Lattice and Hopf algebra of integer relations

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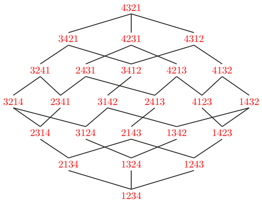
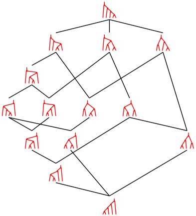
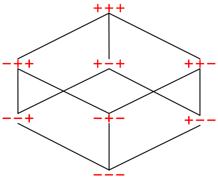
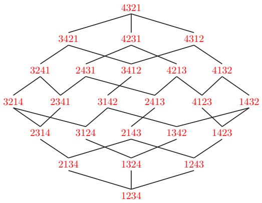
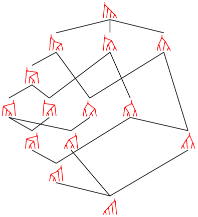
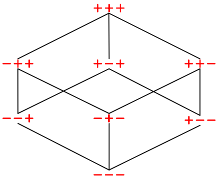
Université Paris-Sud

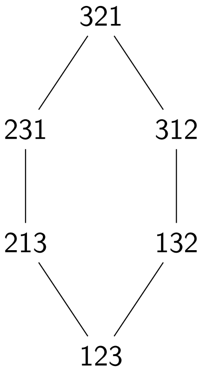
Séminaire Flajolet 21/09/2017

	permutations	binary trees	binary sequences
Combinatorics			
Algebra	Malvenuto-Reutenauer algebra	Loday-Ronco algebra	Descent Hopf algebra

	permutations	binary trees	binary sequences
Combinatorics			
Algebra	Malvenuto-Reutenauer algebra	Loday-Ronco algebra	Descent Hopf algebra

→ All these objects (and more) can be interpreted in terms of *integer posets*.

Weak order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231
312
213
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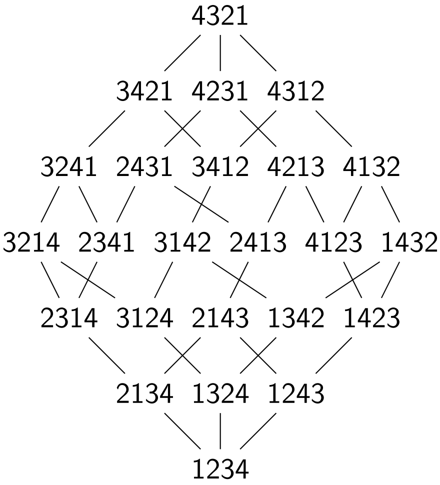
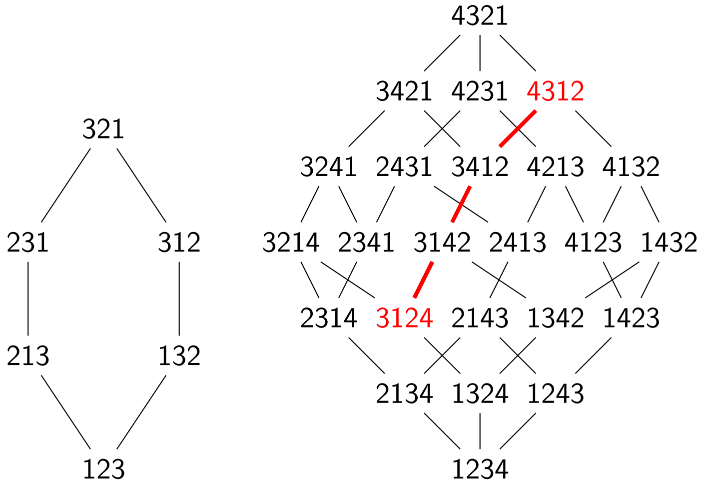
123


4321
3421 4231 4312
3241 2431 3412 4213 4132
3214 2341 3142 2413 4123 1432
2314 3124 2143 1342 1423
2134 1324 1243
1234

Weak order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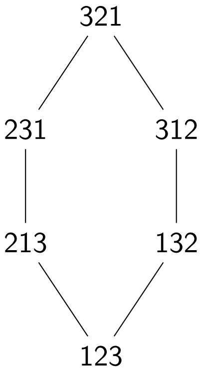
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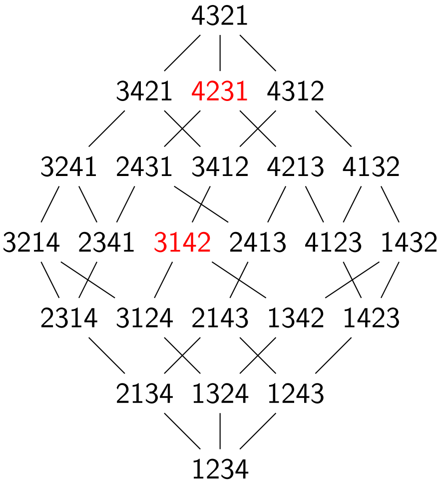
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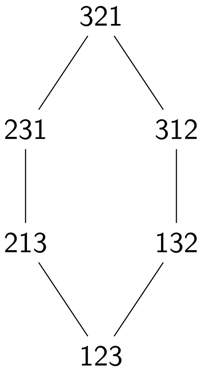
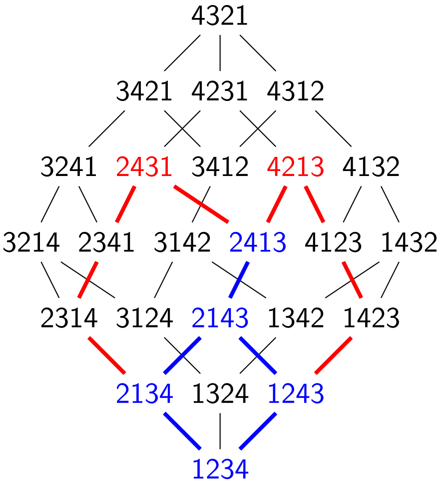
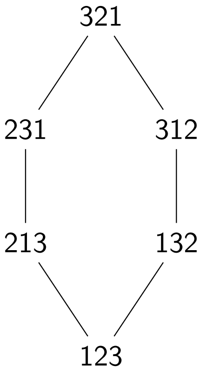
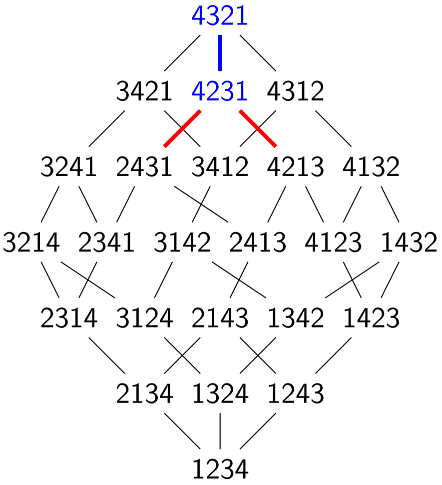




$$2413 \wedge 4213 = 2413$$

Weak order





$2413 \wedge 4213 = 2413$ $\quad$ $2413 \vee 4213 = 4231$

Permutation poset

4312

Permutation poset

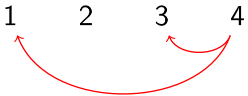
4312

Permutation poset

4312

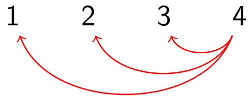
Permutation poset

4312



Permutation poset

4312



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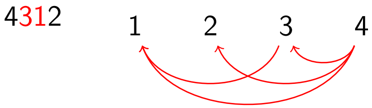

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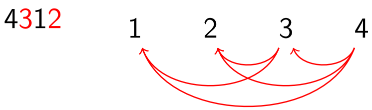

Permutation poset

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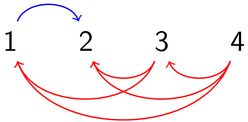

Permutation poset

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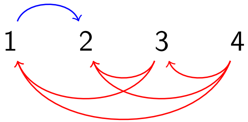

Permutation poset

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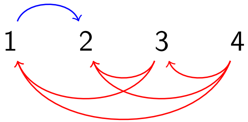
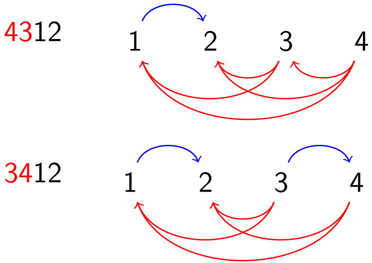


3412

Permutation poset

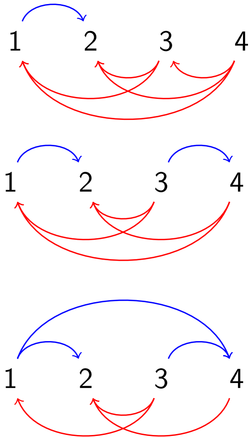


Permutation poset

4312

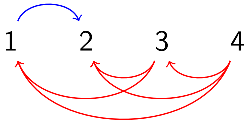
3412

3142



Relation associated to a permutation

$\ldots x \ldots y \ldots \Rightarrow x \mathrm{R} y$ | 4312



Relation associated to a permutation

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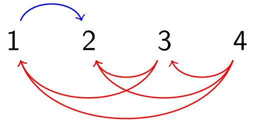
4312

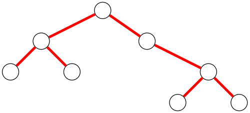


Weak order

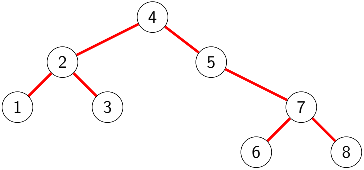
$$\mathrm{R}^{\mathsf{Inc}} = \{ i \,\mathrm{R}\, j, i < j \}$$
$$\mathrm{R}^{\mathsf{Dec}} = \{ j \,\mathrm{R}\, i, i < j \}$$

$$\mathrm{R} \preccurlyeq \mathrm{S} \Leftrightarrow \quad \mathrm{R}^{\mathsf{Inc}} \supseteq \mathrm{S}^{\mathsf{Inc}} \text{ and } \mathrm{R}^{\mathsf{Dec}} \subseteq \mathrm{S}^{\mathsf{Dec}}$$

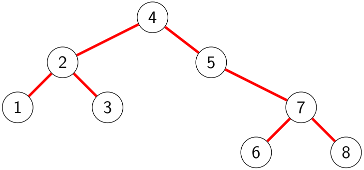
Binary tree poset



Binary tree poset

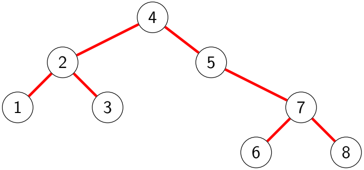


Binary tree poset



1 2 3 4 5 6 7 8

Binary tree poset



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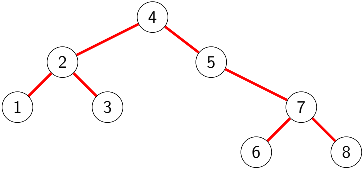

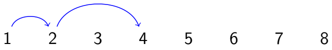

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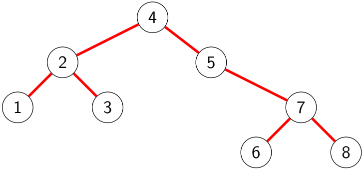

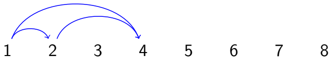

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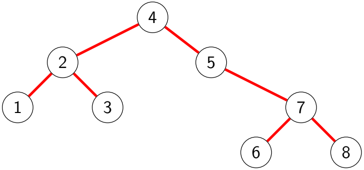

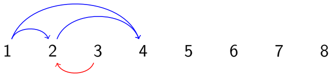

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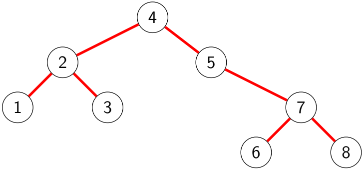

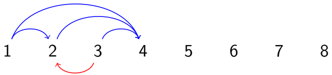

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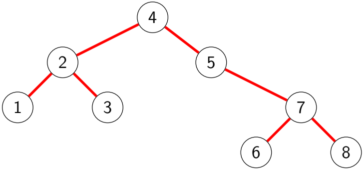

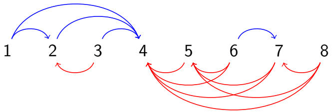

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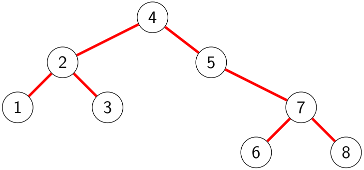
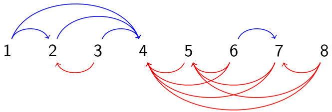




→ Tamari lattice

Binary sequence poset

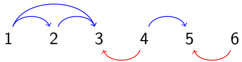
$- - + - +$ 1   2   3   4   5   6

Binary sequence poset

$- - + - +$

Binary sequence poset

$- - + - +$



Binary sequence poset

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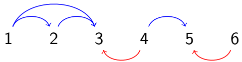
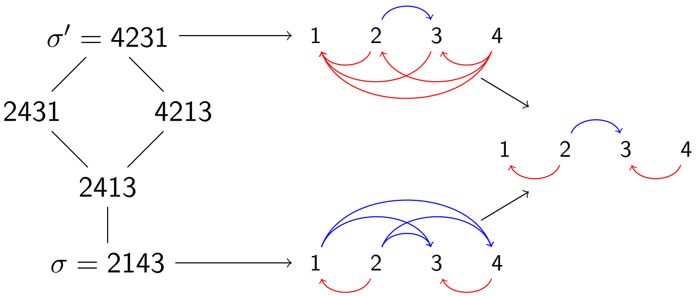


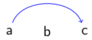
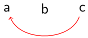
$\rightarrow$ Boolean lattice

Permutation intervals



Interval-poset characterization

Posets (anti-symmetric, transitive) and

	intervals of permutations (weak order)	intervals of binary trees (Tamari lattice)	intervals of binary sequence (boolean lattice)
 $\Rightarrow$			
 $\Rightarrow$			

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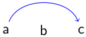
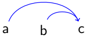
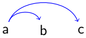
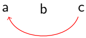
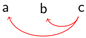
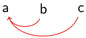
 $\Rightarrow$	 or 		
 $\Rightarrow$	 or 		

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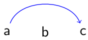
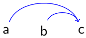
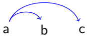
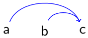
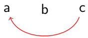
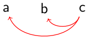
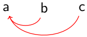
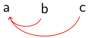
 ⇒	 or 		
 ⇒	 or 		

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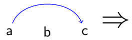
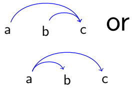
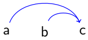
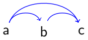
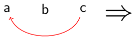
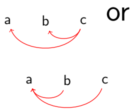
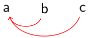
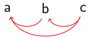
	intervals of permutations (weak order)	intervals of binary trees (Tamari lattice)	intervals of binary sequence (boolean lattice)
			
			

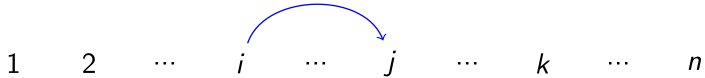
Binary relations on integer

R is a binary relation of size n.

$$1 \quad 2 \quad \cdots \quad i \quad \cdots \quad j \quad \cdots \quad k \quad \cdots \quad n$$

Binary relations on integer

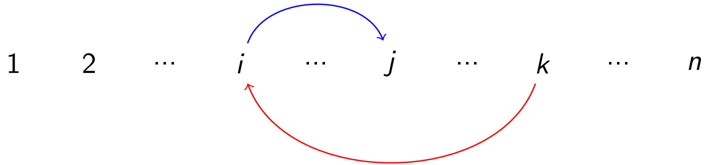
R is a binary relation of size n.



$i \, \mathrm{R} \, j$

Binary relations on integer

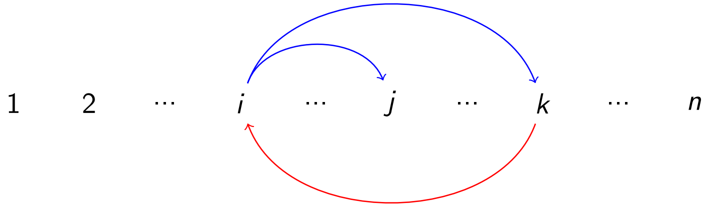
R is a binary relation of size n.



$i \mathrel{\mathrm{R}} j$ $\qquad$ $k \mathrel{\mathrm{R}} i$

Binary relations on integer

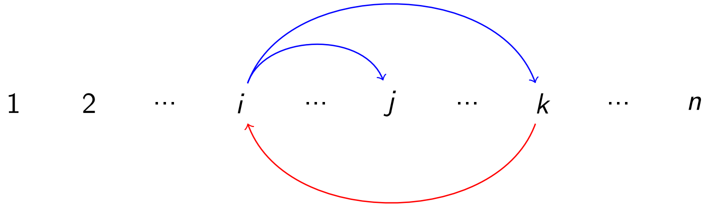
R is a binary relation of size n.



$i \,\mathrm{R}\, j$ $\qquad$ $k \,\mathrm{R}\, i$ $\qquad$ $i \,\mathrm{R}\, k$

Binary relations on integer

R is a binary relation of size n.



$i \,\mathrm{R}\, j$ $\quad$ $k \,\mathrm{R}\, i$ $\quad$ $i \,\mathrm{R}\, k$

Family of size $2^{n(n-1)}$.

Weak order

Let R an integer binary relation

$$\mathrm{R}^{\mathsf{Inc}} = \{i \,\mathrm{R}\, j, i < j\}$$
$$\mathrm{R}^{\mathsf{Dec}} = \{j \,\mathrm{R}\, i, i < j\}$$

Weak order

Let R an integer binary relation

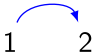
$$\mathrm{R}^{\mathsf{Inc}} = \{i \,\mathrm{R}\, j, i < j\}$$
$$\mathrm{R}^{\mathsf{Dec}} = \{j \,\mathrm{R}\, i, i < j\}$$

Let R and S be two integer binary relations

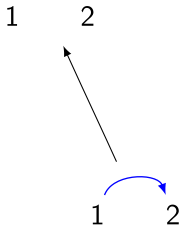
$$\mathrm{R} \preccurlyeq \mathrm{S} \Leftrightarrow \qquad \mathrm{R}^{\mathsf{Inc}} \supseteq \mathrm{S}^{\mathsf{Inc}} \text{ et } \mathrm{R}^{\mathsf{Dec}} \subseteq \mathrm{S}^{\mathsf{Dec}}$$

Binary relations of size 2

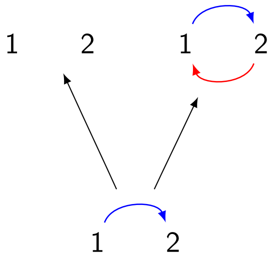
Binary relations of size 2



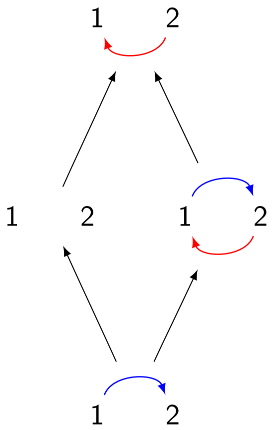
Binary relations of size 2

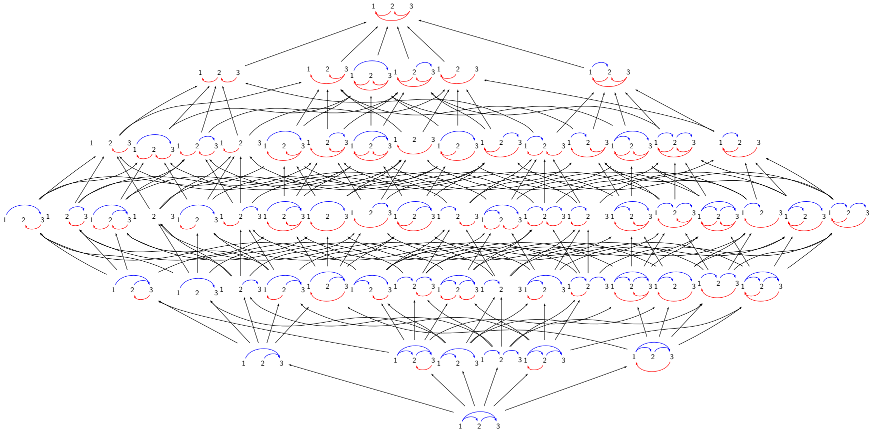


Binary relations of size 2

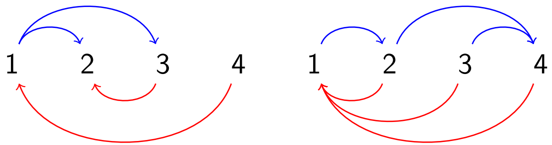


Binary relations of size 2

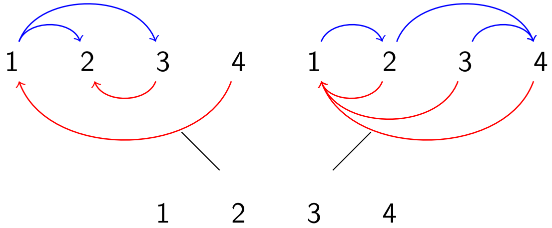




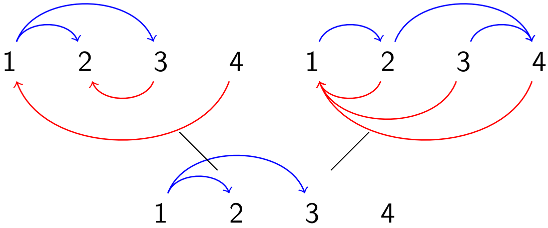
Meet and join



Meet and join



Meet and join



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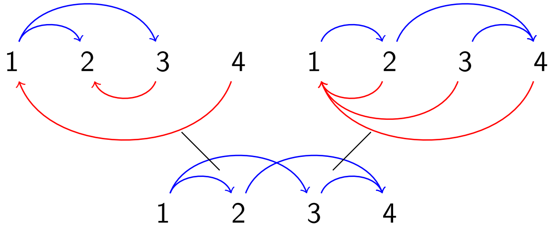
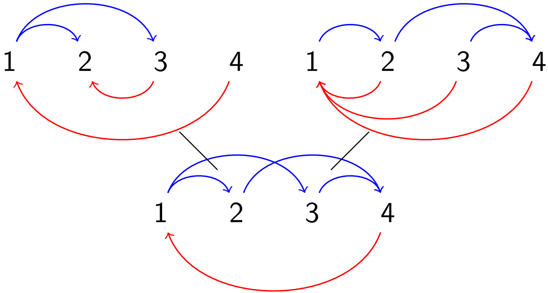
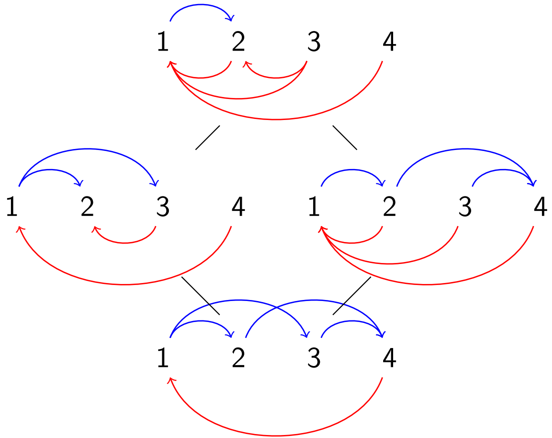


Meet and join

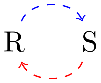


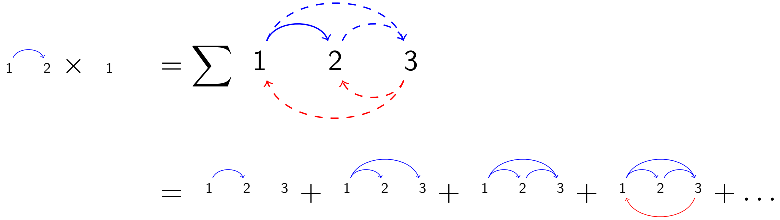
Meet and join

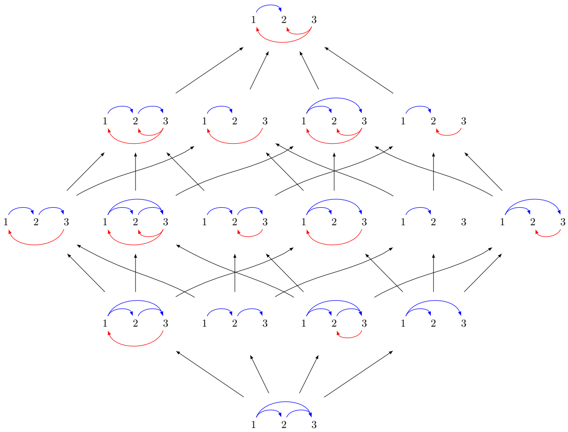


Hopf algebra

$$\mathrm{R} \times \mathrm{S} = \sum$$



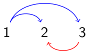

1 2 × 1 = ∑ 1 2 3
= 1 2 3 + 1 2 3 + 1 2 3 + 1 2 3 + ...

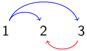


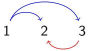
Coproduct

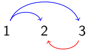
$$\triangle \mathrm{R} = \sum \mathrm{P} \otimes \mathrm{Q}$$

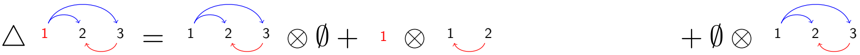
where $\mathrm{R} = \mathrm{P} \cup \mathrm{Q} \cup (\mathrm{P} \rightarrow \mathrm{Q})$

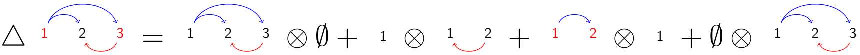
Δ  $=$

$\triangle$ 

$=$ 

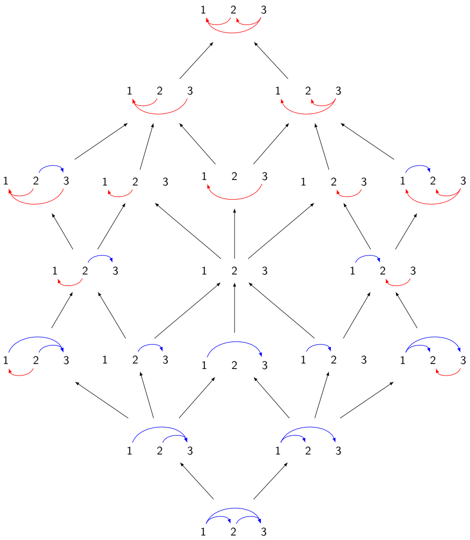
$\otimes \emptyset + \emptyset \otimes$ 




△ 1 2 3 = 1 2 3 ⊗ ∅ + 1 ⊗ 1 2 + 1 2 ⊗ 1 + ∅ ⊗ 1 2 3

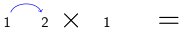
Integer Posets

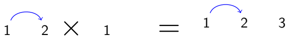
We *restrict* ourselves to transitive antisymmetric relations.

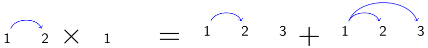


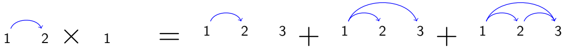
Hopf algebra of integer poset

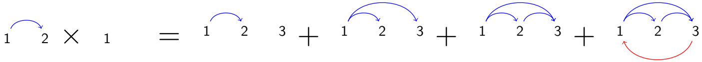
Quotient of the integer relations Hopf algebra
$\mathrm{R} \equiv 0$ if R is not a poset.

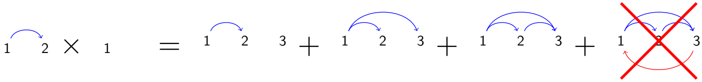

1
2
×
1
=

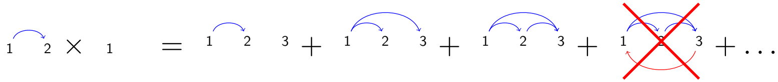

1 2 × 1 = 1 2 3

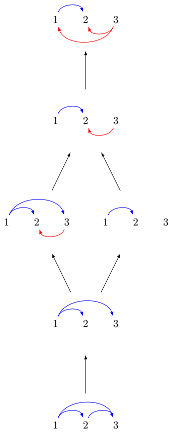

1
2
×
1
=
1
2
3
+
1
2
3


1 2 × 1 = 1 2 3 + 1 2 3 + 1 2 3


1 2 × 1 = 1 2 3 + 1 2 3 + 1 2 3 + 1 2 3


1
2
×
1
=
1
2
3
+
1
2
3
+
1
2
3
+
1
2
3


1
2
×
1
=
1
2
3
+
1
2
3
+
1
2
3
+
1
2
3
+ ...


1 2 3
1 2 3
1 2 3
1 2 3
1 2 3
1 2 3

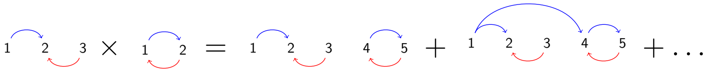
To check: is it well defined?

$P \times 0 = 0$

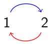
$\triangle 0 = 0$

$$\mathrm{P} \times 0 = 0$$

$$\mathrm{P} \times 0 = 0$$

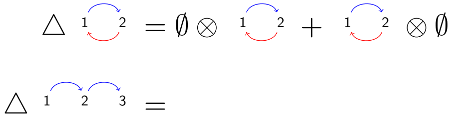


$\triangle 0 = 0$

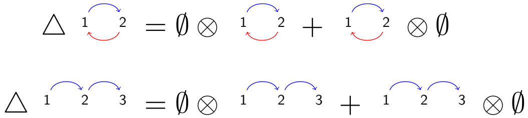
$\triangle$  $=$

$\triangle$  $=$

$$\triangle 0 = 0$$



$\triangle 0 = 0$

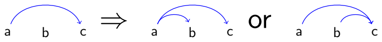


Question: can we restrict to intervals of permutations?

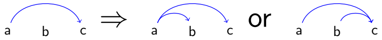
Question: can we restrict to intervals of permutations?

Yes!

Interval-poset of permutations



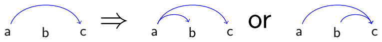
Interval-poset of permutations

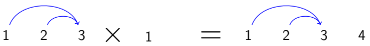


Product

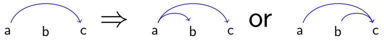
Interval-poset of permutations

Product

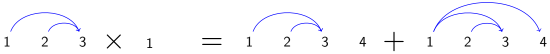




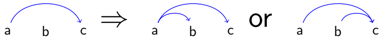
Interval-poset of permutations



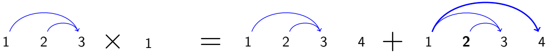
Product



Interval-poset of permutations

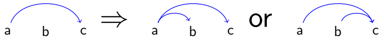


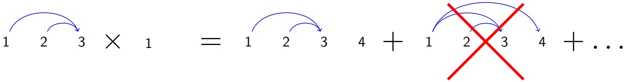
Product



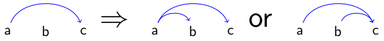
Interval-poset of permutations

Product

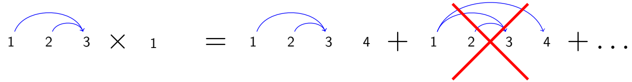




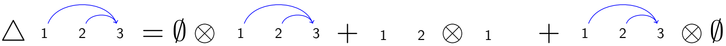
Interval-poset of permutations



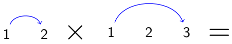
Product



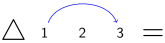
Coproduct



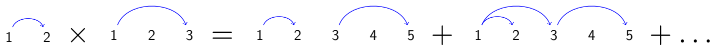
Product with 0



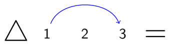
Coproduct of 0



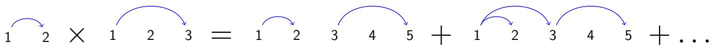
Product with 0



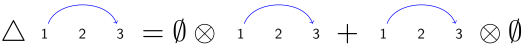
Coproduct of 0



Product with 0



Coproduct of 0



Question: can we restrict to intervals of binary trees (and binary sequences)?

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Not exactly... But

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Not exactly... But

Lattice: ok
Hopf algebra: as a sub algebra of the Hopf algebra on permutations intervals

References

- Châtel, Pilaud, P. *The weak order on integer posets* arXiv:1701.07995
- Pilaud, P. *Permutrees* arXiv:1606.09643
- Pilaud, P. *The Hopf algebra of integer posets* (work in progress)