Good Predictions Are Worth a Few Comparisons

Carine Pivoteau

with Nicolas Auger and Cyril Nicaud

LIGM - Université Paris-Est-Marne-la-Vallée

April 2016

A case study

Find both the min. and the max. of an array of size n.

Naive Algorithm:

5	1	4	3	6	0	2	8	7	9

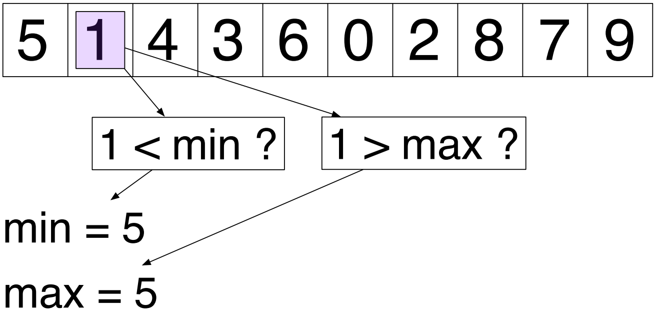
min = 5

max = 5

A case study

Find both the min. and the max. of an array of size n.

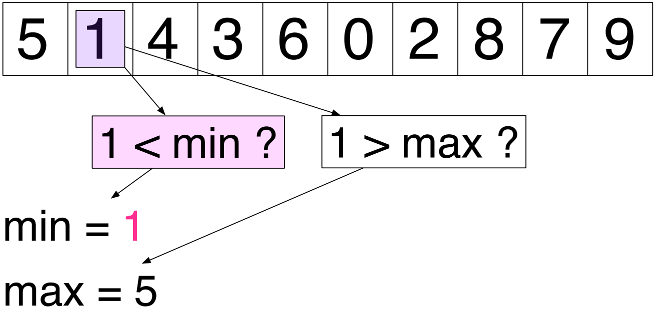
Naive Algorithm:



A case study

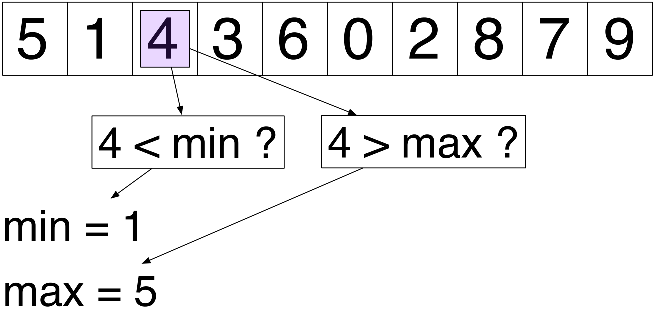
Find both the min. and the max. of an array of size n.

Naive Algorithm:



Find both the min. and the max. of an array of size n.

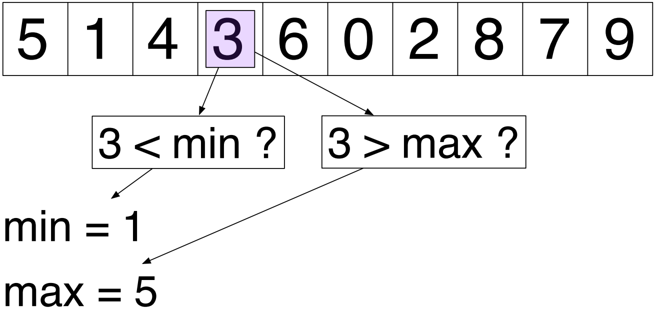
Naive Algorithm:



A case study

Find both the min. and the max. of an array of size n.

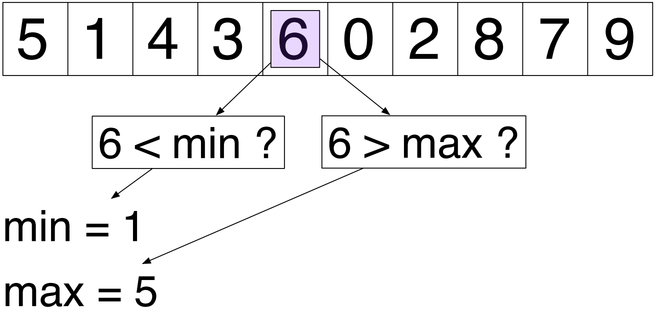
Naive Algorithm:



A case study

Find both the min. and the max. of an array of size n.

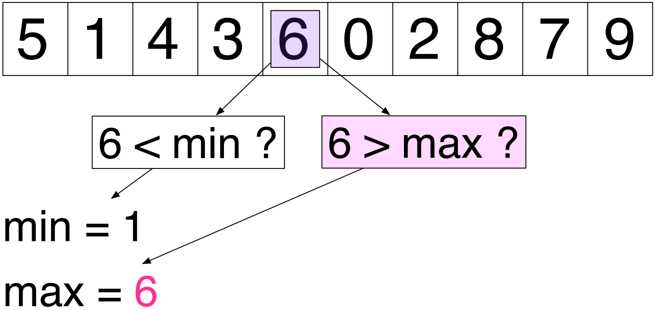
Naive Algorithm:



A case study

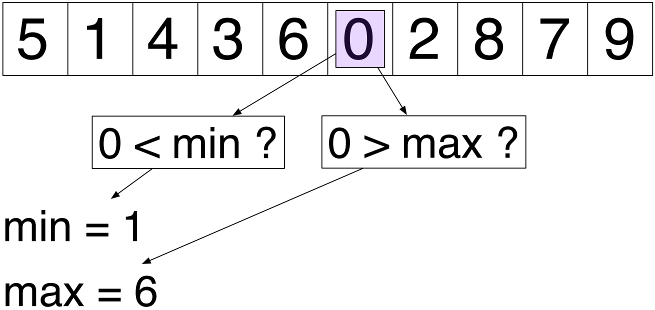
Find both the min. and the max. of an array of size n.

Naive Algorithm:



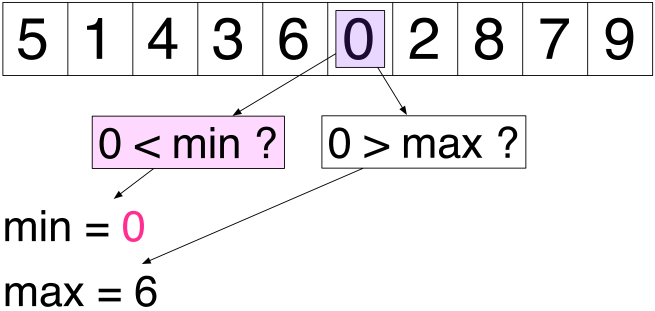
Find both the min. and the max. of an array of size n.

Naive Algorithm:



Find both the min. and the max. of an array of size n.

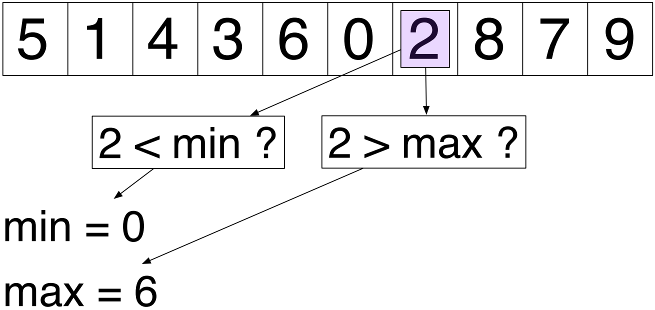
Naive Algorithm:



A case study

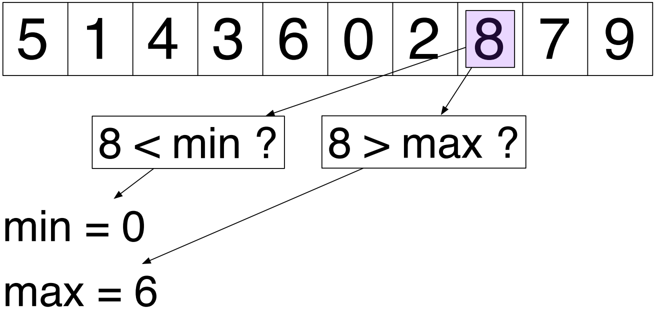
Find both the min. and the max. of an array of size n.

Naive Algorithm:



Find both the min. and the max. of an array of size n.

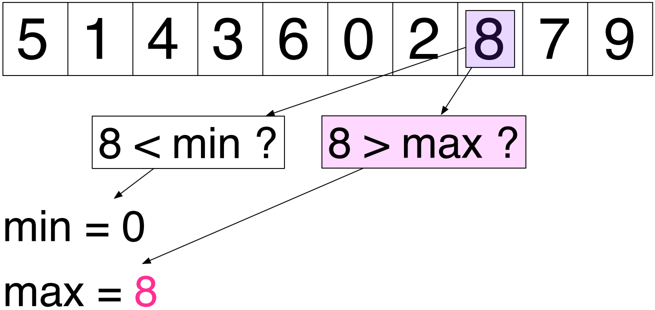
Naive Algorithm:



A case study

Find both the min. and the max. of an array of size n.

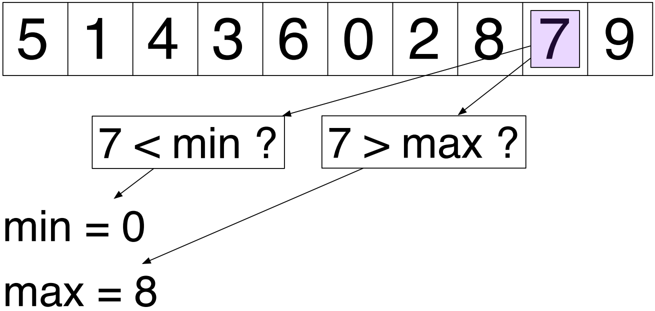
Naive Algorithm:



A case study

Find both the min. and the max. of an array of size n.

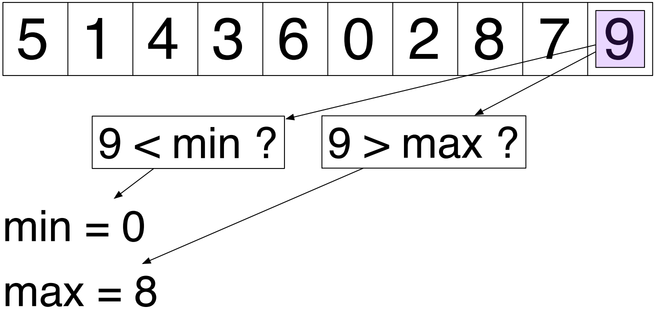
Naive Algorithm:



A case study

Find both the min. and the max. of an array of size n.

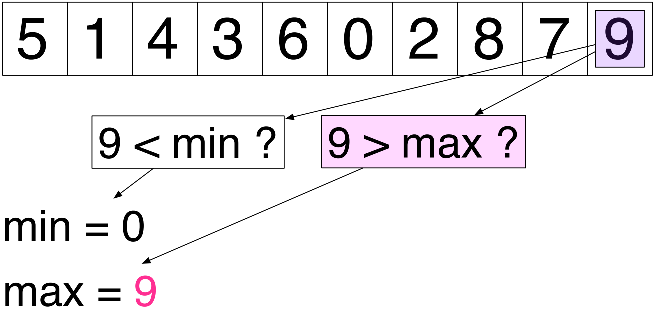
Naive Algorithm:



A case study

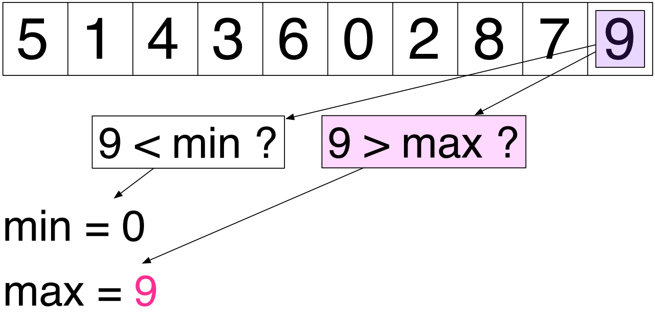
Find both the min. and the max. of an array of size n.

Naive Algorithm:



Find both the min. and the max. of an array of size n.

Naive Algorithm: **$2n$ comparisons**



Can we do better?

A case study

Find both the min. and the max. of an array of size n.

Optimized Algorithm:

5	1	4	3	6	0	2	8	7	9

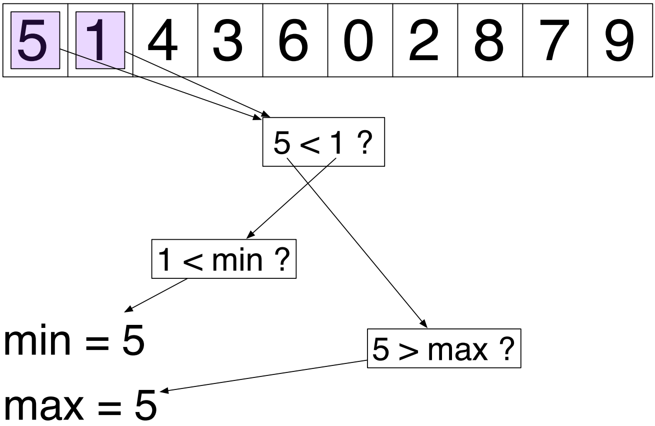
min = 5

max = 5

A case study

Find both the min. and the max. of an array of size n.

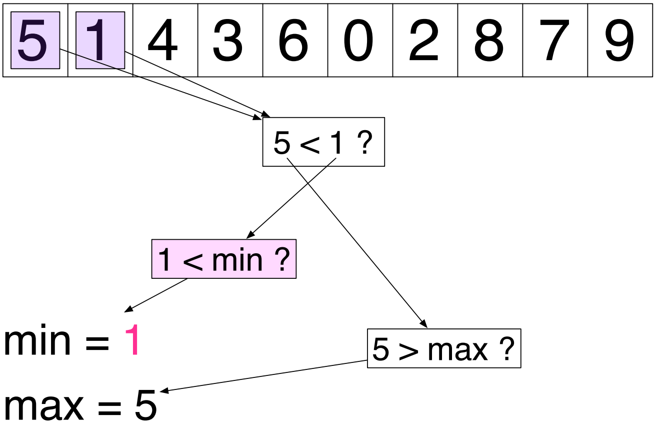
Optimized Algorithm:



A case study

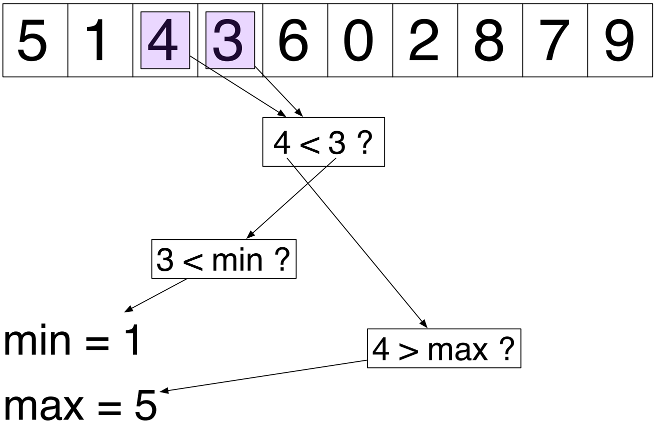
Find both the min. and the max. of an array of size n.

Optimized Algorithm:



Find both the min. and the max. of an array of size n.

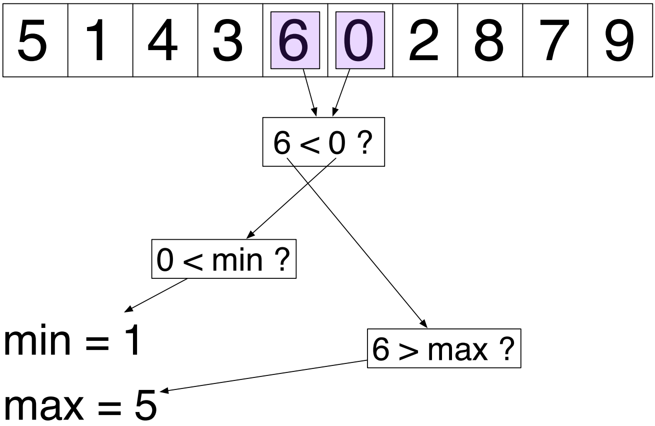
Optimized Algorithm:



A case study

Find both the min. and the max. of an array of size n.

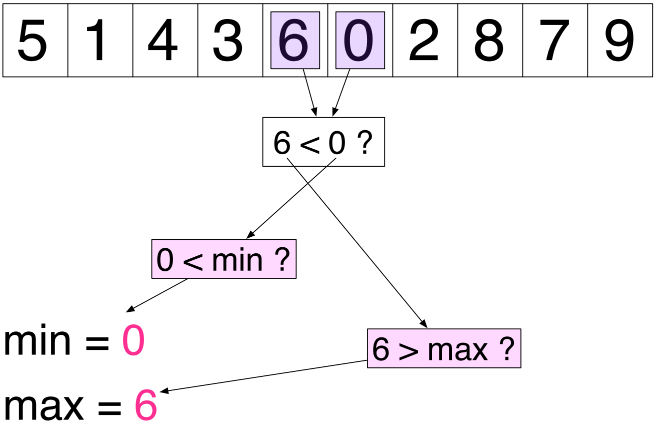
Optimized Algorithm:



A case study

Find both the min. and the max. of an array of size n.

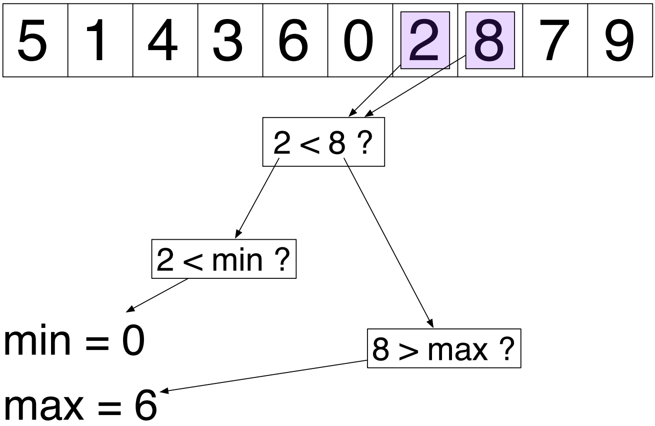
Optimized Algorithm:



A case study

Find both the min. and the max. of an array of size n.

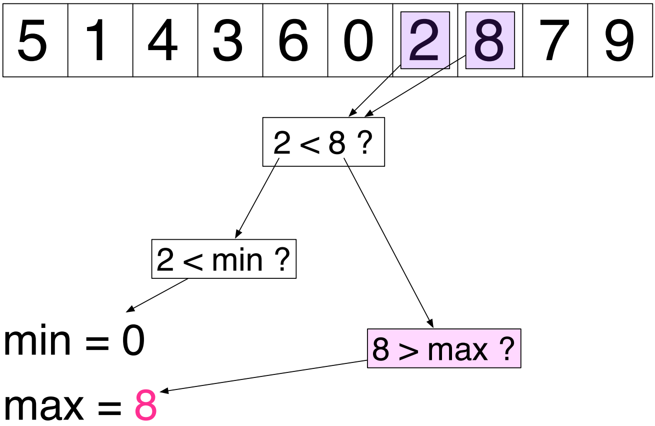
Optimized Algorithm:



A case study

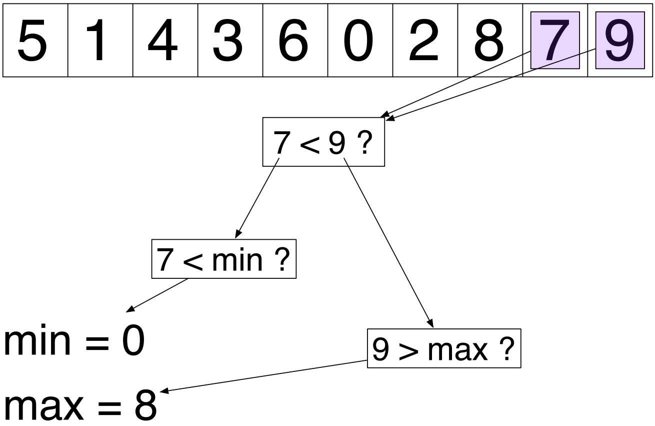
Find both the min. and the max. of an array of size n.

Optimized Algorithm:



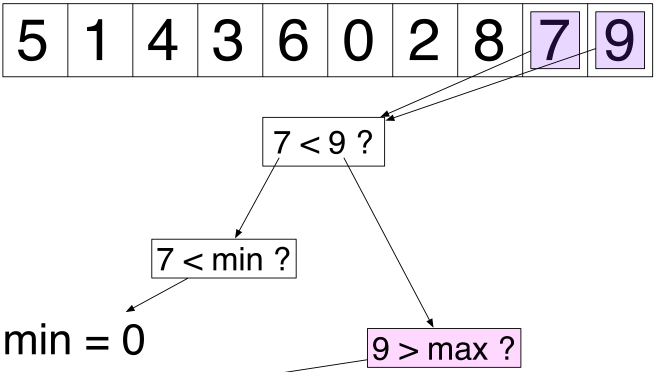
Find both the min. and the max. of an array of size n.

Optimized Algorithm:



Find both the min. and the max. of an array of size n.

Optimized Algorithm:



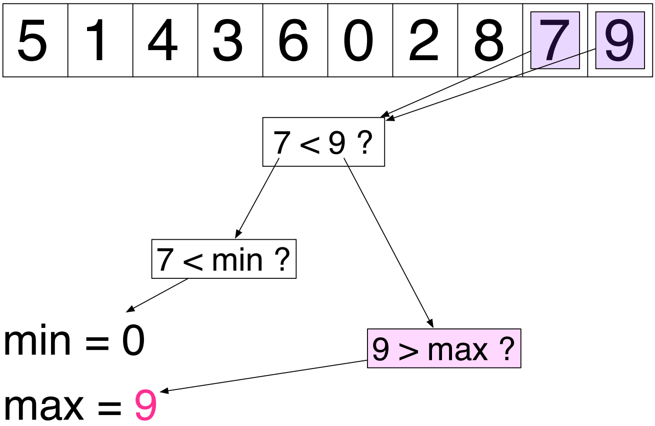
min = 0

max = 9

A case study

Find both the min. and the max. of an array of size n.

Optimized Algorithm: **$3n/2$ comparisons (optimal)**



A case study

Find both the min. and the max. of an array of size n.

Optimized Algorithm: **$3n/2$ comparisons (optimal)**

Naive Algorithm: **$2n$ comparisons**

A case study

Find both the min. and the max. of an array of size n.

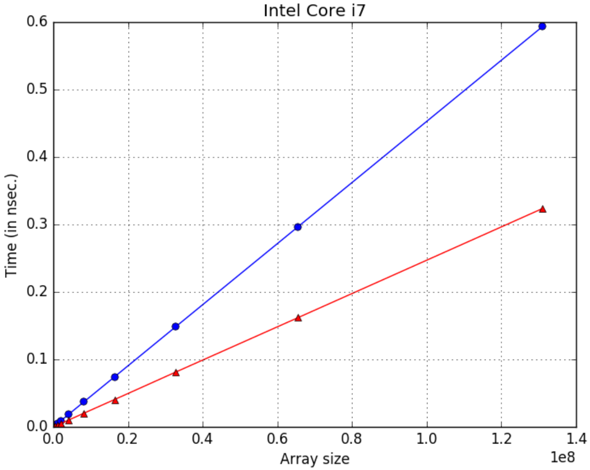
Optimized Algorithm: **$3n/2$ comparisons (optimal)**

Naive Algorithm: **$2n$ comparisons**

In practice, on uniform random data?

A case study

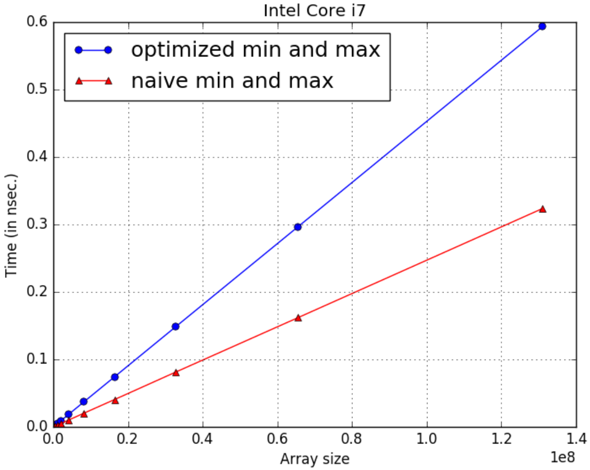
Find both the min. and the max. of an array of size n.



- in C,
- using `gcc -O0`,
- random integers

A case study

Find both the min. and the max. of an array of size n.



- in C,
- using `gcc -O0`,
- random integers

optimized min/max search

```
// RAND_ARRAY: an array of length N
// filled with random integers

  min = RAND_ARRAY[0];
  max = RAND_ARRAY[0];
  for(i=0; i<N; i+=2){ //assume N is even
    a1 = RAND_ARRAY[i];
    a2 = RAND_ARRAY[i+1];
    if (a1 < a2) {
      if (a1 < min) min = a1;
      if (a2 > max) max = a2;
    }
    else {
      if (a2 < min) min = a2;
      if (a1 > max) max = a1;
    }
  }
```

What “really” happens in the processor...

sample of assembly code (`gcc -O0`)

```
  mov esi, dword ptr [rbp - 60]
  cmp esi, dword ptr [rbp - 64]
  jge LBB2_8

  mov eax, dword ptr [rbp - 60]
  cmp eax, dword ptr [rbp - 12]
  jge LBB2_5

  mov eax, dword ptr [rbp - 60]
  mov dword ptr [rbp - 12], eax
LBB2_5:
  mov eax, dword ptr [rbp - 64]
  cmp eax, dword ptr [rbp - 16]
  jle LBB2_7
  ...
```

What “really” happens in the processor...

sample of assembly code (`gcc -O0`)

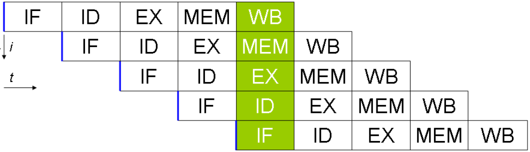
```
  mov esi, dword ptr [rbp - 60]
  cmp esi, dword ptr [rbp - 64]
  jge LBB2_8

  mov eax, dword ptr [rbp - 60]
  cmp eax, dword ptr [rbp - 12]
  jge LBB2_5

  mov eax, dword ptr [rbp - 60]
  mov dword ptr [rbp - 12], eax
LBB2_5:
  mov eax, dword ptr [rbp - 64]
  cmp eax, dword ptr [rbp - 16]
  jle LBB2_7
  ...
```

- Each instruction can be decomposed:

IF	ID	EX	MEM	WB

- Most modern processors are pipelined
- Instructions are parallelized

simple 5 stages pipeline:



What “really” happens in the processor...

sample of assembly code (`gcc -O0`)

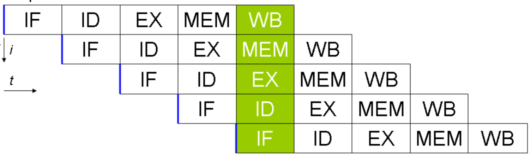
```
    mov  esi, dword ptr [rbp - 28]
    cmp  esi, dword ptr [rbp - 32]
    jge  LBB2_8

    mov  eax, dword ptr [rbp - 28]
    cmp  eax, dword ptr [rbp - 12]
    jge  LBB2_5

    mov  eax, dword ptr [rbp - 28]
    mov  dword ptr [rbp - 12], eax
LBB2_5:
    mov  eax, dword ptr [rbp - 32]
    cmp  eax, dword ptr [rbp - 16]
    jle  LBB2_7

    mov  eax, dword ptr [rbp - 32]
    mov  dword ptr [rbp - 16], eax
LBB2_7:
    jmp  LBB2_14
LBB2_8:
    mov  eax, dword ptr [rbp - 32]
    cmp  eax, dword ptr [rbp - 12]
    jge  LBB2_10

    mov  eax, dword ptr [rbp - 32]
    mov  dword ptr [rbp - 12], eax
LBB2_10:
    mov  eax, dword ptr [rbp - 28]
    cmp  eax, dword ptr [rbp - 16]
    jle  LBB2_14

    mov  eax, dword ptr [rbp - 28]
    mov  dword ptr [rbp - 16], eax

LBB2_14:
    mov  eax, dword ptr [rbp - 4]
    add  eax, 2
    mov  dword ptr [rbp - 4], eax
```

- Each instruction can be decomposed:

IF	ID	EX	MEM	WB

- Most modern processors are pipelined
- Instructions are parallelized

simple 5 stages pipeline:



Branch prediction

Branch predictors are used to avoid stalls on branches!

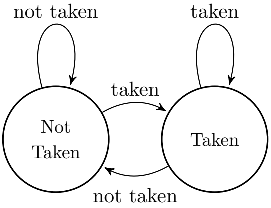
- Conditional instructions (such as the “if” statement) yield branches in the execution of a program
- A **misprediction** can be quite **expensive**!
- The **branch predictor** will guess which branch will be *taken* (T) or not (NT).
- Different schemes: static, dynamic, local, global,. . .

Computer Architecture: A Quantitative Approach (5th ed.), Hennessy & Patterson

Branch prediction

Branch predictors are used to avoid stalls on branches!

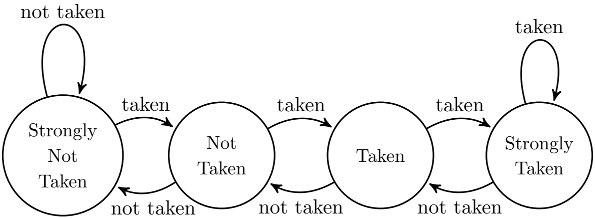
1-bit predictor:



Computer Architecture: A Quantitative Approach (5th ed.), Hennessy & Patterson

Branch predictors are used to avoid stalls on branches!

2-bit predictor:

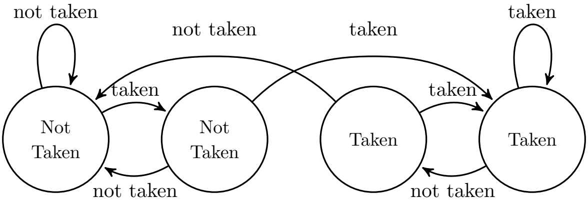


Computer Architecture: A Quantitative Approach (5th ed.), Hennessy & Patterson

Branch prediction

Branch predictors are used to avoid stalls on branches!

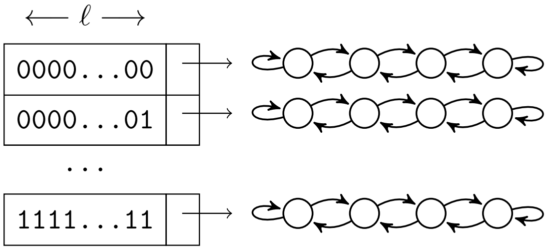
2-bit predictor:



Computer Architecture: A Quantitative Approach (5th ed.), Hennessy & Patterson

Branch predictors are used to avoid stalls on branches!

Global (or mixed) predictor:



Computer Architecture: A Quantitative Approach (5th ed.), Hennessy & Patterson

Branch prediction

Branch predictors are used to avoid stalls on branches!

- Conditional instructions (such as the "if" statement) yield branches in the execution of a program
- A **misprediction** can be quite **expensive**!
- The **branch predictor** will guess which branch will be *taken* (T) or not (NT).
- Different schemes: static, dynamic, local, global,. . .
- Min and max search is very sensitive to branch prediction...

Computer Architecture: A Quantitative Approach (5th ed.), Hennessy & Patterson

Branch prediction

Branch predictors are used to avoid stalls on branches!

- Conditional instructions (such as the “if” statement) yield branches in the execution of a program
- A **misprediction** can be quite **expensive**!
- The **branch predictor** will guess which branch will be *taken* (T) or not (NT).
- Different schemes: static, dynamic, local, global,. . .
- Min and max search is very sensitive to branch prediction...

... though we can avoid this using `CMOV` instructions...

Computer Architecture: A Quantitative Approach (5th ed.), Hennessy & Patterson

Branch prediction

Branch predictors are used to avoid stalls on branches!

- Conditional instructions (such as the "if" statement) yield branches in the execution of a program
- A **misprediction** can be quite **expensive**!
- The **branch predictor** will guess which branch will be *taken* (T) or not (NT).
- Different schemes: static, dynamic, local, global,...
- Min and max search is very sensitive to branch prediction...

... though we can avoid this using `CMOV` instructions...

... but still ...

Computer Architecture: A Quantitative Approach (5th ed.), Hennessy & Patterson

Previous Work

- Brodal & Moruz, 2005 : mispredictions and (adaptive) sorting

Tradeoffs Between Branch Mispredictions and Comparisons for Sorting Algorithms

Gerth Stølting Brodal[1,*] and Gabriel Moruz[1]

BRICS[**], Department of Computer Science, University of Aarhus,
IT Parken, Åbogade 34, DK-8200 Århus N, Denmark
{gerth, gabi}@daimi.au.dk

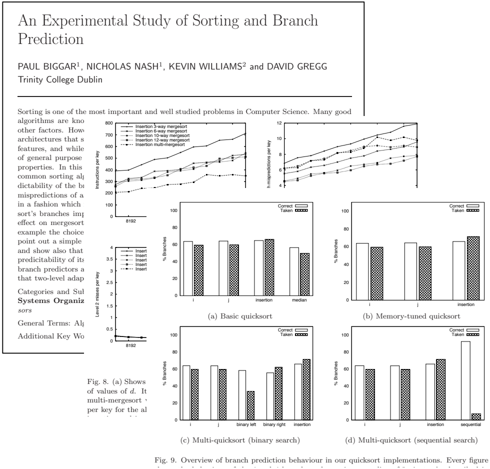
Abstract. Branch mispredictions is an important factor affecting the running time in practice. In this paper we consider tradeoffs between the number of branch mispredictions and the number of comparisons for sorting
algorith
mispred
by adop
tions. F
rithm p
$\Omega(n \log$
of inver
by Esti
col and
mispred

Measure	Comparisons	Branch mispredictions
Dis	$O(dn(1+\log(1+\text{Dis})))$	$\Omega(n\log_d(1+\text{Dis}))$
Exc	$O(dn(1+\text{Exc}\log(1+\text{Exc})))$	$\Omega(n\text{Exc}\log_d(1+\text{Exc}))$
Enc	$O(dn(1+\log(1+\text{Enc})))$	$\Omega(n\log_d(1+\text{Enc}))$
Inv	$O(dn(1+\log(1+\text{Inv}/n)))$	$\Omega(n\log_d(1+\text{Inv}/n))$
Max	$O(dn(1+\log(1+\text{Max})))$	$\Omega(n\log_d(1+\text{Max}))$
Osc	$O(dn(1+\log(1+\text{Osc}/n)))$	$\Omega(n\log_d(1+\text{Osc}/n))$
Reg	$O(dn(1+\log(1+\text{Reg})))$	$\Omega(n\log_d(1+\text{Reg}))$
Rem	$O(dn(1+\text{Rem}\log(1+\text{Rem})))$	$\Omega(n\text{Rem}\log_d(1+\text{Rem}))$
Runs	$O(dn(1+\log(1+\text{Runs})))$	$\Omega(n\log_d(1+\text{Runs}))$
SMS	$O(dn(1+\log(1+\text{SMS})))$	$\Omega(n\log_d(1+\text{SMS}))$
SUS	$O(dn(1+\log(1+\text{SUS})))$	$\Omega(n\log_d(1+\text{SUS}))$

Fig. 4. Lower bounds on the number of branch mispredictions for deterministic comparison based adaptive sorting algorithms for different measures of presortedness, given the upper bounds on the number of comparisons

Previous Work

- Brodal & Moruz, 2005 : mispredictions and (adaptive) sorting
- Biggar *et al*, 2008 : experimental, branch prediction and sorting



Previous Work

- Brodal & Moruz, 2005 : mispredictions and (adaptive) sorting
- Biggar *et al*, 2008 : experimental, branch prediction and sorting
- Sanders and Winkel, 2004 : quicksort variant without branches

Super Scalar Sample Sort

Peter Sanders[1] and Sebastian Winkel[2]

[1] Max Planck Institut für Informatik
Saarbrücken, Germany, sanders@mpi-sb.mpg.de
[2] Chair for Prog. Lang. and Compiler Construction
Saarland University, Saarbrücken, Germany, sewi@cs.uni-sb.de

Abstract. Sample sort, a generalization of quicksort that partitions the input into many pieces, is known as the best practical comparison based sorting algorithm for distributed memory parallel computers. We show that … mic i … cond … facili … final … ber c … Itani … the (… quick …

$t := \langle s_{k/2}, s_{k/4}, s_{3k/4}, s_{k/8}, s_{3k/8}, s_{5k/8}, s_{7k/8}, \ldots \rangle$ //
for $i := 1$ **to** n **do** // locate each element
  $j := 1$ // current tree node := root
  repeat $\log k$ *times* // will be unrolled
    $j := 2j + (a_i > t_j)$ // left or right?
  $j := j - k + 1$ // bucket index
  $|b_j|{+}{+}$ // count bucket size
  $o(i) := j$ // remember oracle

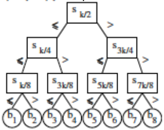


Fig. 2. Finding buckets using implicit search trees. The picture is for $k = 8$. We adopt the C convention that "$x > y$" is one if $x > y$ holds, and zero else.

Previous Work

- Brodal & Moruz, 2005 : mispredictions and (adaptive) sorting
- Biggar *et al*, 2008 : experimental, branch prediction and sorting
- Sanders and Winkel, 2004 : quicksort variant without branches
- Elmasry *et al*, 2012 : mergesort variant without branches

Branch Mispredictions Don't Affect Mergesort*

Amr Elmasry[1], Jyrki Katajainen[1,2], and Max Stenmark[2]

[1] Department of Computer Science, University of Copenhagen
Universitetsparken 1, 2100 Copenhagen East, Denmark
[2] Jyrki Katajainen and Company
Thorsgade 101, 2200 Copenhagen North, Denmark

Abstract. In quicksort, due to branch mispredictions, a skewed pivot-selection strategy can lead to a better performance than the exact-median pivot-selection strategy, e... free. In this paper we investigate t... the behaviour of mergesort. By d... branches, we can avoid most nega... dictions. When sorting a sequence... mergesort performs $n \log_2 n + O(n$... most $O(n)$ branch mispredictions

```
while (p != t1 && q != t2) {
  if (less(*q, *p)) {
    s = q;
    ++q;
  }
  else {
```

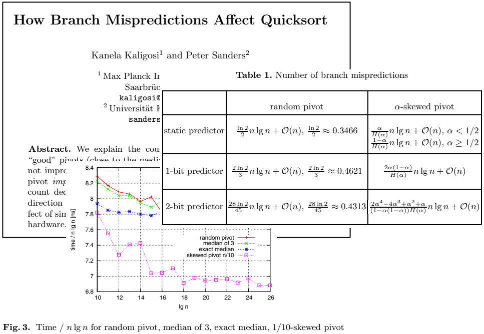
```
test:
  done = (q == t2);
  if (done) goto exit;
entrance:
  x = *p;
  s = p + 1;
  y = *q;
  t = q + 1;
  smaller = less(y, x);
  if (smaller) s = t;
  if (smaller) q = t;
  if (! smaller) p = s;
  if (! smaller) y = x;
  x = *r;
  *r = y;
  --s;
  *s = x;
  ++r;
  done = (p == t1);
  if (! done) goto test;
exit:
```

Table 3. The execution time [ns], the number of conditional branches, and the number of mispredictions, each per $n \log_2 n$, for two in-situ variants of mergesort.

Program	In-situ std::stable_sort				In-situ mergesort			
	Time		Branches	Mispredicts	Time		Branches	Mispredicts
n	Per	Ares			Per	Ares		
2^{10}	49.2	29.7	9.0	2.08	7.3	5.7	1.93	0.26
2^{15}	57.6	35.0	11.1	2.38	7.1	5.6	1.94	0.15
2^{20}	62.7	38.5	12.9	2.53	7.4	5.7	1.92	0.11
2^{25}	68.0	41.3	14.4	2.62	7.6	5.7	1.92	0.09

Previous Work

- Brodal & Moruz, 2005 : mispredictions and (adaptive) sorting
- Biggar *et al*, 2008 : experimental, branch prediction and sorting
- Sanders and Winkel, 2004 : quicksort variant without branches
- Elmasry *et al*, 2012 : mergesort variant without branches
- Kaligosi and Sanders, 2006 : mispredictions and quicksort



How Branch Mispredictions Affect Quicksort

Kanela Kaligosi[1] and Peter Sanders[2]

[1] Max Planck In
Saarbrüc
kaligosi@
[2] Universität K
sanders

Abstract. We explain the cou
"good" pivots (close to the medi
not impro
pivot *imp*
count dec
direction
fect of sin
hardware.

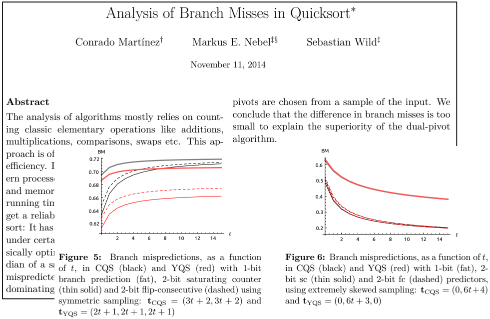
Table 1. Number of branch mispredictions

	random pivot	α-skewed pivot
static predictor	$\frac{\ln 2}{2} n \lg n + \mathcal{O}(n)$, $\frac{\ln 2}{2} \approx 0.3466$	$\frac{\alpha}{H(\alpha)} n \lg n + \mathcal{O}(n)$, $\alpha < 1/2$ $\frac{1-\alpha}{H(\alpha)} n \lg n + \mathcal{O}(n)$, $\alpha \geq 1/2$
1-bit predictor	$\frac{2 \ln 2}{3} n \lg n + \mathcal{O}(n)$, $\frac{2 \ln 2}{3} \approx 0.4621$	$\frac{2\alpha(1-\alpha)}{H(\alpha)} n \lg n + \mathcal{O}(n)$
2-bit predictor	$\frac{28 \ln 2}{45} n \lg n + \mathcal{O}(n)$, $\frac{28 \ln 2}{45} \approx 0.4313$	$\frac{2\alpha^4 - 4\alpha^3 + \alpha^2 + \alpha}{(1-\alpha(1-\alpha))H(\alpha)} n \lg n + \mathcal{O}(n)$

Fig. 3. Time / $n \lg n$ for random pivot, median of 3, exact median, 1/10-skewed pivot

Previous Work

- Brodal & Moruz, 2005 : mispredictions and (adaptive) sorting
- Biggar *et al*, 2008 : experimental, branch prediction and sorting
- Sanders and Winkel, 2004 : quicksort variant without branches
- Elmasry *et al*, 2012 : mergesort variant without branches
- Kaligosi and Sanders, 2006 : mispredictions and quicksort
- Martínez, Nebel and Wild, 2014 : mispredictions and quicksort



Analysis of Branch Misses in Quicksort*

Conrado Martínez[†] Markus E. Nebel[‡§] Sebastian Wild[‡]

November 11, 2014

Abstract

The analysis of algorithms mostly relies on counting classic elementary operations like additions, multiplications, comparisons, swaps etc. This approach is of … efficiency. I… ern process… and memor… running tin… get a reliab… sort: It has … under certa… sically opti… dian of a s… mispredicte… dominating … pivots are chosen from a sample of the input. We conclude that the difference in branch misses is too small to explain the superiority of the dual-pivot algorithm.

Figure 5: Branch mispredictions, as a function of t, in CQS (black) and YQS (red) with 1-bit branch prediction (fat), 2-bit saturating counter (thin solid) and 2-bit flip-consecutive (dashed) using symmetric sampling: $\mathbf{t}_{\mathrm{CQS}} = (3t+2, 3t+2)$ and $\mathbf{t}_{\mathrm{YQS}} = (2t+1, 2t+1, 2t+1)$

Figure 6: Branch mispredictions, as a function of t, in CQS (black) and YQS (red) with 1-bit (fat), 2-bit sc (thin solid) and 2-bit fc (dashed) predictors, using extremely skewed sampling: $\mathbf{t}_{\mathrm{CQS}} = (0, 6t+4)$ and $\mathbf{t}_{\mathrm{YQS}} = (0, 6t+3, 0)$

Previous Work

- Brodal & Moruz, 2005 : mispredictions and (adaptive) sorting
- Biggar *et al*, 2008 : experimental, branch prediction and sorting
- Sanders and Winkel, 2004 : quicksort variant without branches
- Elmasry *et al*, 2012 : mergesort variant without branches
- Kaligosi and Sanders, 2006 : mispredictions and quicksort
- Martínez, Nebel and Wild, 2014 : mispredictions and quicksort
- Brodal and Moruz, 2006 : skewed binary search trees

Skewed Binary Search Trees

Gerth Stølting Brodal[1,*] and Gabriel Moruz[1]

BRICS[**], Department of Computer Science, University of Aarhus, IT Parken, Åbogade 34, DK-8200 Århus N, Denmark. E-mail: {gerth,gabi}@daimi.au.dk

Abstract. It is well-known t
a binary search tree should
shown that a dominating fa
the number of cache faults p
layout of a binary search tr
by several hundred percent. I
branching to the left or righ
same cost, e.g. because of br
study the class of skewed bin
binary search tree the ratio b
size of the tree is a fixed con
trees). In this paper we preser
layouts of static skewed bina
tree is accessed with a unifo
many of the memory layouts we consider skewed binary search trees can
perform better than perfect balanced search trees. The improvements in
the running time are on the order of 15%.

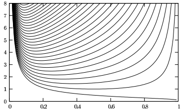


Fig. 1. Bound on the expected cost for a random search, where the cost for visiting the left child is $c_l = 1$ and the cost for processing the right child is $c_r = 0, 1, 2, \ldots, 28$ ($c_r = 0$ being the lowest curve).

Previous Work

- Brodal & Moruz, 2005 : mispredictions and (adaptive) sorting
- Biggar *et al*, 2008 : experimental, branch prediction and sorting
- Sanders and Winkel, 2004 : quicksort variant without branches
- Elmasry *et al*, 2012 : mergesort variant without branches
- Kaligosi and Sanders, 2006 : mispredictions and quicksort
- Martínez, Nebel and Wild, 2014 : mispredictions and quicksort
- Brodal and Moruz, 2006 : skewed binary search trees

Skewed Binary Search Trees

Gerth Stølting Brodal[1,*] and Gabriel Moruz[1]

BRICS**, Department of Computer Science, University of Aarhus, IT Parken, Åbogade 34, DK-8200 Århus N, Denmark. E-mail: {gerth,gabi}@daimi.au.dk

Abstract. It is well-known t
a binary search tree should
shown that a dominating fa
the number of cache faults p
layout of a binary search tr
by several hundred percent.
branching to the left or righ
same cost, e.g. because of br
study the class of skewed bin
binary search tree the ratio b
size of the tree is a fixed con
trees). In this paper we preser
layouts of static skewed bina
tree is accessed with a unifo
many of the memory layouts
perform better than perfect balanced search trees. The improvements in the running time are on the order of 15%.

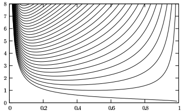


Fig. 1. Bound on the expected cost for a random search, where the cost for visiting the left child is $c_l = 1$ and the cost for processing the right child is $c_r = 0, 1, 2, \ldots, 28$ ($c_r = 0$ being the lowest curve).

Back to simultaneous min and max search

Proposition

Expected number of mispredictions, for the uniform distribution, on arrays of size n:

- **Naive Min Max Search:**

  $\sim 4 \log n$ *for the 1-bit predictor*

  $\sim 2 \log n$ *for the two 2-bit predictors and the 3-bit saturating counter.*

- **Optimized Min Max Search:**

  $\sim n/4 + \mathcal{O}(\log n)$ *for all four predictors.*

Idea of the proof:

- asymptotic analysis of the records in a random permutation,
- use the fundamental bijection that relates the records to the cycles in permutations,
- use classical results on the average number of cycles.

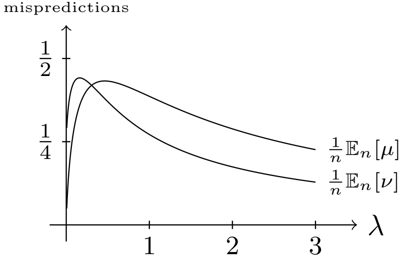
What if the distribution is not uniform?

Definition (Ewens-like distribution for records)

- To any $\sigma \in \mathfrak{S}_n$, we associate a weight $w(\sigma) = \theta^{\text{record}(\sigma)}$.
- Let $W_n = \sum_{\sigma \in \mathfrak{S}_n} w(\sigma) = \theta^{(n)}$ and $\mathbb{P}(\sigma) = \frac{\theta^{\text{record}(\sigma)}}{\theta^{(n)}}$.

with $\theta^{(n)} = \theta(\theta+1)\dots(\theta+n-1)$

Expected number of mispredictions:



μ: naive algorithm
ν: optimized algorithm
$\theta := \lambda n$.

$\mathbb{E}_n[\mu] \sim \mathbb{E}_n[\nu]$ for $\lambda_0 \approx 0.305$.

But optimized performs less comparisons, thus it becomes better before λ_0.

Exponentiation by squaring

Introducing unnecessary tests to speed up

POW(x,n)

```
r = 1;
while (n > 0) {
  // n is odd
  if (n & 1)
    r = r * x;
  n /= 2;
  x = x * x;
}
```

`x` is a floating-point number, `n` is an integer and `r` is the result.

$x^n = (x^2)^{\lfloor n/2 \rfloor} x^{n_0}$

Introducing unnecessary tests to speed up

POW(x,n)

```
r = 1;
while (n > 0) {
  // n is odd
  if (n & 1) ℙ = 1/2
    r = r * x;
  n /= 2;
  x = x * x;
}
```

`x` is a floating-point number, `n` is an integer and `r` is the result.

$x^n = (x^2)^{\lfloor n/2 \rfloor} x^{n_0}$

Introducing unnecessary tests to speed up

POW(x,n)

```
r = 1;
while (n > 0) {
  // n is odd
  if (n & 1)   ℙ = 1/2
    r = r * x;
  n /= 2;
  x = x * x;
}
```

x is a floating-point number, n is an integer and r is the result.

$x^n = (x^2)^{\lfloor n/2 \rfloor} x^{n_0}$

UNROLLED(x,n)

```
r = 1;
while (n > 0) {
  t = x * x;
  // n0 == 1
  if (n & 1)
    r = r * x;
  // n1 == 1
  if (n & 2)
    r = r * t;
  n /= 4;
  x = t * t;
}
```

$x^n = (x^4)^{\lfloor n/4 \rfloor} (x^2)^{n_1} x^{n_0}$

Introducing unnecessary tests to speed up

POW(x,n)

```
r = 1;
while (n > 0) {
  // n is odd
  if (n & 1)          ℙ = 1/2
    r = r * x;
  n /= 2;
  x = x * x;
}
```

x is a floating-point number, n is an integer and r is the result.

$x^n = (x^2)^{\lfloor n/2 \rfloor} x^{n_0}$

UNROLLED(x,n)

```
r = 1;
while (n > 0) {
  t = x * x;
  // n0 == 1
  if (n & 1)          ℙ = 1/2
    r = r * x;
  // n1 == 1
  if (n & 2)          ℙ = 1/2
    r = r * t;
  n /= 4;
  x = t * t;
}
```

$x^n = (x^4)^{\lfloor n/4 \rfloor} (x^2)^{n_1} x^{n_0}$

Introducing unnecessary tests to speed up

POW(x,n)

```
r = 1;
while (n > 0) {
  // n is odd
  if (n & 1) ℙ = 1/2
    r = r * x;
  n /= 2;
  x = x * x;
}
```

x is a floating-point number, n is an integer and r is the result.

$x^n = (x^2)^{\lfloor n/2 \rfloor} x^{n_0}$

UNROLLED(x,n)

```
r = 1;
while (n > 0) {
  t = x * x;
  // n0 == 1
  if (n & 1) ℙ = 1/2
    r = r * x;
  // n1 == 1
  if (n & 2) ℙ = 1/2
    r = r * t;
  n /= 4;
  x = t * t;
}
```

$x^n = (x^4)^{\lfloor n/4 \rfloor} (x^2)^{n_1} x^{n_0}$

GUIDED(x,n)

```
r = 1;
while (n > 0) {
  t = x * x;
  // n1n0! = 00
  if (n & 3){
    if (n & 1)
      r = r * x;
    if (n & 2)
      r = r * t;
  }
  n /= 4;
  x = t * t;
}
```

Introducing unnecessary tests to speed up

POW(x,n)

```
r = 1;
while (n > 0) {
  // n is odd
  if (n & 1)      ℙ = 1/2
    r = r * x;
  n /= 2;
  x = x * x;
}
```

x is a floating-point number, n is an integer and r is the result.

$$x^n = (x^2)^{\lfloor n/2 \rfloor} x^{n_0}$$

UNROLLED(x,n)

```
r = 1;
while (n > 0) {
  t = x * x;
  // n_0 == 1
  if (n & 1)      ℙ = 1/2
    r = r * x;
  // n_1 == 1
  if (n & 2)      ℙ = 1/2
    r = r * t;
  n /= 4;
  x = t * t;
}
```

$$x^n = (x^4)^{\lfloor n/4 \rfloor} (x^2)^{n_1} x^{n_0}$$

GUIDED(x,n)

```
r = 1;
while (n > 0) {
  t = x * x;
  // n_1 n_0 != 00
  if (n & 3){     ℙ = 3/4
    if (n & 1)
      r = r * x;
    if (n & 2)
      r = r * t;
  }
  n /= 4;
  x = t * t;
}
```

Introducing unnecessary tests to speed up

POW(x,n)

```
r = 1;
while (n > 0) {
  // n is odd
  if (n & 1)   ℙ = 1/2
    r = r * x;
  n /= 2;
  x = x * x;
}
```

x is a floating-point number, n is an integer and r is the result.

$x^n = (x^2)^{\lfloor n/2 \rfloor} x^{n_0}$

UNROLLED(x,n)

```
r = 1;
while (n > 0) {
  t = x * x;
  // n0 == 1
  if (n & 1)   ℙ = 1/2
    r = r * x;
  // n1 == 1
  if (n & 2)   ℙ = 1/2
    r = r * t;
  n /= 4;
  x = t * t;
}
```

$x^n = (x^4)^{\lfloor n/4 \rfloor} (x^2)^{n_1} x^{n_0}$

GUIDED(x,n)

```
r = 1;
while (n > 0) {
  t = x * x;
  // n1n0 != 00
  if (n & 3){   ℙ = 3/4
    if (n & 1)   ℙ = 2/3
      r = r * x;
    if (n & 2)   ℙ = 2/3
      r = r * t;
  }
  n /= 4;
  x = t * t;
}
```

Introducing unnecessary tests to speed up

POW(x,n)

```
r = 1;
while (n > 0) {
  // n is odd
  if (n & 1)      ℙ = 1/2
    r = r * x;
  n /= 2;
  x = x * x;
}
```

x is a floating-point number, n is an integer and r is the result.

$x^n = (x^2)^{\lfloor n/2 \rfloor} x^{n_0}$

UNROLLED(x,n)

```
r = 1;
while (n > 0) {
  t = x * x;
  // n0 == 1
  if (n & 1)      ℙ = 1/2
    r = r * x;
  // n1 == 1
  if (n & 2)      ℙ = 1/2
    r = r * t;
  n /= 4;
  x = t * t;
}
```

$x^n = (x^4)^{\lfloor n/4 \rfloor} (x^2)^{n_1} x^{n_0}$

GUIDED(x,n)

```
r = 1;
while (n > 0) {
  t = x * x;
  // n1n0! = 00
  if (n & 3){     ℙ = 3/4
    if (n & 1)    ℙ = 2/3
      r = r * x;
    if (n & 2)    ℙ = 2/3
      r = r * t;
  }
  n /= 4;
  x = t * t;
}
```

- 25 % more comparisons for GUIDED than for UNROLLED

Introducing unnecessary tests to speed up

POW(x,n)

```
r = 1;
while (n > 0) {
  // n is odd
  if (n & 1)     P = 1/2
    r = r * x;
  n /= 2;
  x = x * x;
}
```

x is a floating-point number, n is an integer and r is the result.

$x^n = (x^2)^{\lfloor n/2 \rfloor} x^{n_0}$

UNROLLED(x,n)

```
r = 1;
while (n > 0) {
  t = x * x;
  // n0 == 1
  if (n & 1)     P = 1/2
    r = r * x;
  // n1 == 1
  if (n & 2)     P = 1/2
    r = r * t;
  n /= 4;
  x = t * t;
}
```

$x^n = (x^4)^{\lfloor n/4 \rfloor} (x^2)^{n_1} x^{n_0}$

GUIDED(x,n)

```
r = 1;
while (n > 0) {
  t = x * x;
  // n1n0 != 00
  if (n & 3){     P = 3/4
    if (n & 1)    P = 2/3
      r = r * x;
    if (n & 2)    P = 2/3
      r = r * t;
  }
  n /= 4;
  x = t * t;
}
```

- 25 % more comparisons for GUIDED than for UNROLLED
- GUIDED exponential is 14% faster than the UNROLLED one;

Introducing unnecessary tests to speed up

POW(x,n)

```
r = 1;
while (n > 0) {
   // n is odd
   if (n & 1)   ℙ = 1/2
      r = r * x;
   n /= 2;
   x = x * x;
}
```

x is a floating-point number, n is an integer and r is the result.

$x^n = (x^2)^{\lfloor n/2 \rfloor} x^{n_0}$

UNROLLED(x,n)

```
r = 1;
while (n > 0) {
   t = x * x;
   // n0 == 1
   if (n & 1)   ℙ = 1/2
      r = r * x;
   // n1 == 1
   if (n & 2)   ℙ = 1/2
      r = r * t;
   n /= 4;
   x = t * t;
}
```

$x^n = (x^4)^{\lfloor n/4 \rfloor} (x^2)^{n_1} x^{n_0}$

GUIDED(x,n)

```
r = 1;
while (n > 0) {
   t = x * x;
   // n1n0! = 00
   if (n & 3){   ℙ = 3/4
      if (n & 1)   ℙ = 2/3
         r = r * x;
      if (n & 2)   ℙ = 2/3
         r = r * t;
   }
   n /= 4;
   x = t * t;
}
```

- 25 % more comparisons for GUIDED than for UNROLLED
- GUIDED exponential is 14% faster than the UNROLLED one;
- GUIDED exponential is 29% faster than the classical one;

Introducing unnecessary tests to speed up

POW(x,n)

```
r = 1;
while (n > 0) {
  // n is odd
  if (n & 1) ℙ = 1/2
    r = r * x;
  n /= 2;
  x = x * x;
}
```

x is a floating-point number, n is an integer and r is the result.

$x^n = (x^2)^{\lfloor n/2 \rfloor} x^{n_0}$

UNROLLED(x,n)

```
r = 1;
while (n > 0) {
  t = x * x;
  // n0 == 1
  if (n & 1) ℙ = 1/2
    r = r * x;
  // n1 == 1
  if (n & 2) ℙ = 1/2
    r = r * t;
  n /= 4;
  x = t * t;
}
```

$x^n = (x^4)^{\lfloor n/4 \rfloor} (x^2)^{n_1} x^{n_0}$

GUIDED(x,n)

```
r = 1;
while (n > 0) {
  t = x * x;
  // n1n0! = 00
  if (n & 3){ ℙ = 3/4
    if (n & 1) ℙ = 2/3
      r = r * x;
    if (n & 2) ℙ = 2/3
      r = r * t;
  }
  n /= 4;
  x = t * t;
}
```

- 25 % more comparisons for GUIDED than for UNROLLED
- GUIDED exponential is 14% faster than the UNROLLED one;
- GUIDED exponential is 29% faster than the classical one;
- yet, the number of multiplications is essentially the same.

Guided Pow: average number of mispredictions

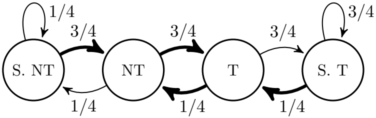
Theorem

Compute x^n, for random n in $\{0, \ldots, N-1\}$.

- **Expected nb. of conditionals:**
  - $\sim \log_2 N$ *for classical and unrolled pow*
  - $\sim \frac{5}{4} \log_2 N$ *for the guided one*
- **Expected nb. of mispredictions:**
  - $\sim \frac{1}{2} \log_2 N$ *for classical and unrolled pow*
  - $\sim \left(\frac{1}{2}\mu(\frac{3}{4}) + \frac{3}{4}\mu(\frac{2}{3})\right) \log_2 N$ *for guided pow*

GUIDED(x,n)

```
r = 1;
while (n > 0) {
  t = x * x;
  // n1n0! = 00
  if (n & 3) {
    if (n & 1)
      r = r * x;
    if (n & 2)
      r = r * t;
  }
  n /= 4;
  x = t * t;
}
```



$$\mu(\tfrac{3}{4}) = \tfrac{3}{10} \text{ and } \mu(\tfrac{2}{3}) = \tfrac{2}{5}$$

Number of mispredictions (Ergodic Th.):

$$\mathbb{E}[M_n] \sim \mathbb{E}[L_n] \times \mu(p)$$

L_n: length of the path in the Markov chain, and $\mu(p) = \sum_{(i,j)\in \text{ mispred}} \pi_p(i) M_p(i,j)$.

Guided Pow: average number of mispredictions

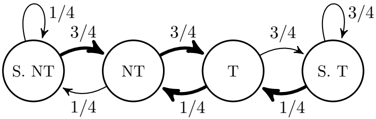
Theorem

Compute x^n, for random n in $\{0, \ldots, N-1\}$.

- **Expected nb. of conditionals:**
  - $\sim \log_2 N$ *for classical and unrolled pow*
  - $\sim \frac{5}{4} \log_2 N$ *for the guided one*
- **Expected nb. of mispredictions:**
  - $\sim \frac{1}{2} \log_2 N$ *for classical and unrolled pow*
  - $\sim 0.45 \log_2 N$ *for guided pow (2-bit pred.)*

GUIDED(x,n)

```
r = 1;
while (n > 0) {
  t = x * x;
  // n1n0! = 00
  if (n & 3) {
    if (n & 1)
      r = r * x;
    if (n & 2)
      r = r * t;
  }
  n /= 4;
  x = t * t;
}
```



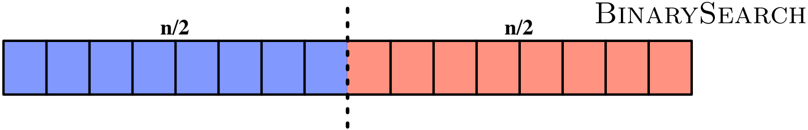
$$\mu(\tfrac{3}{4}) = \tfrac{3}{10} \text{ and } \mu(\tfrac{2}{3}) = \tfrac{2}{5}$$

- 25 % more comparisons than unrolled
- unnecessary if : added mispred.
- other ones : less mispred.

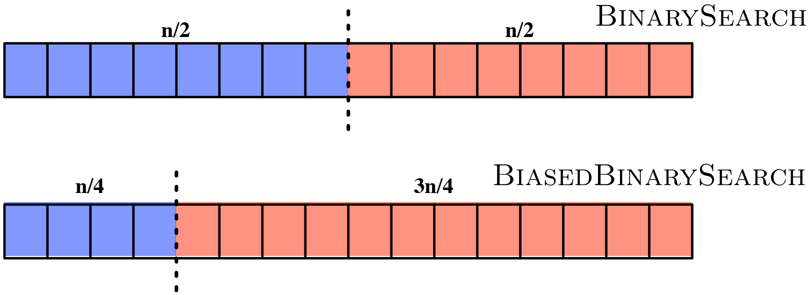
- 5 % less mispred. (2-bit predictor)
- **11 % less mispred.** (3-bit predictor)

Binary Search

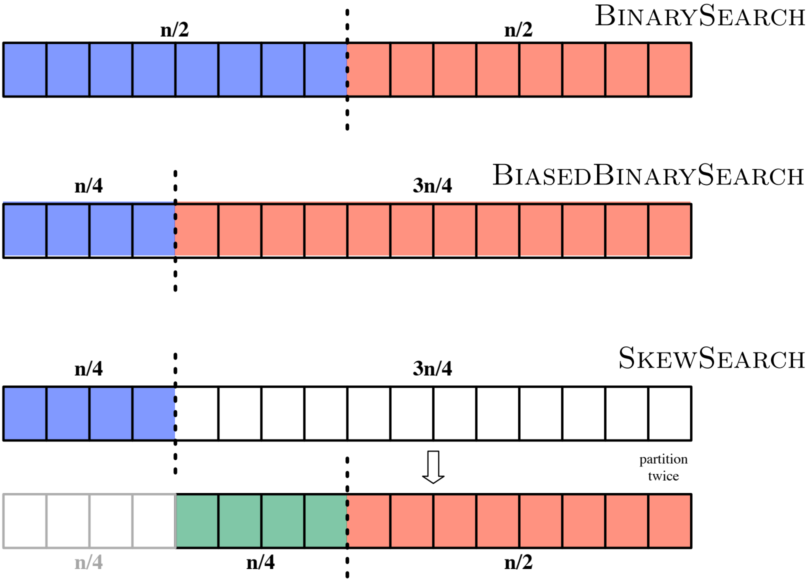
Unbalancing the binary search



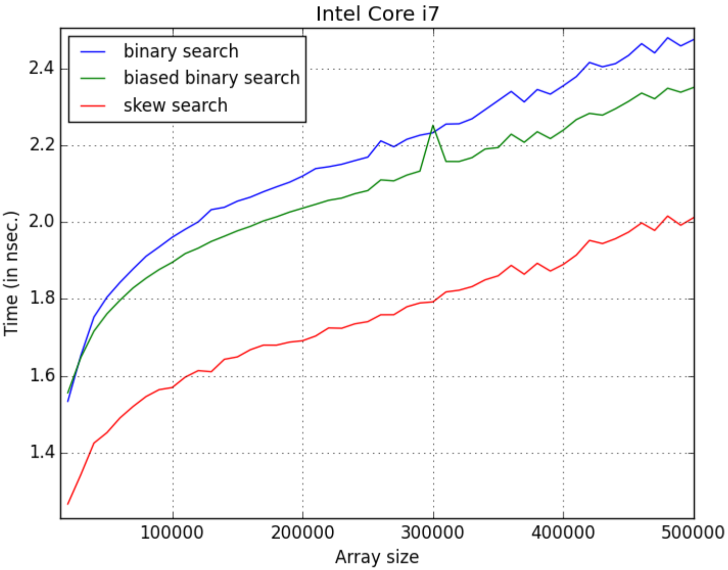
Unbalancing the binary search



Unbalancing the binary search



Unbalancing the binary search



Analysis of the local predictor

Theorem

For arrays of size n filled with random uniform integers. C_n is the number of comparisons and M_n the number of mispredictions.

	BinarySearch	BiasedBinarySearch	SkewSearch
$\mathbb{E}[C_n]$	$\frac{\log n}{\log 2}$	$\frac{4 \log n}{(4 \log 4 - 3 \log 3)}$	$\frac{7 \log n}{(6 \log 2)}$
$\mathbb{E}[M_n]$	$\frac{\log n}{(2 \log 2)}$	$\mu(\frac{1}{4})\mathbb{E}[C_n]$	$\left(\frac{4}{7}\mu(\frac{1}{4}) + \frac{3}{7}\mu(\frac{1}{3})\right)\mathbb{E}[C_n]$

μ is the expected misprediction probability associated with the predictor.

Idea of the proof:

- Get the expected number of times a given conditional is executed by Roura's Master Theorem [Rou01].
- Ensure that our predictors behave *almost* like Markov chains.

Analysis of the local predictor

Theorem

For arrays of size n filled with random uniform integers. C_n is the number of comparisons and M_n the number of mispredictions.

	BinarySearch	BiasedBinarySearch	SkewSearch
$\mathbb{E}[C_n]$	$1.44 \log n$	$1.78 \log n$	$1.68 \log n$
$\mathbb{E}[M_n]$	$0.72 \log n$	$0.53 \log n$	$0.58 \log n$

with a 2-bit saturated counter.

Idea of the proof:

- Get the expected number of times a given conditional is executed by Roura's Master Theorem [Rou01].
- Ensure that our predictors behave *almost* like Markov chains.

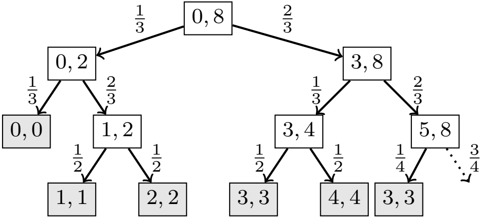
Almost like Markov chains?

Expected number of iterations $L(n)$ of BiasedBinarySearch:

$$L(n) = 1 + \frac{a_n}{n+1} L(a_n) + \frac{b_n}{n+1} L(b_n), \text{ with } a_n = \left\lfloor \frac{n}{4} \right\rfloor + 1, b_n = \left\lceil \frac{3n}{4} \right\rceil$$

and $L(0) = 0$

But $\frac{a_n}{n+1}$ and $\frac{b_n}{n+1}$ are not fixed anymore...

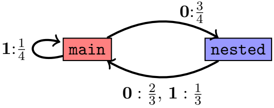


The trick...

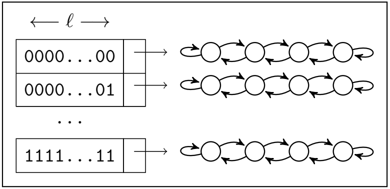
The probability that the path $\mathcal{P}$ taken by BiasedBinarySearch in the decomposition tree differs from the one taken in the ideal tree at one of the first length$(\mathcal{P}) - \sqrt{\log n}$ steps is $\mathcal{O}(\frac{1}{\log n})$.

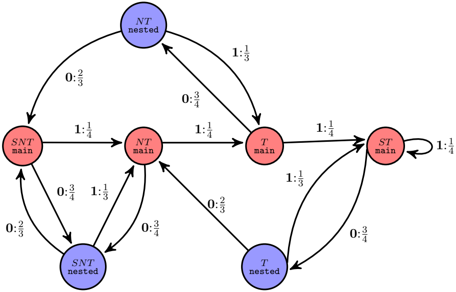
What about a global predictor?

```
d = 0; f = n;
while (d < f){
    m1 = (3*d+f)/4;
    if (T[m1] > x) f = m1;
    else {
        m2 = (d+f)/2;
        if (T[m2] > x){
            f = m2;
            d = m1+1;
        }
        else d = m2+1;
    }
}
return f;
```



Global predictor





Gerth Stølting Brodal and Gabriel Moruz.
Tradeoffs Between Branch Mispredictions and Comparisons for Sorting Algorithms.
In *Algorithms and Data Structures*, volume 3608, pages 385–395. Springer Berlin Heidelberg, 2005.

Gerth Stølting Brodal and Gabriel Moruz.
Skewed Binary Search Trees.
In *Algorithms ESA 2006*, volume 4168, pages 708–719. Springer Berlin Heidelberg, 2006.

Paul Biggar, Nicholas Nash, Kevin Williams, and David Gregg.
An experimental study of sorting and branch prediction.
Journal of Experimental Algorithmics, 12:1, June 2008.

Amr Elmasry, Jyrki Katajainen, and Max Stenmark.
Branch Mispredictions Dont Affect Mergesort.
In *Experimental Algorithms*, volume 7276, pages 160–171. Springer Berlin Heidelberg, Berlin, Heidelberg, 2012.

John L. Hennessy and David A. Patterson.
Computer Architecture, Fifth Edition: A Quantitative Approach.
Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 5th edition, 2011.

Kanela Kaligosi and Peter Sanders.
How Branch Mispredictions Affect Quicksort.
In *Algorithms ESA 2006*, volume 4168, pages 780–791. Springer Berlin Heidelberg, 2006.

Conrado Martínez, Markus E. Nebel, and Sebastian Wild.
Analysis of branch misses in quicksort.
In *Proceedings of the Twelfth Workshop on Analytic Algorithmics and Combinatorics, ANALCO 2015, San Diego, CA, USA, January 4, 2015*, pages 114–128, 2015.

Salvador Roura.
Improved master theorems for divide-and-conquer recurrences.
Journal of the ACM, 48(2):170–205, 2001.