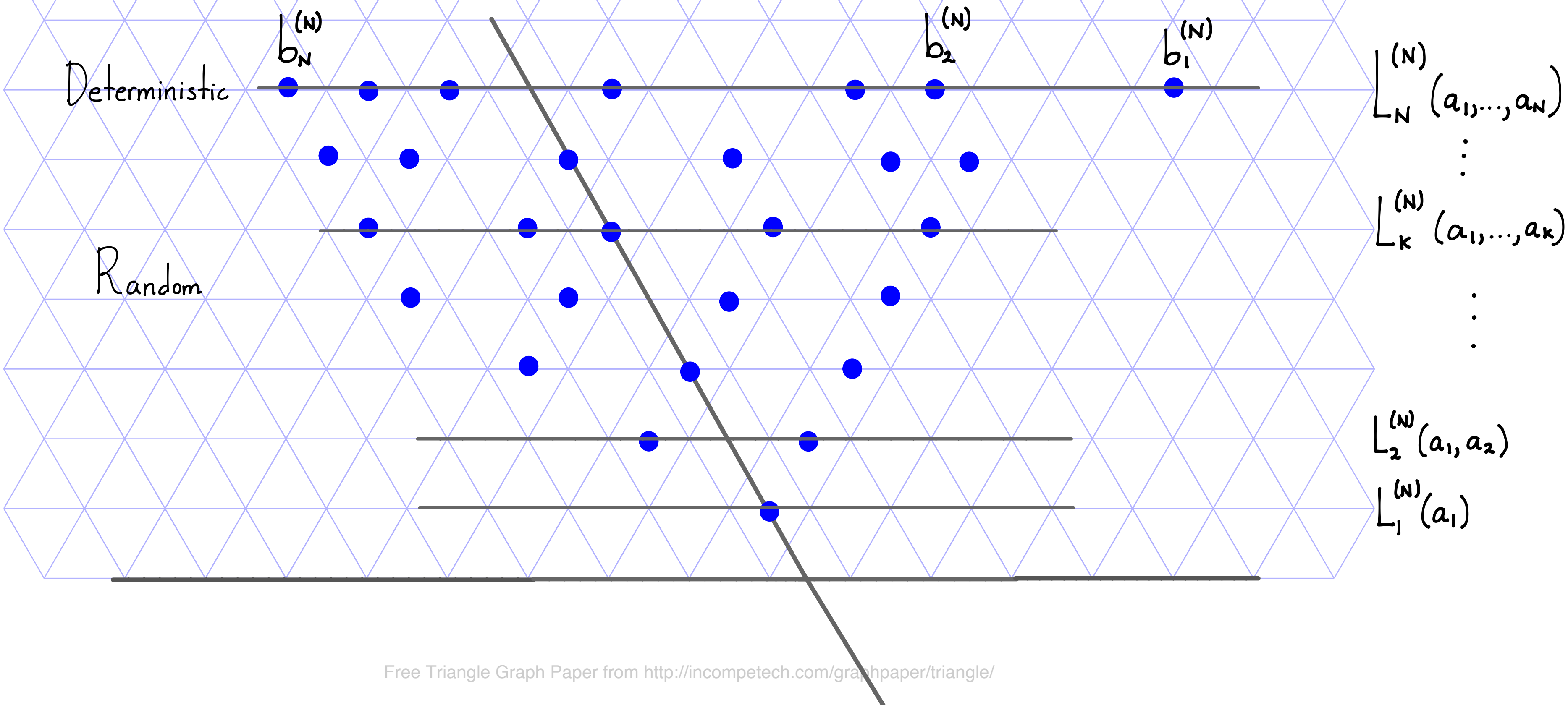
- For each domain $\Omega^{(N)}$, we have hierarchy of N analytic functions.



- Top function: Laplace transform of a pure orbital measure,

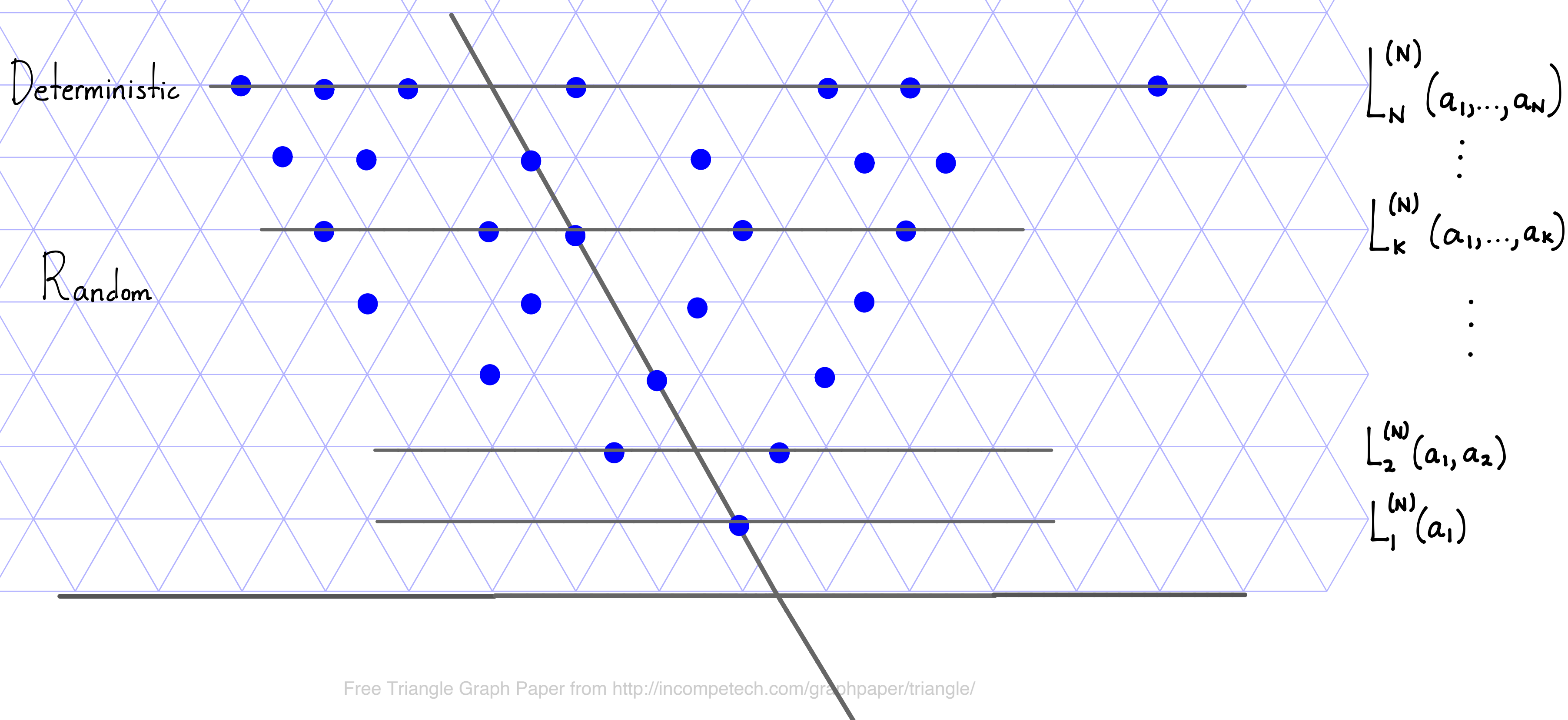
$$L_N^{(N)}(a_1,\dots,a_N) = \int_{U(N)} e^{\mathrm{Tr}\left[\begin{smallmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{smallmatrix}\right] U \left[\begin{smallmatrix} b_1^{(N)} & & \\ & \ddots & \\ & & b_N^{(N)} \end{smallmatrix}\right] U^{-1}} dU.$$

- Lower functions: non-trivial mixtures of orbital measures.



- Key realization: Top function determines all those below it,

$$\boxed{L_k^{(N)}(a_1,\dots,a_k) = \left(\prod_{i=1}^{k} \frac{a_i}{e^{a_i}-1}\right)^{N-k} L_N^{(N)}(a_1,\dots,a_k,0,\dots,0)}$$



- **Fact 1**: Rational irreps of $GL(N)$ are parameterized by N-point particle configurations $\{b_1 > \ldots > b_N\} \subset \mathbb{Z}$.

- **Fact 2**: Kirillov character formula:

$$\frac{\chi^{(b_1,\ldots,b_N)}(e^{a_1},\ldots,e^{a_N})}{\chi^{(b_1,\ldots,b_N)}(1,\ldots,1)} = \prod_{1\le i<j\le N} \frac{a_i - a_j}{e^{a_i} - e^{a_j}} \cdot \int_{U(N)} e^{\mathrm{Tr}\left[\begin{smallmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{smallmatrix}\right] U \left[\begin{smallmatrix} b_1 & & \\ & \ddots & \\ & & b_N \end{smallmatrix}\right] U^{-1}} dU$$

- **Fact 3**: Branching rule:

$$\chi^{(b_1,\ldots,b_N)}(e^{a_1},\ldots,e^{a_{N-1}},1) = \sum_{(c_1,\ldots,c_{N-1}) \prec (b_1,\ldots,b_N)} \chi^{(c_1,\ldots,c_{N-1})}(e^{a_1},\ldots,e^{a_{N-1}}).$$

- Key identity:

$$L_K^{(N)}(a_1,\dots,a_K) = \left(\prod_{i=1}^{K} \frac{a_i}{e^{a_i}-1}\right)^{N-K} \int_{U(N)} e^{\mathrm{Tr}\begin{bmatrix} a_1 & & & & \\ & \ddots & & & \\ & & a_K & & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} U \begin{bmatrix} b_1^{(N)} & & \\ & \ddots & \\ & & b_N^{(N)} \end{bmatrix} U^{-1}} dU.$$

- Reduces problem to "just analysis":

$$\int_{U(N)} e^{\frac{1}{\sqrt{N}}\mathrm{Tr}\begin{bmatrix} a_1 & & & & \\ & \ddots & & & \\ & & a_K & & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} U \begin{bmatrix} b_1^{(N)} & & \\ & \ddots & \\ & & b_N^{(N)} \end{bmatrix} U^{-1}} dU \overset{?}{\sim} e^{\bullet (a_1+\dots+a_K) + \bullet \frac{1}{2}(a_1^2+\dots+a_K^2) + o(1)}$$

- This amounts to the asymptotic analysis of a ubiquitous special function: the Harish-Chandra / Itzykson + Zuber integral.

Free Triangle Graph Paper from http://incompetech.com/graphpaper/triangle/

- HCIZ integral: the function $\mathbb{C} \times \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}$ defined by

$$(z; a_1, \dots, a_N; b_1, \dots, b_N) \mapsto \int_{U(N)} e^{z \operatorname{Tr} \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{bmatrix} U \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_N \end{bmatrix} U^{-1}} dU$$

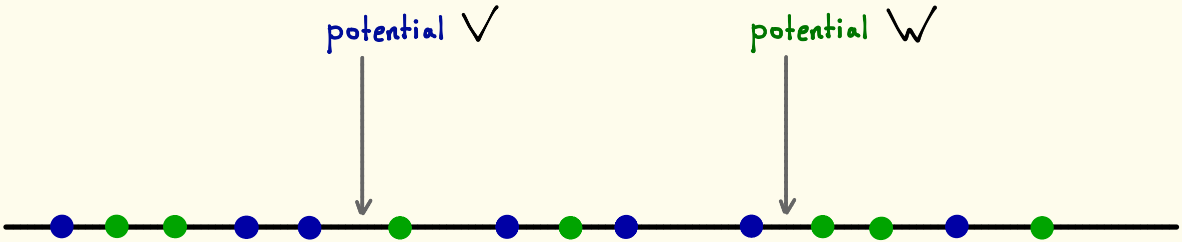
- Harish-Chandra: generalizes the Schur polynomials.

- Itzykson-Zuber: study of the normal 2-matrix model with AB-interaction:

$$\mathbb{P}_N(dA, dB) = \frac{1}{Z_N} e^{\operatorname{Tr}(V(A) + W(B) + zAB)} dA\,dB.$$

coupling constant

Two-Component Coulomb Gas: $e^{-\mathcal{H}}$



$$
\begin{aligned}
\mathcal{H}(z;\, a_1,\dots,a_N;\, b_1,\dots,b_N) = {} & \sum_i V(a_i) - \sum_{i\neq j} \log |a_i - a_j| \\
& + \sum_i W(b_i) - \sum_{i\neq j} \log |b_i - b_j| \\
& + \log \int_{U(N)} e^{z \operatorname{Tr} \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{bmatrix} U \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_N \end{bmatrix} U^{-1}} dU.
\end{aligned}
$$

- Itzykson–Zuber Conjecture (1980): Let

$$
\begin{array}{ccccc}
 & & a_1^{(1)} & & \\
 & a_1^{(2)} & & a_2^{(2)} & \\
a_1^{(3)} & & a_2^{(3)} & & a_3^{(3)} \\
\iddots & & & & \ddots
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccccc}
 & & b_1^{(1)} & & \\
 & b_1^{(2)} & & b_2^{(2)} & \\
b_1^{(3)} & & b_2^{(3)} & & b_3^{(3)} \\
\iddots & & & & \ddots
\end{array}
$$

be two triangular arrays of complex numbers such that the limits

$$\phi_m = \lim_{N\to\infty} \frac{1}{N} p_m\left(\frac{a_1^{(N)}}{\sqrt{N}}, \dots, \frac{a_N^{(N)}}{\sqrt{N}}\right) \quad \text{and} \quad \psi_m = \lim_{N\to\infty} \frac{1}{N} p_m\left(\frac{b_1^{(N)}}{\sqrt{N}}, \dots, \frac{b_N^{(N)}}{\sqrt{N}}\right)$$

Newton power sums

exist for all $m \in \mathbb{N}$. Then, $\exists\, \varepsilon > 0$ such that

$$\frac{1}{N^2} \log \int_{U(N)} e^{z \operatorname{Tr}\left[\begin{smallmatrix} a_1^{(N)} & & \\ & \ddots & \\ & & a_N^{(N)} \end{smallmatrix}\right] U \left[\begin{smallmatrix} b_1^{(N)} & & \\ & \ddots & \\ & & b_N^{(N)} \end{smallmatrix}\right] U^{-1}} dU$$

converges uniformly on compact subsets of $\{z \in \mathbb{C} : |z| < \varepsilon\}$.

- Lots of attempts to solve this problem by hard analysis.

- The HCIZ integral has a natural $S(N)\times S(N)$ symmetry.

- Maclaurin series can be presented in the form

Newton power sums

$$\log \int_{U(N)} e^{z \operatorname{Tr} \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{bmatrix} U \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_N \end{bmatrix} U^{-1}} dU = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha,\beta \vdash d} C_N(\alpha,\beta)\, p_\alpha(a_1,\dots,a_N)\, p_\beta(b_1,\dots,b_N).$$

???

Adolf Hurwitz

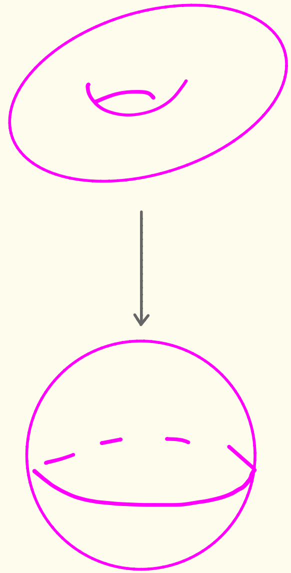
(1) Probably invented the Exponential Formula:

$$W = e^H$$

all structures

connected structures

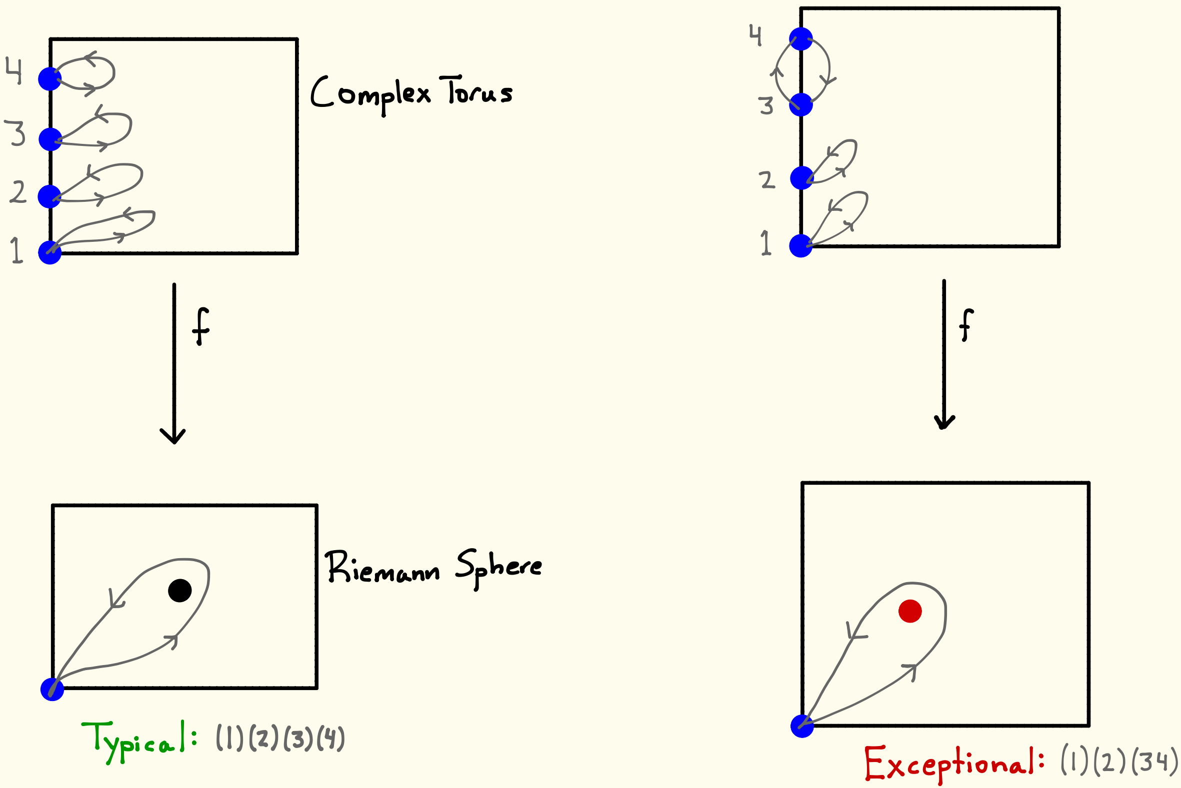
(2) First to "count surfaces":



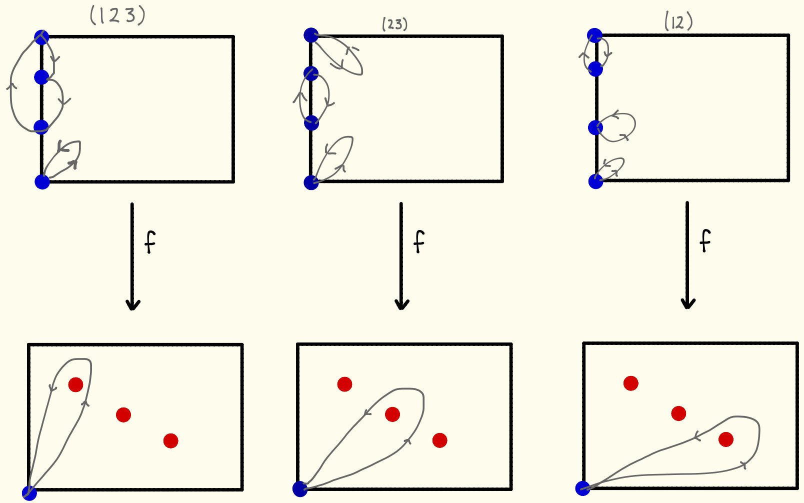
(3) Invented the braid groups:

$$B(n) = \langle \sigma_1, \dots, \sigma_n \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j|>1 \rangle$$

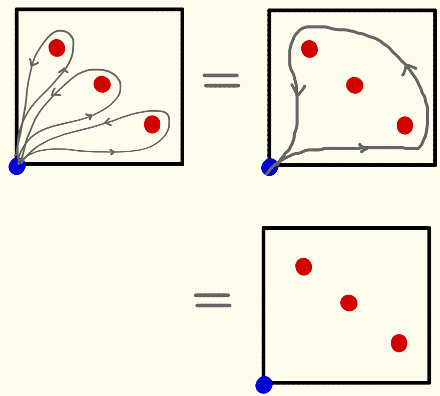
FACT: Every holomorphic function from a compact, connected Riemann surface to the Riemann sphere is a branched covering.



Hurwitz Encoding



$$(123)(23)(12) = (1)(2)(3)(4)$$

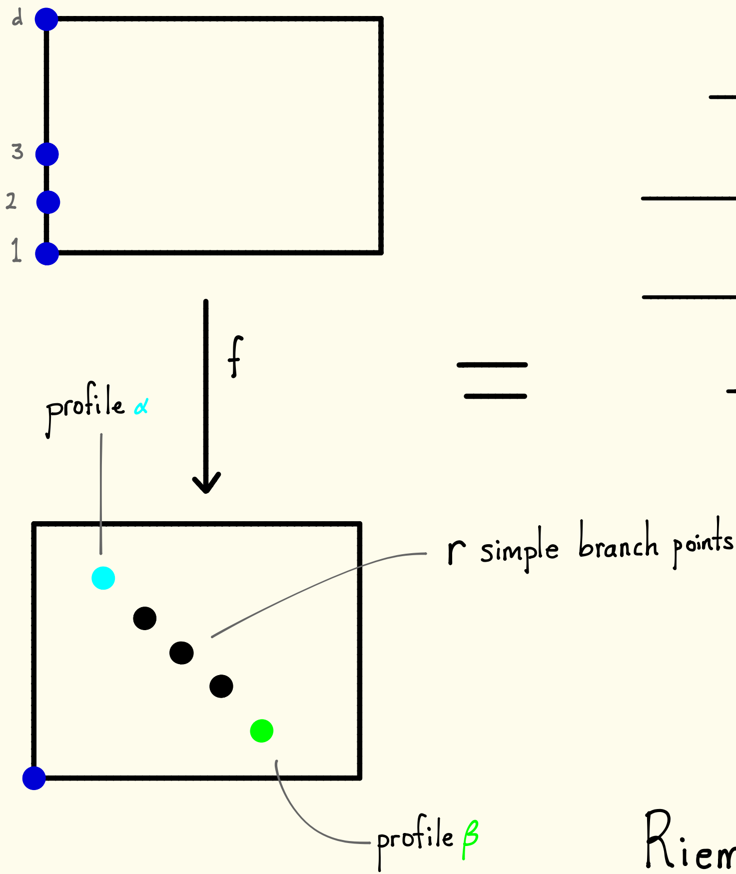


FACT: Given a topological branched covering $f : S \to \mathbb{P}^1$, there is a unique complex structure on S which makes f holomorphic.

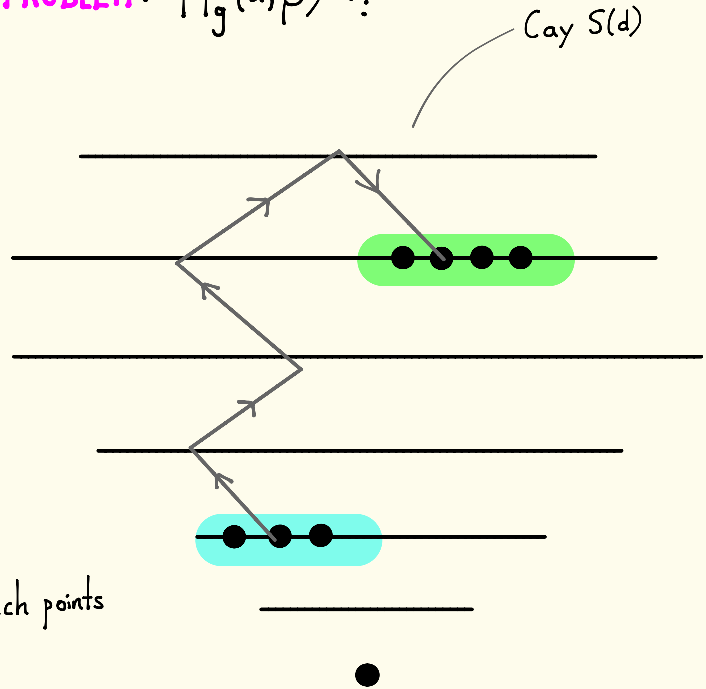
$$\text{GEOMETRY} \xleftrightarrow{\text{(topology)}} \text{COMBINATORICS}$$

$$\left\{ \begin{array}{c} \text{degree } d \text{ branched covers} \\ \text{of } \mathbb{P}^1 \text{ with given ramification} \\ \text{data} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{transitive factorizations of} \\ (1)\ldots(d) \text{ in } S(d) \text{ with factors} \\ \text{in prescribed conjugacy classes} \end{array} \right\}$$

THE HURWITZ PROBLEM: $H_g(\alpha,\beta)=?$



$=$

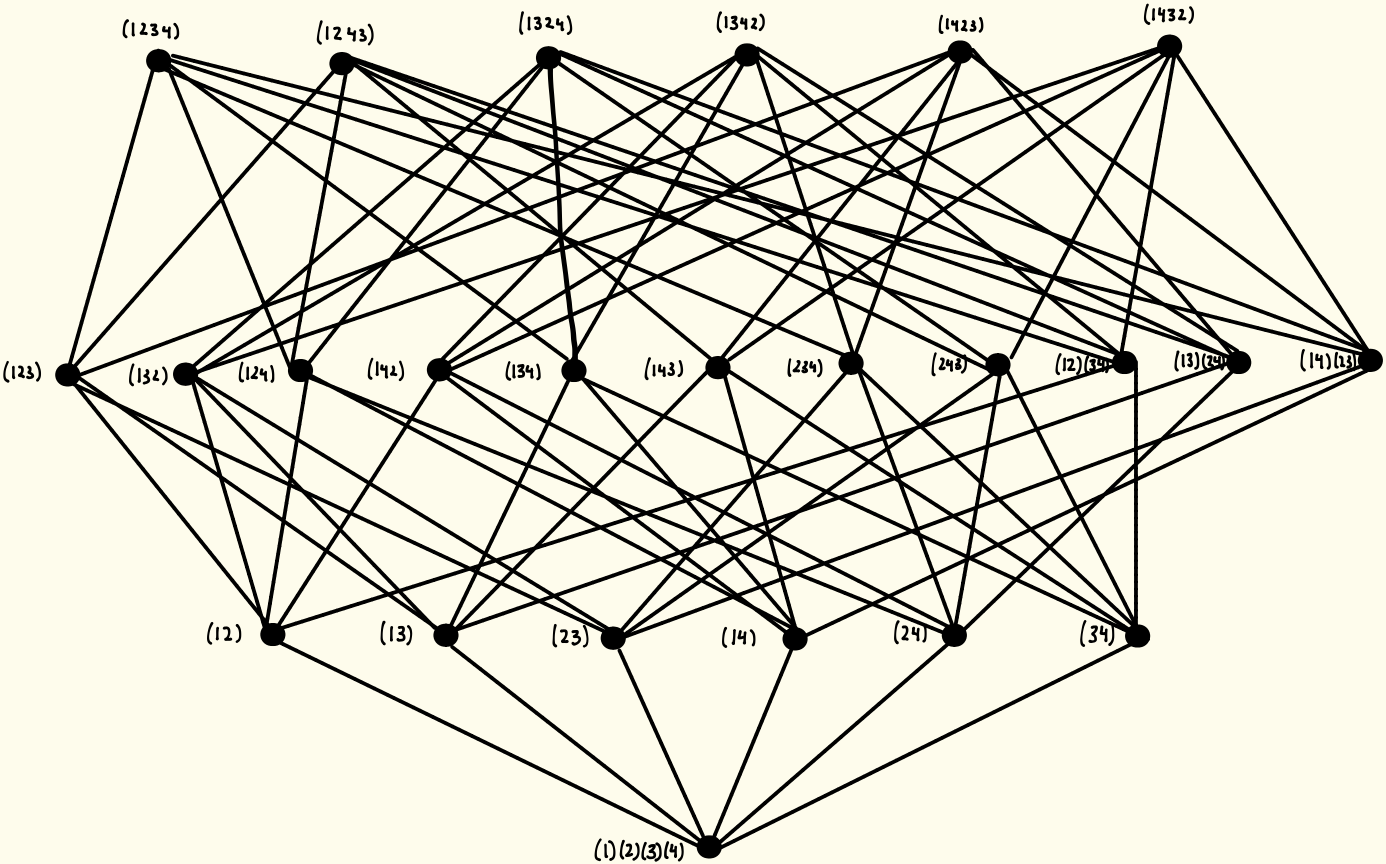


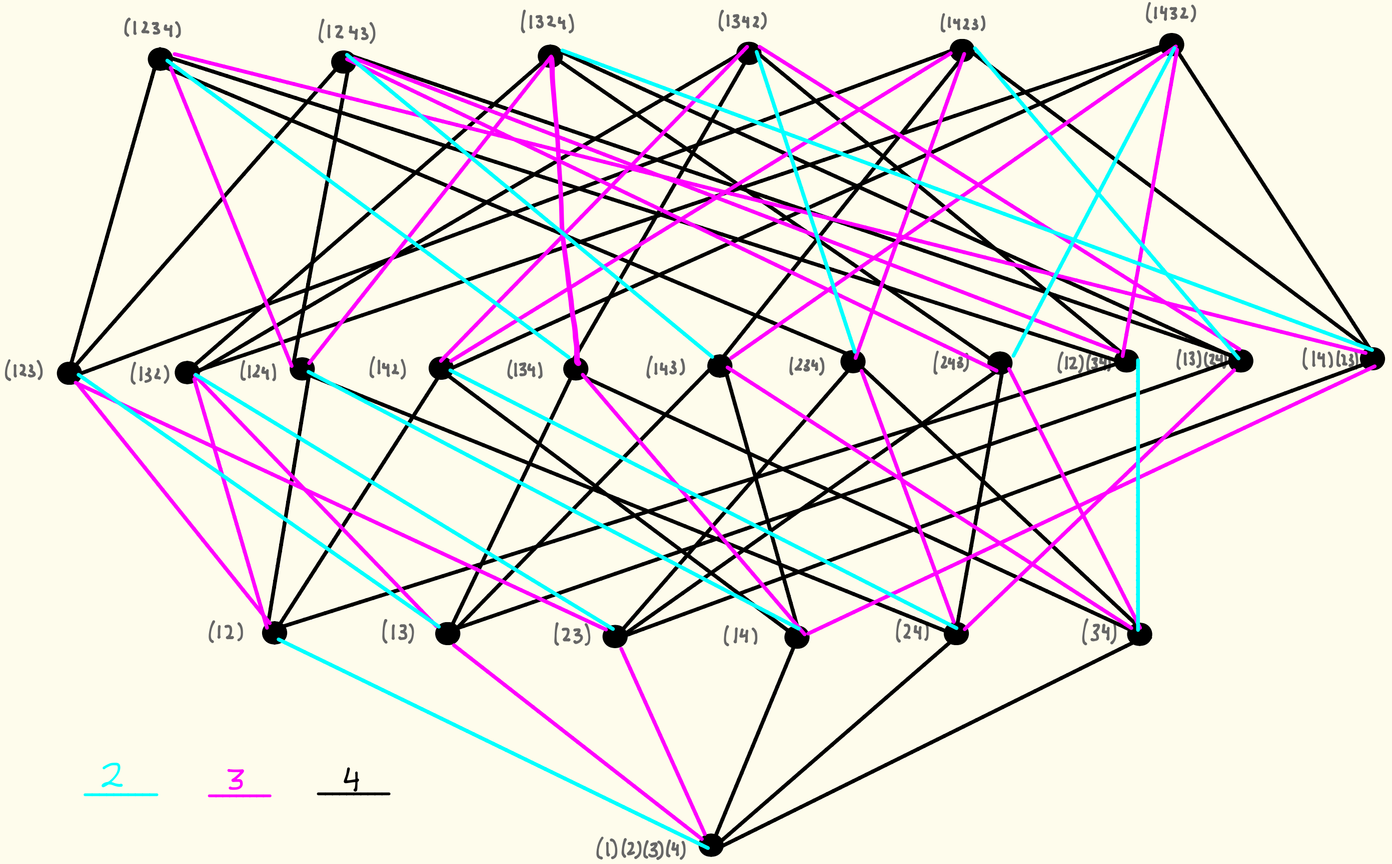
$$\sigma = \rho\,\tau_1 \ldots \tau_r$$

$\langle \rho, \tau_1, \ldots, \tau_r, \sigma \rangle$ transitive

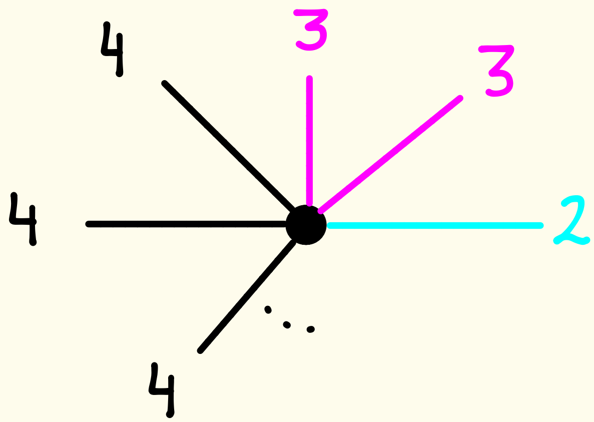
Riemann-Hurwitz: $r = 2g-2+\ell(\alpha)+\ell(\beta)$.

CAYLEY GRAPH OF S(4)



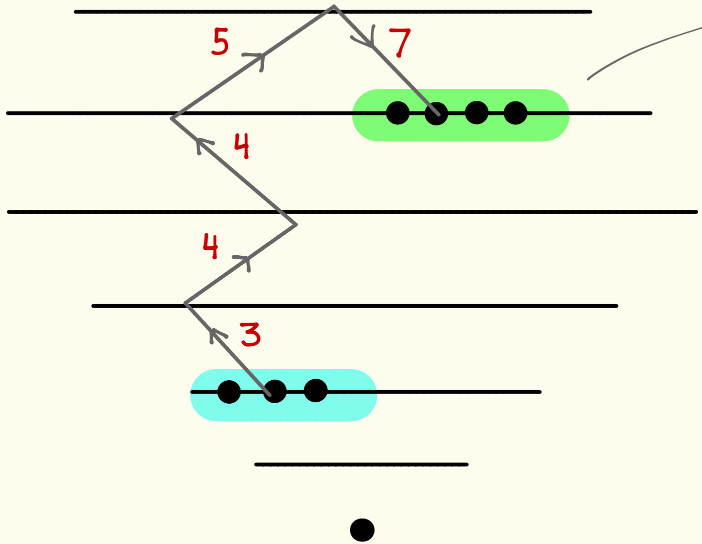

(1234)
(1243)
(1324)
(1342)
(1423)
(1432)
(123)
(132)
(124)
(142)
(134)
(143)
(234)
(243)
(12)(34)
(13)(24)
(14)(23)
(12)
(13)
(23)
(14)
(24)
(34)
(1)(2)(3)(4)
2
3
4

A Fragment of S(d)



$$T = \begin{bmatrix} (12) & (13) & (14) \\ & (23) & (24) \\ & & (34) \end{bmatrix}$$

THE MONOTONE HURWITZ PROBLEM: $\vec{H}_g(\alpha,\beta)=?$



Virtual history of an increasingly snobbish particle.

• By $S(N) \times S(N)$ symmetry,

$$\log \int_{U(N)} e^{z \operatorname{Tr} \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{bmatrix} U \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_N \end{bmatrix} U^{-1}} dU = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta \vdash d} C_N(\alpha, \beta)\, p_\alpha(a_1, \dots, a_N)\, p_\beta(b_1, \dots, b_N).$$

• Goulden, Guay-Paquet, Novak: For any $1 \leq d \leq N$,

$$\boxed{C_N(\alpha, \beta) = (-1)^{\ell(\alpha) + \ell(\beta)} N^{2 - d - \ell(\alpha) - \ell(\beta)} \sum_{g=0}^{\infty} \frac{\vec{H}_g(\alpha, \beta)}{N^{2g}}}$$

- Itzykson - Zuber limit:
  $\exists\, \varepsilon>0$ such that

$$\frac{1}{N^2}\log\int_{U(N)} e^{z\,\mathrm{Tr}\begin{bmatrix} a_1^{(N)} & & \\ & \ddots & \\ & & a_N^{(N)}\end{bmatrix} U \begin{bmatrix} b_1^{(N)} & & \\ & \ddots & \\ & & b_N^{(N)}\end{bmatrix} U^{-1}} dU \longrightarrow \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha,\beta\vdash d} (-1)^{\ell(\alpha)+\ell(\beta)} \vec{H}_0(\alpha,\beta)\,\phi_\alpha \psi_\beta .$$

- Reminder:

$$\phi_m = \lim_{N\to\infty} \frac{1}{N} p_m\left(\frac{a_1^{(N)}}{\sqrt{N}},\dots,\frac{a_N^{(N)}}{\sqrt{N}}\right) \quad \text{and} \quad \psi_m = \lim_{N\to\infty} \frac{1}{N} p_m\left(\frac{b_1^{(N)}}{\sqrt{N}},\dots,\frac{b_N^{(N)}}{\sqrt{N}}\right).$$

Newton power sums

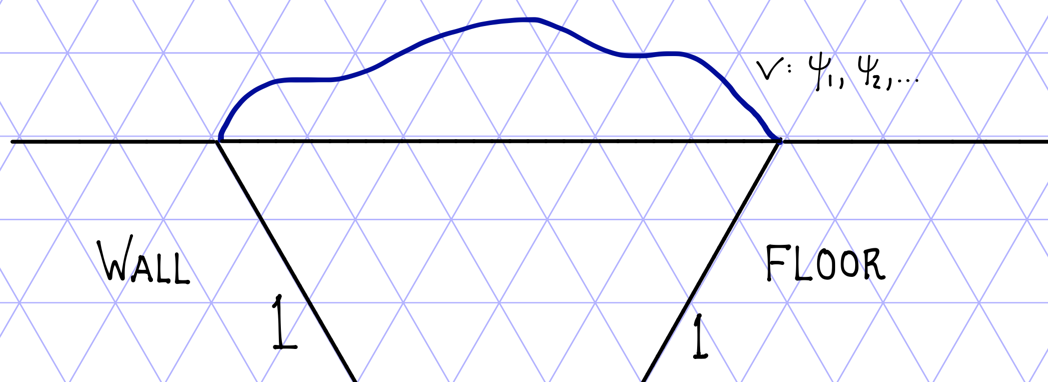
- Back to tiling regime: much simpler!!! We have

$$\underbrace{a_1, \dots, a_k}_{\text{fixed}} \quad \text{and} \quad \begin{array}{ccccc} & & b_1^{(1)} & & \\ & b_1^{(2)} & & b_2^{(2)} & \\ b_1^{(3)} & & b_2^{(3)} & & b_3^{(3)} \\ \iddots & & & & \ddots \end{array}$$

such that $\exists$ limits

$$\psi_m = \lim_{N \to \infty} \frac{1}{N} p_m\left(\frac{b_1^{(N)}}{N}, \dots, \frac{b_N^{(N)}}{N}\right).$$

- Reminder:



- Analytic degeneration: $\exists\, \varepsilon > 0$ such that

$$\frac{1}{N}\log\int_{U(N)} e^{z\operatorname{Tr}\begin{bmatrix} a_1 & & & & \\ & \ddots & & & \\ & & a_k & & \\ & & & 0 & \\ & & & & \ddots\ 0\end{bmatrix} U \begin{bmatrix} b_1^{(N)} & & \\ & \ddots & \\ & & b_N^{(N)}\end{bmatrix} U^{-1}} dU \longrightarrow \sum_{d=1}^{\infty} \frac{z^d}{d!} p_d(a_1,\dots,a_k) \sum_{\beta \vdash d} (-1)^{1+\ell(\beta)} \vec{H}_0(d,\beta)\psi_\beta,$$

uniformly on compact subsets of $\{(z; a_1,\dots,a_k) \in \mathbb{C}\times\mathbb{C}^k : |za_i| < \varepsilon\}$.

- Combinatorial degeneration:

$$\sum_{\beta \vdash d} (-1)^{1+\ell(\beta)} \vec{H}_0(d,\beta)\psi_\beta = (d-1)! \cdot d^{\text{th}} \text{ free cumulant of } \nu$$

$$= \text{an explicit polynomial in } \psi_1,\dots,\psi_d.$$

- Tuning the coupling constant z to $z = N^{-\frac{1}{2}}$ yields $N \to \infty$ asymptotics

$$\log \int_{U(N)} e^{\frac{1}{\sqrt{N}} \operatorname{Tr} \begin{bmatrix} a_1 & & & & \\ & \ddots & & & \\ & & a_k & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} U \begin{bmatrix} b_1^{(N)} & & \\ & \ddots & \\ & & b_N^{(N)} \end{bmatrix} U^{-1}} dU \sim \sum_{d=1}^{\infty} N^{1-\frac{d}{2}} \frac{\kappa_d}{d} p_d(a_1, \dots, a_k),$$

uniformly on compact subsets of $\mathbb{C}^k$.

- This gives us everything we need, and more:

$$\log L_k^{(N)}\left(\frac{a_1}{\sqrt{N}}, \dots, \frac{a_k}{\sqrt{N}}\right) \sim \sum_{d=1}^{\infty} N^{1-\frac{d}{2}} \left(\frac{\kappa_d}{d} - \frac{c_d}{d!}\right) p_d(a_1, \dots, a_k).$$

- Keeping just the first two terms yields the CLT:

$$\log L_k^{(N)}\left(\frac{a_1}{\sqrt{N}}, \dots, \frac{a_k}{\sqrt{N}}\right) = \sqrt{N}\left(\psi_1 - \tfrac{1}{2}\right) p_1(a_1, \dots, a_k) + \left(\psi_2 - \psi_1^2 - \tfrac{1}{12}\right) p_2(a_1, \dots, a_k) + o(1).$$

— END —