Geometric and combinatorial questions on lattice polytopes

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based on joint work with

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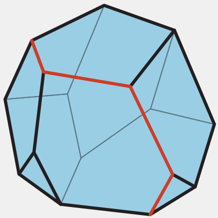
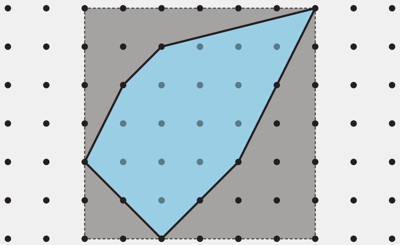
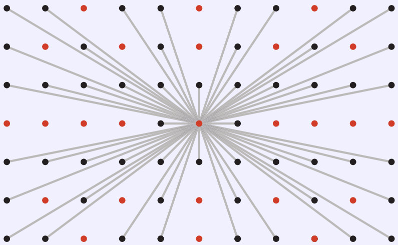






Outline of the lecture

1) Questions on lattice polytopes that arise from

- Linear optimization,
- Combinatorics,
- Physics.

2) Results on the diameter of lattice polytopes and lattice zonotopes

3) Results on the number of vertices of primitive zonotopes

4) The number of the d-dimensional lattice polytopes contained in $[0, k]^d$

5) A graph structure on the set of lattice polytopes

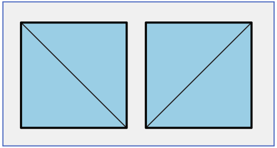
Reasons to study lattice polytopes

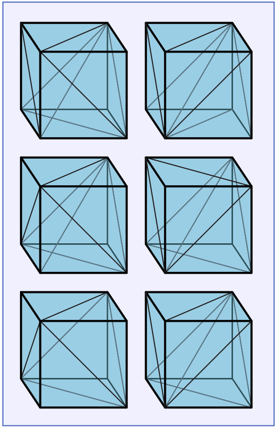
The d-dimensional unit cube $[0,1]^d$ is already an interesting lattice polytope.

d	#T	#T/sym	#S
2	2	1	2
3	74	6	5
4	92 487 256	247 451	16
5	?	?	67
6	?	?	308
7	?	?	1 493
8	?	?	?

#T: number of triangulations of $[0,1]^d$ (A238820/A238821)
#S: simplexity of $[0,1]^d$ (A019503)

De Loera, 1996
P, 2013
Mara, 1976
Cottle, 1982
Hughes, 1993
Hughes-Anderson, 1996





Questions on lattice polytopes: number

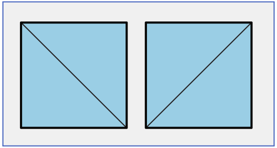
The d-dimensional unit cube $[0,1]^d$ is already an interesting lattice polytope.

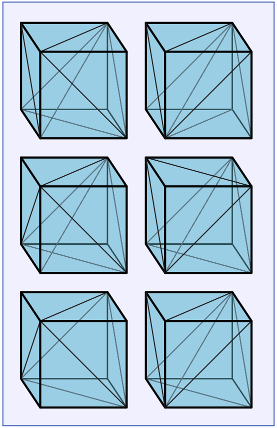
d	#P	#P/sym
2	5	2
3	151	12
4	60 879	347
5	4 292 660 729	1 226 525
6	18 446 743 888 401 503 325	?
8	?	?

#P: number of d-dimensional lattice polytopes in $[0,1]^d$ (A105230)
#P/sym: same as #P, but up to symmetry (A105231)

Aichholzer, 2000 ($2^{32} - 2\,306\,567$)
P, Rakotonarivo 2019 ($2^{64} - 185\,308\,048\,291$)

Theorem: $\lim_{d\to\infty} \frac{\#P}{2^{2^d}} = 1.$

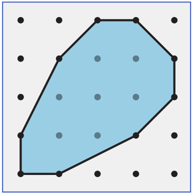




Questions on lattice polytopes: vertices

Question: what is the largest number of vertices of a convex lattice polygon contained in the square $[0,k]^2$?

Question: what is the largest number of vertices $\phi(d,k)$ of a lattice polytope contained in the hypercube $[0,k]^d$?



$\phi(2,4) = 9$

Theorem (Thiele, 1991, Acketa-Žunić 1995): $\lim_{k\to\infty} \frac{\phi(2,k)}{k^{2/3}} = \frac{12}{(2\pi)^{2/3}}$.

Theorem (Bárány-Larman, 1998): the number of vertices of the convex hull of all the lattice points in a d-dimensional ball of diameter k satisfies

$$c_1(d)k^{d\frac{d-1}{d+1}} \leq \#vertices \leq c_2(d)k^{d\frac{d-1}{d+1}}$$

The diameter of a polygon with v vertices is $\lfloor v/2 \rfloor$. When $d > 2$, what about looking at the diameter of lattice polytopes instead?

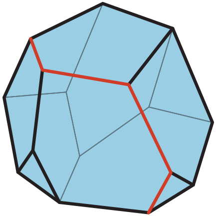
Questions on lattice polytopes: diameter

Question: what is the largest possible diameter $\delta(d, k)$ of a lattice polytope contained in the hypercube $[0, k]^d$?

Linear Optimization: find a point x in $\mathbb{R}^d$ such that

$$Ax \leq b,$$

where $b \in \mathbb{R}^d$ and A is a $n \times d$ matrix.



The set P of the points x is a polyhedron and, if bounded, a polytope.

Simplex method: the point $x \in P$ we search for is such that cx is maximal for some row vector c. That method finds a path in the edge-graph of P.

The diameter of (the edge-graph of) P, $\delta(P)$, is a lower bound on the number of pivots of the simplex method.

Largest possible diameter

Question: what is the largest possible diameter $\delta(d, k)$ of a lattice polytope contained in the hypercube $[0, k]^d$?

Theorem (Naddef, 1989): $\delta(d, 1) = d$.

Theorem (Thiele, 1991, Acketa-Žunić 1995): $\lim_{k \to \infty} \frac{\delta(2, k)}{k^{2/3}} = \frac{6}{(2\pi)^{2/3}}$.

Theorem (Kleinschmid-Onn, 1992): $\delta(d, k) \leq kd$.

Theorem (Del Pia-Michini, 2016): if $k \geq 2$, then $\delta(d, k) \leq kd - \left\lceil \frac{d}{2} \right\rceil$.

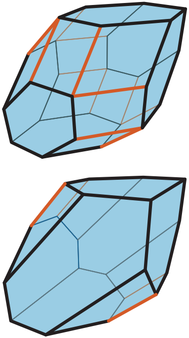
Theorem (Deza-P, 2018): if $k \geq 3$, then $\delta(d, k) \leq kd - \left\lceil \frac{2}{3} d \right\rceil - (k - 3)$.

Largest possible diameter

d \ k	1	2	3	4	5	6	7	8	9	10
2	2	3	4	4	5	6	6	7	8	$\cdots$
3	3	4	6	7	9	10				
4	4	6	8							
5	5	7	10							
$\vdots$	$\vdots$	$\vdots$								
d	d	$\left\lfloor \frac{3}{2} d \right\rfloor$								

↑

All the known values of $\delta(d, k)$



↑

Two of the nine (up to symmetry) lattice polytopes of diameter 6 contained in the cube $[0,3]^3$... among 332 335 207 073.

Naddef, 1989 — $\delta(d, 1) = d$

Thiele, 1991, Acketa-Žunić 1995, Deza-Manoussakis-Onn, 2018

Del Pia-Michini, 2016 — $\delta(d, 2) = \lfloor 3d/2 \rfloor$

Deza-P, 2018 — $\delta(4, 3) = 8$

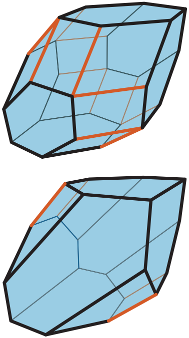
Chadder-Deza, 2017 — $\delta(3, 4) = 7$, $\delta(3, 5) = 9$

Deza-Deza-Guan-P, 2019 — $\delta(3, 6) = \delta(5, 3) = 10$

P-Rakotonarivo, 2019

Largest possible diameter

d \ k	1	2	3	4	5	6	7	8	9	10
2	2	3	4	4	5	6	6	7	8	$\cdots$
3	3	4	6	7	9	10				
4	4	6	8							
5	5	7	10							
$\vdots$	$\vdots$	$\vdots$								
d	d	$\lfloor \frac{3}{2} d \rfloor$								

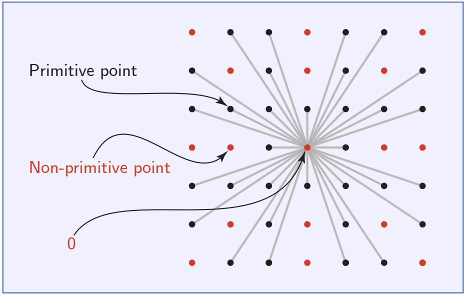


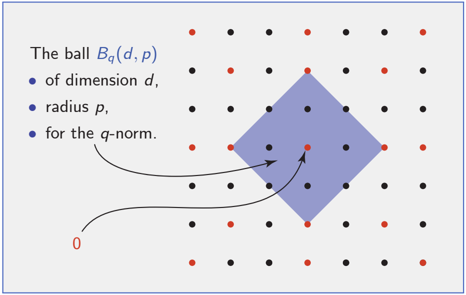
Theorem (Deza-Manoussakis-Onn, 2018): if $k < 2d$, then

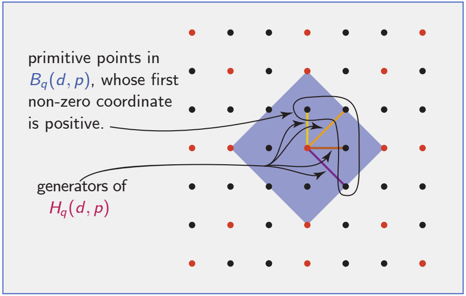
$$\delta(d,k) \geq \left\lfloor \frac{(k+1)d}{2} \right\rfloor$$

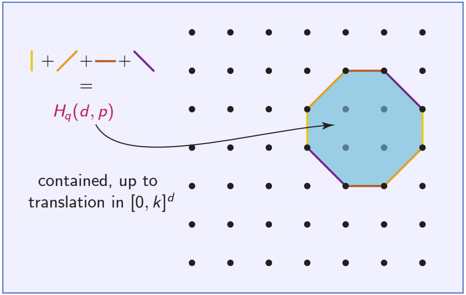
Conjecture (Deza-Manoussakis-Onn, 2018): this is sharp when $k < 2d$. In general, $\delta(d,k)$ is achieved by a lattice zonotope contained in $[0,k]^d$.

Primitive zonotopes (Deza, Manoussakis, Onn, 2018)



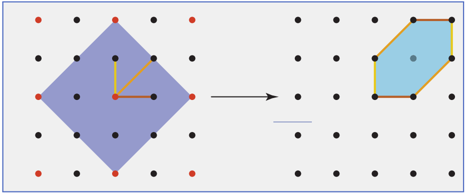


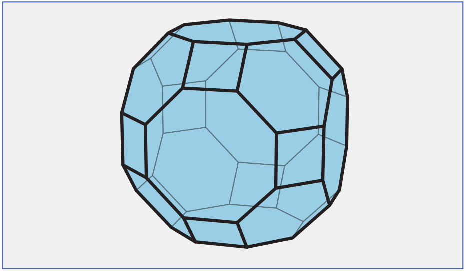




Primitive zonotopes (Deza, Manoussakis, Onn, 2018)

The Minkowski sum of the generators of $H_q(d,p)$ contained in $[0,+\infty[^d$ is another family of primitive zonotopes, denote by $H_q^+(d,p)$.





$H_1(d,2)$ is the type B permutohedron:

- $2^d d!$ vertices,
- diameter d^2,
- contained (up to translation) in the hypercube $[0,2d-1]^d$.

Theorem (Deza-Manoussakis-Onn): $\delta(d,k) \geq \left\lfloor \frac{(k+1)d}{2} \right\rfloor$ when $k < 2d$.

Asymptotic diameter

Theorem (Thiele, 1991, Acketa-Žunić 1995): $\lim_{k\to\infty} \frac{\delta(2,k)}{k^{2/3}} = \frac{6}{(2\pi)^{2/3}}$.

But, when $d > 2$ and k grows large,

$$?? \leq \delta(d,k) \leq k(d-1) \text{ (minus a term that does not depend on } k\text{)}.$$

Call $\delta_Z(d,k)$ the largest possible diameter of a lattice zonotope in $[0,k]^d$.

Theorem (Deza-P-Sukegawa, 2019): For any fixed d,

$$\lim_{k\to\infty} \frac{\delta_Z(d,k)}{k^{\frac{d}{d+1}}} = \left(\frac{2^{d-1}(d+1)^d}{d!\zeta(d)}\right)^{\frac{1}{d+1}}$$

Corollary (Deza-P-Sukegawa, 2019): For any fixed d,

$$\delta(d,k) \geq \left(\frac{2^{d-1}k^d(d+1)^d}{d!\zeta(d)}\right)^{\frac{1}{d+1}} + o(k^{\frac{d}{d+1}}).$$

Asymptotic diameter

Theorem (Deza-P-Sukegawa, 2019):

$$\lim_{p\to\infty} \frac{\delta(H_q(d,p))}{p^d} = \frac{\left(2\Gamma\left(\frac{1}{q}+1\right)\right)^d}{2\Gamma\left(\frac{d}{q}+1\right)\zeta(d)}$$

$$\lim_{p\to\infty} \frac{\delta\left(H_q^+(d,p)\right)}{p^d} = \frac{\Gamma\left(\frac{1}{q}+1\right)^d}{\Gamma\left(\frac{d}{q}+1\right)\zeta(d)}$$

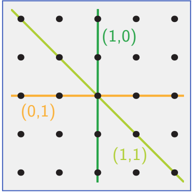
$$\mathrm{vol}(B_q(d,p)) = \frac{\left(2\Gamma\left(\frac{1}{q}+1\right)p\right)^d}{2\Gamma\left(\frac{d}{q}+1\right)} \text{ and } \lim_{p\to\infty} \frac{\#\text{PP in } B_q(d,p)}{\mathrm{vol}(B_q(d,p))} = \frac{1}{\zeta(d)}.$$

Theorem (Deza-P-Sukegawa, 2019): Consider an integer p, and the smallest possible integer k such that $H_1(d,p)$ is contained in the hypercube $[0,k]^d$, up to translation. The largest diameter of a lattice zonotope contained in $[0,k]^d$ is uniquely achieved by $H_1(d,p)$.

Lattice polytopes in theoretical physics

Theoretical physicists are interested in the number $a(d)$ of generalized retarded functions.

$a(d)$ is the number of regions in the arrangement formed by the $2^d - 1$ hyperplanes normal to 0, 1-vectors.



Theorem (Billera et al., 2012):

$$\prod_{i=0}^{d-1} (2^i + 1) \leq a(d) < 2^{d^2}.$$

However, by duality, $a^+(d) = f_0(H^+_\infty(d,1))$

Theorem (Deza-P-Rakotonarivo, 2019): if $d \geq 3$,

$$6 \prod_{i=1}^{d-2} (2^{i+1} + i) \leq a(d) \leq 2(d+4)2^{(d-1)(d-2)}.$$

d	$a(d)$
1	2
2	6
3	32
4	370
5	11 292
6	1 066 044
7	347 326 352
8	419 172 756 930
9	?

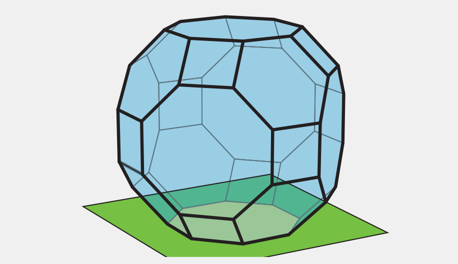
What about $H_\infty(d,1)$?

The number of vertices of $H_\infty(d,p)$ turns up in combinatorial optimization: it is the worst-case complexity of multicriteria matroid optimization

Theorem (Melamed-Onn, 2014): $d!2^d \leq f_0(H_\infty(d,1)) < O(3^{d(d-1)})$.

Theorem (Deza-P-Rakotonarivo, 2019):

$$\prod_{i=0}^{d-1}(3^i+1) \leq f_0(H_\infty(d,1)) < 2(3^{d-1}+1)^{d-1}.$$



$H_\infty(d,1) \cap M = H_\infty(d-1,1) + P$ for some polytope P. There are $3^{d-1}+1$ possible heights for M.

As $f_0(P+Q) \geq f_0(Q)$,

$$\frac{f_0(H_\infty(d,1))}{f_0(H_\infty(d-1,1))} \geq 3^{d-1}+1.$$

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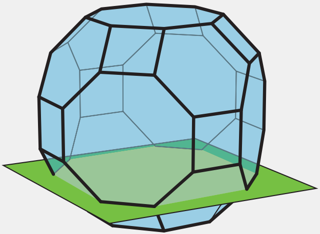

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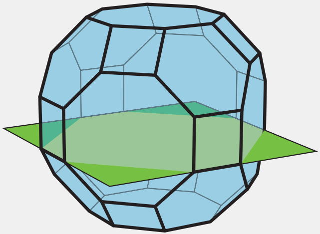

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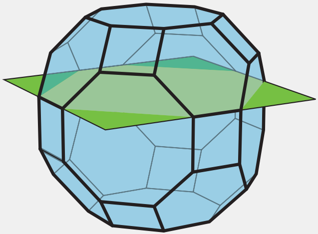

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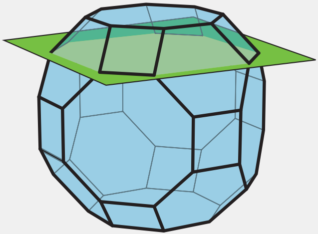

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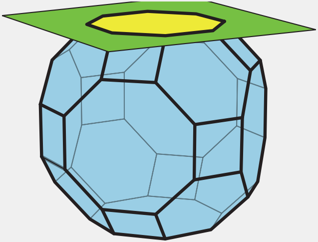

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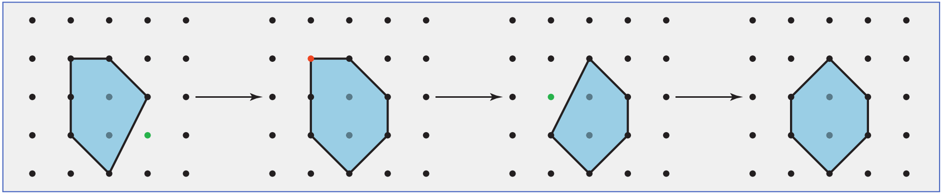
A graph on lattice polytopes

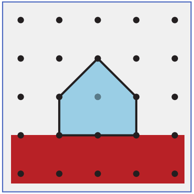
Say a lattice pentagon P and a lattice hexagon H can be transformed into one another by a move when all the vertices of P are vertices of H.



Question: can any lattice pentagon or hexagon be transformed into any other lattice pentagon or hexagon by such moves?

Theorem (David-P-Rakotonarivo, 2018): yes!

If one restricts to the pentagons and hexagons contained in a convex polyhedral region, then the answer is no, even for a "large" (unbounded) region like $\mathbb{R}\times[0,+\infty[$.



A graph on lattice polytopes

General case: two lattice polytopes P and Q can be transformed into one another by an elementary move when they both have the same dimension and their vertex sets differ by exactly one vertex.

General question: can any d-dimensional lattice polytope be transformed into any other by a sequence of moves? In other words, is the graph $\Lambda(d)$ whose vertices are the d-dimensional lattice polytopes and whose edges are the elementary moves connected?

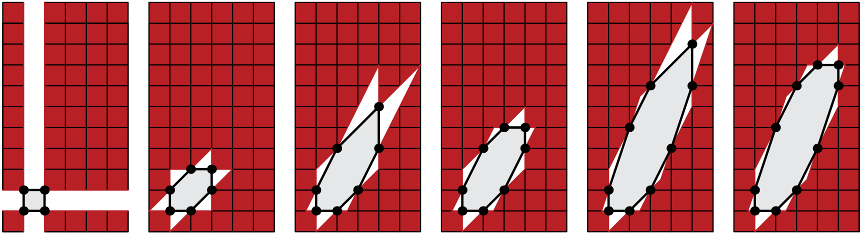
What was false for pentagons and hexagons (connectedness inside a box) is true for polytopes of any fixed dimension d whith $d+1$ and $d+2$ vertices. In particular it is true for triangles and quadrilaterals!

Theorem (David-P-Rakotonarivo, 2018): for any positive k, the subgraph induced in $\Lambda(d)$ by the simplices and the polytopes with $d+2$ vertices contained in the hypercube $[0,k]^d$ is connected

Corollary (David-P-Rakotonarivo, 2018): $\Lambda(d)$ is connected.

A graph on lattice polytopes

In fact, the subgraph induced in $\Lambda(2)$ by the polygons with n and $(n+1)$ vertices is always disconnected, except when $n = 3$ or $n = 5$.



Theorem (David-P-Rakotonarivo, 2018): for any $d \geq 4$, there exist lattice polytopes P whose number n of vertices can be arbitrarily large such that P cannot be transformed into any lattice polytope with $n + 1$ vertices.

Question: When $d = 3$, are there such polytopes with n arbitrarily large?

Question: What are the values of $d \geq 3$ and n such that the subgraph induced in $\Lambda(d)$ by the polytopes with n and $n + 1$ vertices is connected?