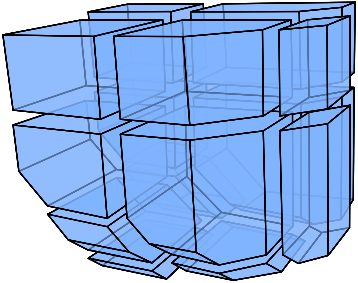


Tropical Catalan Subdivisions

Arnau Padrol
IMJ - PRG
UPMC Paris 06

joint work with

Cesar Ceballos and
Camilo Sarmiento

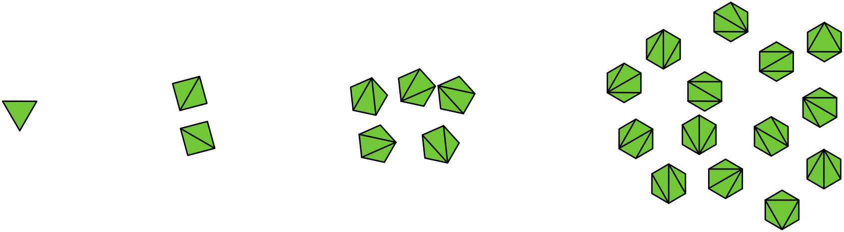
Séminaire Flajolet - 29/09/2016

Tropical Catalan Subdivisions

Catalan numbers

The *Catalan numbers*

$$1, 2, 5, 14, 42, 132, 429, 1430, \ldots$$



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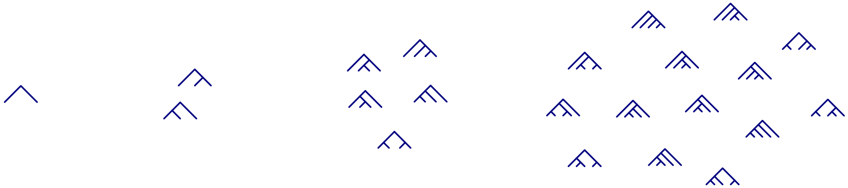

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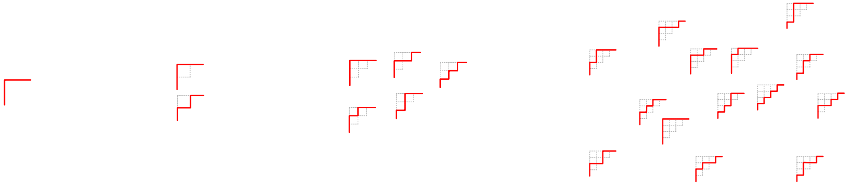
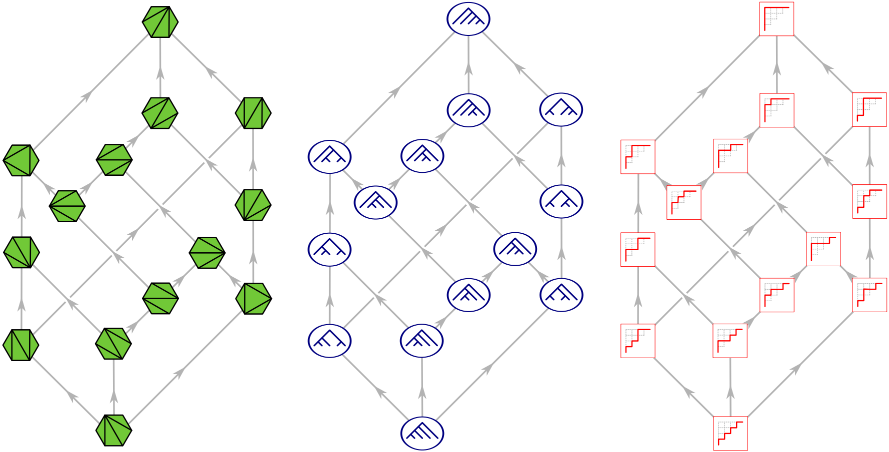
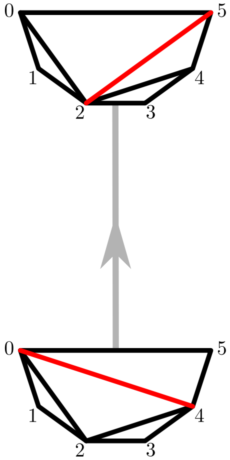


The Tamari lattice

The *Tamari lattice*: a partial order on Catalan families

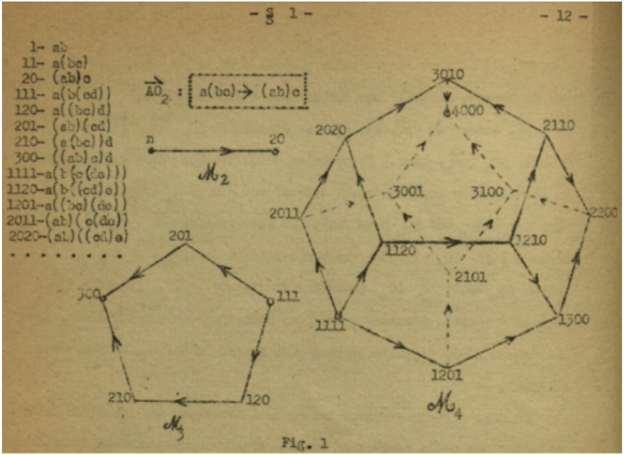


The *Tamari lattice*: a partial order on Catalan families



The Associahedron

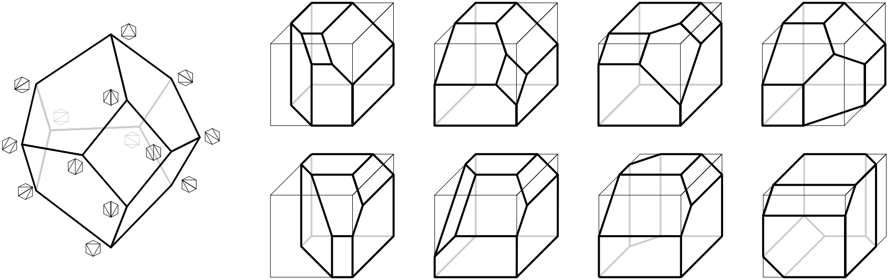
The Tamari lattice can be realized by the graph of the *Associahedron*



[Dov Tamari, Monoïdes préordonnés et chaînes de Malcev, PhD thesis, Université de Paris, (1951)]

The Associahedron

The Tamari lattice can be realized by the graph of the *Associahedron*

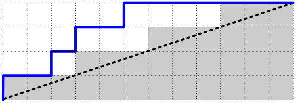


From [Ceballos, Santos, Ziegler, Many non-equivalent realizations of the Associahedron (2014)]

Stasheff ('63), Haiman ('84), Lee ('89), Gelfand-Kapranov-Zelevinsky ('94), Chapoton-Fomin-Zelevinsky ('02), Rote-Santos-Streinu ('03), Loday ('04), Hohlweg-Lange ('07), Postnikov ('09), Ceballos-Santos-Ziegler ('14)... among many others.

The m-Tamari lattice

Fuss-Catalan path: lattice path from $(0, 0)$ to (mn, n) that stays weakly above the main diagonal.



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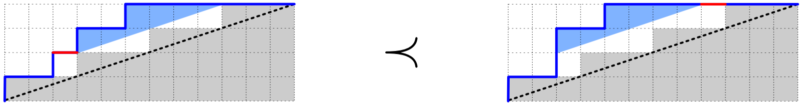
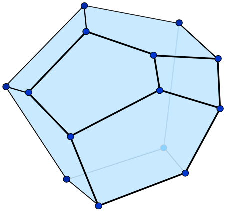


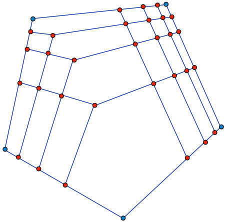
m-Tamari lattice: poset (actually a lattice) on Fuss-Catalan paths determined by this following covering relation

[François Bergeron and Louis-François Préville-Ratelle. Higher trivariate diagonal harmonics via generalized Tamari posets, (2012)]

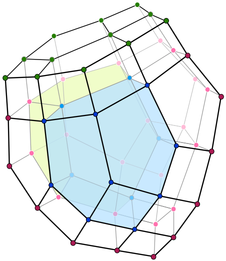
The m-Tamari lattice



1-Tamari $n = 4$



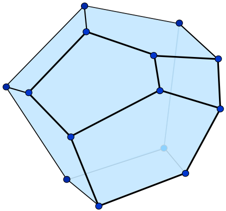
4-Tamari $n = 3$



2-Tamari $n = 4$

[François Bergeron. Combinatorics of r-Dyck paths, r-Parking functions, and the r-Tamari lattices (2012)]

The m-Tamari lattice



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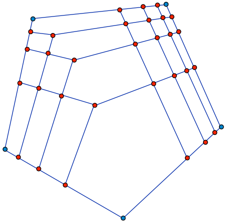
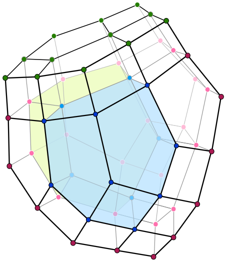


4-Tamari $n = 3$



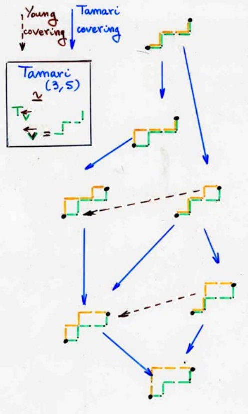
2-Tamari $n = 4$

[François Bergeron. Combinatorics of r-Dyck paths, r-Parking functions, and the r-Tamari lattices (2012)]

There has also been recently a lot of work on geometric (or polytopal) realizations of the associahedron (the Tamari poset for $r = 1$). This leads to the natural question of describing similar constructions for all r-Tamari posets. Figure 6 suggests a tantalizing outlook on this.

The ν-Tamari lattice

ν-Tamari lattice: similar on paths above a given lattice path ν...



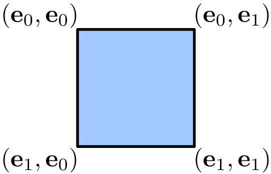
[Louis-François Préville-Ratelle and Xavier Viennot, An extension of Tamari lattices (2014)]

Tropical Catalan Subdivisions

The associahedral triangulation

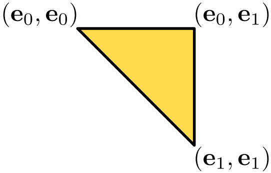
Consider the *product of two simplices*

$$\Delta_n \times \Delta_{\overline{n}} = \operatorname{conv}\left\{(\mathbf{e}_i, \mathbf{e}_{\bar{j}}) \colon 0 \le i, \bar{j} \le n\right\}.$$



We want to triangulate (subdivide into simplicies) the sub-polytope

$$\mathcal{C}_n = \operatorname{conv}\left\{(\mathbf{e}_i, \mathbf{e}_{\bar{j}}) \colon 0 \le i \le \bar{j} \le n\right\}$$



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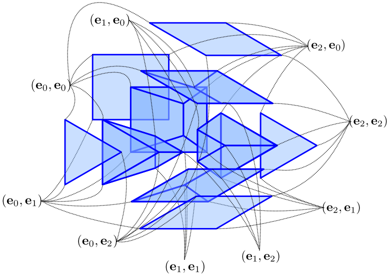

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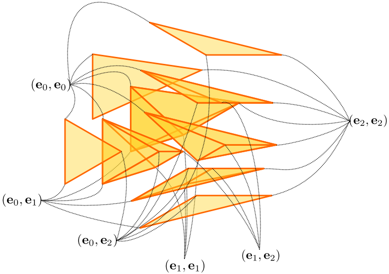

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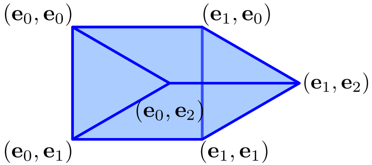

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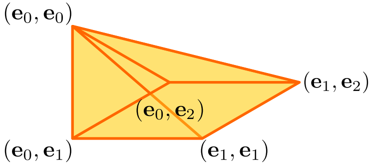
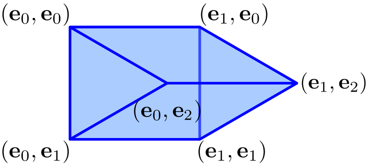


The associahedral triangulation

Consider the *product of two simplices*

$$\Delta_n \times \Delta_{\overline{n}} = \mathrm{conv}\left\{(\mathbf{e}_i, \mathbf{e}_{\bar{j}}) : 0 \le i, \bar{j} \le n\right\}.$$



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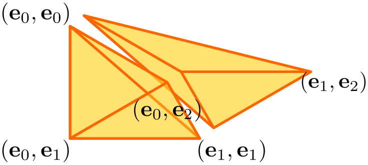
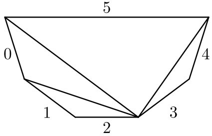
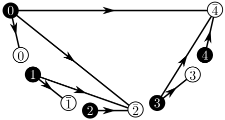
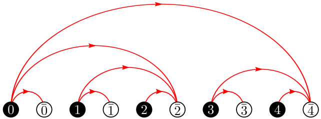


The associahedral triangulation

The cells: indexed by triangulations of an $(n+2)$-gon







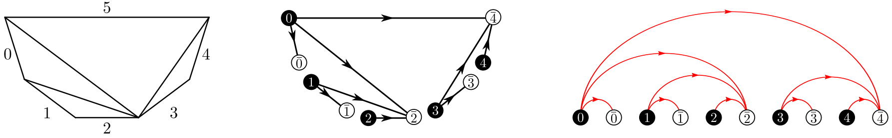
In this example, the cell is:

$$\operatorname{conv}\left\{(\mathbf{e}_0, \mathbf{e}_{\overline{0}}), (\mathbf{e}_0, \mathbf{e}_{\overline{2}}), (\mathbf{e}_0, \mathbf{e}_{\overline{4}}), (\mathbf{e}_1, \mathbf{e}_{\overline{1}}), \ldots, (\mathbf{e}_4, \mathbf{e}_{\overline{4}})\right\}$$

Alternatively: *bipartite non-crossing* trees on $[n] \sqcup [\overline{n}]$.

The associahedral triangulation

The cells: indexed by triangulations of an $(n+2)$-gon



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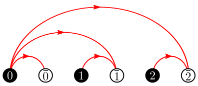
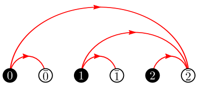
Alternatively: *bipartite non-crossing* trees on $[n] \sqcup [\overline{n}]$.

Fact

- *These cells triangulate the polytope $\mathcal{C}_n \subset \Delta_n \times \Delta_{\overline{n}}$*
- *This triangulation is dual to an associahedron.*
- *The triangulation is regular and flag.*

The associahedral triangulation

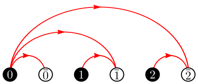


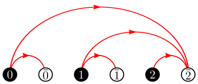


Example

The 1-dimensional associahedron is the dual of a triangulation of a 4-dimensional polytope $\mathcal{C}_2 \subset \Delta_2 \times \Delta_{\overline{2}}$.

The associahedral triangulation





Example

The 1-dimensional associahedron is the dual of a triangulation of a 4-dimensional polytope $\mathcal{C}_2 \subset \Delta_2 \times \Delta_{\overline{2}}$.

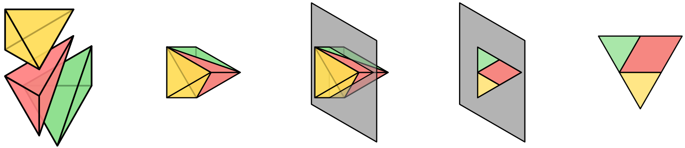
Why do you want to draw a 1-dim edge in 4 dimensions?

This might look like a disadvantage.

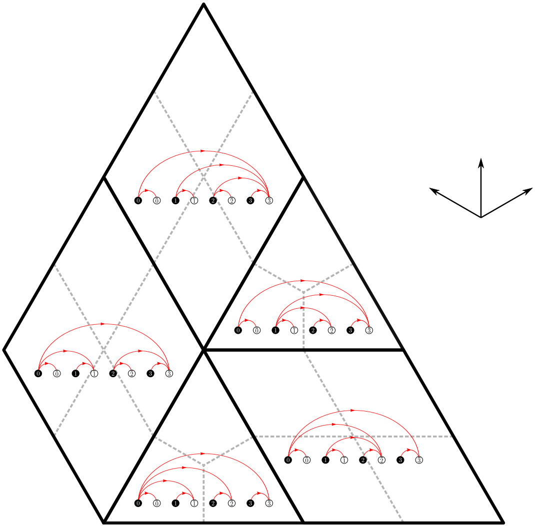
But this approach is actually very powerful.

Theorem (The Cayley trick [Huber-Rambau-Santos ’00])

$$\left\{\begin{matrix}\textit{triangulations}\\ \textit{of } \Delta_n \times \Delta_{\overline{m}}\end{matrix}\right\} \xleftrightarrow{\textbf{Cayley trick}} \left\{\begin{matrix}\textit{fine mixed}\\ \textit{subdivisions of } m\Delta_{n-1}\end{matrix}\right\}$$



An example



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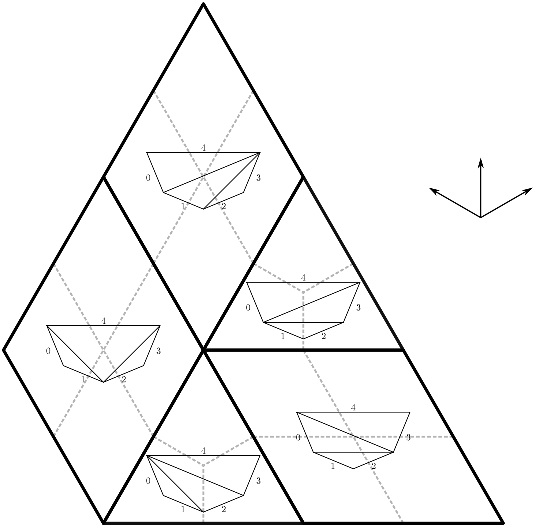


The associahedral triangulation

This triangulation has appeared under different disguises in many independent papers:

- Gelfand–Graev–Postnikov, Combinatorics of hypergeometric functions associated with positive roots, ’97. (As a triangulation of a root polytope)
- Stanley–Pitman, A polytope related to empirical distributions, plane trees, parking functions, and the associahedron, ’02. (As a mixed subdivision of the Stanley – Pitman polytope.)
- Petersen–Pylyavskyy–Speyer, A non-crossing standard monomial theory, ’10. (As a triangulation of a Gelfand-Tseltsin polytope.)
- Santos–Stump–Welker, Noncrossing sets and a Grassmann associahedron. ’14. (As a triangulation of an Order polytope.)
- . . .

The $(I, \bar{J})$-triangulation

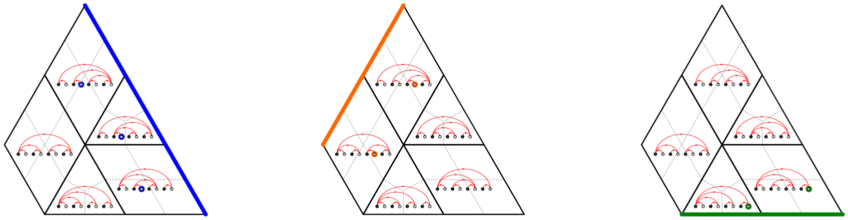
Faces of $\Delta_n \times \Delta_{\bar{n}}$ are of the form

$$\Delta_I \times \Delta_{\bar{J}} = \text{conv}\left\{(\mathbf{e}_i, \mathbf{e}_{\bar{j}}) : i \in I \text{ and } \bar{j} \in \bar{J}\right\}$$

The restriction of the associahedral triangulation to the face

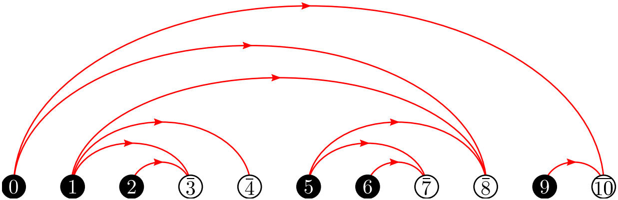
$$\mathcal{C}_{I,\bar{J}} = \text{conv}\left\{(\mathbf{e}_i, \mathbf{e}_{\bar{j}}) : i \in I \text{ and } \bar{j} \in \bar{J}, i \leq \bar{j}\right\}$$

is called the $(I, \bar{J})$-triangulation.



The $(I, \bar{J})$-triangulation

The cells of this restricted triangulation are indexed by *$(I, \bar{J})$-trees*
(bipartite non-crossing trees with support $I \cup \bar{J}$)

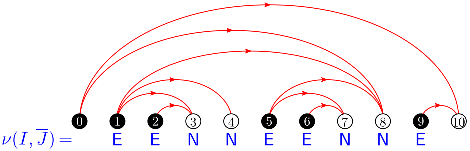


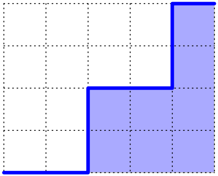
In this example,

$$I = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{5}, \mathbf{6}, \mathbf{9}\} \qquad \bar{J} = \{\overline{\mathbf{3}}, \overline{\mathbf{4}}, \overline{\mathbf{7}}, \overline{\mathbf{8}}, \overline{\mathbf{10}}\}$$

The $(I,\bar{J})$-triangulation

Given such a tree T we associate two paths $\nu(I,\bar{J})$ and $\rho(T)$:





$\nu(I,\bar{J})$ replaces black and white balls by east and north steps respectively.
$\rho(T)$ counts the in-degrees of the white balls.

Note: the path $\rho(T)$ is weakly above ν.

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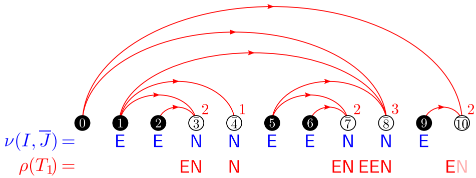

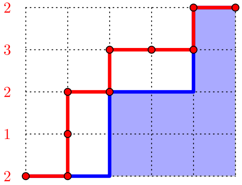

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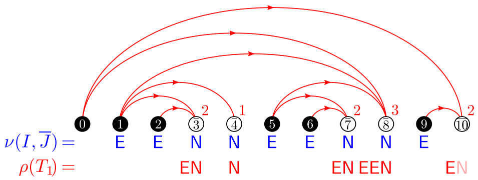

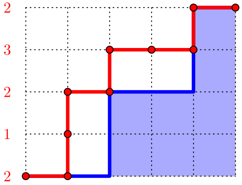

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$\rho(T)$ counts the in-degrees of the white balls.

Note: the path $\rho(T)$ is weakly above ν.

Proposition (CPS)

Let I, J be a partition of $[n]$ with $0 \in I$ and $n \in J$, and $\nu = \nu(I,\bar{J})$.

- *ρ is a bijection from $(I,\bar{J})$-trees to ν-paths.*
- *flips of $(I,\bar{J})$-trees correspond to ν-Tamari covering relations.*

Realizing the ν-Tamari lattice: as the dual of a triangulation

Theorem (CPS)

The ν-Tamari lattice Tam(ν) *can be realized geometrically as the dual of a regular triangulation of a sub-polytope of $\Delta_a \times \Delta_b$.*

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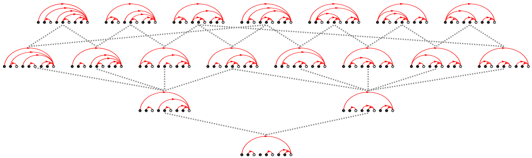
Simplicial complex of $(I, \bar{J})$-forests



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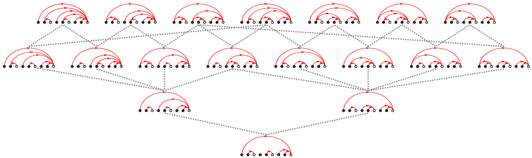


ν-Narayana numbers

The h-vector $(h_0, h_1, \ldots)$ of the simplicial $(I,\bar{J})$-associahedron

$$h_\ell = \text{number of } \nu(I,\bar{J})\text{-paths with exactly } \ell \text{ valleys},$$

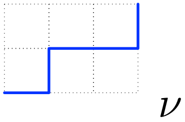
where a valley of a path is an occurrence of EN.

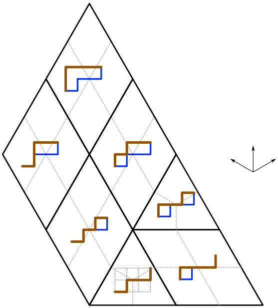
Classical Narayana numbers when $\nu = \{EN\}^n$, rational Narayana numbers when ν is the lowest path above a line with rational slope.

Realizing the ν-Tamari lattice: as the dual of a mixed subdivision

Corollary (CPS)

Let ν be a lattice path from $(0,0)$ to (a,b). $\mathrm{Tam}(\nu)$ *is the dual of a coherent mixed subdivision of a generalized permutahedron.*





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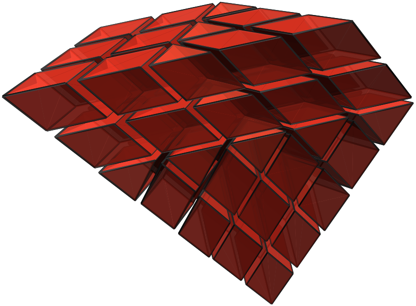

Tropical Catalan Subdivisions

Tropical geometry

Tropical semiring

$(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$

where $x \oplus y := \max(x, y)$ and $x \odot y = x + y$

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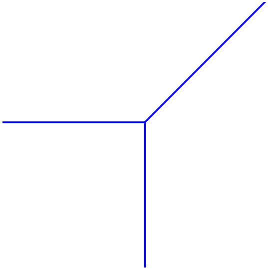
$\rightsquigarrow$

Tropical Geometry
Tropical polynomials, tropical curves,
tropical hyperplanes, tropical lines . . .



A tropical line

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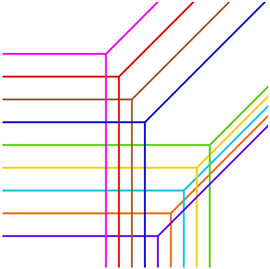

A tropical
arrangement of lines

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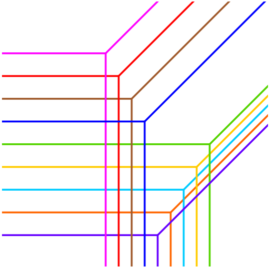

A tropical arrangement of lines

Theorem (Develin-Sturmfels ’04)

$$\left\{ \begin{matrix} \textit{regular triangulations} \\ \textit{of } \Delta_n \times \Delta_{\overline{m}} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \textit{(combintorial types of) generic arrangements of} \\ m \textit{ tropical hyperplanes in } \mathbb{TP}^{n-1} \end{matrix} \right\}$$

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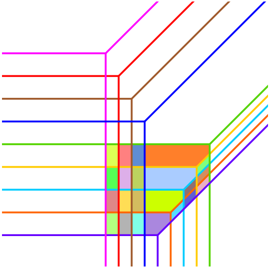
Tropical Geometry
Tropical polynomials, tropical curves, tropical hyperplanes, tropical lines ...



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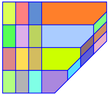

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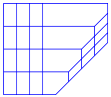

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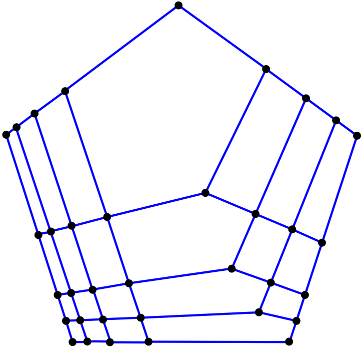
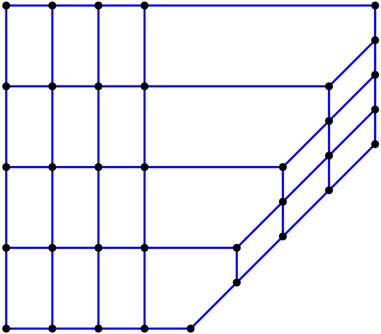
A realization of the ν-Tamari lattice

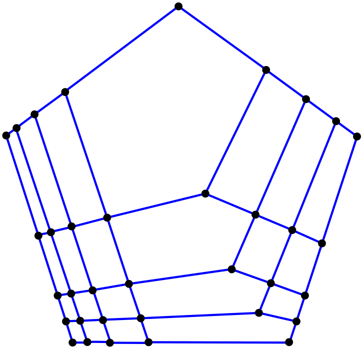


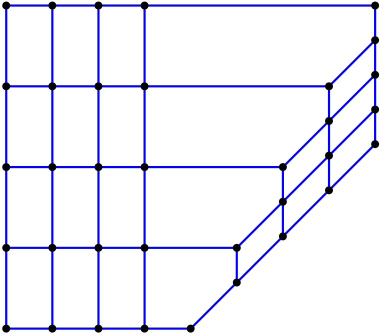


A realization of the ν-Tamari lattice

Theorem (CPS)

Let ν be a lattice path from $(0,0)$ to (a,b). $\text{Tam}(\nu)$ *is the edge graph of a polyhedral complex induced by a tropical hyperplane arrangement.*





The 4-Tamari lattice for $n = 3$.

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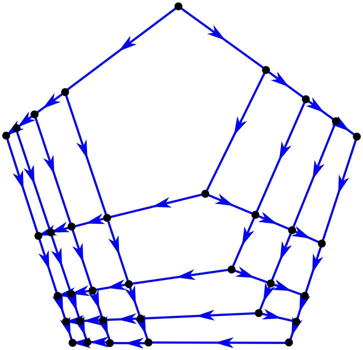

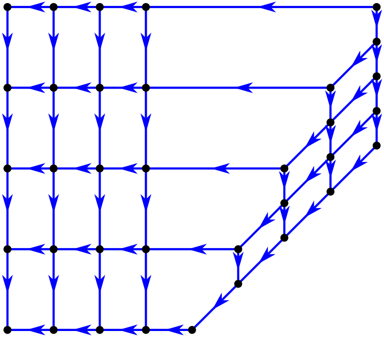

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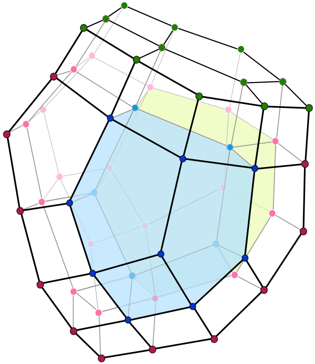

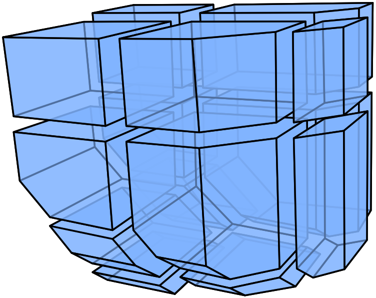

The 2-Tamari lattice for $n = 4$.

A realization of the ν-Tamari lattice

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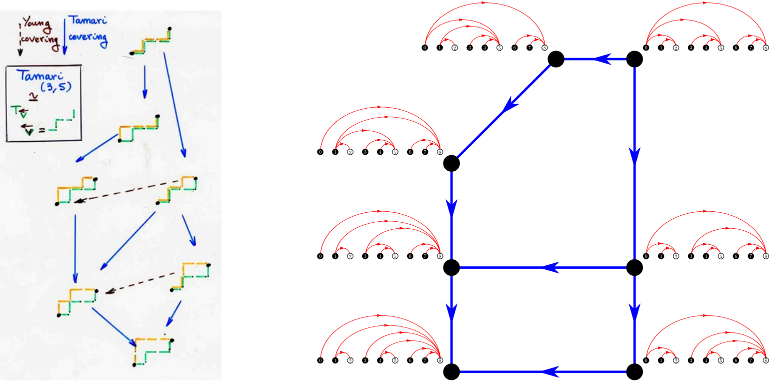
Let ν be a lattice path from $(0,0)$ to (a,b). $\mathrm{Tam}(\nu)$ *is the edge graph of a polyhedral complex induced by a tropical hyperplane arrangement.*



The rational Tamari lattice $\mathrm{Tam}(3,5)$.

A realization of the ν-Tamari lattice

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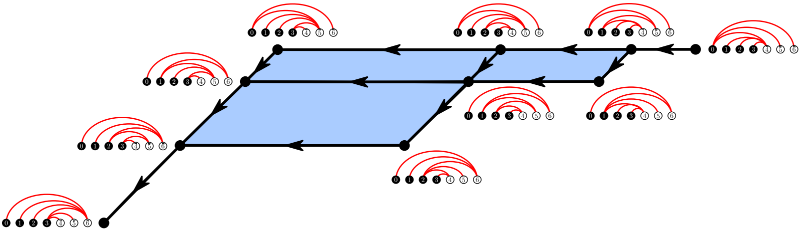

The $(\{0,1,2,3\},\{\overline{4},\overline{5},\overline{6}\})$-Tamari lattice.

A realization of the ν-Tamari lattice

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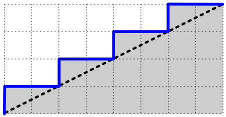
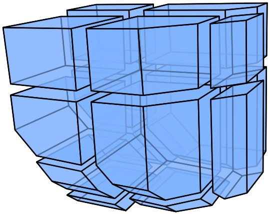
Let ν be a lattice path from $(0,0)$ to (a,b). $\mathrm{Tam}(\nu)$ is the edge graph of a polyhedral complex induced by a tropical hyperplane arrangement.

Theorem

The support of Asso_ν is convex only if ν does not have two (non-inicial) consecutive north steps.
In this case, Asso_ν is a subdivision of a classical associahedron into Cartesian products of associahedra.

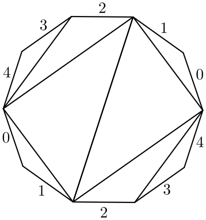


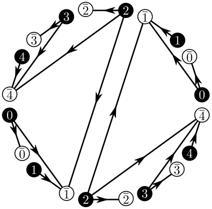


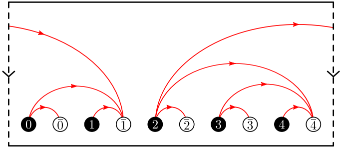
Epilogue: Type B Tropical Catalan Subdivisions

The cyclohedron triangulation

Consider the following trees indexed by cyclic symmetric triangulations of a $(2n+2)$-gon:





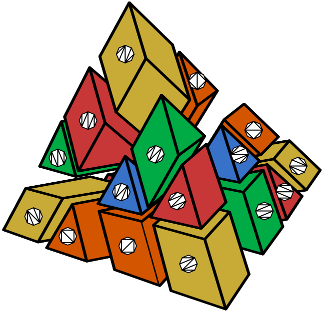


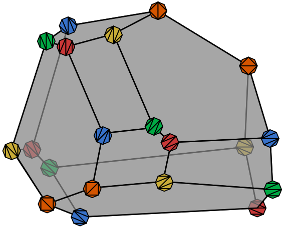
Via *bipartite cyclically non-crossing* trees on $[n] \sqcup [\overline{n}]$.

The cyclohedron triangulation

Theorem (CPS)

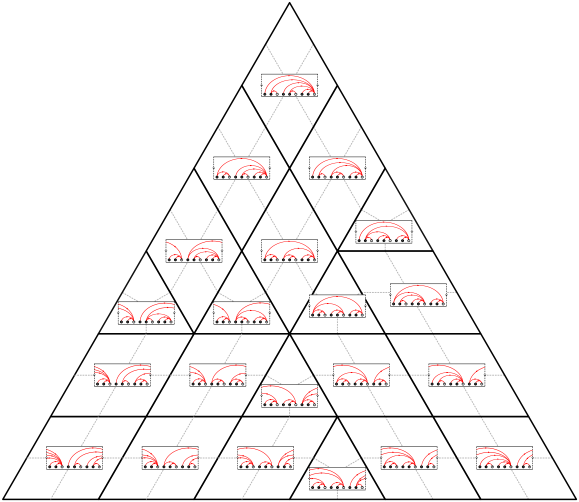
This collection of cells form a regular triangulation of $\Delta_n \times \Delta_{\overline{n}}$ dual to an n-dimensional cyclohedron.





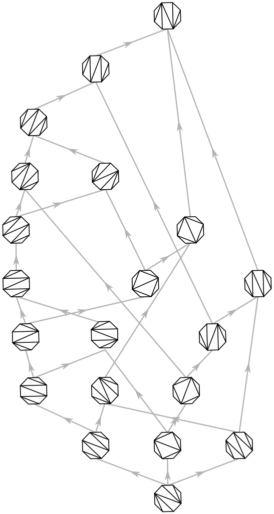
A cyclic $(I, \bar{J})$-triangulation

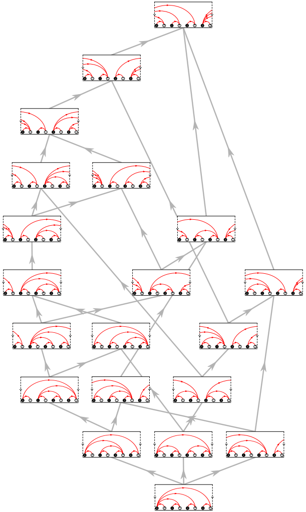
Restricting to faces: a natural definition for *$(I, \bar{J})$-triangulations of type B_n*



The Tamari poset of type B_n

Thomas ’06 and Reading ’06 defined Tamari lattices of type B_n





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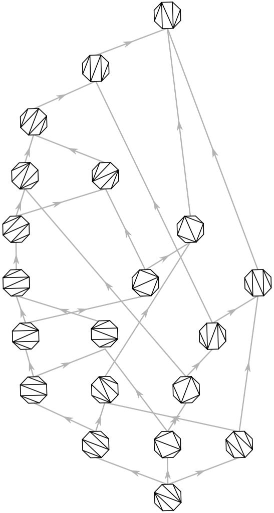
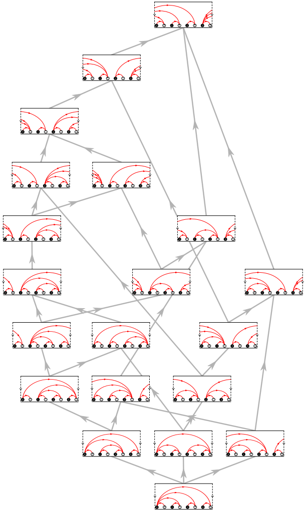
⇝ a natural definition for $(I,\bar{J})$*-Tamari posets of type* B_n





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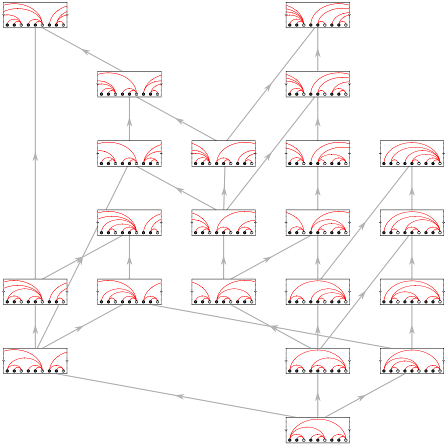
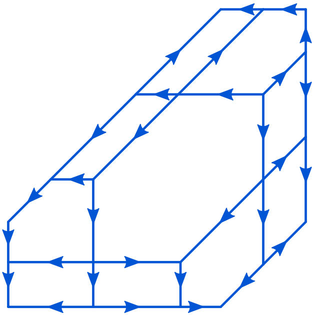


Tropicalizing ⇝ a natural definition for *$(I, \bar{J})$-cyclohedra*
($(I, \bar{J})$ associahedra of type B_n)



$(I,\bar{J})$-cyclohedra of type B_n

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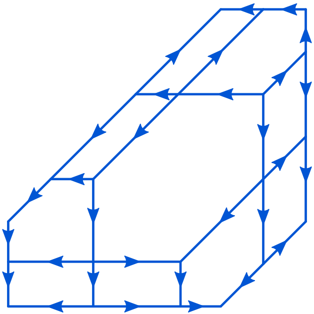

Thank you!