

Lattice and Hopf algebra of integer relations

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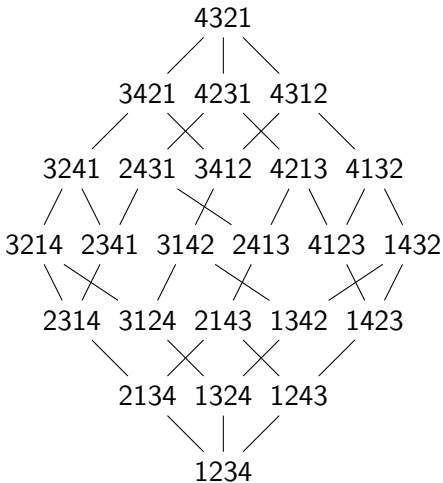
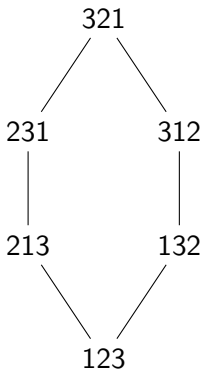
Séminaire Flajolet 21/09/2017

	permutations	binary trees	binary sequences
Combinatorics			
Algebra	Malvenuto-Reutenauer algebra	Loday-Ronco algebra	Descent Hopf algebra

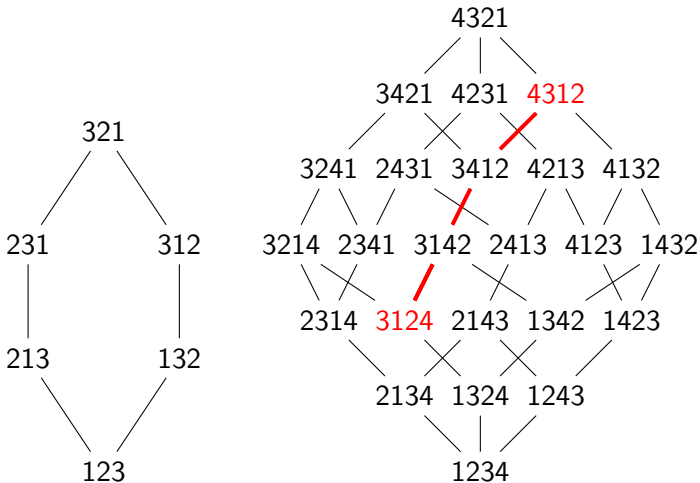
	permutations	binary trees	binary sequences
Combinatorics			
Algebra	Malvenuto-Reutenauer algebra	Loday-Ronco algebra	Descent Hopf algebra

→ All these objects (and more) can be interpreted in terms of *integer posets*.

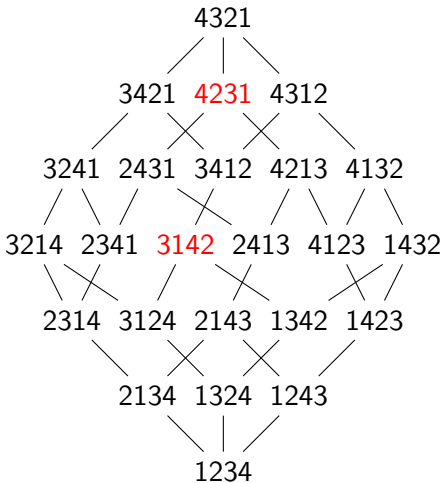
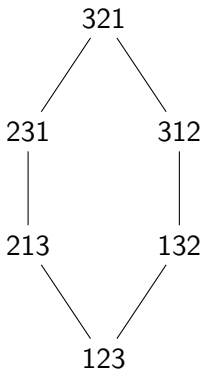
Weak order



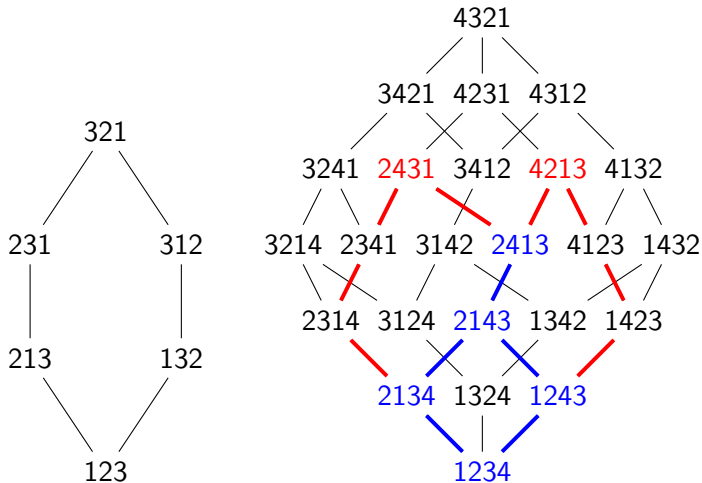
Weak order



Weak order

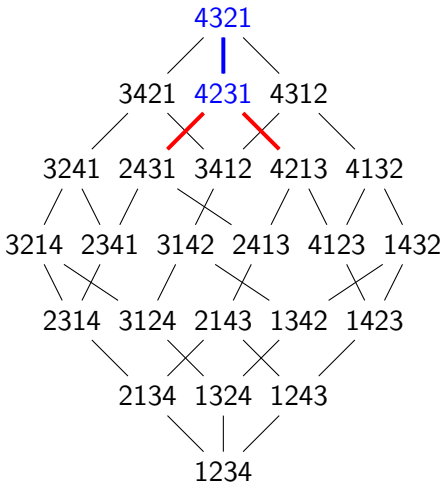
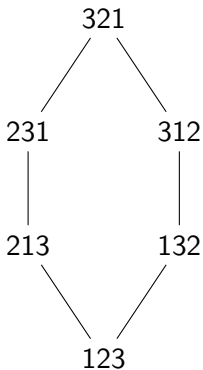


Weak order



$$2413 \wedge 4213 = 2413$$

Weak order



$$2413 \wedge 4213 = 2413$$

$$2413 \vee 4213 = 4231$$

Permutation poset

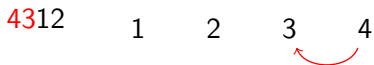
4312

Permutation poset

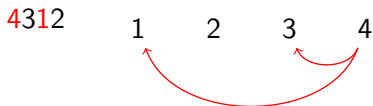
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Permutation poset

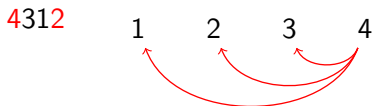
4312 1 2 3 4



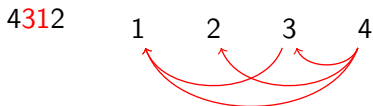
Permutation poset



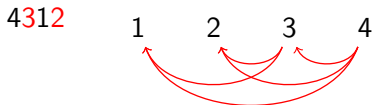
Permutation poset



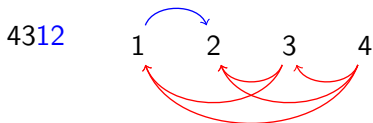
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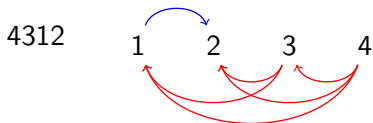
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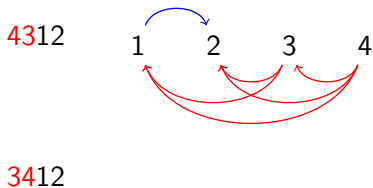
Permutation poset



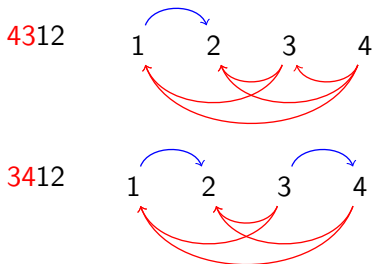
Permutation poset



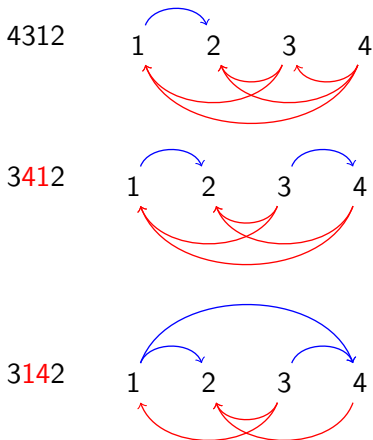
Permutation poset



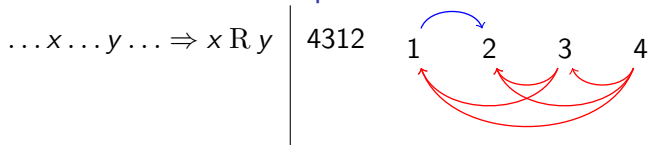
Permutation poset



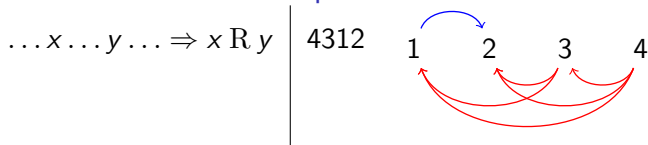
Permutation poset



Relation associated to a permutation



Relation associated to a permutation



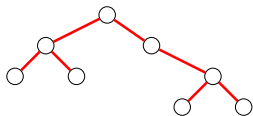
Weak order

$$R^{\text{Inc}} = \{i R j, i < j\}$$

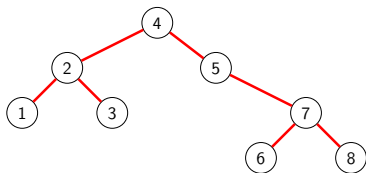
$$R^{\text{Dec}} = \{j R i, i < j\}$$

$$R \preceq S \Leftrightarrow R^{\text{Inc}} \supseteq S^{\text{Inc}} \text{ and } R^{\text{Dec}} \subseteq S^{\text{Dec}}$$

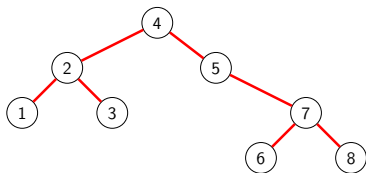
Binary tree poset



Binary tree poset

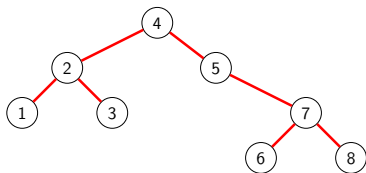


Binary tree poset

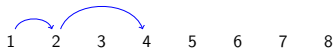
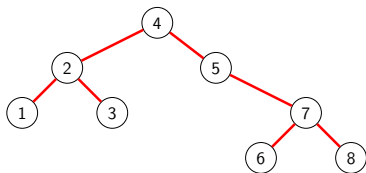


1 2 3 4 5 6 7 8

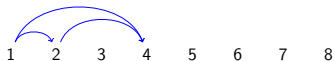
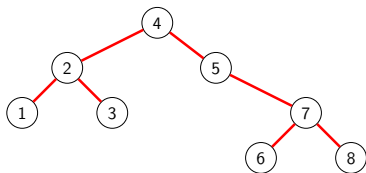
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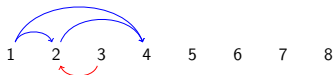
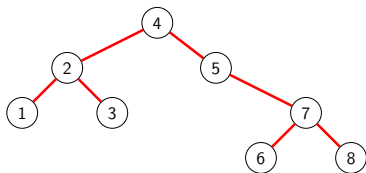
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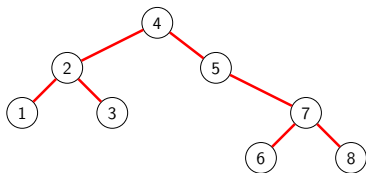
Binary tree poset



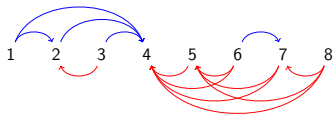
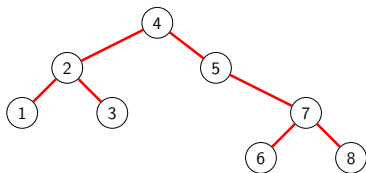
Binary tree poset



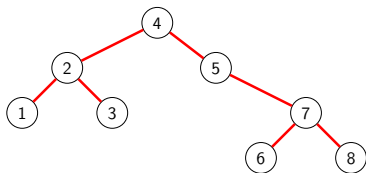
Binary tree poset



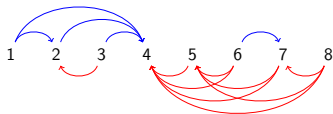
Binary tree poset



Binary tree poset



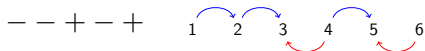
→ Tamari lattice



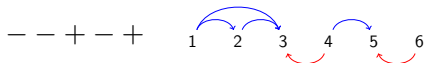
Binary sequence poset

— — + — + 1 2 3 4 5 6

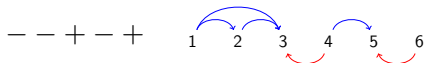
Binary sequence poset



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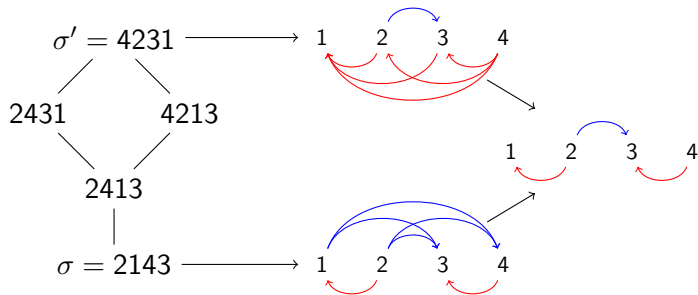


Binary sequence poset



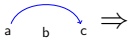

→ Boolean lattice

Permutation intervals



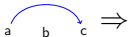
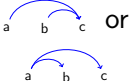

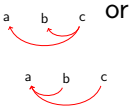
Interval-poset characterization

Posets (anti-symmetric, transitive) and

	intervals of permutations (weak order)	intervals of binary trees (Tamari lattice)	intervals of binary sequence (boolean lattice)
			
			

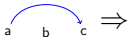
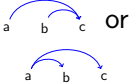


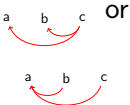

Interval-poset characterization

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	 or		
	 or		

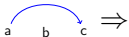





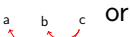

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Interval-poset characterization

Posets (anti-symmetric, transitive) and

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Binary relations on integer

R is a binary relation of size n .

1 2 ... i ... j ... k ... n

Binary relations on integer

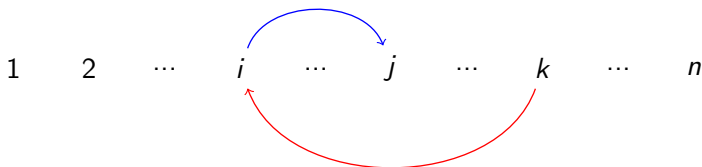
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$$i R j$$

Binary relations on integer

R is a binary relation of size n .

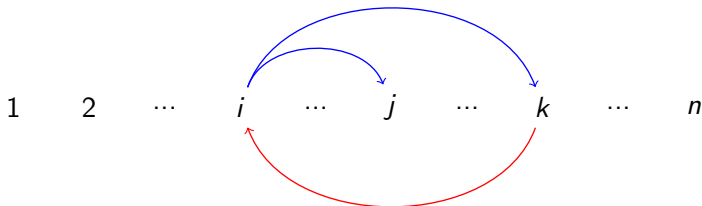


$$i R j$$

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Binary relations on integer

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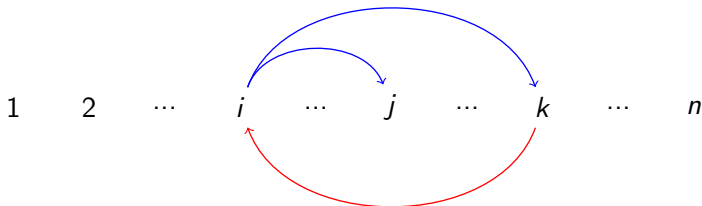
$$i R j$$

$$k R i$$

$$i R k$$

Binary relations on integer

R is a binary relation of size n .



$$i R j$$

$$k R i$$

$$i R k$$

Family of size $2^{n(n-1)}$.

Weak order

Let R an integer binary relation

$$R^{\text{Inc}} = \{i R j, i < j\}$$

$$R^{\text{Dec}} = \{j R i, i < j\}$$

Weak order

Let R an integer binary relation

$$R^{\text{Inc}} = \{i R j, i < j\}$$

$$R^{\text{Dec}} = \{j R i, i < j\}$$

Let R and S be two integer binary relations

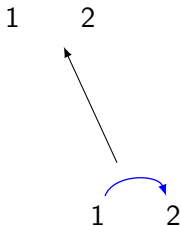
$$R \preceq S \Leftrightarrow R^{\text{Inc}} \supseteq S^{\text{Inc}} \text{ et } R^{\text{Dec}} \subseteq S^{\text{Dec}}$$

Binary relations of size 2

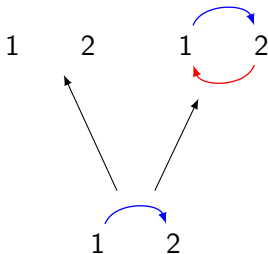
Binary relations of size 2



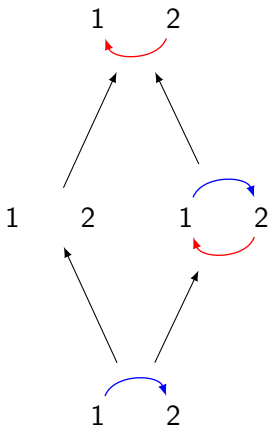
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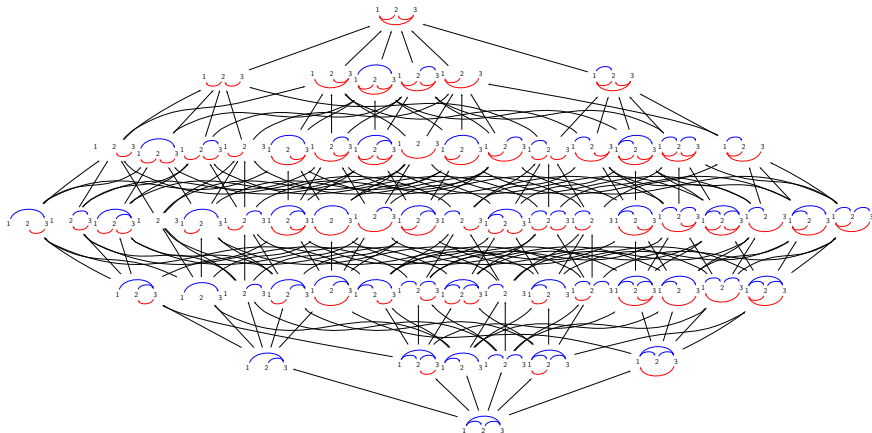


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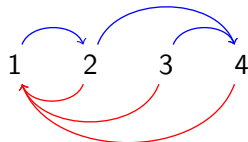
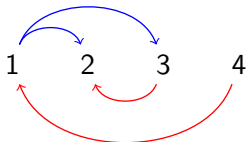


Binary relations of size 2

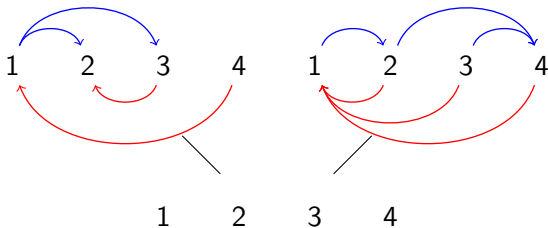




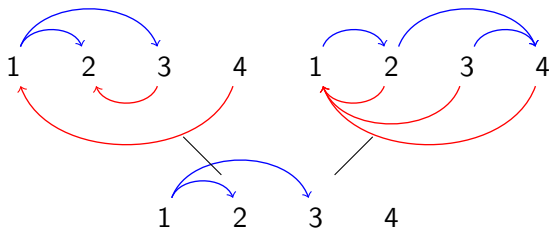
Meet and join



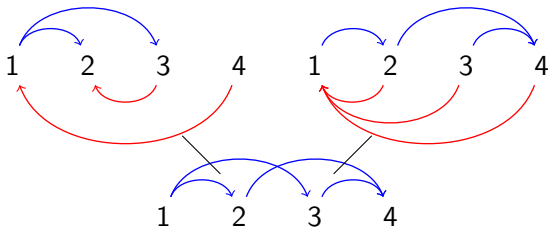
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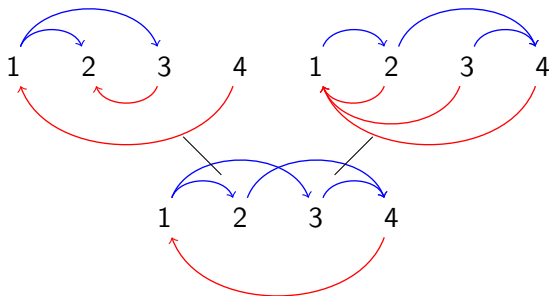
Meet and join



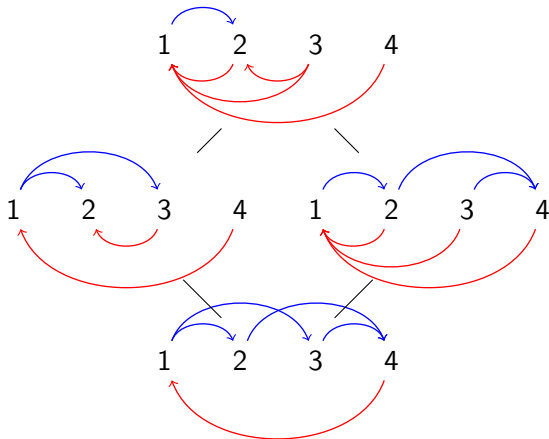
Meet and join



Meet and join



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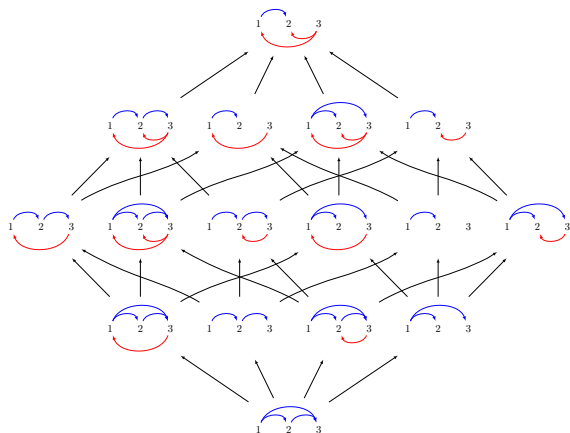


Hopf algebra

$$R \times S = \sum R \circlearrowright S$$

$$\begin{aligned}
 \overset{1}{\curvearrowright} \overset{2}{\times} \overset{1}{\curvearrowright} &= \sum \overset{1}{\curvearrowright} \overset{2}{\curvearrowright} \overset{3}{\curvearrowright} \\
 &= \overset{1}{\curvearrowright} \overset{2}{\curvearrowright} \overset{3}{\curvearrowright} + \overset{1}{\curvearrowright} \overset{2}{\curvearrowright} \overset{3}{\curvearrowright} + \overset{1}{\curvearrowright} \overset{2}{\curvearrowright} \overset{3}{\curvearrowright} + \overset{1}{\curvearrowright} \overset{2}{\curvearrowright} \overset{3}{\curvearrowright} + \dots
 \end{aligned}$$

The diagram shows the multiplication of two elements in a Hopf algebra. The first element is a blue arc from 1 to 2 and a red arc from 2 to 1. The second element is a blue arc from 1 to 2 and a red arc from 2 to 1. The result is a sum of terms, each representing a different way to connect the three nodes 1, 2, and 3. The first term is a blue arc from 1 to 2 and a red arc from 2 to 3. The second term is a blue arc from 1 to 2 and a red arc from 2 to 1. The third term is a blue arc from 1 to 2 and a red arc from 2 to 3. The fourth term is a blue arc from 1 to 2 and a red arc from 2 to 1. The sum continues with more terms.



Coproduct

$$\Delta R = \sum P \otimes Q$$

where $R = P \cup Q \cup (P \rightarrow Q)$

$$\triangleleft \begin{array}{ccc} & \overset{\text{blue}}{\curvearrowright} & \\ 1 & 2 & 3 \\ & \underset{\text{red}}{\curvearrowleft} & \end{array} =$$

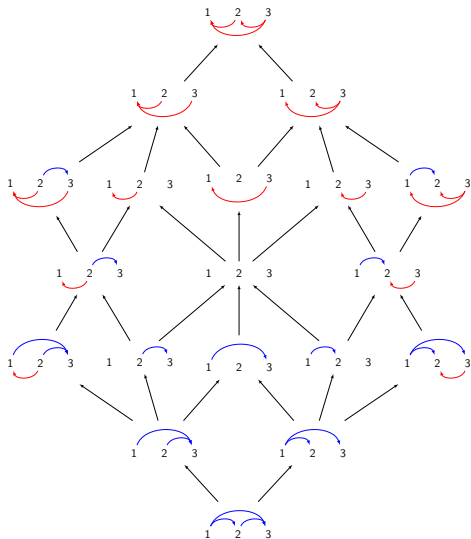
$$\Delta \begin{array}{c} 1 \quad 2 \quad 3 \\ \text{blue arc } 1 \rightarrow 2 \\ \text{red arc } 2 \rightarrow 3 \end{array} = \begin{array}{c} 1 \quad 2 \quad 3 \\ \text{blue arc } 1 \rightarrow 2 \\ \text{red arc } 2 \rightarrow 3 \end{array} \otimes \emptyset + \emptyset \otimes \begin{array}{c} 1 \quad 2 \quad 3 \\ \text{blue arc } 1 \rightarrow 2 \\ \text{red arc } 2 \rightarrow 3 \end{array}$$

$$\Delta \begin{array}{c} 1 \quad 2 \quad 3 \\ \curvearrowright \quad \curvearrowright \\ \curvearrowleft \end{array} = \begin{array}{c} 1 \quad 2 \quad 3 \\ \curvearrowright \quad \curvearrowright \\ \curvearrowleft \end{array} \otimes \emptyset + 1 \otimes \begin{array}{c} 1 \quad 2 \\ \curvearrowleft \end{array} + \emptyset \otimes \begin{array}{c} 1 \quad 2 \quad 3 \\ \curvearrowright \quad \curvearrowright \\ \curvearrowleft \end{array}$$

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Integer Posets

We *restrict* ourselves to transitive antisymmetric relations.



Hopf algebra of integer poset

Quotient of the integer relations Hopf algebra

$\mathbb{R} \equiv 0$ if \mathbb{R} is not a poset.

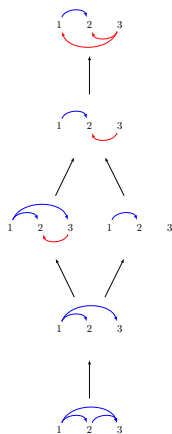
$$1 \overset{\curvearrowright}{\curvearrowleft} 2 \times 1 =$$

$$1 \overset{\curvearrowright}{\rightarrow} 2 \times 1 = 1 \overset{\curvearrowright}{\rightarrow} 2 \quad 3$$

$$\begin{array}{c} \text{1} \quad \text{2} \\ \text{---} \curvearrowright \\ \text{X} \quad \text{1} \end{array} = \begin{array}{c} \text{1} \quad \text{2} \\ \text{---} \curvearrowright \\ \text{3} \end{array} + \begin{array}{c} \text{1} \quad \text{2} \quad \text{3} \\ \text{---} \curvearrowright \quad \curvearrowright \\ \text{1} \quad \text{2} \quad \text{3} \end{array}$$

$$\begin{array}{c} \text{1} \quad \text{2} \\ \text{---} \curvearrowright \\ \text{X} \quad \text{1} \end{array} = \begin{array}{c} \text{1} \quad \text{2} \\ \text{---} \curvearrowright \\ \text{3} \end{array} + \begin{array}{c} \text{1} \quad \text{2} \quad \text{3} \\ \text{---} \curvearrowright \quad \text{3} \\ \text{+} \end{array} + \begin{array}{c} \text{1} \quad \text{2} \quad \text{3} \\ \text{---} \curvearrowright \quad \text{3} \\ \text{+} \end{array}$$

$$\begin{array}{c} \text{1} \\ \text{2} \end{array} \times \begin{array}{c} \text{1} \\ \text{1} \end{array} = \begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array} + \begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array} + \begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array} + \begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array}$$



To check: is it well defined?

$$P \times 0 = 0$$

$$\Delta 0 = 0$$

$$P \times 0 = 0$$

$$1 \xrightarrow{\text{blue}} 2 \xrightarrow{\text{red}} 3 \times \begin{matrix} 1 \xrightarrow{\text{blue}} 2 \\ 2 \xrightarrow{\text{red}} 1 \end{matrix} =$$

$$P \times 0 = 0$$

$$\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array} \times \begin{array}{c} \text{1} \\ \text{2} \end{array} = \begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array} \begin{array}{c} \text{4} \\ \text{5} \end{array} + \begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \\ \text{4} \\ \text{5} \end{array} + \dots$$

$$\Delta 0 = 0$$

$$\Delta \begin{array}{c} 1 \quad 2 \\ \curvearrowright \quad \curvearrowleft \\ \color{red}{\curvearrowright} \quad \color{red}{\curvearrowleft} \end{array} =$$

$$\Delta \begin{array}{c} 1 \quad 2 \quad 3 \\ \color{blue}{\curvearrowright} \quad \color{blue}{\curvearrowright} \end{array} =$$

$$\Delta 0 = 0$$

$$\Delta \begin{array}{c} \text{1} \quad \text{2} \\ \text{---} \text{---} \\ \text{---} \end{array} = \emptyset \otimes \begin{array}{c} \text{1} \quad \text{2} \\ \text{---} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{1} \quad \text{2} \\ \text{---} \text{---} \\ \text{---} \end{array} \otimes \emptyset$$

$$\Delta \begin{array}{c} \text{1} \quad \text{2} \quad \text{3} \\ \text{---} \text{---} \text{---} \\ \text{---} \end{array} =$$

$$\Delta 0 = 0$$

$$\Delta \begin{array}{c} 1 \quad 2 \\ \curvearrowright \\ \curvearrowleft \end{array} = \emptyset \otimes \begin{array}{c} 1 \quad 2 \\ \curvearrowright \\ \curvearrowleft \end{array} + \begin{array}{c} 1 \quad 2 \\ \curvearrowright \\ \curvearrowleft \end{array} \otimes \emptyset$$

$$\Delta \begin{array}{c} 1 \quad 2 \quad 3 \\ \curvearrowright \quad \curvearrowright \\ \curvearrowleft \quad \curvearrowleft \end{array} = \emptyset \otimes \begin{array}{c} 1 \quad 2 \quad 3 \\ \curvearrowright \quad \curvearrowright \\ \curvearrowleft \quad \curvearrowleft \end{array} + \begin{array}{c} 1 \quad 2 \quad 3 \\ \curvearrowright \quad \curvearrowright \\ \curvearrowleft \quad \curvearrowleft \end{array} \otimes \emptyset$$

Question: can we restrict to intervals of permutations?

Question: can we restrict to intervals of permutations?

Yes!

Interval-poset of permutations



Interval-poset of permutations

Product

$$\begin{array}{c} \text{a} \\ \text{b} \\ \text{c} \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} \Rightarrow \begin{array}{c} \text{a} \\ \text{b} \\ \text{c} \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} \text{ or } \begin{array}{c} \text{a} \\ \text{b} \\ \text{c} \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array}$$

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} \times 1 =$$

Interval-poset of permutations

Product

$$\begin{array}{c} \text{a} \\ \text{b} \\ \text{c} \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} \Rightarrow \begin{array}{c} \text{a} \\ \text{b} \\ \text{c} \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} \text{ or } \begin{array}{c} \text{a} \\ \text{b} \\ \text{c} \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array}$$

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} \times 1 = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array}$$

Interval-poset of permutations

Product

$$\begin{array}{c} \text{a} \\ \text{b} \\ \text{c} \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} \Rightarrow \begin{array}{c} \text{a} \\ \text{b} \\ \text{c} \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} \text{ or } \begin{array}{c} \text{a} \\ \text{b} \\ \text{c} \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array}$$

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} \times 1 = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} 4 + \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} 4$$

Interval-poset of permutations

Product

$$\begin{array}{c} \curvearrowright \\ a \quad b \quad c \end{array} \Rightarrow \begin{array}{c} \curvearrowright \\ a \quad b \quad c \end{array} \text{ or } \begin{array}{c} \curvearrowright \\ a \quad b \quad c \end{array}$$

$$\begin{array}{c} \curvearrowright \\ 1 \quad 2 \quad 3 \end{array} \times 1 = \begin{array}{c} \curvearrowright \\ 1 \quad 2 \quad 3 \end{array} 4 + \begin{array}{c} \curvearrowright \\ 1 \quad 2 \quad 3 \end{array} 4$$

Interval-poset of permutations

Product

$$\begin{array}{c} \text{a} \\ \text{b} \\ \text{c} \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} \Rightarrow \begin{array}{c} \text{a} \\ \text{b} \\ \text{c} \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} \text{ or } \begin{array}{c} \text{a} \\ \text{b} \\ \text{c} \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array}$$

$$\begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} \times \begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} = \begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \\ \text{4} \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} + \begin{array}{c} \text{1} \\ \text{2} \\ \text{3} \\ \text{4} \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array} + \dots$$

Interval-poset of permutations

Product

$$\begin{array}{c} \curvearrowright \\ a \quad b \quad c \end{array} \Rightarrow \begin{array}{c} \curvearrowright \\ a \quad b \quad c \end{array} \text{ or } \begin{array}{c} \curvearrowright \\ a \quad b \quad c \end{array}$$

$$\begin{array}{c} \curvearrowright \\ 1 \quad 2 \quad 3 \end{array} \times 1 = \begin{array}{c} \curvearrowright \\ 1 \quad 2 \quad 3 \end{array} 4 + \cancel{\begin{array}{c} \curvearrowright \\ 1 \quad 2 \quad 3 \end{array} 4} + \dots$$

Coproduct

$$\Delta \begin{array}{c} \curvearrowright \\ 1 \quad 2 \quad 3 \end{array} = \emptyset \otimes \begin{array}{c} \curvearrowright \\ 1 \quad 2 \quad 3 \end{array} + \begin{array}{c} 1 \quad 2 \end{array} \otimes 1 + \begin{array}{c} \curvearrowright \\ 1 \quad 2 \quad 3 \end{array} \otimes \emptyset$$

Product with 0

$$1 \overset{\curvearrowright}{\rightarrow} 2 \times 1 \overset{\curvearrowright}{\rightarrow} 2 \rightarrow 3 =$$

Coproduct of 0

$$\triangleleft 1 \overset{\curvearrowright}{\rightarrow} 2 \rightarrow 3 =$$

Product with 0

$$\begin{array}{c} \curvearrowright \\ 1 \quad 2 \end{array} \times \begin{array}{c} \curvearrowright \\ 1 \quad 2 \quad 3 \end{array} = \begin{array}{c} \curvearrowright \\ 1 \quad 2 \end{array} \begin{array}{c} \curvearrowright \\ 3 \quad 4 \quad 5 \end{array} + \begin{array}{c} \curvearrowright \quad \curvearrowright \\ 1 \quad 2 \quad 3 \quad 4 \quad 5 \end{array} + \dots$$

Coproduct of 0

$$\triangleleft \begin{array}{c} \curvearrowright \\ 1 \quad 2 \quad 3 \end{array} =$$

Product with 0

$$\begin{array}{c} \curvearrowright \\ 1 \quad 2 \end{array} \times \begin{array}{c} \curvearrowright \\ 1 \quad 2 \quad 3 \end{array} = \begin{array}{c} \curvearrowright \\ 1 \quad 2 \end{array} \begin{array}{c} \curvearrowright \\ 3 \quad 4 \quad 5 \end{array} + \begin{array}{c} \curvearrowright \quad \curvearrowright \\ 1 \quad 2 \quad 3 \end{array} \begin{array}{c} \curvearrowright \\ 4 \quad 5 \end{array} + \dots$$

Coproduct of 0

$$\triangle \begin{array}{c} \curvearrowright \\ 1 \quad 2 \quad 3 \end{array} = \emptyset \otimes \begin{array}{c} \curvearrowright \\ 1 \quad 2 \quad 3 \end{array} + \begin{array}{c} \curvearrowright \\ 1 \quad 2 \quad 3 \end{array} \otimes \emptyset$$

Question: can we restrict to intervals of binary trees (and binary sequences)?

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Not exactly... But

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Not exactly... But

Lattice: ok

Hopf algebra: as a sub algebra of the Hopf algebra on permutations intervals

References

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