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Polytopes, shuffles, quivers and flags
Transcription by A. Sportiello

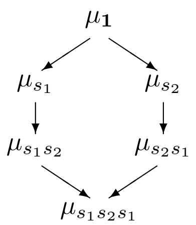
Abstract

[by the Author] There are two geometries that show remarkable similarities: that of quiver varieties and that of affine flag varieties. By work of Braverman-Gaitsgory and Gaussent-Littelmann and Kashiwara-Saito and Kamnitzer-Baumann one sees the crystals, in the sense of Kashiwara, coming from both quivers and flags. In the picture of Leclerc-Geiss-Schroer one sees how elements of the shuffle algebra come from quiver varieties. In joint work with A. Ghitza and S. Kannan we are seeing shuffle elements coming from affine flag varieties. Following my recent joint work Ghitza and Kannan, I will explain the purely combinatorial approach for seeing the moment polytopes and the shuffle elements.

1 Introduction

The beginning of my collaboration with Ghitza and Kannan was the confluence of our different expertises, with the aim of investigating possible relations among *global sections of the flag variety* $H^0(G/B, \mathcal{L}_\lambda)$, on which Kannan has worked for a long time, and *MV-cycles of type* λ.

We will now define these objects, but before this let us first introduce the related concept of *MV polytope*. A good picture of such a structure is the classical diagram for the Bruhat order of the symmetric group over 3 elements[1]



This object is not still a polytope in the sense we are interested in (we have to associate "lengths" to the edges, and, to have a MV polytope, these lengths have to satisfy certain relations). Furthermore, we haven't still made clear in which sense it is a polytope, and not just a polygon, i.e. where is the "interior".

[1] Here and in the following, s_i denotes the transposition of i and $i+1$.

We will come back on this later on, but for the moment just keep this picture in mind, of "how a MV polytope looks like".

It is known that MV polytopes have a 1-to-1 relation with *column-strict Young Tableaux*, that is, tableaux looking like

1	1	1	2	3
2	2	3		
3	4			

This relation has been object of my interests for a long time. In particular, I tried to extend it to other Lie groups and root systems. However, in the following we will not specially make use of the relation between MV polytopes and column-strict tableaux. We will instead concentrate on the relation of these polytopes with three other "objects": MV cycles Z_b, Pre-projective algebra modules Λ_b, and quiver Hecke algebra modules L_b. In a sense, an appropriate "shadow" operation on these objects, consisting in extracting a character, naturally leads to consider MV polytopes.

While this "shadow" procedure has been long investigated for the last two cases (in particular, by Kamnitzer and Baumann for pre-projective algebra modules, and by Lusztig, Kashiwara, Khovanov-Lauda and others for quiver Hecke algebra modules), the parallel construction for MV cycles was investigated so far, again by Kamnitzer, only up to a certain extent, and one of our original contributions to the picture is on this subject.

The character formulas are valued in an algebra of words (possibly, including a parameter q). A remarkable aspect of the topic at hand is that, not obviously, they take value in a rather "tiny" subalgebra of the free algebra, the *Shuffle Algebra* $\mathbb{C}[N]$, that we go to define in the following section.

The character $\mathrm{ch}(L_b)$ is related to the construction of the so-called *dual canonical basis* in the shuffle algebra, and is primarily related to quantum groups. The character $\mathrm{ch}(\Lambda_b)$ is related to the construction of the *dual semi-canonical basis*. Also the character $\mathrm{ch}(Z_b)$ leads to the construction of a basis, that we will call *dual MV basis*.

2 The Shuffle Algebra

Let $\mathcal{F}$ be the free associative algebra generated by the letters $f_1, \ldots, f_n$. Thus, the generic monomials are nothing but finite *words* in this alphabet. The *shuffle product* $\circ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ is defined through its action on pairs of monomials. For u and v words of length k and ℓ respectively,

$$u \circ v := \sum_{\sigma \in \mathfrak{S}_{k+\ell}/\mathfrak{S}_k \times \mathfrak{S}_\ell} \sigma(uv)\,, \tag{1}$$

that is, we sum over all reorderings of the symbols $u_1 \cdots u_k v_1 \cdots v_\ell$, which preserve the ordering of u_i's among themselves, and of v_j's among themselves. For example, $ab \circ cd$ consists of $\binom{4}{2} = 6$ summands

$$ab \circ cd = abcd + acbd + acdb + cabd + cadb + cdab$$

that, if u and v contain the same letters, can get combined, e.g.

$$ab \circ ba = abba + abba + abab + baba + baab + baab = 2(abba + baab) + abab + baba\,.$$

The shuffle algebra $\mathbb{C}[N]$ is the $\circ$-subalgebra generated by $f_1, \ldots, f_N$. Note that it is indeed a non-trivial subalgebra, as not all words can be generated, but only certain combinations.

In our perspective, the generators $f_1, \ldots, f_N$ will corresponds to the co-roots $\alpha_i^\vee$ in the root system of a Lie Algebra. You can consider the example of A_n root systems, by taking $\alpha_i \equiv \alpha_i^\vee = e_i - e_{i+1}$, where e_i are vectors of a canonical basis of $\mathbb{R}^{n+1}$. The Weyl group is the symmetric group, in which transpositions s_i and the reversal operator w_0 (the "bottom" element in the Bruhat lattice), are faithfully represented as

$$s_i = \begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & \ddots & & \\ \textit{i-th row} & & 0 & 1 & \\ & 0 & 1 & 0 & \\ & & & & \ddots \end{pmatrix}; \qquad w_0 = \begin{pmatrix} & & & & 1 \\ & 0 & & 1 & \\ & & & \cdot^{\cdot^{\cdot}} & \\ & & 1 & & \\ & 1 & & 0 & \\ 1 & & & & \end{pmatrix}.$$

The MV polytope corresponding to $\mathrm{ch}(L_b)$ is the convex hull of a collection of terms, corresponding to the set of possible paths. For example, if we have generators $\alpha_1^\vee$ and $\alpha_2^\vee$, the character is $\mathrm{ch}(L_b) = f_1 f_2 \circ f_2 f_1 = abba + abba + abab + baba + baab + baab$.

[I don't get the construction in the generic case. And by the way, $f_1 f_2$ is not in the shuffle algebra!]

The polytope b is then constructed by taking the convex hull of these six terms, where the two generators are represented as south-west and south-east arrows, and words are represented through concatenation of the elementary steps. In particular, from the terms above we obtain a sort of "filled hexagon", that **[in which sense?]** corresponds to the hexagonal diagram shown at the beginning.

3 Quiver Hecke Algebra Modules

The *Khovanov-Lauda-Rouquier*, or *quiver Hecke algebra* R_d has generators

$$\{e_u\}_{u \text{ words of length } d}\,, \quad y_1, \ldots, y_d, \quad \psi_1, \ldots, \psi_d\,.$$

The words u of length d will be taken as basis vectors $|u\rangle$ of a linear space, and the generators e_u are one-dimensional projectors, in particular they satisfy $\sum_u e_u = 1$ and $e_u e_v = \delta_{uv} e_u$. Generators y_j are the analogous of Murphy elements in the symmetric group, in particular they commute among themselves, while the ψ_j's are the analogous of simple transpositions. There are several other relations in the definition of the algebra, but let me skip a precise definition here. What I want to stress is the fact that R_d is $\mathbb{Z}$-graded. This should sound surprising, as it is notoriously hard to give a grading structure to the comparison counterpart, the symmetric group.

More precisely, for M a graded R_d-module,

$$M = \bigoplus_{i \in \mathbb{Z}} M[i] = \bigoplus_{u} \bigoplus_{i \in \mathbb{Z}} e_u M[i] \,, \tag{2}$$

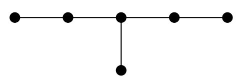
the character is

$$\mathrm{ch}(M) = \sum_{i} \sum_{u} \dim(e_u M[i]) \; q^i \; |u\rangle \,, \tag{3}$$

i.e. a linear combination of words, which is an element of the q-shuffle algebra (an appropriate q-deformation of the shuffle algebra, with $q = 1$ corresponding to the case defined above).

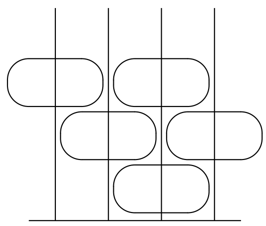
A result contained in a paper by Kleshchev and myself is the classification of the homogeneous simple R_d-modules.

One must consider Dynkin diagrams, such, e.g., E_6

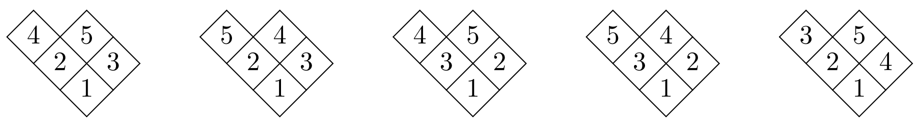


and interpret them as adjacency structures, for the construction of "heaps" (in the case of simply-laced diagrams, while in the non–simply-laced cases a more complicated 'folding' procedure is involved).

For example, for A_4 we could have the heap



The module associated to this heap has dimension 5, which corresponds to the 5 possible 'histories' in the heap construction, conveniently encoded through (rotated) standard Young Tableaux



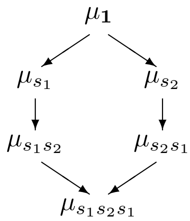
Then, the character $\mathrm{ch}(L_b)$ associated to the module is a linear combination of words associated to these diagrams, where, in the word, the i-th letter is f_j if, in the diagram, the entry i is in the j-th diagonal. In our example,

$$\mathrm{ch}(L_b) = f_3 f_2 f_4 f_1 f_3 + f_3 f_2 f_4 f_3 f_1 + f_3 f_4 f_2 f_1 f_3 + f_3 f_4 f_2 f_3 f_1 + f_3 f_2 f_1 f_4 f_3 \,.$$

Thus, this construction for the character of quiver modules is related to standard Young tableaux. As we will motivate in the following, quiver Hecke algebra modules are related to MV polytopes, that in turns are related to column-strict tableaux. Although apparently similar objects, these two families of tableaux quite seldomly get mixed in natural combinatorial constructions, and the subject at hand is a remarkable exception to this feature.

4 MV polytopes

A MV polytope b is defined as the convex hull of its vertices. The vertices are in bijection with the elements of a Weyl group W_0, $V = \{\mu_w\}_{w \in W_0}$. Recall the $\mathfrak{S}_3$ example we gave at the beginning



Consider a minimal-length path from the top ($\mathbf{1}$) to the bottom (w_0) of the lattice, $w_0 = s_{i_1} s_{i_1} \cdots s_{i_N}$. The *i-perimeter*, or *Lusztig parametrization* of b is the datum of a set of integer lengths along this path,

$$\mathrm{per}_i(b) = (\ell_1, \ell_2, \ldots, \ell_N) \tag{4}$$

with, disregarding for a moment the rest of the diagram,

$$\mu_{\mathbf{1}} \xrightarrow{\ell_1} \mu_{s_{i_1}} \xrightarrow{\ell_2} \mu_{s_{i_1} s_{i_2}} \longrightarrow \cdots \xrightarrow{\ell_N} \mu_{w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}}$$

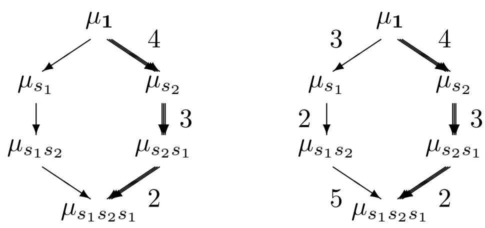
The set of lengths on any other path (that we could call "j-perimeter") is then determined completely, through a construction of "local deformations" whose elementary moves are Coxeter moves $R_{i\ i+1\ i}^{i+1\ i\ i+1}$ (i and $i+1$ are names for adjacent sites on the Dynkin diagram, allusive to the A_n case) and $R_{i\ j}^{j\ i}$ for i, j not adjacent on the Dynkin diagram, corresponding to the relations in the algebra $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$ for i and j not adjacent. The local transformations are

$$R_{i\ j}^{j\ i}(\ell_\alpha, \ell_{\alpha+1}) = (\ell_{\alpha+1}, \ell_\alpha)\,; \tag{5}$$

$$R_{i\ i+1\ i}^{i+1\ i\ i+1}(\ell_\alpha, \ell_{\alpha+1}, \ell_{\alpha+2}) = \big(\ell_\alpha + \ell_{\alpha+1} - \hat{\ell}, \hat{\ell}, \ell_\alpha + \ell_{\alpha+1} - \hat{\ell}\big) \qquad \hat{\ell} = \min(\ell_\alpha, \ell_{\alpha+2})\,. \tag{6}$$

It is a nice exercise to check that indeed also the second operation is an involution. In our example, taking as i-path the right-most one and as perimeter

$(\ell_1, \ldots, \ell_3) = (4, 3, 2)$, we obtain for the only other path the sequence $(3, 2, 5)$:



We are now ready to define completely a MV polytope: it is a polytope, whose vertices are constructed from a Weyl group in the way described above, together with a set of lengths satisfying the relations above.

We can give alternate generators $\tilde{f}_i$, called *crystal operators*, that have a simple increment action on the i-perimeter. If s_{i_a} is used at the $a - th$ step of the i-path construction of w_0, and b has parametrization $(\ell_1, \ell_2, \ldots, \ell_N)$, then the new polytope $\tilde{f}_{i_a} b$ has i-perimeter

$$\mathrm{per}_i(\tilde{f}_{i_a} b) = (\ell_1, \ell_2, \ldots, \ell_{a-1}, \ell_a + 1, \ell_{a+1}, \ldots, \ell_N)\,. \tag{7}$$

Thus, defining the polytope b_+ composed of a single node and no intervals (corresponding to the trivial group containing only the identity), any polytope b can be constructed through an appropriate sequence of crystal operators

$$b = \tilde{f}_{i_1}^{c_1} \tilde{f}_{i_2}^{c_2} \cdots \tilde{f}_{i_N}^{c_N} b_+ \tag{8}$$

[I don't get this... if the operator increases all the lengths of the steps using the same transposition, how do you get different lengths, e.g., for the first and third step in our example? what is the sequence of $\tilde{f}$ that grows our example of $\mathfrak{S}_3$ with $(4, 3, 2)$ starting from b_+ ?]

5 MV cycles

Call $\mathbb{C}((t))$ the set of Laurent formal power series with complex coefficients,

$$\mathbb{C}((t)) = \{a_{-c} t^{-c} + a_{-c+1} t^{-c+1} + \ldots\}_{a_i \in \mathbb{C},\, -c \in \mathbb{Z}}\,. \tag{9}$$

We will then consider a group G of matrices with coefficients in this field. For simplicity just assume that $G = \mathrm{GL}_{n+1}\big(\mathbb{C}((t))\big)$, but our construction would work also for generic Kac-Moody Algebras. Within G, consider the subgroup of lower-triangular matrices with 1's on the diagonal,

$$\mathcal{U}^- = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \right\} \subseteq G$$

with generators

$$y_i(at^j) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & 0 & & \\ i\text{-th row} & & at^j & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

Furthermore, use the shortcut $\mathcal{K}$ for our field $\mathbb{C}((t))$ [2]

Define the diagonal matrices $t_{\lambda^\vee} := \mathrm{diag}(t^{\lambda_i})$, with $\lambda_i \in \mathbb{Z}$.

Clearly, $G/\mathcal{K}$ is the loop Grassmannian, and Cartan or Iwasara decompositions are essential tools to have control over it. The first choice corresponds to describe G as

$$G = \bigsqcup_{\lambda^\vee} \mathcal{K} t_{\lambda^\vee} \mathcal{K}\,, \tag{10}$$

while the second choice gives

$$G = \bigsqcup_{\mu^\vee} \mathcal{U}^- t_{\mu^\vee} \mathcal{K}\,. \tag{11}$$

The *Mirković-Vilonen cycles* of type $\lambda^\vee$ and weight $\mu^\vee$, (i.e., the MV cycles), are the irreducible components Z_b in the space (closure operation is understood)

$$Z_b \in \mathrm{Irr}\big(\mathcal{K} t_{\lambda^\vee} \mathcal{K} \cap \mathcal{U}^- t_{\mu^\vee} \mathcal{K}\big)\,. \tag{12}$$

They are a natural object if one is interested in the question of which cosets are contained in the intersection. In particular, the Z_b's contain the information on their subgroups.

It is a result of Kamnitzer that MV-cycles are indexed by MV-polytopes b. Furthermore, by Baumann and Gaussent we have a parallel between the "growth" construction on the polytope, and an analogous construction on the cycle. If

$$b = \tilde{f}_{i_1}^{c_1} \tilde{f}_{i_2}^{c_2} \cdots \tilde{f}_{i_N}^{c_N} b_+\,, \tag{13}$$

then, a single step of the growth process is expressed in terms of the generators $y_i(at^j)$

$$Z_b = y_{i_1}\big(t^{e_1} \mathbb{C}[t^{-1}]^*_{c_1}\big)\,. \tag{14}$$

[2] **[I actually missed the definition of $\mathcal{K}$]...** after some desperate google search (for "loog grassmannian"), I found:
Sergey Arkhipov; Roman Bezrukavnikov; Victor Ginzburg
Quantum groups, the loop Grassmannian, and the Springer resolution,
J. Amer. Math. Soc. **17** (2004), 595-678.
`http://www.ams.org/journals/jams/2004-17-03/S0894-0347-04-00454-0/`
that, on bottom of page 41, states: — Let $\mathcal{K} = \mathbb{C}((z))$ be the field of formal Laurent power series, and $\mathcal{O} = \mathbb{C}[[z]] \subset \mathcal{K}$ its ring of integers, that is, the ring of formal power series regular at $z = 0$. Write $G^\vee(\mathcal{K})$, resp. $G^\vee(\mathcal{O})$, for the set of $\mathcal{K}$-rational, resp. $\mathcal{O}$-rational, points of $G^\vee$. The coset space $\mathrm{Gr} := G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$ is called the *loop Grassmannian.* — With the hope that this article follows the same notations as the speaker, I will thus assume this in the following.

Here e_j is just the scalar product

$$e_j = \langle \alpha_{i_j}, -c_{j+1}\alpha^\vee_{i_{j+1}} - \cdots - c_N \alpha^\vee_{i_N} \rangle \tag{15}$$

and the space $\mathbb{C}[t^{-1}]^*_c$ is the linear space of polynomials of degree c in t^{-1}, with no constant term. Thus, it has dimension c. In particular, the dimension of a Z_b cycle is calculated in a straightforward way.

Thus, let Z_b be a cycle of dimension d. The analogous of the growth process is called here a *composition series*. It is a choice $(i_1, \ldots, i_d; j_1, \ldots, j_d)$ such that Z_b is the space

$$Z_b = \left\{ y_{i_1}(a_1 t^{j_1}) \cdots y_{i_d}(a_d t^{j_d}) \mathcal{K} \right\}_{a_1, \ldots, a_d \in \mathbb{C}^*} . \tag{16}$$

Then, the character of Z_b is the generating function of the composition series of b

$$\mathrm{ch}(Z_b) = \sum_{(i_1, \ldots, i_d; j_1, \ldots, j_d)} f_{i_1} \cdots f_{i_d} \,, \tag{17}$$

which is again an element in the Shuffle Algebra.